Pseudo-Maximum Likelihood Estimation of ARCH(∞) Models

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Abstract

Strong consistency and asymptotic normality of the Gaussian pseudo-maximum likelihood estimate of the parameters in a wide class of ARCH(∞) processes are established. We require the ARCH weights to decay at least hyperbolically, with a faster rate needed for the central limit theorem than for the law of large numbers. Various rates are illustrated in examples of particular parameterizations in which our conditions are shown to be satisfied.

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1. INTRODUCTION

ARCH(∞) processes comprise a wide class of models for conditional heteroscedasticity in time series. Consider, for $t \in \mathbb{Z} = \{0, \pm 1, ...\}$, the equations

$$x_t = \sigma_t \varepsilon_t, \tag{1}$$

$$\sigma_t^2 = \omega_0 + \sum_{j=1}^{\infty} \psi_{0j} x_{t-j}^2, \tag{2}$$

where

$$\omega_0 > 0, \psi_{0j} > 0 \, (j \geq 1), \ \sum_{j=1}^{\infty} \psi_{0j} < \infty, \tag{3}$$

and $\{\varepsilon_t\}$ is a sequence of independent identically distributed ($i.i.d.$) unobservable real-valued random variables. Any strictly stationary solution x_t to (1) and (2) will be called an ARCH(∞) process. We consider a parametric version, in which we know functions $\psi_j(\zeta)$ of the $r \times 1$ vector ζ, for $r < \infty$, such that for some unknown ζ_0

$$\psi_j(\zeta_0) = \psi_{0j}, \quad j \geq 1. \tag{4}$$

Also, ω_0 is unknown and x_t is unobservable but we observe

$$y_t = \mu_0 + x_t \tag{5}$$

for some unknown μ_0.

ARCH(∞) processes, extending the ARCH(m), $m < \infty$, process of Engle (1982) and the GARCH(n, m) process of Bollerslev (1986), were considered by Robinson (1991) as a class of parametric alternatives in testing for serial independence of y_t. Empirical evidence of Whistler (1990), Ding, Granger and Engle (1993) has suggested the possibility of long memory autocorrelation in the squares of financial data. Taking (contrary to the first requirement in (3)) $\omega_0 = 0$, such long memory in x_t^2 driven by (1) and (2) was considered by Robinson (1991), the ψ_{0j} being the autoregressive weights of a fractionally integrated process, implying $\sum_{j=1}^{\infty} \psi_{0j} = 1$; see also Ding

and Granger (1996). For such ψ_{0j}, and the same objective function as was employed to generate the tests of Robinson (1991), Koulikov (2003) established asymptotic statistical properties of estimates of ζ_0. On the other hand, under our assumption $\omega_0 > 0$, Giraitis, Kokoszka and Leipus (2000) found that such ψ_{0j} are inconsistent with covariance stationarity of x_t, which holds when $\sum_{j=1}^{\infty} \psi_{0j} < 1$. Finite variance of x_t implies summability of coefficients of a linear moving average in martingale differences representation of x_t^2; see Zaffaroni (2004). In this paper we do not assume finite variance of x_t, but rather that x_t has a finite fractional moment of degree less than 2. The first requirement in (3) was shown by Kazakevicius and Leipus (2002) to be necessary for existence of an x_t satisfying (1) and (2). The intermediate requirement in (3) is sufficient but not necessary for a.s. positivity of σ_t^2, and is imposed here to facilitate a clearer focus on ψ_{0j} which decay, possibly slowly, but never vanish.

We wish to estimate the $(r+2) \times 1$ vector $\theta_0 = (\omega_0, \mu_0, \zeta_0')'$ on the basis of observations y_t, $t = 1, ..., T$, the prime denoting transposition. The case when μ_0 is known, e.g. $\mu_0 = 0$, is covered by a simplified version of our treatment. If the y_t are instead unobserved regression errors then $\mu_0 = 0$ but we would need to replace x_t by residuals in what follows; the details of this extension would be relatively straightforward. Another relatively straightforward extension would cover simultaneous estimation of regression parameters, ω_0 and ζ_0, after replacing μ_0 by a more general parametric function; as in (1), (2) and (5), efficiency gain is afforded by simultaneous estimation.

Under stronger restrictions than $\sum_{j=1}^{\infty} \psi_{0j} < 1$, Giraitis and Robinson (2001) considered discrete-frequency Whittle estimation of ζ_0, based on the squared observations y_t^2 (with μ_0 known to be zero), this being asymptotically equivalent to constrained least squares regression of y_t^2 on the y_{t-s}^2, $s > 0$, a method employed in special cases of (2) by Engle (1982) and Bollerslev (1986). In these the spectral density of y_t^2, when it exists, has a convenient closed form. This property, along with availability

of the fast Fourier transform, makes discrete-frequency Whittle based on the y_t^2 a computationally attractive option for point estimation, even in very long financial time series. However it has a number of disadvantages, as discussed by Giraitis and Robinson (2001): it is not only asymptotically inefficient under Gaussian ε_t, but never asymptotically efficient; it requires finiteness of fourth moments of y_t for consistency and of eighth moments for asymptotic normality, which are sometimes considered unacceptable for financial data; its limit covariance matrix is relatively complicated to estimate; it is less well motivated in ARCH models than in stochastic volatility and nonlinear moving average models such as those of Taylor (1986), Robinson and Zaffaroni (1997, 1998), Harvey (1998), Breidt et al (1998), Zaffaroni (2003), where the actual likelihood is computationally relatively intractable, whilst Whittle estimation also plays a less special role in the short-memory-in-y_t^2 ARCH models of Giraitis and Robinson (2001) than in the long-memory-in-y_t^2 models of the previous five references, where it entails automatic 'compensation' for possible lack of square-integrability of the spectrum of y_t^2. Mikosch and Straumann (2002) have shown that a finite fourth moment is necessary for consistency of Whittle estimates, and that convergence rates are slowed by fat tails in ε_t.

For Gaussian ε_t a widely-used approximate maximum likelihood estimate is defined as follows. Denote by $\theta = (\omega, \mu, \zeta')'$ any admissible value of θ_0 and define

$$\begin{aligned} x_t(\mu) &= y_t - \mu, \\ \sigma_t^2(\theta) &= \omega + \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}^2(\mu), \end{aligned}$$

for $t \in \mathbb{Z}$, and

$$\bar{\sigma}_t^2(\theta) = \omega + \sum_{j=1}^{t-1} \psi_j(\zeta) x_{t-j}^2(\mu) 1(t \geq 2),$$

for $t \geq 1$, where $1(.)$ denotes the indicator function. Define also

$$\begin{aligned} q_t(\theta) &= \frac{x_t^2(\mu)}{\sigma_t^2(\theta)} + \ln \sigma_t^2(\theta), \quad \bar{q}_t(\theta) = \frac{x_t^2(\mu)}{\bar{\sigma}_t^2(\theta)} + \ln \bar{\sigma}_t^2(\theta), \quad 1 \leq t \leq T, \\ Q_T(\theta) &= T^{-1} \sum_{t=1}^{T} q_t(\theta), \quad \bar{Q}_T(\theta) = T^{-1} \sum_{t=1}^{T} \bar{q}_t(\theta), \\ \tilde{\theta}_T &= \arg\min_{\theta \in \Theta} Q_T(\theta), \quad \hat{\theta}_T = \arg\min_{\theta \in \Theta} \bar{Q}_T(\theta), \end{aligned}$$

where Θ is a prescribed compact subset of $\mathbb{R}^{r+2}$. The quantities with over-bar are introduced due to y_t being unobservable for $t \leq 0$; $\tilde{\theta}_T$ is uncomputable. Because we do not assume Gaussianity in the asymptotic theory, we refer to $\hat{\theta}_T$ as a pseudo-maximum likelihood estimate (PMLE).

We establish strong consistency of $\hat{\theta}_T$ and asymptotic normality of $T^{\frac{1}{2}}\left(\hat{\theta}_T - \theta_0\right)$, as $T \to \infty$, for a class of $\psi_j(\zeta)$ sequences. In case of the first property this is accomplished by first showing strong consistency of $\tilde{\theta}_T$ and then that $\hat{\theta}_T - \tilde{\theta}_T \to 0$, a.s. In case of the second we likewise first show it for $T^{\frac{1}{2}}\left(\tilde{\theta}_T - \theta_0\right)$ and then show that $\hat{\theta}_T - \tilde{\theta}_T = o_p(T^{-\frac{1}{2}})$, but the latter property, and thus the asymptotic normality of $T^{\frac{1}{2}}(\hat{\theta}_T - \theta_0)$, is achieved only under a restricted set of possible ζ_0 values, and this seems of practical concern in relation to some popular choices of the $\psi_j(\zeta)$. These results are presented in the following section, along with a description of regularity conditions and partial proof details. The structure of the proof is similar in several respects to earlier ones for the GARCH case of (2), especially that of Berkes, Horvath and Kokoszka (2003). Sections 3 and 4 apply the results to particular parametric models.

2. ASSUMPTIONS AND MAIN RESULTS

Our assumptions are as follows.

Assumption $A(q)$, $q \geq 2$. *The* ε_t *are i.i.d. random variables with* $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E\left|\varepsilon_0\right|^q < \infty$ *and probability density function* $f(\varepsilon)$ *satisfying*

$$f(\varepsilon) = O\left(L(\left|\varepsilon\right|^{-1})\left|\varepsilon\right|^b\right), \text{ as } \varepsilon \to 0,$$

for $b > -1$ *and a function* L *that is slowly varying at the origin.*

Assumption B**.** *There exist* $\omega_L, \omega_U, \mu_L, \mu_U$ *such that* $0 < \omega_L < \omega_U < \infty$, $-\infty < \mu_L < \mu_U < \infty$, *and a compact set* $\Upsilon \in R^r$ *such that* $\Theta = [\omega_L, \omega_U] \times [\mu_L, \mu_U] \times \Upsilon$.

Assumption C. θ_0 *is an interior point of* Θ.

