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An R&D-Based Model of Multi-Sector Growth

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Abstract
We develop a multi-sector general equilibrium model in which productivity growth is driven by the production of sector-specific knowledge. In the model, we find that long run differences in total factor productivity growth across sectors are independent of the parameters of the knowledge production function except for one, which we term the fertility of knowledge. Differences in R&D intensity are also independent of most other parameters. The fertility of knowledge in the capital sector is central to the growth properties of the model economy.

JEL Codes: D24, D92, O31, O41,
Keywords: Endogenous technical change, multisector growth, fertility of knowledge, total factor productivity, R&D intensity, investment-specific technical change.

Pigmaei gigantum humeris impositi plusquam ipsi gigantes vident. — Bernard of Chartres, circa 1100.

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1 Introduction

Total factor productivity (TFP) growth rates vary widely and persistently across industries.[1] However, economic growth rates for aggregate variables are known to be fairly constant in the long run. What industry-level parameters might lie behind the observed differences in TFP growth rates? How can one reconcile the observation of a stable aggregate growth path with diverging productivity at the industry level?

This paper answers these questions using a general equilibrium model of endogenous productivity change. In the model, TFP growth is the outcome of research activity, which produces economically useful knowledge. Industries differ in terms of a number of factors that affect the returns to R&D activity, including the determinants of the demand for their goods, and the parameters of the production function for new knowledge. We investigate which of these factors lead to cross-industry differences in TFP growth rates and R&D intensity, and study under what conditions the economy possesses an aggregate balanced growth path (BGP).

Our framework is a multi-sector growth model and contains a number of factors that have been thought to affect the returns to R&D.[2] In addition, we consider one new potential source of cross-industry variation: the extent to which old knowledge is useful for the production of new knowledge. We call this the *fertility* of knowledge. In a class of one-sector growth models, Jones (1995) shows that the value of this parameter must be less than unity if the aggregate economy is to possess a balanced growth path that is consistent with the data. Otherwise, the economy displays "scale effects": larger countries grow faster, and countries with growing populations would display accelerating growth.[3]

Allowing for R&D in a multi-sector framework, we find that a BGP requires restrictions only on the fertility parameter of the industries that produce capital goods. Allowing capital to be an input in the production of knowledge, this restriction is that "average" fertility in these sectors must be bounded from above by a number strictly less than one. The upper bound is a function of the capital share of costs.

Strikingly, we find that the equilibrium ranking of TFP growth rates across sectors depends only on cross-industry differences in the fertility of knowledge. While the result is stark, the role of the fertility parameter in a one-sector context provides some

[1] For example, among capital goods, Jorgenson et al (2005) estimate that annual TFP growth rates in the United States over the period 1960-2004 range from -1.0% per year in Structures to 10.4% in Computers and Office Equipment.

[2] A small sample of the related literature includes Romer (1990), Aghion and Howitt (1992) and Jones (1995) on R&D-based models of economic growth, and Pakes and Schankerman (1984), Jaffe (1986) and Cohen and Levin (1989) on R&D intensity.

[3] In the terminology of Jones (2005a), these are "strong" scale effects.

intuition. In a one-sector model, eliminating aggregate scale effects by introducing decreasing returns to knowledge also eliminates the effects of other parameters and policy variables on equilibrium rates of economic growth. In a multi-sector context, a similar result applies to the parameters that determine the returns to R&D. These parameters may affect sectoral differences in R&D intensity, and even the size of each sector, but only the fertility of knowledge affects the rate at which inputs accumulate relative to each other – including knowledge.[4] The key to this result, however, is not that fertility is limited in each sector, but that TFP growth in each sector must be stable over time.

Our results address some fundamental issues regarding the nature of aggregate growth and in particular growth accounting exercises. A notable feature of postwar US data is a secular decline in relative price of capital, and this phenomenon is widely attributed investment-specific technical change: productivity improvements that affect primarily the capital goods sector. Greenwood et al (1997) and Cummins and Violante (2002) ascribe approximately 60% of economic growth to investment-specific technical change and suggest that this is a significant channel for postwar US economic growth. Investment-specific technical change in our model can arise endogenously when the fertility of knowledge is higher in the durable goods sector.

In a calibrated version of the model, we find that the model economy is broadly consistent with the distribution of income and employment across activities, and with the ranking of TFP growth rates across types of capital. The fertility of knowledge varies widely across sectors, but we find fertility values to be typically quite low, except in the fastest-growing sectors.

In related work, Klenow (1996) studies the determinants of cross-sectoral TFP growth differences in a version of Romer's (1990) model, in which R&D results in a growing number of intermediates. However, that model has scale effects at the industry level, and this is what largely drives the theoretical results. Acemoglu and Guerrieri (2006) develop a two-sector endogenous growth model in which growth is driven by the production of knowledge. Their model does not identify the technological determinants of R&D intensity across industries but focuses instead on the role of different capital shares across industries. In Krusell (1998), R&D leads to investment-specific technical change, as it can do here. However, R&D only occurs in the capital goods sector, so it is unsuitable for cross-industry comparisons. Vourvachaki (2006) also develops an endogenous growth model with two final good sectors, but again R&D only occurs in one sector.

