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# Strengthening economic linkages between Leeds and Manchester: feasibility and implications (structural model appendix)

## Report

# Strengthening Economic Linkages between Leeds and Manchester: Feasibility and Implications (Structural Model Appendix)

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**Abstract**

This technical Appendix describes the structural model used as part of the Spatial Economic Research Centre's work for the Northern Way on linkages between the Manchester and Leeds City Regions. A summary of the research, as well as a full report of the findings, can be found on the web-sites of both SERC and the Northern Way.

**Keywords:** heterogeneous firms; general equilibrium; monopolistic competition; gravity equations; variable demand elasticities

**JEL Classification:** F12; F15; F17

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# Description of the structural model

The goal of this Appendix is to describe the model used for simulations of productivity and wage changes in different counterfactuals as part of SERC's research for the Northern Way on the impact of strengthening economic linkages between Leeds and Manchester. The model has been developed by Behrens et al. (2008) to which the reader might refer for further details.

# 1 Closed economy

Consider a closed economy with a final consumption good, provided as a continuum of horizontally differentiated varieties. We denote by $\Omega$ the endogenously determined set of available varieties, with measure $N$. There are $L$ consumers, each of whom supplies inelastically one unit of labor, which is the only factor of production.

## 1.1 Preferences and demands

All consumers have identical preferences which display 'love of variety' and give rise to demands with variable elasticities. Following Behrens and Murata (2007), the utility maximization problem of a representative consumer is given by:

$$\max_{q(j),\, j\in\Omega} \; U \equiv \int_{\Omega} \left[1 - \mathrm{e}^{-\alpha q(j)}\right] \mathrm{d}j \qquad \text{s.t.} \quad \int_{\Omega} p(j) q(j) \mathrm{d}j = E, \tag{1}$$

where $E$ denotes expenditure; $p(j) > 0$ and $q(j) \geq 0$ stand for the price and the per capita consumption of variety $j$; and $\alpha > 0$ is a parameter. As shown by Behrens and Murata (2007), solving (1) yields the following demand functions:

$$q(i) = \frac{E}{N\overline{p}} - \frac{1}{\alpha}\left\{ \ln\left[\frac{p(i)}{N\overline{p}}\right] + h \right\}, \quad \forall i \in \Omega, \tag{2}$$

where

$$\overline{p} \equiv \frac{1}{N}\int_{\Omega} p(j)\mathrm{d}j \quad \text{and} \quad h \equiv -\int_{\Omega} \ln\left[\frac{p(j)}{N\overline{p}}\right]\frac{p(j)}{N\overline{p}}\mathrm{d}j$$

denote the average price and the differential entropy of the price distribution, respectively. Since marginal utility at zero consumption is bounded, the demand for a variety need not be positive. Indeed, as can be seen from (2), the demand for variety $i$ is positive if and only if its price is lower than the reservation price $p^d$. Formally,

$$q(i) > 0 \quad \iff \quad p(i) < p^d \equiv N\overline{p}\,\mathrm{e}^{\frac{\alpha E}{N\overline{p}} - h}. \tag{3}$$

Note that the reservation price $p^d$ is a function of the price aggregates $\overline{p}$ and $h$. Combining expressions (2) and (3) allows us to express the demand for variety $i$ concisely as follows:

$$q(i) = \frac{1}{\alpha}\ln\left[\frac{p^d}{p(i)}\right]. \tag{4}$$

## 1.2 Technology and market structure

The labor market is assumed to be perfectly competitive so that all firms take the wage rate $w$ as given. Prior to production, each firm engages in research and development, which requires a fixed amount $F$ of labor paid at the market wage. Each firm discovers its marginal labor requirement $m(i) \geq 0$ only after making this irreversible investment. We assume that $m(i)$ is drawn from a common and known, continuously differentiable distribution $G$. Since research and development costs are sunk, a firm will survive (i.e., remain active) in the market provided it can charge a price $p(i)$ above marginal cost $m(i)w$.

Each surviving firm sets its price to maximize operating profit

$$\pi(i) = L\big[p(i) - m(i)w\big]q(i), \tag{5}$$

where $q(i)$ is given by (4). Since there is a continuum of firms, no individual firm has any impact on $p^d$ so that the first-order conditions for (operating) profit maximization are given by:

$$\ln\left[\frac{p^d}{p(i)}\right] = \frac{p(i) - m(i)w}{p(i)}, \quad \forall i \in \Omega. \tag{6}$$

A price distribution satisfying (6) is called a *price equilibrium*. Multiplying both sides of (6) by $p(i)$, integrating over $\Omega$, and using (4) yield the average price as follows:

$$\overline{p} = \overline{m}w + \frac{\alpha E}{N}, \tag{7}$$

where $\overline{m} \equiv (1/N)\int_\Omega m(j)\mathrm{d}j$ denotes the average marginal labor requirement of the surviving firms. Observe that expression (7) displays pro-competitive effects, i.e., the average price is decreasing in the mass of surviving firms $N$.

Equations (4) and (6) imply that $q(i) = (1/\alpha)[1 - m(i)w/p(i)]$, which allows us to derive the upper and lower bounds for the marginal labor requirement. The maximum output is given by $q(i) = 1/\alpha$ at $m(i) = 0$. The minimum output is given by $q(i) = 0$ at $p(i) = m(i)w$, which by (6) implies that $p(i) = p^d$. Therefore, the cutoff marginal labor requirement is defined as $m^d \equiv p^d/w$. A firm that draws $m^d$ is indifferent between producing and not producing, whereas all firms with a draw below (resp., above) $m^d$ remain in (resp., exit from) the market.