Assumption D. *For all* $j \geq 1$,

$$\begin{aligned} \inf_{\zeta \in \Upsilon} \psi_j(\zeta) &> 0; && (6) \\ \sup_{\zeta \in \Upsilon} \psi_j(\zeta) &\leq K j^{-\underline{d}-1}, \text{ for some } \underline{d} > 0; && (7) \\ \psi_{0j} &\leq K\psi_{0k} \text{ for } 1 \leq k \leq j, && (8) \end{aligned}$$

where K *throughout denotes a generic, positive constant.*

Assumption E**.** *There exists a strictly stationary and ergodic solution* x_t *to (1) and (2), and for some*

$$\rho \in ((\underline{d}+1)^{-1}, 1), \tag{9}$$

with $\underline{d}$ *as in Assumption D, we have*

$$E\left|x_0\right|^{2\rho} < \infty. \tag{10}$$

Assumption $F(l)$. *For all* $j \geq 1$, $\psi_j(\zeta)$ *has continuous* k*th derivative on* Υ *such that, with* ζ_i *denoting the* i*-th element of* ζ,

$$\left|\frac{\partial^k \psi_j(\zeta)}{\partial\zeta_{i_1}...\partial\zeta_{i_k}}\right| \leq K\psi_j(\zeta)^{1-\eta} \tag{11}$$

for all $\eta > 0$ *and all* $i_h = 1, ..., r$, $h = 1, ..., k$, $k \leq l$.

Assumption G. *For each* $\zeta \in \Upsilon$ *there exist integers* $j_i = j_i(\zeta)$, $i = 1, ..., r$, *such that* $1 \leq j_1(\zeta) < ... < j_r(\zeta) < \infty$ *and*

$$rank\left\{\Psi_{(j_1,...,j_r)}(\zeta)\right\} = r,$$

where

$$\Psi_{(j_1,...,j_r)}(\zeta) = \left\{\psi_{j_1}^{(1)}(\zeta), ..., \psi_{j_r}^{(1)}(\zeta)\right\}, \quad \psi_j^{(1)}(\zeta) = \frac{\partial \psi_j(\zeta)}{\partial \zeta}.$$

Assumption H**.** *There exists*

$$d_0 > \frac{1}{2} \tag{12}$$

such that

$$\psi_{0j} \leq K j^{-1-d_0}, \tag{13}$$

and (10) holds for

$$\rho \in (4/(2d_0 + 3), 1). \tag{14}$$

Assumption $A(q)$ allows some asymmetry in ε_t, but implies the less primitive condition (which does not even require existence of a density) employed in a similar context by Berkes, Horvath and Kokoszka (2003). Assumptions B and C are standard. The inequalities (7) and (13) together imply $d_0 \geq \underline{d}$, whilst (8) with (3) is milder than monotonicity but implies $\psi_{0j} = o(j^{-1})$ as $j \to \infty$. We take $\eta > 0$ in Assumption $F(l)$ because $\psi_j(\zeta) < 1$ for all large enough j, by (7). Assumption G is crucial to the proof of consistency, being used in Lemmas 9 and 10 to show that, in the limit, θ_0 globally minimizes $Q_T(\theta)$; it also ensures non-singularity of the matrix H_0 in Proposition 2 and Theorem 2 below. This and other assumptions are discussed in Sections 3 and 4 in connection with some parameterizations of interest,

We present asymptotic results for the uncomputable $\tilde{\theta}_T$ as propositions, those for $\hat{\theta}_T$ as theorems. All these, and the Corollaries in Sections 3 and 4 and Lemmas in Section 5, assume (1)-(5).

Proposition 1 *For some $\delta > 0$, let Assumptions $A(2+\delta)$, B, C, D, E, $F(1)$ and G, hold. Then*

$$\tilde{\theta}_T \to \theta_0 \ \ a.s. \ \ as \ T \to \infty.$$

Proof. The proof follows as in, e.g., Jennrich (1969, Theorem 6) from uniform a.s. convergence, over Θ, of $Q_T(\theta)$ to $Q(\theta) = Eq_0(\theta)$ established in Lemma 7, the fact that $Q_T(\hat{\theta}_T) \leq Q_T(\theta)$, and Lemma 10. ■

Theorem 1 *For some $\delta > 0$, let Assumptions $A(2+\delta)$, B, C, D, E, $F(1)$ and G hold. Then*

$$\hat{\theta}_T \to \theta_0 \ a.s. \ as \ T \to \infty. \tag{15}$$

Proof. From Lemmas 7 and 8, $\bar{Q}_T(\theta)$ converges uniformly to $Q(\theta)$ a.s., whence the proof is as indicated for Proposition 1. ■

Denote by κ_j the j-th cumulant of ε_t and introduce

$$G_0 = (2+\kappa_4)M - 2\kappa_3(N+N') + P, \quad H_0 = M + \frac{1}{2}P,$$

where

$$M = E(\tau_0\tau_0'), \quad N = E(\sigma_0^{-1}\tau_0)e_2', \quad P = E(\sigma_0^{-2})e_2e_2',$$

for $\tau_0 = \tau_0(\theta_0)$, $\tau_t(\theta) = (\partial/\partial\theta)\log\sigma_t^2(\theta)$, and e_2 the second column of the $(r+2)\times(r+2)$ identity matrix. In case μ_0 is known (for example, to be zero), we omit the second row and column from M, and have instead $G_0 = (2+\kappa_4)M$, $H_0 = M$. In case ε_t is Gaussian, $\kappa_3 = \kappa_4 = 0$, so $G_0 = 2H_0 = 2M + P$.

Proposition 2 *Let Assumptions $A(4)$, B, C, D, E, $F(3)$ and G hold. Then*

$$T^{\frac{1}{2}}\left(\tilde{\theta}_T - \theta_0\right) \to_d N(0, H_0^{-1}G_0\,H_0^{-1}), \ \ as \ T \to \infty.$$

Proof. Write

$$Q_T^{(1)}(\theta) = \frac{\partial Q_T(\theta)}{\partial\theta} = T^{-1}\sum_{t=1}^{T} u_t(\theta),$$

where

$$u_t(\theta) = \tau_t(\theta)(1 - \chi_t^2(\theta)) + \sigma_t^{-2}(\theta)\nu_t(\theta),$$

with

$$\chi_t(\theta) = \frac{x_t^2(\mu)}{\sigma_t^2(\theta)}, \quad \nu_t(\theta) = \frac{\partial x_t 2(\mu)}{\partial\theta} = -2x_t(\mu)e_2.$$

By the mean value theorem

$$0 = Q_T^{(1)}(\tilde{\theta}_T) = Q_T^{(1)}(\theta_0) + \widetilde{H}_T(\tilde{\theta}_T - \theta_0), \tag{16}$$

where $\widetilde{H}_T$ has as its *ith* row the *ith* row of $H_T(\theta) = T^{-1}\sum_{t=1}^{T} h_t(\theta)$ evaluated at $\theta = \tilde{\theta}_T^{(i)}$, where $h_t(\theta) = (\partial^2/\partial\theta\partial\theta')\, Q_T(\theta)$, $\left\|\tilde{\theta}_T^{(i)} - \theta_0\right\| \leq \left\|\tilde{\theta}_T - \theta_0\right\|$, where we define $\|A\| = \{tr(A'A)\}^{\frac{1}{2}}$ for any real matrix A. Now $u_t(\theta_0) = \tau_t(\theta_0)(1 - \varepsilon_t^2) - 2e_2\varepsilon_t/\sigma_t$ is, by Lemmas 2, 3 and 7, a stationary ergodic martingale difference vector with finite variance, so from Brown (1971) and the Cramer-Wold device, $T^{\frac{1}{2}}Q_T^{(1)}(\theta_0) \to_d N(0, G_0)$ as $T \to \infty$. Finally, by Lemma 7 and Theorem 1, $\widetilde{H}_T \to_p H_0$, whence the proof is completed in standard fashion. ■

Define

$$\begin{aligned}
\bar{u}_t(\theta) &= \frac{\partial \bar{q}_t(\theta)}{\partial\theta}, \bar{g}_t(\theta) = \bar{u}_t(\theta)\bar{u}_t'(\theta), \quad \bar{h}_t(\theta) = \frac{\partial^2 \bar{q}_t(\theta)}{\partial\theta\partial\theta'}, \\
\bar{G}_T(\theta) &= T^{-1}\sum_{t=1}^{T}\bar{g}_t(\theta), \quad \bar{H}_T(\theta) = T^{-1}\sum_{t=1}^{T}\bar{h}_t(\theta).
\end{aligned}$$

Theorem 2 *Let Assumptions A(4), B, C, D, E, F(3), G and H hold. Then*

$$T^{\frac{1}{2}}\left(\hat{\theta}_T - \theta_0\right) \to_d N(0, H_0^{-1}G_0H_0^{-1}) \ \ as\ T \to \infty, \tag{17}$$

and $H_0^{-1}G_0H_0^{-1}$ is strongly consistently estimated by $\bar{H}_T^{-1}(\hat{\theta}_T)\bar{G}_T(\hat{\theta}_T)\bar{H}_T^{-1}(\hat{\theta}_T)$.