Section 2 describes the structure of the model, and Section 3 studies analytically

[4] This is consistent with the observation that R&D intensity varies much more in cross sections than do TFP growth rates. See Klenow (1996) and Burnside (1996).

its long run behavior. Section 4 explores the quantitative implications of our results. Section 5 provides with more discussion on the knowledge production function. Section 6 concludes.

2 Economic Environment

The economy consists of z sectors, where firms in sectors $i \in \{1, ..m-1\}$ produce consumption goods, whereas firms in sectors $j \in \{m, ...z\}$ produce investment goods. Each good comes in a continuum of differentiated varieties, and is produced using capital and labor as physical inputs. The productivity of a firm in any given sector depends upon the quantity of technical knowledge or "ideas" at its disposal. New knowledge is produced as a result of individual firm activity, and spillovers from other firms.

2.1 Households

Time is discrete and indexed by t. There is a continuum of households of measure $N_t = g_N^t$. Households have preferences over a finite number of different goods $i \in \{1, ..m-1\}$, each of which comes in a continuum of varieties $h \in [0,1]$. The life-time utility of a household is

$$\sum_{t=0}^{\infty} (\beta g_N)^t \frac{c_t^{1-\theta} - 1}{1-\theta} \tag{1}$$

$$c_t = \prod_{i=1}^{m-1} \left(\frac{c_{it}}{\omega_i} \right)^{\omega_i}, \; c_{it} = \left(\int_0^1 c_{iht}^{\frac{\mu_i - 1}{\mu_i}} dh \right)^{\frac{\mu_i}{\mu_i - 1}} \qquad i = 1, ..m-1 \tag{2}$$

where β is the discount factor, $\beta g_N < 1, \theta > 0, \mu_i > 1, \omega_i > 0$ and $\sum_i \omega_i = 1$. The parameter μ_i is the elasticity of substitution across different varieties of good i, and $1/\theta$ is the intertemporal elasticity of substitution. The life-time utility is a logarithmic function of c_t when $\theta = 1$. We use lower case letters to denote per-capita variables. In most of what follows, i indexes a consumption sector or good, j indexes a type of capital, and h indexes a firm in any given sector.

Each household member is endowed with one unit of labor and k_t units of capital. They earn income by renting capital and labor to firms, and by earning profits from the firms. Their budget constraint is

$$\sum_{i=1}^{m-1} \int p_{iht} c_{iht} dh + \sum_{j=m}^{z} \int p_{jht} x_{jht} dh \leq w_t + R_t k_t + \pi_t \tag{3}$$

where x_{jht} is investment in variety h of capital good j, p_{iht} is the price of variety h of good i, w_t and R_t are rental prices of labor and capital, and $N_t\pi_t \equiv \sum_{i=1}^{z} \int_0^1 \Pi_{iht} dh$ equals total profits from firms.

The capital accumulation equation is $g_N k_{t+1} = x_t + (1-\delta_k) k_t$ The composite investment good x_t is produced via a Cobb-Douglas function of all capital types j, so the elasticity of substitution across capital goods j is equal to one, while the elasticity of substitution across different varieties of capital good j is equal to $\mu_j > 1$:

$$x_t = \prod_{j=m}^{z} \left(\frac{x_{jt}}{\kappa_j}\right)^{\kappa_j}, \; x_{jt} = \left[\int x_{jht}^{(\mu_j-1)/\mu_j} dh\right]^{\mu_j/(\mu_j-1)} \qquad j = m, ..z \tag{4}$$

where $\kappa_j > 0$ and $\sum_j \kappa_j = 1$. We define the price index for the consumption composite c_t and the investment composite respectively as:

$$p_{ct} \equiv \frac{\sum_{i=1}^{m-1} \int_0^1 p_{iht} c_{iht} dh}{c_t}; \qquad p_{xt} \equiv \frac{\sum_{j=m}^{z} \int_0^1 p_{jht} x_{jht} dh}{x_t}. \tag{5}$$

2.2 Firms

The production function for variety h of good i is

$$Y_{iht} = T_{iht} K_{iht}^{\alpha} N_{iht}^{1-\alpha} \tag{6}$$

where $\alpha \in (0,1)$, T_{iht} is knowledge, Y_{iht} is output, K_{iht} is capital and N_{iht} is labor.

Let F_{iht} be knowledge produced by firm h in sector $i \leq h$. Knowledge accumulates over time according to the function

$$T_{ih,t+1} = F_{iht} + (1-\delta_T) T_{iht} \tag{7}$$

where new knowledge is produced according to the function

$$F_{iht} = A_i \left(T_{iht}^{1-\sigma_i} T_{it}^{\sigma_i}\right)^{\phi_i} Q_{iht}^{\alpha} L_{iht}^{1-\alpha} \tag{8}$$

where Q_{iht} and L_{iht} are capital and labor used in production of knowledge, and $T_{it} = \int_0^1 T_{iht} dh$. The knowledge production function has constant returns to rivalrous inputs (capital and labor). In this case, following the empirical literature, the equilibrium stock of knowledge equals the depreciated stock of R&D spending. We study the implications of diminishing returns (or duplication in research) in Section (5).[5]

[5] The capital share is the same as in the final goods production function. This allows us to derive closed-form solutions: more importantly, it implies that the production function differs across sectors only in terms of the TFP index T_{ih}, which is the focus of the paper.