Since firms differ only by their marginal labor requirement, we can express all firm-level variables in terms of $m$. Solving (6) by using the Lambert $W$ function, defined as $\varphi = W(\varphi)\mathrm{e}^{W(\varphi)}$, the profit-maximizing prices and quantities, as well as operating profits, can be expressed as follows:

$$p(m) = \frac{mw}{W}, \quad q(m) = \frac{1}{\alpha}(1 - W), \quad \pi(m) = \frac{Lmw}{\alpha}\left(W^{-1} + W - 2\right), \tag{8}$$

where we suppress the argument $\mathrm{e}m/m^d$ of $W$ to alleviate notation. It is readily verified that $W' > 0$ for all non-negative arguments and that $W(0) = 0$ and $W(\mathrm{e}) = 1$ (see Section A.1 for the derivation of (8) and the properties of $W$). Hence, $0 \leq W \leq 1$ if $0 \leq m \leq m^d$. The expressions in (8) show that a firm with a draw $m^d$ charges a price equal to marginal cost, faces zero demand, and earns zero profit. Since $W' > 0$, we readily obtain $\partial p(m)/\partial m > 0$, $\partial q(m)/\partial m < 0$ and $\partial \pi(m)/\partial m < 0$. In words, firms with better draws charge lower prices, sell larger quantities, and earn higher operating profits than firms with worse draws.

### 1.3 Equilibrium

We now state the equilibrium conditions for the closed economy, which consist of zero expected profits and labor market clearing. First, given the mass of entrants $N^E$, the mass of surviving firms can be written as $N = N^E G(m^d)$. Using (5), the zero expected profit condition for each firm is given by:

$$L \int_0^{m^d} \left[p(m) - mw\right] q(m) \mathrm{d}G(m) = Fw, \tag{9}$$

which, combined with (8), can be rewritten as

$$\frac{L}{\alpha} \int_0^{m^d} m \left(W^{-1} + W - 2\right) \mathrm{d}G(m) = F. \tag{10}$$

As the left-hand side of (10) is strictly increasing in $m^d$ from 0 to $\infty$, there always exists a unique equilibrium cutoff (see Section A.2). Furthermore, the labor market clearing condition is given by:[1]

$$N^E \left[ L \int_0^{m^d} m q(m) \mathrm{d}G(m) + F \right] = L, \tag{11}$$

which, combined with (8), can be rewritten as

$$N^E \left[ \frac{L}{\alpha} \int_0^{m^d} m \left(1 - W\right) \mathrm{d}G(m) + F \right] = L. \tag{12}$$

Given the equilibrium cutoff $m^d$, equation (12) can be uniquely solved for $N^E$.

How does population size affect entry and firms' survival probabilities? Using the equilibrium conditions (10) and (12), we can show that a larger $L$ leads to more entrants $N^E$ and a smaller cutoff $m^d$, respectively (see Section A.3). The effect of population size on the mass of surviving firms $N$ is in general ambiguous. However, under the commonly made assumption that firms' productivity draws $1/m$ follow a Pareto distribution

$$G(m) = \left(\frac{m}{m^{\max}}\right)^k,$$

[1]Note that by using (9) and the budget constraint $N^E \int_0^{m^d} p(m) q(m) \mathrm{d}G(m) = E$, we obtain $EL/(wN^E) = L \int_0^{m^d} m q(m) \mathrm{d}G(m) + F$ which, together with (11), yields $E = w$ in equilibrium.

with upper bound $m^{\max} > 0$ and shape parameter $k \geq 1$, we can show that $N$ is increasing in $L$.[2] Using this distributional assumption, we readily obtain closed-form solutions for the equilibrium cutoff and mass of entrants:

$$m^d = \left[\frac{\alpha F \left(m^{\max}\right)^k}{\kappa_2 L}\right]^{\frac{1}{k+1}} \quad \text{and} \quad N^E = \frac{\kappa_2}{\kappa_1 + \kappa_2}\frac{L}{F},$$

where $\kappa_1$ and $\kappa_2$ are positive constants that solely depend on $k$ (see Sections B.1 and B.2).[3] The mass of surviving firms is then given as follows:

$$N = \frac{\kappa_2^{\frac{1}{k+1}}}{\kappa_1 + \kappa_2}\left(\frac{\alpha}{m^{\max}}\right)^{\frac{k}{k+1}}\left(\frac{L}{F}\right)^{\frac{1}{k+1}},$$

which is increasing in population size $L$. One can further check that $N$ is decreasing in the fixed labor requirement $F$ and in the upper bound $m^{\max}$. Finally, since $\overline{m} = [k/(k+1)]m^d$ holds when productivity follows a Pareto distribution, a larger population also maps into higher average productivity $1/\overline{m}$.

# 2 Open economy

We now turn to the open economy case. As dealing with two regions only marginally alleviates the notational burden, we first derive the equilibrium conditions for the general case with $K$ asymmetric regions that we use when taking our model to the data. We then present some clear-cut analytical results for the special case of two asymmetric regions in order to guide the intuition for the general case.

## 2.1 Preferences and demands

Preferences are analogous to the ones described in the previous section. Let $p_{sr}(i)$ and $q_{sr}(i)$ denote the price and the per capita consumption of variety $i$ when it is produced in region $s$ and consumed in region $r$. It is readily verified that the demand functions in the open economy case are given as follows:

$$q_{sr}(i) = \frac{E_r}{N_r^c \overline{p}_r} - \frac{1}{\alpha}\left\{\ln\left[\frac{p_{sr}(i)}{N_r^c \overline{p}_r}\right] + h_r\right\}, \quad \forall i \in \Omega_{sr},$$

where $N_r^c$ is the mass of varieties consumed in region $r$; $\Omega_{sr}$ denotes the set of varieties produced in region $s$ and consumed in region $r$; and

$$\overline{p}_r \equiv \frac{1}{N_r^c}\sum_s \int_{\Omega_{sr}} p_{sr}(j)\mathrm{d}j \quad \text{and} \quad h_r \equiv -\sum_s \int_{\Omega_{sr}} \ln\left[\frac{p_{sr}(j)}{N_r^c \overline{p}_r}\right]\frac{p_{sr}(j)}{N_r^c \overline{p}_r}\mathrm{d}j$$

[2] The Pareto distribution has been extensively used in the previous literature on heterogeneous firms (e.g., Bernard *et al.*, 2007; Helpman *et al.*, 2008; Melitz and Ottaviano, 2008).