Proof. We have

$$0 = \bar{Q}_T^{(1)}(\hat{\theta}_T) = \bar{Q}_T^{(1)}(\theta_0) + \ \widehat{H}_T(\hat{\theta}_T - \theta_0),$$

where $\bar{Q}_T^{(1)}(\theta) = (\partial/\partial\theta)Q_T(\theta)$ and $\hat{H}_T$ has as its *ith* row the *ith* row of $\bar{H}_T(\theta)$ evaluated at $\theta = \hat{\theta}_T^{(i)}$, for $\left\|\hat{\theta}_T^{(i)} - \theta_0\right\| \leq \left\|\hat{\theta}_T - \theta_0\right\|$. Thus from (16)

$$\hat{\theta}_T - \widetilde{\theta}_T = (\widetilde{H}_T^{-1} - \widehat{H}_T^{-1})\bar{Q}_T^{(1)}(\theta_0) - \widetilde{H}_T^{-1}\left\{\bar{Q}_T^{(1)}(\theta_0) - Q_T^{(1)}(\theta_0)\right\},$$

where the inverses exist a.s. for all sufficiently large T by Lemma 9. In view of Proposition 2 and Lemma 8, (17) follows on showing that

$$\bar{Q}_T^{(1)}(\theta_0) - Q_T^{(1)}(\theta_0) = o_p(T^{-1/2}).$$

The left hand side can be written $(B_{1T} + B_{2T} + B_{3T})/T$, where

$$B_{1T} = \sum_{t=1}^{T} \varepsilon_t^2 \, b_{1t}, \; B_{2T} = -\sum_{t=1}^{T} (\varepsilon_t^2 - 1) b_{2t}, \; B_{3T} = -2e_2 \sum_{t=1}^{T} \varepsilon_t b_{3t},$$

with

$$b_{1t} = -\frac{\bar{\sigma}_t^{2(1)}(\sigma_t^2 - \bar{\sigma}_t^2)}{\bar{\sigma}_t^4}, \; b_{2t} = \frac{\sigma_t^{2(1)}}{\sigma_t^2} - \frac{\bar{\sigma}_t^{2(1)}}{\bar{\sigma}_t^2}, \; b_{3t} = \frac{\sigma_t^2 - \bar{\sigma}_t^2}{\bar{\sigma}_t^2 \sigma_t},$$

for $\bar{\sigma}_t^2 = \bar{\sigma}_t^2(\theta_0)$, $\sigma_t^{2(1)} = \sigma_t^{2(1)}(\theta_0)$, $\bar{\sigma}_t^{2(1)} = \bar{\sigma}_t^{2(1)}(\theta_0)$, with $\sigma_t^{2(1)}(\theta) = (\partial/\partial\theta)\sigma_t^2(\theta)$, $\bar{\sigma}_t^{2(1)}(\theta) = (\partial/\partial\theta)\bar{\sigma}_t^2(\theta)$. We show that $B_{iT} = o_p(T^{1/2})$, $i = 1, 2, 3$. For the remainder of this proof we drop the zero subscript in ψ_{0j}.

Consider first B_{1T}. We have

$$\bar{\sigma}_t^{2(1)} = \left(1, -2\sum_{j=1}^{t-1} \psi_j x_{t-j}, \sum_{j=1}^{t-1} \psi_j^{(1)} x_{t-j}^2\right)', \tag{18}$$

where $\psi_j^{(1)} = \psi_j^{(1)}(\zeta_0)$. From Assumption $F(1)$

$$\left\|\bar{\sigma}_t^{2(1)}\right\| \leq 1 + 2\sum_{j=1}^{t-1} \psi_j \left|x_{t-j}\right| + K\sum_{j=1}^{t-1} \psi_j^{1-\eta} x_{t-j}^2,$$

for all $\eta > 0$. Now

$$\sum_{j=1}^{t-1} \psi_j \left|x_{t-j}\right| \leq \left(\sum_{j=1}^{t-1} \psi_j x_{t-j}^2\right)^{1/2} \left(\sum_{j=1}^{\infty} \psi_j\right)^{1/2} \leq K\bar{\sigma}_t,$$

so since $\bar{\sigma}_t \geq \omega_L > 0$

$$\bar{\sigma}_t^{-2} \sum_{j=1}^{t-1} \psi_j \left|x_{t-j}\right| \leq K\bar{\sigma}_t^{-1} < \infty.$$

From (8),

$$\sum_{j=1}^{t-1} \psi_j^{1-\eta} x_{t-j}^2 \leq K\psi_t^{-\eta}\bar{\sigma}_t^2.$$

It follows that

$$\left\|\bar{\sigma}_t^{2(1)}\right\| / \bar{\sigma}_t^2 \leq K\psi_t^{-\eta}. \tag{19}$$

On the other hand, by the c_r-inequality (Loève (1977, p.157)) and (10)

$$E(\sigma_t^2 - \bar{\sigma}_t^2)^\rho \leq K \sum_{j=t}^{\infty} \psi_j^\rho E\left|x_{t-j}\right|^{2\rho} \leq K \sum_{j=t}^{\infty} \psi_j^\rho. \tag{20}$$

Thus by (8) and (14)

$$E\left\|b_{1t}\right\|^\rho \leq K\psi_t^{-\eta\rho} \sum_{j=t}^{\infty} \psi_j^\rho \leq K \sum_{j=t}^{\infty} \psi_j^{\rho(1-\eta)} \leq Kt^{1-\rho(d_0+1)(1-\eta)}, \tag{21}$$

choosing $\eta < 1 - 1/\{\rho(d_0+1)\}$, which (14) enables. Applying the c_r-inequality again,

$$E\|B_{1T}\|^\rho \leq K \sum_{t=1}^{T} E\left|\varepsilon_0\right|^{2\rho} E\left\|b_{1t}\right\|^\rho.$$

Applying (21), this is $O(1)$ when $\rho > 2/(d_0+1)$, whilst when $\rho \leq 2/(d_0+1)$ we may choose η so small to bound it by

$$KT^{2-\rho(d_0+1)(1-\eta)} \leq KT^{\rho/2-\{1+2(d_0+1)(1-\eta)\}[\rho/2-2/\{1+2(d_0+1)(1-\eta)\}]} = o(T^{\rho/2}),$$

using (12) (which requires (13)), and arbitrariness of η. Thus $B_{1T} = o_p(T^{\frac{1}{2}})$ by Markov's inequality.

Consider B_{2T}. By independence of ε_t and b_{2t}, by c_r-inequality when $\rho \leq \frac{1}{2}$, and by the inequality of von Bahr and Esseen (1965) and the fact that the ε_t^2 are i.i.d. with mean 1 when $\rho > \frac{1}{2}$,

$$E\left\|B_{2T}\right\|^{2\rho} \leq K \sum_{t=1}^{T} (E\left|\varepsilon_0\right|^{4\rho} + 1) E\left\|b_{2t}\right\|^{2\rho} \leq K \sum_{t=1}^{T} (E\left\|b_{4t}\right\|^{2\rho} + E\left\|b_{5t}\right\|^{2\rho}),$$

where

$$b_{4t} = \frac{\sigma_t^{2(1)} - \bar{\sigma}_t^{2(1)}}{\sigma_t^2},\ b_{5t} = \frac{\bar{\sigma}_t^{2(1)}(\sigma_t^2 - \bar{\sigma}_t^2)}{\bar{\sigma}_t^2 \sigma_t^2}.$$

Thus, from Assumptions $F(1)$ and H,

$$\begin{aligned}
\|b_{4t}\| &\leq \left(2\sum_{j=t}^{\infty}\psi_j\left|x_{t-j}\right| + \sum_{j=t}^{\infty}\left\|\psi_j^{(1)}\right\| x_{t-j}^2\right)/\sigma_t^2. \\
&\leq \sigma_t^{-2}\left[2\left\{\sum_{j=t}^{\infty}\psi_j\right\}^{1/2} + \left\{\sum_{j=t}^{\infty}\left(\left\|\psi_j^{(1)}\right\|^2/\psi_j\right)x_{t-j}^2\right\}^{1/2}\right]\left\{\sum_{j=t}^{\infty}\psi_j x_{t-j}^2\right\}^{1/2} \\
&\leq K\left\{\left(\sum_{j=t}^{\infty}j^{-d_0-1}\right)^{1/2} + \left(\sum_{j=t}^{\infty}\psi_j^{1-2\eta}x_{t-j}^2\right)^{1/2}\right\} \\
&\leq K\left[t^{-d_0/2} + \left\{\sum_{j=t}^{\infty}j^{-(d_0+1)(1-2\eta)}x_{t-j}^2\right\}^{1/2}\right],
\end{aligned}$$

so

$$E\left\|b_{4t}\right\|^{2\rho} \leq Kt^{-\rho d_0} + K\sum_{j=t}^{\infty}j^{-(d_0+1)\rho(1-2\eta)} \leq Kt^{1-(d_0+1)\rho(1-2\eta)},$$

for sufficiently small η. Thus $\sum_{t=1}^{T}E\left\|b_{4t}\right\|^{2\rho}$ is $O(1)$ for $\rho > 2/(d_0+1)$, whilst for $\rho \leq 2/(d_0+1)$ it is bounded by

$$KT^{2-(d_0+1)\rho(1-2\eta)} \leq KT^{\rho-(d_0+2)\{\rho-2/(d_0+2)\}+2(d_0+1)\rho\eta} = o(T^{\rho})$$

from (14) and arbitrariness of η. Also $\|b_{5t}\| \leq K\left\|\bar{\sigma}_t^{2(1)}/\bar{\sigma}_t^2\right\|(\sigma_t^2 - \bar{\sigma}_t^2)^{1/2}$, so from (19) and (20) we have $E\left\|b_{5t}\right\|^{2\rho} \leq Kt^{1-(d_0+1)\rho(1-2\eta)}$, and proceeding as before

$$\sum_{t=1}^{T}E\left\|b_{5t}\right\|^{2\rho} = o(T^{\rho}),$$

and thence $B_{2T} = o_p(T^{1/2})$.

Next

$$E\left\|B_{3T}\right\|^{2\rho} \leq KE\left|\sum_{t=1}^{T}\varepsilon_t b_{3t}\right|^{2\rho} \leq K\sum_{t=1}^{T}E\left|\varepsilon_0\right|^{2\rho}Eb_{3t}^{2\rho},$$

applying c_r-inequality when $\rho \leq \frac{1}{2}$ and Von Bahr and Esseen (1965) when $\rho > \frac{1}{2}$.

Now $b_{3t} \leq (\sigma_t^2 - \bar{\sigma}_t^2)^{1/2} \bar{\sigma}_t^{-2}$ so from (20),

$$\begin{aligned} E \left\| B_{3T} \right\|^{2\rho} &\leq K \sum_{t=1}^{T} \sum_{j=t}^{\infty} \psi_j^{\rho} \\ &\leq K \left\{ (1(\rho > 2/(d_0+1)) + (\ln T) 1(\rho = 2/(d_0+1)) + T^{2-\rho(d_0+1)} 1(\rho < 2/(d_0+1)) \right\} \\ &= o(T^{\rho}), \end{aligned}$$

much as before. Thence $B_{3T} = o_p(T^{1/2})$.