The literature has focused on several factors that may determine cross-industry R&D intensity differences. A survey by Cohen and Levin (1989) identifies them broadly as: market size, technological opportunity, and spillovers. In our model, industries may differ in terms of five parameters: $\mu_i, \omega_i, A_i, \sigma_i$ and ϕ_i. The first two, μ_i and ω_i, are preference parameters, that determine the scope and sensitivity of potential returns to R&D. The remaining parameters are technological, and determine the structure of the knowledge production function. Pakes and Schankerman (1984) identify technological opportunity as a constant in the ideas' production function, similar to our parameter A_i. Our formulation of knowledge spillovers σ_i is as in Krusell (1998).

In the model, knowledge is a persistent stock which may be increased by research. The amount of new knowledge generated by research is a function of the quantity of inputs used, and the quantity of pre-existing knowledge. Parameter $\sigma_i \in [0, 1)$ indicates the extent to which firms in sector i benefit from the knowledge of their competitors.[6] We denote $\gamma_{iht} \equiv T_{iht+1}/T_{iht}$ as the growth factor of T_{ih}. Parameter $\phi_i \in \mathbb{R}$ captures the extent to which pre-existing knowledge is useful for the production of new knowledge. We will be say that if $\phi_i > \phi_{i'}$ then sector i is more *fertile* than sector i'.

In a one-sector context, our notion of "fertility" is referred to as the "intertemporal knowledge spillover," or the "measure of decreasing returns to ideas." Its effect on the knowledge stock when positive is referred to as "standing on shoulders," and when negative as "fishing out." In the former case, knowledge becomes easier to produce as it accumulates, whereas in the latter case it becomes more difficult. One of our innovations is to present fertility as a potential source of cross-industry variation and, as a result, we feel it is appropriate to introduce new and more concise terminology.

Each sector $i \leq z$ is monopolistically competitive. Taking its demand function as given, a firm h in sector i chooses optimally its level of production and R&D inputs in order to maximize the discounted stream of real profits,

$$\sum_{t=0}^{\infty} \lambda_t \frac{\Pi_{iht}}{p_{ct}} \tag{9}$$

where λ_t is the discount factor at time t, with $\lambda_0 = 1$, and $\lambda_t = \prod_{s=1}^{t} \frac{1}{1+r_t}$ for $t \geq 1$,

[6] In the model there are no *cross-sectoral* spillovers. We study such spillovers in Section (5). Empirical research finds mostly weak spillovers across firms within narrow, related sectors, but does not find measurable spillovers across broad sectors – see Bernstein and Nadiri (1988) and (1989) and Schultz (2006).

where r_t is the real interest rate. The firm's profits are given by

$$\Pi_{iht} = p_{iht}Y_{iht} - w_t\left(N_{iht} + L_{iht}\right) - R_t\left(K_{iht} + Q_{iht}\right) \tag{10}$$

3 Decentralized Equilibrium

Definition 1 *A decentralized equilibrium consists of sequences of prices and of allocations such that:*

1. *Given the sequence of prices* $\left\{\left\{(p_{iht})_{h\in[0,1]}\right\}_{i=1}^{z}, R_t, w_t\right\}_{t=0,1..}$, *households choose* $\left\{\left\{(c_{iht}, x_{iht})_{h\in[0,1]}\right\}_{i=1}^{z}\right\}_{t=0,1,...}$ *to maximize their discounted stream of utility* (1)*;*

2. *Given the sequence of input prices* $\{R_t, w_t\}_{t=0,1..}$, *and taking their demand functions as given, firms choose* $\left\{\left\{(K_{iht}, N_{iht}, Q_{iht}, L_{iht})_{h\in[0,1]}\right\}_{i=1}^{z}\right\}_{t=0,1...}$ *to maximize discounted stream of profit* (9)*;*

3. *The sequence of input prices* $\{R_t, w_t\}_{t=0,..}$ *satisfies the capital and labor market clearing conditions in all periods:*

$$\sum_{i=1}^{z}\int_0^1 \left(K_{iht} + Q_{iht}\right) = K_t, \ \sum_{i=1}^{z}\int_0^1 \left(N_{iht} + L_{iht}\right) = N_t \tag{11}$$

Our objective is to understand dynamics across broad sectors of the economy and not across different varieties of any given good. Therefore, we focus on symmetric equilibria across varieties within each sector i and suppress the firm index h henceforth.

The equilibrium results are summarized in the following claims. All proofs are given in the Appendix.

Lemma 1 *In equilibrium,*

$$\frac{K_{it}}{N_{it}} = \frac{Q_{jt}}{L_{jt}} \qquad \forall i, j, \text{ and } \frac{p_{it}}{p_{jt}} = \frac{T_{jt}\left(1 - 1/\mu_j\right)}{T_{it}\left(1 - 1/\mu_i\right)}. \tag{12}$$

The intuition is that competitive input markets require marginal rates of technical substitution across labor and capital to be equalized across firms. The assumption of the Cobb-Douglas production function with equal capital shares across activities then delivers the results. Under (12), the capital market-clearing condition then implies that the aggregate capital-labor ratio k is also the capital-labor ratio for both production and R&D activities. This facilitates the proof of the following aggregation result.