[3] For this solution to be consistent, we must ensure that $m^d \leq m^{\max}$, i.e., $m^{\max} \geq \alpha F/(\kappa_2 L)$.

denote the average price and the differential entropy of the price distribution of all varieties consumed in region $r$. As in the closed economy case, the demand for domestic variety $i$ (resp., foreign variety $j$) is positive if and only if the price of variety $i$ (resp., variety $j$) is lower than the reservation price $p_r^d$. Formally,

$$q_{rr}(i) > 0 \iff p_{rr}(i) < p_r^d \quad \text{and} \quad q_{sr}(j) > 0 \iff p_{sr}(j) < p_r^d,$$

where $p_r^d \equiv N_r^c \overline{p}_r \mathrm{e}^{\alpha E_r/(N_r^c \overline{p}_r) - h_r}$ is a function of the price aggregates $\overline{p}_r$ and $h_r$. The demands for domestic and foreign varieties can then be concisely expressed as follows:

$$q_{rr}(i) = \frac{1}{\alpha} \ln \left[ \frac{p_r^d}{p_{rr}(i)} \right] \quad \text{and} \quad q_{sr}(j) = \frac{1}{\alpha} \ln \left[ \frac{p_r^d}{p_{sr}(j)} \right]. \tag{13}$$

## 2.2 Technology and market structure

Technology and the entry process are identical to the ones described in Section 2. We assume that shipments from $r$ to $s$ are subject to trade costs $\tau_{rs} > 1$ for all $r$ and $s$, that markets are segmented, and that firms are free to price discriminate.

Firms in region $r$ independently draw their productivities from a region-specific distribution $G_r$. Assuming that firms incur trade costs in terms of labor, the operating profit of firm $i$ in $r$ is given by:

$$\pi_r(i) = \sum_s \pi_{rs}(i) = \sum_s L_s q_{rs}(i) \left[ p_{rs}(i) - \tau_{rs} m_r(i) w_r \right]. \tag{14}$$

Each firm maximizes (14) with respect to its prices $p_{rs}(i)$ separately. Since it has no impact on the price aggregates and on the wages, the first-order conditions are given by:

$$\ln \left[ \frac{p_s^d}{p_{rs}(i)} \right] = \frac{p_{rs}(i) - \tau_{rs} m_r(i) w_r}{p_{rs}(i)}, \quad \forall i \in \Omega_{rs}. \tag{15}$$

We first solve for the average price in region $r$. To do so, multiply (15) by $p_{rs}(i)$, use (13), integrate over $\Omega_{rs}$, and finally sum the resulting expressions to obtain

$$\overline{p}_r \equiv \frac{1}{N_r^c} \sum_s \int_{\Omega_{sr}} p_{sr}(j) \mathrm{d}j = \frac{1}{N_r^c} \sum_s \tau_{sr} w_s \int_{\Omega_{sr}} m_s(j) \mathrm{d}j + \frac{\alpha E_r}{N_r^c}, \tag{16}$$

where the first term is the average of marginal delivered costs in region $r$. Expression (16) shows that $\overline{p}_r$ is decreasing in the mass $N_r^c$ of firms competing in region $r$, which is similar to the result on pro-competitive effects established in the closed economy case.

Equations (13) and (15) imply that $q_{rs}(i) = (1/\alpha)[1 - \tau_{rs} m_r(i) w_r / p_{rs}(i)]$, which shows that $q_{rs}(i) = 0$ at $p_{rs}(i) = \tau_{rs} m_r(i) w_r$. It then follows from (15) that $p_{rs}(i) = p_s^d$. Hence, a firm located in $r$ with draw $m_{rs}^x \equiv p_s^d / (\tau_{rs} w_r)$ is just indifferent between selling and not selling in region $s$. All firms with draws below $m_{rs}^x$ are productive enough to sell to region $s$. In what follows, we refer to $m^x_{ss} \equiv m^d_s$ as the *domestic cutoff* in region $s$, whereas $m^x_{rs}$ with $r \neq s$ is the *export cutoff*. Export and domestic cutoffs are linked as follows:

$$m^x_{rs} = \frac{\tau_{ss}}{\tau_{rs}} \frac{w_s}{w_r} m^d_s. \tag{17}$$

Expression (17) reveals how trade costs and wage differentials affect firms' ability to break into foreign markets. When wages are equalized ($w_r = w_s$) and internal trade is costless ($\tau_{ss} = 1$), all export cutoffs must fall short of the domestic cutoffs since $\tau_{rs} > 1$. In that case, breaking into any foreign market is always harder than selling domestically. However, in the presence of wage differentials and internal trade costs, the domestic and the foreign cutoffs can no longer be clearly ranked. The usual ranking, namely that exporting to $s$ is more difficult than selling domestically in $s$, prevails only when $\tau_{ss} w_s < \tau_{rs} w_r$.

The first-order conditions (15) can be solved as in the closed economy case. Switching to notation in terms of $m$, the profit-maximizing prices and quantities, as well as operating profits, are given by:

$$p_{rs}(m) = \frac{\tau_{rs} m w_r}{W}, \quad q_{rs}(m) = \frac{1}{\alpha}\left(1 - W\right), \quad \pi_{rs} = \frac{L_s \tau_{rs} m w_r}{\alpha}(W^{-1} + W - 2), \tag{18}$$

where $W$ denotes the Lambert $W$ function with argument $\mathrm{e}\tau_{rs} m w_r / p^d_s$, which we suppress to alleviate notation. It is readily verified that more productive firms again charge lower prices, sell larger quantities, and earn higher operating profits.

Observe that in an open economy, the masses of varieties consumed and produced in each region need not be the same. Given a mass of entrants $N^E_r$, only $N^p_r = N^E_r G_r\left(\max_s \{m^x_{rs}\}\right)$ firms survive, namely those which are productive enough to sell at least in one market. Finally, the mass of varieties consumed in region $r$ is given by

$$N^c_r = \sum_s N^E_s G_s(m^x_{sr}). \tag{19}$$

## 2.3 Equilibrium

The zero expected profit condition for each firm in region $r$ is given by

$$\sum_s L_s \int_0^{m^x_{rs}} \left[p_{rs}(m) - \tau_{rs} m w_r\right] q_{rs}(m) \mathrm{d}G_r(m) = F_r w_r, \tag{20}$$

where $F_r$ is the region-specific fixed labor requirement. Furthermore, each labor market clears in equilibrium, which requires that

$$N^E_r \left[\sum_s L_s \tau_{rs} \int_0^{m^x_{rs}} m q_{rs}(m) \mathrm{d}G_r(m) + F_r\right] = L_r. \tag{21}$$

Last, trade is balanced for each region:

$$N_r^E \sum_{s \neq r} L_s \int_0^{m_{rs}^x} p_{rs}(m) q_{rs}(m) \mathrm{d}G_r(m) = L_r \sum_{s \neq r} N_s^E \int_0^{m_{sr}^x} p_{sr}(m) q_{sr}(m) \mathrm{d}G_s(m).$$

As in the foregoing section, we can restate the equilibrium conditions using the Lambert $W$ function (see Section C for details).