It remains to consider the last statement of the theorem, which follows on standard application of Propositions 1 and 2, Theorem 1 and Lemmas 7 and 8. ■

In earlier versions of this paper we checked the conditions in case of GARCH(n, m) models, in which the $\psi_j(\zeta)$ decay exponentially, and we allow the possibility that the GARCH coefficients lie in a subspace of dimension less than $m + n$; the details are available from the authors on request. However, the literature on asymptotic theory for estimates of GARCH models is now extensive, recent references including Berkes, Horvath and Kokoszka (2003), Comte and Lieberman (2003), Ling and McAleer (2003) and Francq and Zakoian (2004), along with investigations of the properties of the models themselves, see recently Basrak, Davis and Mikosch (2002), Kazakevicius and Leipus (2002), Mikosch and Starica (2000). We focus instead on alternative models which have received less attention, and for which our theoretical framework is primarily intended. We introduce the generating function

$$\psi(z; \zeta) = \sum_{j=1}^{\infty} \psi_j(\zeta) z^j, \quad |z| \leq 1. \tag{22}$$

3. FRACTIONAL GARCH MODELS

A slowly decaying class of ARCH(∞) weights was considered by Robinson (1991), Ding and Granger (1996), Koulikov (2003), generated by

$$\psi(z;\zeta) = 1 - (1-z)^{\zeta}, \quad 0 < \zeta < 1, \tag{23}$$

where $r = 1$ and formally

$$(1-z)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} z^j, \quad |z| \leq 1,\ d > 0. \tag{24}$$

In these references $\omega_0 = 0$ was assumed in (2), but we assume $\omega_0 > 0$ and generalize (23) as follows. Introduce the functions $a_j = a_j(\zeta)$, $b_j = b_j(\zeta)$ and, for $m \geq 1$, $n \geq 0$, $n + m \geq r$,

$$a(z;\zeta) = \sum_{j=1}^{m} a_j z^j, \quad b(z;\zeta) = 1 - \sum_{j=1}^{n} b_j z^j 1(n \geq 1), \tag{25}$$

and for all $\zeta \in \Upsilon$

$$a_j > 0, \quad j = 1, ..., m; \quad b_j > 0, \quad j = 1, ..., n; \tag{26}$$

$$b(z;\zeta) \neq 0, \quad |z| \leq 1; \tag{27}$$

$$a(z;\zeta) \text{ and } b(z;\zeta) \text{ have no common zeros in } z. \tag{28}$$

Now take $\psi(z;\zeta)$ (22) to be given by

$$\psi(z;\zeta) = \frac{a(z;\zeta)\{1-(1-z)^d\}}{z b(z;\zeta)}, \tag{29}$$

with $d = d(\zeta)$ satisfying

$$d \in (0,1). \tag{30}$$

We call x_t based on (29) a fractional GARCH, FGARCH(n, d_0, m), process, for $d_0 = d(\zeta_0)$.

Corollary 1 *Let $\psi(z;\zeta)$ be given by (29) and (25) with $m \geq 1$, $n \geq 0$, and let d and the a_j, b_j be continuously differentiable. For some $\delta > 0$ let Assumptions $A(2+\delta)$, B, C and E hold, with all $\zeta \in \Upsilon$ satisfying (26)-(28), (30) and*

$$rank\left\{\frac{\partial}{\partial\zeta}\left(a_1,...,a_m,b_1,...,b_n,d\right)\right\} = r.$$

Then (15) is true. Let also d and the a_j, b_j be thrice continuously differentiable. Assumption A(4) hold, and assume $d_0 > \frac{1}{2}$ and that (10) holds under (14). Then (17) is true.

Proof. Denoting by c_j $(j \geq 1)$ and d_j $(j \geq 0)$ the coefficients of z^j in the expansions of $a(z;\zeta)/b(z;\zeta)$, $z^{-1}\{1-(1-z)^d\}$ respectively, we have $\psi_j(\zeta) = \sum_{k=0}^{j-1} c_{j-k}d_k$, $j \geq 1$. From Berkes, Horvath and Kokoszka (2003) the c_j are bounded above and below by positive, exponentially decaying sequences when $n \geq 1$, and are all non-negative when $n = 0$. Since the d_j are all positive, it follows that (6) holds. Also, Stirling's approximation indicates that $j^{-d-1}/K \leq d_j \leq Kj^{-d-1}$, so the $\psi_j(\zeta)$ satisfy the same inequalities. Compactness of Υ, smoothness of d and (30) imply $d(\zeta) \geq \underline{d}$, to check (7). The above argument indicates that $\psi_{0j} \leq Kj^{-d_0-1} \leq Kk^{-d_0-1} \leq K\psi_{0k}$ for $j > k \geq 1$, so (8) holds, and thus Assumption D. With regard to (11), note that $(\partial/\partial d)\psi(z;\zeta) = -\{a(z;\zeta)/b(z;\zeta)\}z^{-1}(1-z)^d\ln(1-z)$, where the coefficient of z^j in $-z^{-1}(1-z)^d\ln(1-z)$ is $\sum_{k=1}^{j} k^{-1}d_{j-k} \leq K(\ln j)j^{-d-1} \leq Kj^{-(d+1)(1-\eta)} \leq K\psi_j^{1-\eta}(\zeta)$ for any $\eta > 0$. Derivatives with respect to the a_j, b_j are dominated, and higher derivatives can be dealt with similarly, to complete the checking of Assumption $F(l)$. To check Assumption G, suppress reference to ζ in a, b, ψ and

$$\phi(z) = b(z)^{-1}\{1-(1-z)^d\}, \quad \gamma(z) = b(z)^{-1}a(z),$$

and note that

$$\begin{aligned}\frac{\partial\psi(z)}{\partial a_j} &= z^{j-1}\phi(z), \quad j=1,...,m,\\ \frac{\partial\psi(z)}{\partial b_j} &= z^{j-1}\gamma(z)\phi(z), \quad j=1,...,n,\\ \frac{\partial\psi(z)}{\partial d} &= -\frac{\gamma(z)}{z}(1-z)^d\log(1-z).\end{aligned}$$

Choose $j_i(\zeta)=i$ for $i=1,...,m+n$, $\zeta\in\Upsilon$, leaving $j_{m+n+1}(\zeta)$ to be determined subsequently. Fix ζ and write $U=\Psi_{(j_i,...,j_r)}(\zeta)$, partitioning it in the ratio $m+n:1$ and calling its (i,j)th submatrix U_{ij}. We first show that the $(m+n)\times(m+n)$ matrix U_{11} is non-singular. Write R for the $n\times(m+n)$ matrix with (i,j)th element γ_{j-i}, and S for the $(m+n)\times(m+n)$ matrix with (i,j)th element ϕ_{j-i+1}, where $\phi_j=\gamma_j=0$ for $j\le 0$, and for $j>0$, ϕ_j and γ_j are respectively given by

$$\phi(z)=\sum_{j=1}^{\infty}\phi_j z^j, \quad \gamma(z)=\sum_{j=1}^{\infty}\gamma_j z^j,$$

these series converging absolutely for $|z|\le 1$ in view of (30). Noting that $\psi_j^{(1)}$ is given by $(\partial/\partial\zeta)\psi(z)=\sum_{j=1}^{\infty}\psi_j^{(1)}z^j$, we find that the first m rows of U_{11} can be written $(I_m,O)S$, where I_m is the m-rowed identity matrix, O is the $m\times n$ matrix of zeroes, and, when $n\ge 1$, the last n rows of U_{11} can be written RS. Now S is upper-triangular with non-zero diagonal elements. Thus for $n=0$, $U_{11}=S$ is non-singular. For $n\ge 1$, U_{11} is non-singular if and only if the $n\times n$ matrix, R_2, having (i,j)th element γ_{m+j-i} and consisting of the last n columns of R, is non-singular. This is not so if and only if the γ_j, $j=m,...,m+n-1$, are generated by a homogeneous linear difference equation of degree $n-1$, that is if there exist scalars $\lambda_0,\lambda_1,...,\lambda_{n-1}$, not all zero, such that

$$\lambda_0\gamma_j-\sum_{i=1}^{n-1}\lambda_i\gamma_{j-i}=0, \qquad j=m,...,m+n-1.$$

But it follows from (25) and (27) that they are generated by the linear difference

equation

$$\gamma_j - \sum_{i=1}^{n-1} b_i \gamma_{j-i} = \pi_j, \qquad j = m, ..., m+n-1,$$

where $\pi_m = a_m + b_n \gamma_{m-n}$, $\pi_j = b_n \gamma_{j-n}$ for $j = m+1, ..., m+n-1$. Since $b_n \neq 0$ the π_j are all zero if and only if $\gamma_{m-n} = -a_m/b_n$ and $\gamma_j = 0$ for $j = m+1-n, ..., m-1$. But this implies $\gamma_m = 0$ also, and thence $\gamma_j = 0$, all $j \geq m-n+1$. For $m \leq n$ this is inconsistent with the requirement $a_j > 0$, $j = 1, ..., m$, and for $m > n$ it implies a has a factor b, which is inconsistent with (28). Thus U_{11} is non-singular when $n \geq 1$. Non-singularity of U follows if $U_{22} \neq U_{21} U_{11}^{-1} U_{12}$. For large enough $j_{m+n+1} = j_{m+n+1}(\zeta)$ this must be true because U_{22} decays like $(\ln j_{m+n+1}) j_{m+n+1}^{-d-1}$, whereas the elements of U_{12} are $O\left(\beta^{j_{m+n+1}}\right)$ for some $\beta \in (0,1)$. This Assumption G is true, and thence (13). Clearly (13) is true so under the additional conditions so is Assumption H, and thence (17). ■

For $m = 1$, $n = 0$, (29) reduces to (23) when $a_1 = 1$, whilst when $a_1 \in (0,1)$ it gives model (4.24) of Ding and Granger (1996). The important difference between these two cases is that the covariance stationarity condition $\psi(1; \zeta_0) < 1$ is satisfied in the second but not in the first. In general with (29), as with the GARCH model, x_t is covariance stationary when $a(1; \zeta_0) < b(1; \zeta_0)$ but not otherwise. We compare (29) with

$$\psi(z; \zeta) = 1 - \frac{\{1 - a(z; \zeta)\}}{b(z; \zeta)} (1-z)^d, \tag{31}$$

with d again satisfying (30) and a and b again given as in (25), though we now allow $m = 0$, meaning $a(z; \zeta) \equiv 0$. Thus with $m = n = 0$, (31) reduces to (23). ARCH(∞) models with ψ given by (31) were proposed by Baillie, Bollerslev and Mikkelsen (1996), and called FIGARCH(n, d_0, m). In general, though (31) also gives hyperbolically decaying ψ_{0j}, it differs in some notable respects. Application of (26)-(28) ensures positivity of $\psi_j(\zeta)$ in case of FGARCH and facilitates the above proof, but sufficient conditions in FIGARCH are less apparent in general, though Baillie,

Bollerslev and Mikkelsen (1996) indicated that they can be obtained. Also, unlike FGARCH, FIGARCH x_t never has finite variance.