Lemma 2 *Let* $1/\mu_x = \sum_{i=m}^{z} \kappa_i \left(1/\mu_i\right)$*. The investment sectors* $j \in \{m, ..., z\}$ *can be aggregated into one sector with a production function*

$$N_t x_t \equiv T_{xt} \left(\sum_{j=m}^{z} K_{jt}\right)^{\alpha} \left(\sum_{j=m}^{z} N_{jt}\right)^{1-\alpha} \tag{13}$$

where the knowledge index T_{xt} *equals*

$$T_{xt} = \left[\prod_{j=m}^{z} \left(\frac{1-1/\mu_j}{1-1/\mu_x}\right)\right] \prod_{j=m}^{z} T_{jt}^{\kappa_j} \tag{14}$$

The result derives the production function for the investment composite x_t from the optimal input allocations for sectors $j \in \{m, ..., z\}$. It also shows that the markup for the composite investment good is a weighted average of the markups in each investment sector. We denote $\gamma_{xt} = T_{x,t+1}/T_{xt}$:

$$\gamma_{xt} = \prod_{j=m}^{z} \gamma_{jt}^{\kappa_j}. \tag{15}$$

Lemma 2 is an aggregation result which allows us to relate the rental price of capital to the TFP index of composite investment goods, which simplifies the Euler condition for the consumer in the following Lemma. Let q_t be the relative price of capital and G_t the gross return on investment in terms of capital goods. Then,

$$q_t = \frac{p_{xt}}{p_{ct}} \qquad G_t \equiv 1 - \delta_k + \frac{R_t}{p_{xt}}. \tag{16}$$

Lemma 3 *The Euler condition for the consumer satisfies*

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \frac{q_{t+1}}{q_t} G_{t+1} \tag{17}$$

where the equilibrium physical gross return of investment is:

$$G_t = 1 - \delta_k + \alpha T_{xt} k_t^{\alpha-1} \left(1 - \frac{1}{\mu_x}\right). \tag{18}$$

The key of the proof is to derive R_t/p_{xt} as function of T_{xt} and p_{xt}/p_{jt} follows (12). We next turn to the dynamic decisions for the firm. Define $n_i \equiv N_i/N$ and $l_i \equiv L_i/N$ as the employment shares for production and R&D activities in sector i.

Lemma 4 *The firm's dynamic optimization implies*

$$G_{t+1}\frac{\gamma_{it}^{\phi_i}}{\gamma_{xt}} = A_i k_{t+1}^{\alpha} N_{it+1} T_{it+1}^{\phi_i - 1} + \phi_i (1-\sigma_i)\left(\gamma_{it+1} - 1 + \delta_T\right) + 1 - \delta_T, \quad \forall i \tag{19}$$

Equation (19) is derived from the optimal T_{iht+1} chosen by the firm. If χ_{iht} is the shadow price for knowledge T_{iht+1}, then

$$\chi_{iht} = \underbrace{\left[\frac{\lambda_{t+1}p_{iht+1}}{p_{ct+1}}\frac{\partial Y_{iht+1}}{\partial T_{iht+1}}\right]}_{\text{future production}} + \underbrace{\left[\chi_{iht+1}\frac{\partial F_{iht+1}}{\partial T_{iht+1}}\right]}_{\text{future research}} + \underbrace{\left[\chi_{iht+1}\left(1-\delta_T\right)\ \right]}_{\text{future knowledge}} \tag{20}$$

The equation reflect three benefits of the production of more knowledge: (1) more efficient *production of goods and services* at time $t+1$, (2) more efficient *production of knowledge* at time $t+1$, and (3) more future knowledge. The equilibrium χ_{iht} is determined by equating the marginal benefit and marginal cost of research efforts (Q_{iht} and L_{iht}), e.g. for L_{iht}

$$\chi_{iht}\frac{\partial F_{iht}}{\partial L_{iht}} = \frac{\lambda_t}{p_{ct}}\frac{\partial \Pi_{iht+1}}{\partial L_{iht}} \tag{21}$$

The key to equation (19) is that some of these terms depend on the *rate* at which technology improves – due to changes in the relative prices of final goods and changes in shadow price of knowledge, whereas others depend on its *level* – due to the current cost of resources. Rational researchers must thus make a trade-off between both dynamic and static concerns. If the economy displays a balanced growth path, the dynamic elements of this decision problem must be constant. For the terms that are expressed in *levels* to also be constant, however, requires the growth rates of their component variables to offset each other. In the case of equation (19), this suggests that, the long run growth rate of TFP might be related to the (endogenous) growth rate of the capital stock, to the growth rate of the population, and to coefficients that affect the balance between these growth rates – in this case α and fertility ϕ_i.