In what follows, we assume that productivity draws $1/m$ follow a Pareto distribution with identical shape parameters $k \geq 1$. However, to capture differences in local technological possibilities, we allow the upper bounds to vary across regions, i.e., $G_r(m) = (m/m_r^{\max})^k$. A lower $m_r^{\max}$ implies that firms in region $r$ have a higher probability of drawing a better productivity. Under the Pareto distribution, the equilibrium conditions can be greatly simplified. First, using the expressions in Sections B.1 and C.1, labor market clearing requires that

$$N_r^E \left[ \frac{\kappa_1}{\alpha \left(m_r^{\max}\right)^k} \sum_s L_s \tau_{rs} \left( \frac{\tau_{ss}}{\tau_{rs}} \frac{w_s}{w_r} m_s^d \right)^{k+1} + F_r \right] = L_r. \tag{22}$$

Second, using the expressions in Sections B.2 and C.2, zero expected profits imply that

$$\mu_r^{\max} \equiv \frac{\alpha F_r \left(m_r^{\max}\right)^k}{\kappa_2} = \sum_s L_s \tau_{rs} \left( \frac{\tau_{ss}}{\tau_{rs}} \frac{w_s}{w_r} m_s^d \right)^{k+1}, \tag{23}$$

where $\mu_r$ is a simple monotonic transformation of the upper bounds. Last, using the expressions in Sections B.3 and C.3, balanced trade requires that

$$\frac{N_r^E w_r}{\left(m_r^{\max}\right)^k} \sum_{s \neq r} L_s \tau_{rs} \left( \frac{\tau_{ss}}{\tau_{rs}} \frac{w_s}{w_r} m_s^d \right)^{k+1} = L_r \sum_{s \neq r} \tau_{sr} \frac{N_s^E w_s}{\left(m_s^{\max}\right)^k} \left( \frac{\tau_{rr}}{\tau_{sr}} \frac{w_r}{w_s} m_r^d \right)^{k+1}. \tag{24}$$

The $3K$ conditions (22)–(24) depend on $3K$ unknowns: the wages $w_r$, the masses of entrants $N_r^E$, and the domestic cutoffs $m_r^d$. The export cutoffs $m_{rs}^x$ can then be computed using (17). Combining (22) and (23) immediately shows that

$$N_r^E = \frac{\kappa_2}{\kappa_1 + \kappa_2} \frac{L_r}{F_r}. \tag{25}$$

The mass of entrants in region $r$ therefore positively depends on that region's size $L_r$ and negatively on its fixed labor requirement $F_r$.

Adding the term in $r$ that is missing on both sides of (24), and using (23) and (25), we obtain the following equilibrium relationship:

$$\frac{1}{\left(m_r^d\right)^{k+1}} = \sum_s L_s \tau_{rr} \left( \frac{\tau_{rr}}{\tau_{sr}} \frac{w_r}{w_s} \right)^k \frac{1}{\mu_s^{\max}}. \tag{26}$$

Expressions (23) and (26) summarize how wages, upper bounds, cutoffs, trade costs and population sizes are related in general equilibrium.

## 2.4 Two-region case

Our model allows for clear-cut comparative static results with two asymmetric regions. Using (23)–(25), an equilibrium can be characterized by a system of three equations with three unknowns (the two cutoffs $m_1^d$ and $m_2^d$, and the relative wage $w_1/w_2$) as follows:

$$\left(\frac{w_1}{w_2}\right)^{2k+1} = \left(\frac{\tau_{21}}{\tau_{12}}\right)^k \left(\frac{\tau_{22}}{\tau_{11}}\right)^{k+1} \left(\frac{m_2^d}{m_1^d}\right)^{k+1} \left(\frac{\mu_2^{\max}}{\mu_1^{\max}}\right) \tag{27}$$

$$\mu_r^{\max} = L_r \tau_{rr} \left(m_r^d\right)^{k+1} + L_s \tau_{rs} \left(\frac{\tau_{ss}}{\tau_{rs}} \frac{w_s}{w_r} m_s^d\right)^{k+1}, \tag{28}$$

for $r = 1, 2$ and $s \neq r$. Equation (28) for regions 1 and 2 can readily be solved for the cutoffs as a function of the relative wage $\omega \equiv w_1/w_2$:

$$(m_1^d)^{k+1} = \frac{\mu_1^{\max}}{L_1 \tau_{11}} \frac{1 - \rho \left(\frac{\tau_{22}}{\tau_{12}}\right)^k \omega^{-(k+1)}}{1 - \left(\frac{\tau_{11}\tau_{22}}{\tau_{12}\tau_{21}}\right)^k} \quad \text{and} \quad (m_2^d)^{k+1} = \frac{\mu_2^{\max}}{L_2 \tau_{22}} \frac{1 - \rho^{-1} \left(\frac{\tau_{11}}{\tau_{21}}\right)^k \omega^{k+1}}{1 - \left(\frac{\tau_{22}\tau_{11}}{\tau_{21}\tau_{12}}\right)^k}, \tag{29}$$

where $\rho \equiv \mu_2^{\max}/\mu_1^{\max}$ captures relative technological possibilities. A larger $\rho$ (given $F_r$) implies, ceteris paribus, that firms in region 2 face a higher probability of drawing a worse productivity than those in region 1. Substituting the cutoffs (29) into (27) yields after some simplification

$$\text{LHS} \equiv \omega^k = \rho \frac{L_1}{L_2} \left(\frac{\tau_{21}}{\tau_{12}}\right)^k \frac{\rho \tau_{11}^{-k} - \tau_{21}^{-k} \omega^{k+1}}{\tau_{22}^{-k} \omega^{k+1} - \rho \tau_{12}^{-k}} \equiv \text{RHS}. \tag{30}$$

Assume that intraregional trade is less costly than interregional trade, i.e., $\tau_{11} < \tau_{21}$ and $\tau_{22} < \tau_{12}$. Then, the RHS of (30) is decreasing in $\omega$ on its relevant domain, whereas the LHS is increasing in $\omega$. Hence, there exists a unique equilibrium such that the equilibrium relative wage $\omega^*$ is bounded by relative trade costs $\tau_{22}/\tau_{12}$ and $\tau_{21}/\tau_{11}$, relative technological possibilities $\rho$, and the shape parameter $k$ (see Section A.4).