The requirement $d_0 > \frac{1}{2}$ for the central limit theorem in Corollary 2 would also be imposed in a corresponding result for FIGARCH. This is automatically satisfied in GARCH(n, m) models but if only $d_0 \in (0, \frac{1}{2}]$ in (13) is possible in the general setting of Section 3, it appears that the asymptotic bias in $\hat{\theta}_T$ is of order at least $T^{-\frac{1}{2}}$, whereas that for $\tilde{\theta}_T$ is always $o(T^{-\frac{1}{2}})$. Assumption G copes with the replacement of $\sigma_t^2(\theta)$ by $\bar{\sigma}_t^2(\theta)$, the truncation error varying inversely with d_0. Inspection of the proof of Theorem 2 indicates that this bias problem is due to the term $H^{-1}B_{1T}$. The factor $\sigma_t^2 - \bar{\sigma}_t^2$ in b_{1t} is non-negative, and if j^{-d_0-1} is an exact rate for ψ_{0j}, $\sigma_t^2 - \bar{\sigma}_t^2$ exceeds t^{-d_0}/K as $t \to \infty$ with probability approaching one. So far as the factor $\bar{\sigma}_t^{2(1)}/\sigma_t^4$ in b_{1t} is concerned, the second element of $\bar{\sigma}^{2(1)}$ (see (18)) has zero mean, but the first is positive, and though the $\psi_j^{(1)}$ can have elements of either sign, whenever $d_0 \leq \frac{1}{2}$ it seems unlikely that the last r elements of B_{1T} can be $o_p(T^{\frac{1}{2}})$. Nor is there scope for relaxing (12) by strengthening other conditions. With regard to implications for choice of ρ, when $d_0 \geq 2\underline{d} + \frac{1}{2}$, (14) entails no restriction over (9).

Though results of Giraitis, Kokoszka and Leipus (2000) indicate existence of a stationary solution of (1)-(3) when $\psi(1; \zeta_0) < 1$, Kazakevicius and Leipus (2003) have questioned the existence of strictly stationary FIGARCH processes, and thus about the relevance of Assumption E here. The same reservations can be expressed about FGARCH when $a(1; \zeta_0) \geq b(1; \zeta_0)$, and more generally about ARCH(∞) processes with $\psi(1; \zeta_0) \geq 1$. A sufficient condition for (10) can be deduced as follows. Recursive substitution gives

$$\sigma_t^2 \leq K + K \sum_{l=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \psi_{0j_1} \cdots \psi_{0j_l} \varepsilon_{t-j_1}^2 \varepsilon_{t-j_1-j_2}^2 \cdots \varepsilon_{t-j_1 \cdots - j_l}^2 \right),$$

so by the c_r-inequality

$$\sigma_t^{2\rho} \leq K + K \sum_{l=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \psi_{0j_1}^{\rho} \cdots \psi_{0j_l}^{\rho} \left|\varepsilon_{t-j_1}\right|^{2\rho} \left|\varepsilon_{t-j_1-j_2}\right|^{2\rho} \cdots \left|\varepsilon_{t-j_1\cdots-j_l}\right|^{2\rho} \right).$$

Thus from Lemma 2

$$E\left|x_t\right|^{2\rho} < E\left|\sigma_t\right|^{2\rho} \leq K + K \sum_{l=0}^{\infty} \left(E\left|\varepsilon_0\right|^{2\rho} \sum_{j=1}^{\infty} \psi_{0j}^{\rho} \right)^l.$$

The last bound is finite if and only if

$$E\left|\varepsilon_0\right|^{2\rho} \sum_{j=1}^{\infty} \psi_{0j}^{\rho} < 1. \tag{32}$$

Thus (10) holds if there is a ρ satisfying (9) and (32). Recursive substitution, and the c_r-inequality, were also used by Nelson (1990, Corollary) to upper-bound $E\left|\sigma_t\right|^{2\rho}$ in the GARCH$(1,1)$ case, but he employed the simple dynamic structure available there, and (35) does not reduce to his necessary and sufficient condition.

If $\psi(1;\zeta_0) < 1$, (32) adds nothing because we already know that $Ex_0^2 < \infty$ here, but if $\psi(1;\zeta_0) \geq 1$ the second factor on the left of (32) exceeds 1 and increases with ρ, and the question is either the first factor, which is less than 1 and decreases with ρ (due to $A(q)$), can over-compensate. Analytic verification of (32) for given ζ_0, ρ seems in general infeasible, and numerical verification highly problematic when the ψ_j decay slowly. However, consider the family of densities

$$f(\varepsilon) = \exp\left[-\left\{\alpha(\gamma)\left|\varepsilon\right|\right\}^{1/\gamma} \right] / \left\{2\gamma\Gamma(\gamma)\alpha(\gamma)\right\} \tag{33}$$

for $\gamma > 0$, where $\alpha(\gamma) = \left\{\Gamma(\gamma)/\Gamma(3\gamma)\right\}^{\frac{1}{2}}$ (also used by Nelson (1991) to model the innovation of the exponential GARCH model). We have $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$ as necessary, Assumption $A(q)$ is satisfied for all $q > 0$, and $E\left|\varepsilon_0\right|^{2\rho} = \Gamma((2\rho + 1)\gamma)/\{\Gamma(\gamma)^{1-\rho}\Gamma(3\gamma)^{\rho}\}$. In case $\gamma = 0.5$, (33) is the normal density, for which $\hat{\theta}_T$ is asymptotically efficient. Here $E\left|\varepsilon_0\right|^{2\rho} = 2^{\rho}\Gamma(\rho + .5)/\sqrt{\pi}$, and numerical calculations

for FIGARCH$(0, d_0, 0)$ cast doubt on (32). In case $\gamma = 1$, (33) is the Laplace density, with $E\left|\varepsilon_0\right|^{2\rho} = 2^{\rho-1}\Gamma(2\rho+1)$. As γ increases, $E\left|\varepsilon_0\right|^{2\rho}$ can be made small for fixed $\rho < 1$, for example with $\rho = 0.95$ it is 0.64 when $\gamma = 10$ and 0.42 when $\gamma = 20$.

4. GENERALIZED EXPONENTIAL AND HYPERBOLIC MODELS

FGARCH(n, d_0, m) (and FIGARCH(n, d_0, m)) processes require $d_0 \in (0, 1)$. For $d = 1$, (29) reduces to the GARCH(n, m) case $\psi(z;\zeta) = a(z;\zeta)/b(z;\zeta)$, and for $d > 1$ at least one coefficient in the expansion of (23) is negative, leading to the possibility of negative $\psi_j(\zeta)$. Because FGARCH $\psi_j(\zeta)$ decay like j^{-d-1}, a large mathematical gap is left relative to GARCH processes. Even if exponential decay is anticipated, there is a case for more direct modelling of the $\psi_j(\zeta)$ than provided by GARCH, since it is the $\psi_j(\zeta)$ and their derivatives that must be formed in point and interval estimation based on the PMLE.

Consider the choices

$$\psi_j(\zeta) = \sum_{i=1}^{m} \Gamma(f_i+1)^{-1} e_i d^{f_i+1} j^{f_i} e^{-dj}, \tag{34}$$

$$\psi_j(\zeta) = \sum_{i=1}^{m} \Gamma(f_i+1)^{-1} e_i d \ln^{f_i}(j+1)(j+1)^{-d-1}, \tag{35}$$

where $d = d(\zeta)$ and the $e_i = e_i(\zeta)$, $f_i = f_i(\zeta)$ are such that Υ satisfies

$$d \in (0, \infty), \tag{36}$$

$$e_i > 0, \quad i = 1, ..., m, \tag{37}$$

$$0 \leq f_1 \leq \cdots \leq f_m < \infty, \tag{38}$$

with $2m + 1 \geq r$. Given (1)-(4) and (22), we call x_t generated by (34) a generalized exponential, GEXP(m), process, and x_t generated by (35) a generalized hyperbolic,

GHYP(m), process. Condition (38) is sufficient but not necessary for $\psi_j(\zeta) > 0$, all $j \geq 1$. By choosing m large enough in (34) or (35), any finite $\psi(1;\zeta)$ can be arbitrarily well approximated, but (34) and (35) can also achieve parsimony. For real $x \geq 1$, $x^f e^{-dx}$ and $(\ln x)^f x^{-d-1}$ decay monotonically if $f = 0$, and for $f > 0$ have single maxima at f/d and $e^{f/(d+1)}$respectively. Thus with $m = 1$ and $f_1 = 0$ we have monotonic decay in (34) and (35); otherwise both can exhibit lack of monotonicity, whilst eventually decaying exponentially or hyperbolically. The scale factors in (34) and (35) are so expressed because $x^f e^{-dx}$ and $(\ln x)^f x^{-d-1}$ integrate, over $(0,\infty)$, to $\Gamma(f+1)/d^{f+1}$ and $\Gamma(f+1)/d$, respectively, so that $\psi(1;\zeta) \doteq \sum_{i=1}^{m} e_i$ in both cases, but the approximation may not be very close and the "integrated" case is less easy to distinguish than in GARCH and FGARCH models (though it would be possible to alternatively scale the weights, by infinite sums, to achieve equality).

The following Corollary covers (34) and (35) simultaneously, and implies the special case when the f_i are specified *a priori*, for example to be non-negative integers; strictly speaking, when the true value of f_1 is unknown, Assumption C prevents it from being zero.