3.1 Aggregate Balanced growth path

We look for a balanced growth path (BGP) where aggregate variables are growing at the same constant rate but sectoral TFP growth rates are different.[7] More precisely,

[7] Note that if γ_i are the same across sectors, then Lemma 1 implies that relative prices are constant, then the model essentially reduces to a one-sector growth model.

such a BGP requires a constant consumption-capital ratio which in our model is the expression $c/(qk)$. As in a one-sector model, certain conditions are required for solutions to be interior solutions, and for transversality conditions to be satisfied. We distinguish between these two sets of conditions for balanced growth in a multi-sector model by stating them in separate Propositions.

Define ϕ_x and ϕ_c such that

$$\frac{1}{1-\phi_x} = \sum_{j=m}^{z} \frac{\kappa_j}{1-\phi_j}; \quad \frac{1}{1-\phi_c} = \sum_{i=1}^{m-1} \frac{\omega_i}{1-\phi_i} \tag{22}$$

and define Φ and Υ as:

$$\Phi \equiv \left(1-\phi_x - \frac{\alpha}{1-\alpha}\right)^{-1}; \quad \Upsilon = \frac{1-\phi_x}{1-\phi_c} + \frac{\alpha}{1-\alpha} \tag{23}$$

Proposition 1 *Suppose there exists an equilibrium with $l_i, n_i > 0$ that satisfy the transversality conditions for T_i and k. If*

$$\Phi, \Upsilon > 0 \tag{24}$$

then there exists a unique aggregate balanced growth path. Along this path c/q and k grow by a constant factor $(\gamma_x^*)^{1/(1-\alpha)}$, and c grows by a constant factor $(\gamma_x^*)^{\Upsilon}$, where

$$\gamma_x^* = g_N^{\Phi}. \tag{25}$$

Finally, knowledge T_i grows by a factor γ_i^* where

$$(\gamma_i^*)^{1-\phi_i} = (\gamma_x^*)^{1-\phi_x} \qquad \forall i. \tag{26}$$

Proposition 2 *Along the balanced growth path, $l_i, n_i > 0$, and the transversality conditions for T_i and k are satisfied if*

$$\begin{aligned} \beta &< (1/g_N)^{1+(1-\theta)\Phi\Upsilon}, \quad \textit{and} \\ \forall i, \quad \phi_i(1-\sigma_i) &< 1, \quad g_N^{\Phi/(1-\phi_i)} \geq (1-\delta_T)^{1/(1-\phi_x)} \end{aligned} \tag{27}$$

Corollary 1 *A BGP exists with R&D in all sectors provided conditions* (24) *and* (27) *are satisfied.*

The key of the proof is to observe from (18) that k grows by a factor $\gamma_x^{1/(1-\alpha)}$, which by (15) is constant if γ_i is constant in all the capital sectors. We then use the optimal R&D condition (19) to derive the condition for constant values of γ_j. The upped bound on β is required for the transversality conditions. Then (19) implies $\phi_i (1 - \sigma_i) < 1$ is sufficient for positive n_i. Finally, the knowledge accumulation equation (7) implies that l_i is positive if future knowledge exceeds the stock of existing knowledge after depreciation, i.e. $\gamma_i \geq 1 - \delta_T$, then (26) implies that $g_N^{1/(1-\phi_i)} \geq (1 - \delta_T)^{1/(1-\phi_x)}$ is required for positive l_i. It is worth noting that positive l_i, and in general the existence of an aggregate balanced growth path, does not imply positive TFP growth when $\delta_T > 0$.

A comparison of our main result with Jones (1995) is useful. Jones (1995) presents a one-sector model, in which the growth equation (25) holds but with $\Phi = \lambda/(1 - \phi)$,where λ is a positive constant and ϕ is the one-sector model fertility parameter. So his condition for existence of a BGP is $\phi < 1$. In our model balanced growth restricts only the weighted average of fertility parameters across *capital* goods: $\phi_x < 1$ is necessary for $\Phi > 0$ (although not sufficient). No such restrictions are required on the fertility parameters in consumption goods sectors, or on the average for the whole economy, since $\phi_c < 1$ is not necessary for $\Upsilon > 0$. Also, fertility need not be equal across individual sectors or bounded by unity in each and every sector. These results can reconcile the theoretical predictions with estimates of aggregate fertility parameters, such as those by Porter and Stern (2005) and Abdih and Joutz (2006), who find aggregate parameters larger than unity.

Another difference between our results and Jones's is due to the inclusion of capital in our production function for knowledge. If $\alpha > 0$ the fertility parameter ϕ_x is bounded below unity by a number that depends on the capital share. The key is that new ideas increase the productivity of the sector that produces capital – which is itself an input into the production of ideas – so decreasing returns to ideas alone is not sufficient for the economy to have a BGP.

Thus, we find in general that the restrictions on the fertility parameters imposed by balanced growth depend on whether there are several sectors independently conducting research, on whether there is any exogenous depreciation of knowledge, on whether there are knowledge spillovers, and on whether capital is used in the production of knowledge. When all these are absent our restrictions converge to Jones's restriction.

3.2 Properties along the Balanced growth path

We now discuss other properties along the BGP. We first show that it satisfies the Kaldor (1961) stylized facts. We then discuss the implications of the model for sectoral TFP growth and for sectoral R&D intensity under balanced growth.