Since the RHS of (30) is decreasing in $\omega$, the comparative static results are straightforward to derive. Assume that $\tau_{21} = \tau_{12}$ and $\tau_{11} = \tau_{22}$. In Section A.5 we show that, everything else equal: (i) the larger region has the higher wage; (ii) higher internal trade costs in one region reduce its relative wage; (iii) better access for one region to the other raises its relative wage; (iv) wages converge as bilateral trade barriers fall; (v) the larger region has the lower cutoff and the higher utility; and (vi) the cutoff decreases and the utility increases as bilateral trade barriers fall.

## 2.5 Welfare

To see that tougher selection or more diversity in consumption map into welfare gains, notice that since $\mathrm{e}^{-\alpha q_{sr}(m)} = p_{sr}(m)/p_r^d$ by (13), the indirect utility in region $r$ is given by

$$U_r = \sum_s N_s^E \int_0^{m_{sr}^x} \left[1 - \mathrm{e}^{-\alpha q_{sr}(m)}\right] \mathrm{d}G_s(m) = N_r^c \left(1 - \frac{\overline{p}_r}{p_r^d}\right).$$

Using expression (16), one can verify that $\overline{p}_r = [k/(k+1)]p_r^d + \alpha w_r/N_r^c$, which allows us to express the indirect utility as $U_r = N_r^c/(k+1) - \alpha/(\tau_{rr} m_r^d)$. Since $N_r^c$ is defined as in (19), and making use of the fact that expression (26) holds in equilibrium, we can rewrite the indirect utility as follows:

$$U_r = \left[\frac{1}{(k+1)\kappa_3} - 1\right] \frac{\alpha}{\tau_{rr} m_r^d}. \tag{31}$$

Hence, welfare is inversely proportional to the cutoff $m_r^d$. Alternatively, the equilibrium utility can be written as $U_r = [1/(k+1) - \kappa_3]N_r^c$., i.e., welfare changes in region $r$ are proportional to changes in the mass of varieties available for consumption.

# 3 Estimation and counterfactuals

In this section we take the model with $K$ asymmetric regions to the data. To this end, we first derive two sets of general equilibrium constraints. Using data on wages, GDP per worker, population, firms' productivity dispersion, and generalized transport costs for UK local authorities and city regions, we then structurally estimate trade frictions. With all the elements of the model in our hands, we are finally able to simulate productivity and wage changes across UK regions stemming from different policies.

## 3.1 Gravity equation system

The value of exports from region $r$ to region $s$ is given by

$$X_{rs} = N_r^E L_s \int_0^{m_{sr}^x} p_{rs}(m) q_{rs}(m) \mathrm{d}G_r(m).$$

Using equations (18), (25), and the Pareto distribution for $G_r(m)$, we obtain the following gravity equation:[4]

$$\frac{X_{rs}}{L_r L_s} = \tau_{rs}^{-k} \tau_{ss}^{k+1} (w_s/w_r)^{k+1} w_r \left(m_s^d\right)^{k+1} (\mu_r^{\max})^{-1}. \tag{32}$$

As can be seen from (32), exports depend on bilateral trade costs $\tau_{rs}$, internal trade costs in the destination $\tau_{ss}$, origin and destination wages $w_r$ and $w_s$, destination productivity $m_s^d$, and origin technological possibilities $\mu_r^{\max}$. A higher relative wage $w_s/w_r$ raises the value of exports as firms in $r$ face relatively lower production costs, whereas a higher absolute wage $w_r$ raises the value of exports by increasing export prices $p_{rs}$. Furthermore, a larger $m_s^d$ raises the value of exports since firms located in the destination are on average less productive. Last, a lower $\mu_r^{\max}$ implies

[4] Contrary to standard practice in the gravity literature, we do not move the GDPs but instead move the population sizes to the left-hand side. Applying the former approach to our model would amount to assuming that wages are exogenous in the gravity estimation, which is not the case in general equilibrium.

that firms in region $r$ have higher expected productivity, which raises the value of their exports. Conditions (23) and (26) allow us to derive the following general equilibrium constraints:

$$\frac{1}{(m_s^d)^{k+1}} = \sum_v L_v \tau_{vs}^{-k} \tau_{ss}^{k+1} \left(\frac{w_s}{w_v}\right)^k (\mu_v^{\max})^{-1} \qquad s = 1, 2, \dots K \tag{33}$$

$$\mu_r^{\max} = \sum_v L_v \tau_{rv}^{-k} \tau_{vv}^{k+1} \left(\frac{w_v}{w_r}\right)^{k+1} \left(m_v^d\right)^{k+1} \qquad r = 1, 2. \dots K \tag{34}$$

The *gravity equation system* consists of the gravity equation (32) and the $2K$ general equilibrium constraints (33) and (34) that summarize the interactions between the $2K$ endogenous variables, namely the wages and cutoffs.

## 3.2 Data and estimation procedure

To estimate the gravity equation system (32)–(34) like in Behrens et al. (2008) data on trade flows across regions are needed. However, such data are not available for the UK. An alternative strategy, that we adopt in what follows, is to use the $2K$ general equilibrium constraints (33) and (34) only.