Corollary 2 *Let $\psi(z;\zeta)$ be given by (22) and (34) or (35) with $m \geq 1$ and let d and the e_i, f_i be continuously differentiable. For some $\delta > 0$ let Assumptions $A(2+\delta)$, B, C and E hold, with all $\zeta \in \Upsilon$ satisfying (36)-(38) and*

$$rank\left\{\frac{\partial}{\partial\zeta}\left(e_1, f_1, ..., e_m, f_m, d\right)\right\} = r.$$

Then (15) is true. Let also d and the e_i, f_i be thrice continuously differentiable and Assumption $A(4)$ hold, and assume $d_0 = d(\zeta_0) > \frac{1}{2}$ in case of (35) and that (10) holds under (14). Then (17) is true.

Proof. Given (36)-(38) and the proof of Corollary 1, the verification of Assumptions D and $F(l)$ is straightforward. We check Assumption G for (35) only, a very similar

type of proof holding for (34). We have

$$\psi_j^{(1)} = \left[\begin{array}{c} E\left(u'_{1j}, ..., u'_{mj}\right)' \\ v_j \end{array} \right] d(j+1)^{-d-1},$$

where

$$\begin{aligned} u_{ij} &= (\ln\ln(j+1) - (\partial/\partial f_i)\ln\Gamma(f_{i+1}), 1)' \ln^{f_i}(j+1), \quad i = 1, ..., r, \\ v_j &= -\sum_{i=1}^{m} e_i \Gamma(f_{i+1})^{-1} \ln^{f_i+1}(j+1), \end{aligned}$$

and E is the diagonal matrix whose $(2i-1)$th diagonal element is e_i, and whose even diagonal elements are all 1. Fixing ζ, we show first that the leading $(r-1) \times (r-1)$ submatrix of $\Psi_{(j_1,...,j_r)}(\zeta)$ has full rank, equivalently that U_m has full rank, where, for $i = 1, ..., m$ the $(2i) \times (2i)$ matrix U_i has (k, ℓ)th 2×1 sub-vector u_{kj_ℓ}, $k = 1, ..., i$, $\ell = 1, ..., 2i$. Suppose, for some $i = 1, ..., m-1$ and given $j_1, ..., j_{2i}$, that U_i has full rank, and partition the rows and columns of U_{i+1} in the ratio $2i$: 2, calling its (k, ℓ)th submatrix $U_{k\ell}$ (so $U_{11} = U_i$). Take $j_{2i+2} = j_{2i+1}^2$. Because $\ln\ln x$ strictly increases in $x > 1$, it follows that U_{22} is nonsingular and $\left\|U_{22}^{-1}\right\| = O\left(\ln\ln j_{2i+1} \ln^{-f_{i+1}} j_{2i+1}\right)$. Noting that $\|U_{12}\| = O\left(\ln\ln j_{2i+1} \ln^{f_i} j_{2i+1}\right)$, whilst U_{11} and U_{21} depend only on $j_1, ..., j_{2i}$, we can choose j_{2i+1} such that $U_{11} - U_{12}U_{22}^{-1}U_{21}$ differs negligibly from U_{11}. Thus U_{i+1} has full rank. Since, for $f_1 \geq 0$, U_1 has full rank (for example when $j_1 = 1$, $j_2 = 2$), it follows by induction that U_m has full rank. Since v_j is dominated by a term of order $\ln^{f_m+1} j$, whilst $\|u_{ij}\| = O\left(\ln\ln j \ln^{f_i} j\right)$, a similar argument shows that j_r can then be chosen large enough, to complete verification of Assumption G, and thence (15). We conclude (17) as in the proof of Corollary 1. ■

5. TECHNICAL LEMMAS

Define

$$\sigma_t^{*2}(\theta) = \omega + \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}^2, \quad \sigma_t^{*2} = \omega_U + \sum_{j=1}^{\infty} \sup_{\zeta \in \Upsilon} \psi_j(\zeta) x_{t-j}^2.$$

Lemma 1 *Under Assumptions B and D, for all* $\theta \in \Theta$, $t \in \mathbb{Z}$

$$K^{-1}\sigma_t^{*2}(\theta) \leq \sigma_t^2(\theta) \leq K\sigma_t^{*2}(\theta) \ \ a.s.$$

Proof. A simple extension of Lee and Hansen (1994, Lemma 1). ■

Lemma 2 *Under Assumptions A(2), B, C, D and E, for all* $t \in \mathbb{Z}$

$$\begin{aligned} E\left|x_t\right|^{2\rho} &< E\sigma_t^{2\rho} \leq E\sup_{\theta\in\Theta}\sigma_t^{2\rho}(\theta) \leq KE\sigma_t^{*2\rho} \leq KE\left|x_t\right|^{2\rho} \leq K \qquad (39) \\ \inf_{\theta\in\Theta}\sigma_t^2(\theta) &> 0, \quad \sup_{\theta\in\Theta}\sigma_t^2(\theta) < K\sigma_t^{*2} < \infty \ \ a.s., \qquad (40) \\ E\sup_{\theta\in\Theta}\left|\ln\sigma_t^2(\theta)\right| &\leq K. \qquad (41) \end{aligned}$$

Proof. With respect to (39), the first inequality follows from Jensen's inequality, the second is obvious, the third follows from Lemma 1, the fourth follows from the c_r-inequality, (7) and (9), whilst the last one is due to (10). The proof of (40) uses Lemma 1, $\sigma_t^2(\theta) \geq \omega_L$, (10) and Loève (1977, p.121). To prove (41), $|\ln x| \leq x + x^{-1}$ for $x > 0$ and Lemma 2 give

$$E\sup_{\theta\in\Theta}\left|\ln\sigma_t^2(\theta)\right| \leq \rho^{-1}E\sup_{\theta\in\Theta}\sigma_t^{2\rho}(\theta) + E\left\{\inf_{\theta\in\Theta}\sigma_t^2(\theta)\right\}^{-1} \leq K.$$

■

Lemma 3 *Under Assumptions D, E and F(l), for all* $\theta \in \Theta$, $\sigma_t^2(\theta)$, $q_t(\theta)$ *and their first* l *derivatives are strictly stationary and ergodic.*

Proof. Follows straightforwardly from the assumptions. ■

Lemma 4 *Under Assumption A(2), for positive integer* $k < (b+1)n/2$,

$$E\left(\sum_{t=1}^{n}\varepsilon_t^2\right)^{-k} < \infty. \tag{42}$$

Proof. Denote by $M_X(t) = E(e^{tX})$ the moment-generating function of a random variable X. By Cressie *et al* (1981) the left side of (42) is proportional to

$$\begin{aligned}\int_0^\infty t^{k-1}M_{\sum\varepsilon_t^2}(-t)dt &= \int_0^\infty t^{k-1}M_{\varepsilon_0^2}^n(-t)dt \\ &\leq \int_0^1 t^{k-1}dt + \int_1^\infty t^{k-1}M_{\varepsilon_0^2}^n(-t)dt. \end{aligned}\tag{43}$$

It suffices to show that the last integral is bounded. For all $\delta > 0$, there exists $\eta > 0$ such that $L(\varepsilon^{-1}) \leq \varepsilon^{-\delta}$, $\varepsilon \in (0,\eta)$, so

$$M_{\varepsilon_0^2}(-t) = \int_{-\infty}^{\infty} e^{-t\varepsilon^2}f(\varepsilon)d\varepsilon \leq K\int_0^\eta e^{-t\varepsilon^2}\varepsilon^{b-\delta}d\varepsilon + 2e^{-t\eta^2}.$$

The last integral is bounded by

$$Kt^{(\delta-b-1)/2}\int_0^\infty e^{-\varepsilon}\varepsilon^{(\delta-b-1)/2}d\varepsilon \leq Kt^{(\delta-b-1)/2}.$$

Thus (43) is finite if $k+n(\delta-b-1)/2 < 0$, that is, since δ is arbitrary, if $k < (b+1)n/2$.

■

The previous version of the paper included a longer, independently obtained, proof of the following lemma which we have been able to shorten in one respect by using an idea of Berkes, Horvath and Kokoszka (2003) in a corresponding lemma covering the GARCH(n,m) case.

Lemma 5 *Under Assumptions* $A(q)$, B, C and D, for $p < q/2$,

$$E\sup_{\theta\in\Theta}\left(\frac{\sigma_t^2}{\sigma_t^2(\theta)}\right)^p \leq K < \infty.$$

Proof. We have

$$\sigma_t^2 = \omega_0 + \psi_{01} x_{t-1}^2 + \sum_{j=2}^{\infty} \psi_{0j} x_{t-j}^2 \leq \omega_0 + \psi_{01} \sigma_{t-1}^2 \varepsilon_{t-1}^2 + K\sigma_{t-1}^2$$

from (8). Thus $\sigma_t^2/\sigma_{t-1}^2 \leq K\left(1+\varepsilon_{t-1}^2\right)$ and thence, for fixed $j \geq 1$, $\sigma_t^2/\sigma_{t-j}^2 \leq K h_{tj}$, where $h_{tj} = \Pi_{i=1}^{j}\left(1+\varepsilon_{t-i}^2\right)$. For any $M < \infty$

$$\begin{aligned} \frac{\sigma_t^2}{\sigma_t^2(\theta)} &\leq \frac{K\sigma_t^2}{\sigma_t^{*2}(\theta)} \leq K\left(\frac{\omega}{\sigma_t^2} + \sum_{j=1}^{M} \psi_j(\zeta)\,\varepsilon_{t-j}^2\,\frac{\sigma_{t-j}^2}{\sigma_t^2}\right)^{-1} \\ &\leq \frac{K h_{tM} / \left\{\inf_{\zeta\in\Upsilon} \inf_{j=1,\ldots,M} \psi_j(\zeta)\right\}}{\sum_{j=1}^{M} \varepsilon_{t-j}^2}. \end{aligned}$$

The proof can now be completed much as in the proof of Lemma 5.1 of Berkes, Horvath and Kokoszka (2003), using Hölder's inequality as there but employing our Lemma 4 and taking $M > 2pq/[(b+1)(q-2p)]$. ■