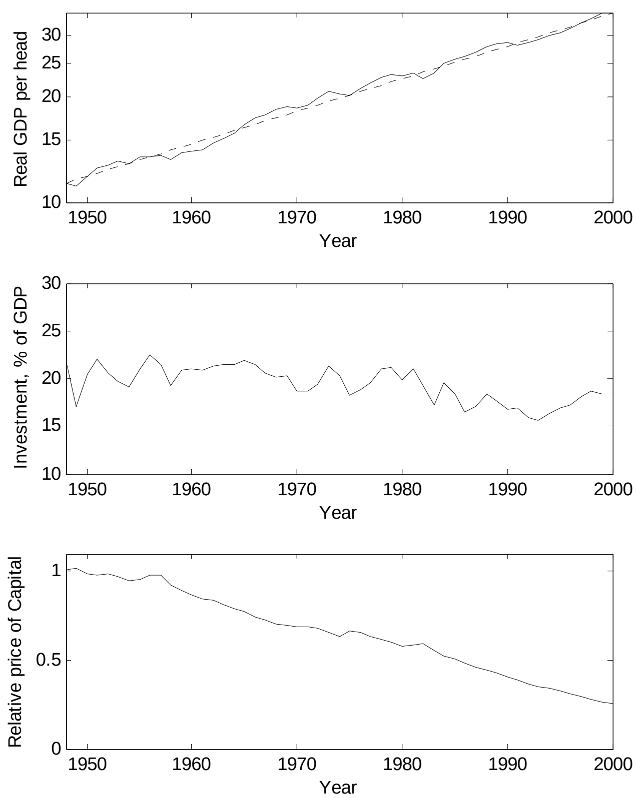

Figure 1: US Real GDP per person (constant 2000 dollars, log scale), investment-ouput ratio, and the quality-adjusted relative price of capital in the United States, 1953-2000. Sources: United States Bureau of Economic Analysis, and Cummins and Violante (2002). Real GDP per head is plotted against an exponential time trend.

Define y as total output per capita in consumption units:

$$y \equiv \sum_{i=1}^{z} \frac{p_i T_i k^\alpha n_i}{p_c} \tag{28}$$

Proposition 3 *Along the BGP, the consumption-output ratio, R&D expenditure shares and the real interest rate are constants.*

Proposition 4 *Along the BGP, sectoral TFP growth rates satisfy*

$$\gamma_i^{1-\phi_i} = \gamma_j^{1-\phi_j} \quad \forall i \tag{29}$$

Thus, for any two sectors i, j *with* $\gamma_i, \gamma_j \geq 1$, *we have* $\gamma_i \geq \gamma_j$ *if and only if* $\phi_i \geq \phi_j$.

Condition (29) follows immediately from (26). It implies that along the aggregate balanced growth path TFP growth rates can be different across sectors but they must be "balanced" according to their fertility. Moreover, for any two sectors with positive TFP growth, the more fertile sector grows faster. Note that (26) implies that sector i has positive TFP growth along the aggregate balanced growth path only when its fertility is less than unity.

Corollary 2 *Along the BGP, the relative price of capital grows by a constant factor:*

$$g_q = \frac{q_{t+1}}{q_t} = (\gamma_x^*)^{(1-\phi_x)/(1-\phi_c)-1}. \tag{30}$$

Corollary (2) follows from Lemma 1 and Proposition 4. In the model, the relative price of capital is falling as in Figure 1 when TFP growth in the "aggregate" consumption sector is slower than TFP growth in the "aggregate" capital sector. If productivity growth is positive in both, then the relative price of capital is falling if and only if $\phi_x > \phi_c$: it must be easier to build on previous knowledge in the durable goods sector than in non-durables.

Along the balanced growth path, the R&D expenditure share is the same as R&D employment share within any sector i. Thus, we refer to $l_i / (l_i + n_i)$ as "R&D intensity" in any sector i.

Proposition 5 *Along the balanced growth path,*

$$\frac{n_i}{l_i} = \frac{G\gamma_i^{\phi_i}/\gamma_x - (1-\delta_T)}{\gamma_i - (1-\delta_T)} - \phi_i (1-\sigma_i) \qquad \forall i. \tag{31}$$

Thus, for any two sectors i, j *with* $\gamma_i, \gamma_j \geq 1$, *if* $\phi_i \geq \phi_j$ *and* $\phi_i (1-\sigma_i) \geq \phi_j (1-\sigma_j)$ *then* $l_i / (l_i + n_i) \geq l_j / (l_j + n_j)$.

Proposition 4 implies that the ranking of sectoral growth rates depends on only one parameter – fertility – and Proposition 5 implies that the ranking of sectoral R&D intensities depends on only two parameters – fertility and spillovers. Spillovers matter because, to the individual firm, the *private fertility* of knowledge (i.e. the intertemporal spillover they perceive) is $(1-\sigma_i)\,\phi_i$, which decreases with σ_i.[8] Consequently, TFP growth rates and R&D intensity need not be correlated in the model. On the other hand, if the values of σ_i do not vary much across industries, then R&D intensity and TFP growth may be correlated in the model – because both are related to the underlying parameter ϕ_i, not because R&D intensity causes TFP growth.