Looking at general equilibrium constrains, one can notice that some variables are easily observable while others needs to be estimated. Data on wages $w_r$ and populations $L_r$ across UK regions are indeed easy to obtain and we borrow them from, respectively, the Annual Survey of Hours and Earnings (ASHE) and the Office for National Statistics (ONS). Data refers to the year 2006. As for productivities $m_r^d$ we use, in a fully consistent way with respect to the model, local GDP per worker in 2006 reconstructed from NUTS3 GDP data provided by Eurostat and total employment figures provided by ONS.[5] Indeed, under the Pareto distribution, the domestic cutoff in each region is proportional to the inverse of the average firm productivity, i.e., $m_r^d = [(k+1)/k]\overline{m}_r$. Since labor is the only production input, a firm productivity in the model corresponds to its value added per worker while the sum of firms' value added equals local GDP. Finally as a measure of the degree of firms' heterogeneity in productivity, the parameter $k$, we use the the rather standard value of 2.[6]

In order to close the model, we are still left with trade frictions ($\tau_{rs}$) and technological possibilities ($\mu_r^{\max}$). Trade frictions, in a broad sense, represents all impediments to doing business in different locations. As standard in the gravity literature (see Anderson and van Wincoop, 2004) we assume that such costs are related to distance. In particular, we assume $\tau_{rs} \equiv d_{rs}^{\gamma}$ where $d_{rs}$

[5]Data on GDP at the local authority level are not available and have been reconstructed from NUTS3 data. In a first step, NUTS3 GDP per worker has been regressed on local wages, employment density, and employment density squared producing an $R^2$ of 0.8092. Using the estimated coefficients of such a regressions, GDP at the local authority level has been estimated from available data on wages, employment density, and employment density squared referring to the same spatial scale.

[6]See Del Gatto et al. (2006).

is the generalized transportation cost (GTC) between region $r$ and $s$ and $\gamma$ is a parameter to be estimated.[7] GTC within a region is either directly available or, for missing observations, it has been assumed to be half of the GTC between a region and its first order contiguity neighbors.

Technological possibilities $\mu_r^{\max}$ represents the competitiveness of a region once 'discounted' for its size, population, and accessibility to other regions. Clearly, data for such a variable do not exist and need to be reconstructed from the model. In order to estimate $\mu_r^{\max}$, as well as the $\gamma$ for all regions, we use the following iterative procedure:

1. We start with an initial guess for $\gamma$ and use (34) to back out values for $\mu_r^{\max}$. Let us denote such values as $\widehat{\mu_r^{\max}}$
2. Using $\widehat{\mu_r^{\max}}$ in (33) we estimate $\gamma$ by non-linear lest squares to produce and estimate $\hat{\gamma}$
3. Using $\hat{\gamma}$ in (34) we obtain new values for $\widehat{\mu_r^{\max}}$.
4. Iterate steps 2 and 3 until convergence. Convergence is achieved whenever the value $\hat{\gamma}$ in two consecutive iterations is lower than $10^{-6}$.

By applying such a procedure we obtain a significant (at the 5% confidence level) $\hat{\gamma}$=-0.0366 with the $R^2$ of the non-linear least squares estimation of (33) being 0.8049.

## 3.3 Counterfactual simulations

In order to simulate the impact of changes in trade costs and/or population on productivity and wages we proceed as follows. We first compute the values of productivity $1/\left(m_s^d\right)$ obtained after the convergence of our iterative procedure using (33): $\widehat{1/\left(m_s^d\right)}$. These values of productivity are then plugged into (34) in order to obtain consistent equilibrium values of wages $\widehat{w_v/w_r}$. Such values of wages are then used in (33) to get another guess for productivity. We iterate such procedure until both $\widehat{1/\left(m_s^d\right)}$ and $\widehat{w_v/w_r}$ satisfy conditions (33) and (34)

In order to back out the impact of transport policies (defined as changes of some trade costs values $\tau_{rs}$) and/or housing policies (defined as changes of some values of local populations Ł$_r$) we solve for the new values of wages and GDP per worker that satisfy the 2K system of equations (33) and (34). GDP per worker changes induced by a given policy can be fully recovered. However, as the model is invariant with respect to the absolute level of wages (i.e. only relative wages matters), regional wage changes induced by a given policy needs to be evaluated with

[7] In particular, we use the weighted average of road GTC and the train GTC. Weights are given by the share of journeys of 5 miles and more made with (respectively) car and train. Such weights, provided by the 2006 National Travel Survey, are equal to 0.8684 and 0.1316. GTC road are based on the year 1998 while GTC train are based on the year 2004. We use the change in the retail price index over the period 1998-2004 to make GTC road comparable to GTC train. See Appendix 3 of the SERC report for details on sources and construction methodology of GTC data.

respect to a reference region. We choose the city-region of Aberdeen as ‘numeraire’ because is rather small and peripheral with respect to the study area of Leeds and Manchester and is therefore likely to be only marginally affected by the policies under evaluation. Such relative wage changes are thus ‘quasi absolute wage changes’. Indeed, in all simulations, GDP per worker changes in Aberdeen are negligible.

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# A: Proofs and computations

**A.1. Derivation of (8) and properties of $W$.** Using $p^d = m^d w$, the first-order conditions (6) can be rewritten as

$$\ln\left[\frac{m^d w}{p(m)}\right] = 1 - \frac{mw}{p(m)}.$$

Taking the exponential of both sides and rearranging terms, we have

$$\mathrm{e}\frac{m}{m^d} = \frac{mw}{p(m)}\mathrm{e}^{\frac{mw}{p(m)}}.$$

Noting that the Lambert W function is defined as $\varphi = W(\varphi)\mathrm{e}^{W(\varphi)}$ and setting $\varphi = \mathrm{e}m/m^d$, we obtain $W(\mathrm{e}m/m^d) = mw/p(m)$, which implies $p(m)$ as given in (8). The derivations of $q(m)$ and $\pi(m)$ then follow straightforwardly.

Turning to the properties of the Lambert $W$ function, we clearly see that $\varphi = W(\varphi)\mathrm{e}^{W(\varphi)}$ implies that $W(\varphi) \geq 0$ for all $\varphi \geq 0$. Taking logarithms on both sides and differentiating yield

$$W'(\varphi) = \frac{W(\varphi)}{\varphi[W(\varphi)+1]} > 0$$

for all $\varphi > 0$. Finally, we have $0 = W(0)\mathrm{e}^{W(0)}$, which implies $W(0) = 0$; and $\mathrm{e} = W(\mathrm{e})\mathrm{e}^{W(\mathrm{e})}$, which implies $W(\mathrm{e}) = 1$.