Lemma 6 *Under Assumptions A(2), B, C, D, E and F(l), for all $p > 0$ and $k \leq l$,*

$$E \sup_{\theta\in\Theta} \left| \frac{1}{\sigma_t^2(\theta)} \frac{\partial^k \sigma_t^2(\theta)}{\partial\theta_{i_1}\ldots\partial\theta_{i_k}} \right|^p < \infty, \tag{44}$$

$$E \sup_{\theta\in\Theta} \left| \frac{1}{\bar{\sigma}_t^2(\theta)} \frac{\partial^k \bar{\sigma}_t^2(\theta)}{\partial\theta_{i_1}\ldots\partial\theta_{i_k}} \right|^p < \infty. \tag{45}$$

Proof. Take $i_1 \leq i_2 \leq \ldots \leq i_k$. First assume $i_1 \geq 3$, whence, for given k and $i_1, \ldots i_k$

$$\frac{\partial^k \sigma_t^2(\theta)}{\partial\theta_{i_1}\ldots\partial\theta_{i_k}} = \sum_{j=1}^{\infty} \xi_j(\zeta) x_{t-j}^2(\mu),$$

where $\xi_j(\zeta) = \partial^k \psi_j(\zeta)/\partial\zeta_{i_1-2}\ldots\partial\zeta_{i_k-2}$. Now

$$\left| \sum_{j=1}^{\infty} \xi_j(\zeta) x_{t-j}^2(\mu) \right| \leq 2 \sum_{j=1}^{\infty} \left|\xi_j(\zeta)\right| (x_{t-j}^2 + \mu^2),$$

so using Lemma 1

$$\left|\frac{1}{\sigma_t^2(\theta)}\frac{\partial^s\sigma_t^2(\theta)}{\partial\theta_{i_1}...\partial\theta_{i_k}}\right| \leq \frac{2\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|x_{t-j}^2}{\sigma_t^{*2}(\theta)} + K\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|.$$

It suffices to take $p > 1$. By Hölder's inequality

$$\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|x_{t-j}^2 \leq \left\{\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|^{p/\rho}\psi_j(\zeta)^{1-p/\rho}x_{t-j}^2\right\}^{\rho/p}\left\{\sum_{j=1}^{\infty}\psi_j(\zeta)x_{t-j}^2\right\}^{1-\rho/p},$$

so

$$\left\{\frac{\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|x_{t-j}^2}{\sigma_t^{*2}(\theta)}\right\}^p \leq K\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|^p\psi_j(\zeta)^{\rho-p}\left|x_{t-j}\right|^{2\rho}.$$

By Assumption $F(l)$, for all $\eta > 0$,

$$\sup_{\zeta\in\Upsilon}\left|\xi_j(\zeta)\right|^p\psi_j(\zeta)^{\rho-p} \leq K\sup_{\zeta\in\Upsilon}\psi_j(\zeta)^{\rho-\eta p} \leq Kj^{-(\underline{d}+1)(\rho-\eta p)}.$$

Since $\rho(\underline{d}+1) > 1$, we may choose η such that $(\underline{d}+1)(\rho-p\eta) > 1$. Thus

$$E\sup_{\theta\in\Theta}\left\{\frac{\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|x_{t-j}^2}{\sigma_t^{*2}(\theta)}\right\}^p < \infty.$$

The above proof implies that also

$$\sup_{\zeta\in\Upsilon}\left\{\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|\right\}^p < \infty,$$

whence the proof of (44) with $i_1 \geq 3$ is concluded. Next take $i_1 = 2$. If $i_2 > 2$

$$\frac{\partial^k\sigma_t^2(\theta)}{\partial\theta_{i_1}...\partial\theta_{i_k}} = -2\sum_{j=1}^{\infty}\xi_j(\zeta)x_{t-j}(\mu), \tag{46}$$

where now $\xi_j(\zeta) = \partial^{k-1}\psi_j(\zeta)/\partial\zeta_{i_2-2}...\partial\zeta_{i_k-2}$, whilst if $i_2 = 2$, $i_3 > 2$

$$\frac{\partial^k\sigma_t^2(\theta)}{\partial\theta_{i_1}...\partial\theta_{i_k}} = -2\sum_{j=1}^{\infty}\xi_j(\zeta),$$

where now $\xi_j(\zeta) = \partial^{k-2}\psi_j(\zeta)/\partial\zeta_{i_3-2}...\partial\zeta_{i_k-2}$. In the first of these cases the proof is seen to be very similar to that above after noting that by the Cauchy inequality (46) is bounded by

$$K\left\{\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|x_{t-j}^2\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|\right\}^{1/2}+K\sum_{j=1}^{\infty}\left|\xi_j(\zeta)\right|,$$

whilst in the second it is more immediate; we thus omit the details. We are left with the cases $i_1 = i_2 = i_3 = 2$ and $i_1 = 1$, both of which are trivial. The details for (45) are very similar (the truncations in numerator and denominator match), and are thus omitted. ■

Define

$$g_t(\theta) = u_t(\theta)u_t'(\theta),\ G_T(\theta) = T^{-1}\sum_{t=1}^{T} g_t(\theta).$$

Lemma 7 *For some $\delta > 0$, under Assumptions $A(2+\delta)$, B, C, D, E and $F(1)$,*

$$\sup_{\theta\in\Theta}\ |Q_T(\theta) - Q(\theta)| \to 0\ \ a.s. \quad as\ T\to\infty, \tag{47}$$

and $Q(\theta)$is continuous in θ. If also Assumption $F(2)$ holds,

$$\sup_{\theta\in\Theta}\ \|G_T(\theta) - G(\theta)\| \to 0\ \ a.s. \quad as \quad T\to\infty, \tag{48}$$

and $G(\theta)$ is continuous in θ. If also Assumption $F(3)$ holds,

$$\sup_{\theta\in\Theta}\ \|H_T(\theta) - H(\theta)\| \to 0\ \ a.s. \quad as \quad T\to\infty, \tag{49}$$

and $H(\theta)$ is continuous in θ.

Proof. To prove (47), note first that by Lemmas 1, 2, 3 and 5

$$\sup_{\Theta}E\left|q_0(\theta)\right| \le \sup_{\Theta}E\left|\log\sigma_0^2(\theta)\right| + \sup_{\Theta}E\chi_0(\theta) < \infty.$$

Thus by ergodicity

$$Q_T(\theta) \to Q(\theta) \qquad a.s.,$$

for all $\theta \in \Theta$. Then uniform convergence follows on establishing the equicontinuity property

$$\sup_{\tilde{\theta}:\|\tilde{\theta}-\theta\|<\varepsilon} \left|Q_T(\tilde{\theta}) - Q_T(\theta)\right| \to 0, \qquad a.s.,$$

as $\varepsilon \to 0$, and continuity of $Q(\theta)$. By the mean value theorem it suffices to show that

$$\sup_{\Theta} \left\| \frac{\partial Q_T(\theta)}{\partial \theta} \right\| + \sup_{\Theta} \left\| \frac{\partial Q(\theta)}{\partial \theta} \right\| < \infty, \quad a.s.,$$

which, by Lòeve (1977, p. 121) and identity of distribution, is implied by $E\sup_{\Theta} \|u_0(\theta)\| < \infty$. By the definition of $u_t(\theta)$, and $x_t^2(\mu) \leq K(x_t^2+1)$, $\|\nu_t(\theta)\| \leq 2(|x_t|+1)$, we have

$$\|u_t(\theta)\| \leq K\left[\|\tau_t(\theta)\|\left\{1+\varepsilon_t^2\frac{\sigma_t^2}{\sigma_t^2(\theta)}\right\} + |\varepsilon_t|\frac{\sigma_t}{\sigma_t(\theta)} + 1\right].$$

Thus $E\sup_{\Theta}\|u_0(\theta)\|$ is bounded by a constant times

$$E\sup_{\Theta}\|\tau_0(\theta)\| + \left[E\sup_{\Theta}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)}\right\}^p\right]^{1/p}\left[E\sup_{\Theta}\|\tau_0(\theta)\|^{p/(p-1)}\right]^{1-1/p}$$
$$+E\sup_{\Theta}\left\{\frac{\sigma_0}{\sigma_0(\theta)}\right\}+1$$

for all $p > 1$. On choosing $p < 1 + \delta/2$, this is finite, by Lemmas 5 and 6. (Our use of Lemmas 5 and 6 is similar to Berkes, Horvath and Kokoszka's (2003) use of their Lemmas 5.1 and 5.2 in the GARCH(n,m) case.) This completes the proof of (47). Then (48) and (49) follow by applying analogous arguments to those above, and so we omit the details; indeed (48) and (49) are only used in the proof of consistency of $\bar{G}_T(\hat{\theta}_T)$, $\bar{H}_T(\hat{\theta}_T)$ for G_0, H_0, where convergence over only a neighbourhood of θ_0 would suffice. ■

Lemma 8 *Under Assumptions* $A(2+\delta)$, B, C, D, E *and* $F(1)$,

$$\sup_{\theta\in\Theta}\left|Q_T(\theta) - \bar{Q}_T(\theta)\right| \to 0 \ \ a.s. \quad \text{as } T \to \infty. \tag{50}$$

If also Assumption $F(2)$ holds,

$$\sup_{\theta\in\Theta} \left\|G_T(\theta) - \bar{G}_T(\theta)\right\| \to 0 \ \ a.s. \quad as\ T \to \infty. \tag{51}$$

If also Assumption $F(3)$ holds,

$$\sup_{\theta\in\Theta} \left\|H_T(\theta) - \bar{H}_T(\theta)\right\| \to 0 \ \ a.s. \quad as\ T \to \infty. \tag{52}$$

Proof. We have $\hat{Q}_T(\theta) - Q_T(\theta) = A_T(\theta) + B_T(\theta)$, where

$$A_T(\theta) = T^{-1}\sum_{t=1}^{T} \ln\left[\frac{\bar{\sigma}_t^2(\theta)}{\sigma_t^2(\theta)}\right], \qquad B_T(\theta) = T^{-1}\sum_{t=1}^{T} x_t^2(\mu)\left\{\bar{\sigma}_t^{-2}(\theta) - \sigma_t^{-2}(\theta)\right\}.$$