Some empirical studies do find thatother factors may be related to R&D intensity, although results can vary from paper to paper. Considering the relationship between their findings and ours suggests further directions for exploration.

First, Cohen and Levin (1989) provide an extensive survey of the literature on the determinants of R&D intensity. A point that stands out from their survey and from the literature overall is the difficulty of identifying both adequate measures of knowledge and adequate indicators of its correlates. For example, patent counts (sometimes weighted by citations) are a commonplace measure of knowledge output. The survey of Griliches (1990) contains a number of caveats regarding the use of patent data for such purposes, particularly that a central determinant of the behavior of patent and citation-weighted patent series may lie in the institutions that regulate the patenting process itself. Caballero and Jaffe (1993) find a decline in the fertility of research in a one-sector model, but also note that this finding is related to an increasing propensity to cite. Samaniego (in press) provides an example of how the use of patent data may bias results if there is a systematic shift in the propensity to patent (or cite), as this would cause a divergence between the true stock of knowledge and its proxy. More generally, an incorrect empirical model of knowledge will lead to biased estimates of its relationship to and influence upon economic factors (including its own production function). Our approach contrasts with that of the related literature because it does not require a measure of the knowledge stock, but rather identifies the properties of the production function for knowledge on the basis of its long run growth implications, within the model framework.

Second, the related empirical work mostly lacks a notion of ϕ_i.[9] If ϕ_i is hard to identify separately from other candidate determinants of R&D or if it is correlated with them, then its omission may introduce biased estimates. For example,

[8] Proposition 5 implies a negative relationship between intra-industry spillovers and R&D intensity controlling for other variables (the $\phi_i' s$). This is consistent with the findings of Bernstein and Nadiri (1989).

[9] Exceptions include Caballero and Jaffe (1993) and Kortum (1997).

Pakes and Schankerman (1984) are unable to separately identify opportunity and appropriability in their model, but ascribe a large role in R&D intensity differences to their combination, as reflected in an industry fixed effect. However, if the empirical counterpart of ϕ_i is not independently identified, it may also enter the fixed effect. Alternatively, it could be that the determinants of ϕ_i are the same determinants of the other parameters – for example, if ϕ_i is related to the size or density of the network of researchers in a given field, it could be correlated with measures of market size, without there being any direct link between market size and R&D intensity through demand side channels. It would be interesting to estimate R&D intensity with a more complete set of industry variables, including a measure of ϕ_i. As noted, Caballero and Jaffe (1993) do estimate an aggregate equivalent of ϕ_i in a structural framework, but they do not report cross-industry estimates nor explore the implications of their findings for R&D intensity.

4 Quantitative Analysis

We now turn to the quantitative implications of our model for TFP growth rates and R&D intensities. First, we would like to see whether a BGP exists for parametrizations that are consistent with the data. Second, given the link between TFP growth rates and fertility in the model, we would like to see what values of ϕ_i the model suggests are consistent with the data. Finally, the model implies that differences in R&D intensity depend only on fertility ϕ_i and spillovers σ_i, and it is of interest to see which of the two might be the quantitatively dominant factor.

Along the balanced growth path, TFP growth rates in the model depend on parameters α, g_N and ϕ_i, while R&D intensities depend, in addition, on δ_T and σ_i. We calibrate these parameters to post-war US data.[10]. First, we consider a special case where the economy has only one capital sector, denoted x, and one consumption sector, denoted c. Thus, $z = m = 2$. As discussed in the introduction, a large literature studies investment-specific technical change, so the behavior of the model economy in this case may shed light on the phenomenon. Second, we consider a version in which there are many capital goods sectors, so that $z > m = 2$.

[10] Wherever possible, we use data for the period 1947-2000, as this is the time period over which Cummins and Violante (2002) report the data that we will use to impute cross-sectoral differences in TFP growth as described below.

4.1 On TFP growth rates

For the 2-sector economy, Lemma 1 relates the changes in relative prices to changes in TFP growth rates, Proposition 1 links TFP growth in the capital sector to aggregate growth, and Proposition 4 links TFP growth in the capital sector to the consumption sector.

To begin, we set $\alpha = 0.35$. Values between 0.3 and 0.4 are common, so we adopt an intermediate value. The US NIPA indicate that $g_y = 1.022$ in consumption units, and the US Census Bureau that $g_N = 1.012$. In the model, g_y also represents the growth of real consumption, so $g_y = \gamma_x^{1/(1-\alpha)} g_q$ where $g_q = \gamma_c/\gamma_x$ is growth in the relative price of capital. The value of g_q will be important for our quantitative results. As a benchmark, we use the quality-adjusted relative prices reported by Cummins and Violante (2002), and explore other values below. They find that $g_q = 1.026^{-1}$, so that $\gamma_x = 1.0313$ and $\gamma_c = 1.0052$. These values are all that are required to derive implicit fertility values in the model economy.