**A.2. Existence and uniqueness of the equilibrium cutoff $m^d$.** We show that there exists a unique equilibrium cutoff $m^d$. To see this, apply the Leibniz integral rule to the left-hand side of (10) and use $W(\mathrm{e}) = 1$ to obtain

$$\frac{\mathrm{e}L}{\alpha(m^d)^2}\int_0^{m^d} m^2\left(W^{-2}-1\right)W'\mathrm{d}G(m) > 0,$$

where the sign comes from $W' > 0$ and $W^{-2} \geq 1$ for $0 \leq m \leq m^d$. Hence, the left-hand side of (10) is strictly increasing. This uniquely determines the equilibrium cutoff $m^d$, because

$$\lim_{m^d\to 0}\int_0^{m^d} m\left(W^{-1}+W-2\right)\mathrm{d}G(m) = 0 \quad \text{and} \quad \lim_{m^d\to\infty}\int_0^{m^d} m\left(W^{-1}+W-2\right)\mathrm{d}G(m) = \infty.$$

**A.3. Market size, the equilibrium cutoff, and the mass of entrants.** Differentiating (10) and using the Leibniz integral rule, we readily obtain

$$\frac{\partial m^d}{\partial L} = -\frac{\alpha F\left(m^d\right)^2}{\mathrm{e}L^2}\left[\int_0^{m^d} m^2\left(W^{-2}-1\right)W'\mathrm{d}G(m)\right]^{-1} < 0,$$

because $W' > 0$ and $W^{-2} \geq 1$ for $0 \leq m \leq m^d$. Differentiating (12) with respect to $L$ yields

$$\frac{\partial N^E}{\partial L} = \frac{F(N^E)^2}{L^2}\left\{1 - \frac{\mathrm{e}L^3}{\alpha F\left(m^d\right)^2}\left[\int_0^{m^d} m^2 W'\mathrm{d}G(m)\right]\frac{\partial m^d}{\partial L}\right\} > 0,$$

where the sign comes from $\partial m^d/\partial L < 0$ as established in the foregoing.

**A.4. Existence and uniqueness in the two-region case.** Under our assumptions on trade costs, the RHS of (30) is non-negative if and only if $\underline{\omega} < \omega < \overline{\omega}$, where $\underline{\omega} \equiv \rho^{1/(k+1)}\left(\tau_{22}/\tau_{12}\right)^{k/(k+1)}$ and $\overline{\omega} \equiv \rho^{1/(k+1)}\left(\tau_{21}/\tau_{11}\right)^{k/(k+1)}$. Furthermore, the RHS is strictly decreasing in $\omega \in (\underline{\omega}, \overline{\omega})$ with $\lim_{\omega\to\underline{\omega}+}$ RHS $= \infty$ and $\lim_{\omega\to\overline{\omega}-}$ RHS $= 0$. The LHS of (30) is, on the contrary, strictly increasing in $\omega \in (0,\infty)$. Hence, there exists a unique equilibrium $\omega^* \in (\underline{\omega}, \overline{\omega})$.

**A.5. Market size, trade frictions, and wages.** (i) First, $\omega^*$ is increasing in $L_1/L_2$ as an increase in $L_1/L_2$ raises the RHS of (30) without affecting the LHS. This implies that if the two regions have equal technological possibilities ($\rho = 1$) and face symmetric trade costs ($\tau_{12} = \tau_{21}$ and $\tau_{11} = \tau_{22}$), the larger region has the higher relative wage.

(ii) Higher internal trade costs in one region reduce the relative wage of that region, because

$$\frac{\partial(\text{RHS})}{\partial\tau_{11}} < 0 \quad \text{iff} \quad \omega^* > \underline{\omega} \quad \text{and} \quad \frac{\partial(\text{RHS})}{\partial\tau_{22}} > 0 \quad \text{iff} \quad \omega^* < \overline{\omega}.$$

(iii) Better access to the foreign market raises the domestic relative wage, whereas better access to the domestic market reduces the domestic relative wage because

$$\frac{\partial(\text{RHS})}{\partial\tau_{12}} < 0 \quad \text{iff} \quad \omega^* < \overline{\omega} \quad \text{and} \quad \frac{\partial(\text{RHS})}{\partial\tau_{21}} > 0 \quad \text{iff} \quad \omega^* > \underline{\omega}.$$

(iv) Assuming that $\tau_{12} = \tau_{21} = \tau$ and that $\tau_{11} = \tau_{22} = t$, one can verify that

$$\frac{\partial(\text{RHS})}{\partial\tau} = -\frac{k\rho t^k}{\tau^{k+1}}\frac{L_1}{L_2}\frac{\rho^2 - \omega^{2(k+1)}}{[\omega^{k+1} - \rho(t/\tau)^k]^2}\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} 0 \quad \text{for} \quad \left\{\begin{array}{c} \underline{\omega} < \rho^{\frac{1}{k+1}} < \omega^* < \overline{\omega} \\ \underline{\omega} < \omega^* = \rho^{\frac{1}{k+1}} < \overline{\omega} \\ \underline{\omega} < \omega^* < \rho^{\frac{1}{k+1}} < \overline{\omega} \end{array}\right\}. \tag{35}$$

Note that when regions are of equal size, but have different upper bounds ($\rho > 1$), the first case of (35) applies since $\omega^* > \rho^{1/(k+1)}$ in equilibrium. To see this, evaluate (30) at $\omega = \rho^{1/(k+1)}$ and recall that $\tau_{21} = \tau_{12} = \tau$ and $L_1 = L_2$. The LHS is equal to $\rho^{k/(k+1)}$, which falls short of the RHS given by $\rho$ (since $\rho > 1$ and $k \geq 1$). Since the LHS is increasing and the RHS is decreasing, it must be that $\omega^* > \rho^{1/(k+1)}$. Hence, lower trade costs reduce the relative wage of the more productive region. Furthermore, when regions have the same upper bounds but different sizes ($L_1 > L_2$), we obtain $\omega^* > \rho^{k/(k+1)} = 1$, so that the first case of (35) applies again. (v) Assuming that $\tau_{12} = \tau_{21} = \tau$ and that $\tau_{11} = \tau_{22} = t$ and using (29), one can verify that

$$\left(\frac{m_1^d}{m_2^d}\right)^{k+1} = \frac{L_2}{\rho L_1}\left[\frac{1 - \rho\left(\frac{t}{\tau}\right)^k \omega^{-(k+1)}}{1 - \rho^{-1}\left(\frac{t}{\tau}\right)^k \omega^{k+1}}\right]. \tag{36}$$