Because

$$\sigma_t^2(\theta) = \bar{\sigma}_t^2(\theta) + \sum_{j=0}^{\infty}\psi_{j+t}(\zeta)x_{-j}^2(\mu),$$

$\ln(1+x) \le x$ for $x > 0$, and $\sigma_t^2(\theta) \ge \omega_L > 0$, it follows that

$$\begin{aligned} |A_T(\theta)| &\le KT^{-1}\sum_{t=1}^{T}\left\{\sigma_t^2(\theta) - \bar{\sigma}_t^2(\theta)\right\} \le KT^{-1}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\psi_j(\zeta)x_{t-j}^2(\mu) \\ &\le KT^{-1}\sum_{t=0}^{\infty}\left\{\sum_{j=t+1}^{t+T}\psi_j(\zeta)\right\}x_{-t}^2(\mu). \end{aligned}$$

Now from (7)

$$\sup_{\zeta\in\Upsilon}\sum_{j=t+1}^{t+T}\psi_j(\zeta) \le K\sum_{j=t+1}^{t+T} j^{-\underline{d}-1} \le K\min(t+1,T)(t+1)^{-\underline{d}-1}.$$

Thus

$$\sup_{\Theta} A_T(\theta) \le KT^{-1}\sum_{t=0}^{T}(t+1)^{-\underline{d}}(x_{-t}^2+1) + K\sum_{t=T}^{\infty} t^{-\underline{d}-1}(x_{-t}^2+1). \tag{53}$$

From c_r-inequality, (9) and (10), $\sum_{t=1}^{\infty}(t+1)^{-\underline{d}-1}x_{-t}^2$ has finite ρ-th moment, and thus, by Loève (1977, p.121), is a.s. finite. Thus the second term of (53) tends to

zero a.s. as $T \to \infty$ whilst the first does so for the same reasons combined with the Kronecker lemma. Next

$$\begin{aligned} |B_T(\theta)| &\leq KT^{-1}\sum_{t=1}^{T}\chi_t(\theta)\sum_{j=t}^{\infty}\psi_j(\zeta)x_{t-j}^2(\mu) \\ &\leq KT^{-1}\sum_{t=1}^{T}\chi_t(\theta)\sum_{j=t}^{\infty} j^{-\underline{d}-1}(x_{t-j}^2+1). \end{aligned} \tag{54}$$

From previous remarks, $\sum_{j=t}^{\infty} j^{-\underline{d}-1}(x_{t-j}^2+1) \to 0$, a.s. Also, for each θ, a.s.

$$T^{-1}\sum_{t=1}^{T}\chi_t(\theta) \to E\chi_0(\theta) \leq K\left\{E\left(\frac{\sigma_0^2}{\sigma_0^2(\theta)}\right)+1\right\} \leq K$$

by ergodicity and Lemma 5. Thus (54) $\to$ 0 a.s. by the Toeplitz lemma. The convergence is uniform in θ because, from the proof of Lemma 7, for all $\theta \in \Theta$,

$$\sup_{\tilde{\theta}:\left\|\tilde{\theta}-\theta\right\|<\varepsilon}\left\|\chi_0(\tilde{\theta})-\chi_0(\theta)\right\| \to 0 \quad a.s.,$$

as $\varepsilon \to 0$. This completes the proof of (50). We omit the proofs of (51) and (52) as they involve the same kind of arguments. ■

Lemma 9 *For some $\delta > 0$, under Assumptions $A(2+\delta)$, B, C, D, E , $F(1)$ and G, $M(\theta)$ is finite and positive definite for all $\theta \in \Theta$.*

Proof. Fix $\theta \in \Theta$. Finiteness of $M(\theta)$ follows from Lemma 6. Positive definiteness follows (by an argument similar to that of Lumsdaine (1996) in the GARCH(1, 1) case) if, for all non-null $(r+2)\times 1$ vectors λ, $\lambda' M(\theta)\lambda = E\{\lambda'\tau_0(\theta)\}^2 > 0$, that is, that

$$\lambda'\tau_0(\theta)\sigma_0^2(\theta) \neq 0 \quad a.s. \tag{55}$$

since $0 < \sigma_0^2(\theta) < \infty$ a.s. Define

$$\begin{aligned}
\tau_{t\omega}(\theta) &= \frac{\partial}{\partial\omega}\ln\sigma_t^2(\theta) = \sigma_t^{-2}(\theta), \\
\tau_{t\mu}(\theta) &= \frac{\partial}{\partial\gamma}\ln\sigma_t^2(\theta) = -2\sigma_t^{-2}(\theta)\sum_{j=1}^{\infty}\psi_j(\zeta)x_{t-j}(\mu), \\
\tau_{t\zeta}(\theta) &= \frac{\partial}{\partial\zeta}\ln\sigma_t^2(\theta) = \sigma_t^{-2}(\theta)\sum_{j=1}^{\infty}\psi_j^{(1)}(\zeta)x_{t-j}^2(\mu),
\end{aligned}$$

so that $\tau_t(\theta) = (\tau_{t\omega}(\theta),\ \tau_{t\mu}(\theta),\ \tau_{t\zeta}'(\theta))'$. Write $\lambda = (\lambda_1, \lambda_2, \lambda_3')'$, where λ_1 and λ_2 are scalar and λ_3 is $r \times 1$. Consider first the case $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$. Suppose (55) does not hold. Then we must have

$$\sum_{j=1}^{\infty}\lambda_3'\psi_j^{(1)}(\zeta)x_{t-j}^2(\mu) = 0, \quad a.s.$$

If $\lambda_3'\psi_1^{(1)}(\zeta) \neq 0$ it follows that

$$(\sigma_{t-1}\varepsilon_{t-1} + \mu_0 - \mu)^2 = -\left\{\lambda_3'\psi_j^{(1)}(\zeta)\right\}^{-1}\sum_{j=2}^{\infty}\lambda_3'\psi_j^{(1)}(\zeta)x_{-j}^2(\mu). \tag{56}$$

Since $\sigma_{t-1} > 0$ a.s. the left side involves the non-degenerate random variable ε_{t-1}, which is independent of the right side, so (56) cannot hold. Thus $\lambda_3'\psi_j^{(1)}(\zeta) = 0$. Repeated application of this argument indicates that, for all ζ, $\lambda_3'\psi_j^{(1)}(\zeta) = 0$, $j = 1, ..., j_r(\zeta)$. This is contradicted by Assumption G, so (56) cannot hold. Next consider the case $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. If (56) does not hold we must have

$$\sum_{j=1}^{\infty}\psi_j(\zeta)x_{t-j}(\mu) = 0, \quad a.s. \tag{57}$$

Let k be the smallest integer such that $\psi_k(\zeta) \neq 0$. Then (57) implies

$$\varepsilon_{t-k} = \sigma_{t-k}^{-1}(\theta)\left\{\mu - \mu_0 - \psi_k^{-1}(\zeta)\sum_{j=k+1}^{\infty}\psi_j(\theta)x_{t-j}(\mu)\right\}.$$

But the left side is nondegenerate and independent of the right side, so (57) cannot hold. Next consider the case $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. If (55) is not true then, taking

$\lambda_2 = 1$, we must have

$$\sum_{j=1}^{\infty} \left\{ \lambda_3' \psi_j^{(1)}(\zeta) x_{t-j}(\mu) - 2\psi_j(\zeta) \right\} x_{t-j}(\mu) = 0, \quad a.s. \tag{58}$$

Let k be the smallest integer such that either $\lambda_3' \psi_k^{(1)}(\zeta) \neq 0$ or $\psi_k(\zeta) \neq 0$; the preceding argument indicates that there exists such k. Then we have

$$\left\{ 2\psi_k(\zeta) - \lambda_3' \psi_k^{(1)}(\zeta)(\sigma_{t-k}\varepsilon_{t-k} + \mu_0 - \mu) \right\} \{\sigma_{t-k}\varepsilon_{t-k} + \mu_0 - \mu\}$$

$$= \sum_{j=k+1}^{\infty} \left\{ \lambda_3' \psi_j^{(1)}(\zeta) x_{t-j}(\mu) - 2\psi_j(\zeta) \right\} x_{t-j}(\mu), \quad a.s.$$

The left side is a.s. non-zero and involves the non-degenerate random variable ε_{t-k}, which is independent of the right side, so (58) cannot hold. We are left with the cases where $\lambda_1 \neq 0$. Taking $\lambda_1 = -1$ and noting that $\sigma_t^2(\theta)\tau_{t\omega}(\theta) \equiv 1$, the preceding arguments indicate that there exist no λ_2 and λ_3 such that

$$\lambda_2 \sigma_t^2(\theta)\tau_{t\mu}(\theta) + \lambda_3' \sigma_t^2(\theta)\tau_{t\zeta}(\theta) = 1, \quad a.s.$$

■

Lemma 10 For some $\delta > 0$, under Assumptions $A(2+\delta)$, B, C, D, E, $F(1)$ and H,

$$\inf_{\substack{\theta \in \Theta \\ \theta \neq \theta_0}} Q(\theta) > Q(\theta_0).$$

Proof. We have

$$Q(\theta) - Q(\theta_0) = E\left[\frac{\sigma_0^2}{\sigma^2(\theta)} - \ln\left\{ \frac{\sigma_0^2}{\sigma^2(\theta)} \right\} - 1 \right] + (\mu - \mu_0)^2 E\left[\frac{1}{\sigma_0^2(\theta)} \right].$$

The second term on the right hand side is zero only when $\mu = \mu_0$ and is positive otherwise. Because $x - \ln x - 1 \geq 0$ for $x > 0$, with equality only when $x = 1$, it remains to show that

$$\ln \sigma_0^2(\theta) = \ln \sigma_0^2 \ \ a.s., \ \text{ some } \theta \neq \theta_0. \tag{59}$$

By the mean value theorem, (59) implies that $(\theta - \theta_0)'\tau_0(\bar{\theta}) = 0$ a.s., for $\theta \neq \theta_0$ and some $\bar{\theta}$ such that $\left\|\bar{\theta} - \theta_0\right\| \leq \|\theta - \theta_0\|$. But by Lemma 9 there is no such $\bar{\theta}$. ■

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