Furthermore, (30) can be rewritten as

$$\omega^k = \frac{\rho L_1}{L_2}\left[\frac{1 - \rho^{-1}\left(\frac{t}{\tau}\right)^k \omega^{k+1}}{1 - \rho\left(\frac{t}{\tau}\right)^k \omega^{-(k+1)}}\right]\frac{\rho}{\omega^{k+1}} \qquad \Longleftrightarrow \qquad \omega^{2k+1} = \rho\left(\frac{m_2^d}{m_1^d}\right)^{k+1},$$

where we use (36) to obtain the equivalence. Now assume that $\rho = 1$ and that $L_1 > L_2$. Then, we know from (i) that $\omega > 1$, which implies $m_1^d < m_2^d$. It is then readily verified from (31) that the larger region has the higher utility. (vi) Let $\tau_{12} = \tau_{21} = \tau$ and $\tau_{11} = \tau_{22} = t$. Imposing symmetry between the two regions, i.e., $\rho = 1$ and $\omega = 1$, and using (29), one can verify that $(m_r^d)^{k+1} = \mu_r^{\max}/\{L_r t[1 + (t/\tau)^k]\}$, thus implying that the cutoff decreases as bilateral trade barriers fall. It is then readily verified from (31) that the utility increases as bilateral trade barriers fall.

# Section B: Integrals involving the Lambert $W$ function

To derive closed-form solutions for various expressions throughout the paper we need to compute integrals involving the Lambert $W$ function. This can be done by using the change in variables suggested by Corless *et al.* (1996, p.341). Let

$$z \equiv W\left(\mathrm{e}\,\frac{m}{I}\right), \quad \text{so that} \quad \mathrm{e}\,\frac{m}{I} = z\mathrm{e}^z, \quad \text{where} \quad I = m_r^d, m_{rs}^x,$$

where subscript $r$ can be dropped in the closed economy. The change in variables then yields $\mathrm{d}m = (1+z)\mathrm{e}^{z-1} I \mathrm{d}z$, with the new integration bounds given by 0 and 1. Under our assumption of a Pareto distribution for productivity draws, the change in variables allows to rewrite integrals in simplified form.

**B.1.** First, consider the following expression, which appears when integrating firms' outputs:

$$\int_0^I m\left[1 - W\left(\mathrm{e}\frac{m}{I}\right)\right] \mathrm{d}G_r(m) = \kappa_1 \left(m_r^{\max}\right)^{-k} I^{k+1},$$

where $\kappa_1 \equiv k\mathrm{e}^{-(k+1)} \int_0^1 (1 - z^2)\left(z\mathrm{e}^z\right)^k \mathrm{e}^z \mathrm{d}z > 0$ is a constant term which solely depends on the shape parameter $k$.

**B.2.** Second, the following expression appears when integrating firms' operating profits:

$$\int_0^I m\left[W\left(\mathrm{e}\frac{m}{I}\right)^{-1} + W\left(\mathrm{e}\frac{m}{I}\right) - 2\right] \mathrm{d}G_r(m) = \kappa_2 \left(m_r^{\max}\right)^{-k} I^{k+1},$$

where $\kappa_2 \equiv k\mathrm{e}^{-(k+1)} \int_0^1 (1+z)\left(z^{-1} + z - 2\right)\left(z\mathrm{e}^z\right)^k \mathrm{e}^z \mathrm{d}z > 0$ is also a constant term which solely depends on the shape parameter $k$.

**B.3.** Finally, the following expression appears when integrating firms' revenues:

$$\int_0^I m\left[W\left(\mathrm{e}\,\frac{m}{I}\right)^{-1} - 1\right] \mathrm{d}G_r(m) = \kappa_3 \left(m_r^{\max}\right)^{-k} I^{k+1},$$

where $\kappa_3 \equiv k\mathrm{e}^{-(1+k)} \int_0^1 \left(z^{-1} - z\right)\left(z\mathrm{e}^z\right)^k \mathrm{e}^z \mathrm{d}z > 0$ is a constant term which solely depends on the shape parameter $k$. Using the expressions for $\kappa_1$ and $\kappa_2$, one can verify that $\kappa_3 = \kappa_1 + \kappa_2$.

# Section C: Equilibrium in the open economy

In this Section we restate the open economy equilibrium conditions of Section 3 using the Lambert W function.

**C.1.** Using (18), the labor market clearing condition can be rewritten as follows:

$$N_r^E \left\{ \frac{1}{\alpha} \sum_s L_s \tau_{rs} \int_0^{m_{rs}^x} m \left[ 1 - W\left( \mathrm{e} \frac{m}{m_{rs}^x} \right) \right] \mathrm{d}G_r(m) + F_r \right\} = L_r. \tag{37}$$

**C.2.** Plugging (18) into (20), zero expected profits require that

$$\frac{1}{\alpha} \sum_s L_s \tau_{rs} \int_0^{m_{rs}^x} m \left[ W\left( \mathrm{e} \frac{m}{m_{rs}^x} \right)^{-1} + W\left( \mathrm{e} \frac{m}{m_{rs}^x} \right) - 2 \right] \mathrm{d}G_r(m) = F_r. \tag{38}$$

As in the closed economy case, the zero expected profit condition depends solely on the cutoffs $m_{rs}^x$ and is independent of the mass of entrants.

**C.3.** Finally, the trade balance condition is given by

$$\begin{aligned} N_r^E w_r \sum_{s \neq r} L_s \tau_{rs} \int_0^{m_{rs}^x} m \left[ W\left( \mathrm{e} \frac{m}{m_{rs}^x} \right)^{-1} - 1 \right] \mathrm{d}G_r(m) \\ = L_r \sum_{s \neq r} N_s^E \tau_{sr} w_s \int_0^{m_{sr}^x} m \left[ W\left( \mathrm{e} \frac{m}{m_{sr}^x} \right)^{-1} - 1 \right] \mathrm{d}G_s(m). \end{aligned} \tag{39}$$

Applying the region-specific Pareto distributions $G_r(m) = (m/m_r^{\max})^k$ to (37)–(39) yields, after some algebra and using the results of Section B, expressions (22)–(24) given in the main text.