Equations (25) and (26) imply that $\phi_x = 0.075$, and $\phi_c = -4.53$. Thus, knowledge in the capital goods sector weakly enhances the ability to produce new knowledge, whereas in the consumption goods sector the production of new knowledge becomes more difficult to produce as it accumulates.

To assess the sensitivity of the results to our choice of relative prices, we performed two experiments. First, we repeated the procedure, this time using the official relative price of capital as reported by the Bureau of Economic Analysis. This price decreases by 0.8% on average over the period in question, which implies a smaller divergence of TFP across sectors. The calibrated value of γ_x is correspondingly lower, as is the value of ϕ_x. Second, we examined a range of values of g_q both larger and smaller than those suggested by the data, to see how robust the basic growth parameters of the model are to these values. Relative prices ranged between 1 – so that the relative price of capital is constant – and 1.04, above which the implied ϕ_c is too high for n_c/l_c in (31) to be positive.[11] Results are reported in Table (1). When the official price deflators are used instead of quality-adjusted prices, the value of ϕ_x drops and becomes negative, although it remains close to zero. ϕ_c is larger than before, since productivity growth in that sector is not as low. On the other hand, when the relative price of capital decreases by about 4% annually, TFP growth in non-durables almost shuts down, so ϕ_c become more negative. ϕ_x is positive, but

[11] It is worth underlining that, for parameters satisfying Proposition 1, the economy always has a BGP. However, this does not imply that, for any given values of (α, g_n, g_y), there exist parameters that can match *any* value of g_q. Indeed, a positive finding of our model is that, for the empirically relevant range of g_q, a parametrization always exists that is consistent with a BGP.

bounded below 17%.

γ_x/γ_c	γ_x	γ_c	ϕ_x	ϕ_c	Source
1.000	1.014	1.014	-0.38	-0.38	Lower bound
1.008	1.02	1.011	-0.16	-0.97	BEA
1.018	1.026	1.008	0.000	-2.26	Intermediate value
1.026	1.031	1.005	0.075	-4.53	CV (2002)
1.040	1.040	1.001	0.164	-78.7	Upper bound

Table (1) – Model economy under different assumptions on the relative price of capital. Values of γ_x/γ_c correspond to the range over which the equilibrium is consistent with a decreasing relative price of capital.

We conclude that, in the baseline model, the values of ϕ_x that are consistent with the data are small or close to zero, and that the corresponding values of ϕ_c are negative. Although we are not aware of any fertility estimates that distinguish among sectors, these small values are consistent with the findings of several authors. For example, using U.S. aggregate data for the period 1950-1993, Jones (2002) obtains a negative value for "aggregate fertility,"[12]. Using data on individual researchers and their patents, Jones (2005b) also argues that a persistent base of knowledge may in fact be a *burden* if current innovators must first acquire the same or greater expertise than past innovators. Kortum (1997) argues that a negative value of aggregate fertility is consistent with a number of medium-term features of the data, such as the relationship between productivity and R&D input over the productivity slowdown. Some estimates of ϕ are larger: see Porter and Stern (2000) or Abdih and Joutz (2006). However, they are based on aggregate patent counts, which could yield biased results if the link between patents and knowledge changes over time – see Grilliches (1990), Lanjouw and Schankerman (1999) and Samaniego (in press) for some of these concerns. Moreover, they do not distinguish between sectors, which vary in their propensity to patent new ideas. If the propensity to patent is higher in industries with high values of ϕ_i, estimates of fertility will be biased due to a compositional effect. This is likely if knowledge in many sectors is managerial or commercial in nature, as discussed below.

To further illustrate the implications of the model, we consider the case where there are several types of capital: $z > m = 2$. The data indicate a range of TFP

[12] Table A1 of Jones (2002) implies $\phi = -15.7$ when the knowledge production function has *constant* returns to rivalrous inputs (labor in his case). When *diminishing* returns are allowed, he obtains values that are as high as $\phi = -2$.

growth rates across capital goods, and it is of interest to see how these translate into differences in sectoral parameters. For these results, we use the benchmark value of $g_q = 1.026$ based on Cummins and Violante (2002). Jorgenson et al (2005) report TFP growth measures for several capital goods industries. However, it is not clear that their methodology of TFP measurement allows these figures to be mapped directly into the current framework.[13] Instead, we impute TFP growth rates using Lemma 1, which implies a relationship between relative prices and relative TFP. Cummins and Violante (2002) also provide data on the quality-adjusted relative price of capital for a comprehensive list of durable good types, which can be used to compute the long-run TFP growth rate for that good implied by Lemma 1.[14]

Results are reported in Table (2). Based on relative prices, TFP growth rates across capital types range from 20% for computers and office equipment to 1.3% for structures. Thus, the model economy is consistent with a wide dispersion of TFP growth rates. A prediction of the model (and of many multi-sector models) is that changes in the relative prices of goods should be related to the TFP growth rates of the sectors that produce those goods, as measured by other methods. Hence, Table (2) also reproduces the TFP growth rates from Jorgenson et al (2005), for the period $1960 - 2004$, and compares them to the TFP growth rates implied by the model and the relative prices. It is to be expected that the values generated by the model are higher: the price data used to compute model TFP are quality-adjusted, whereas Jorgenson et al (2005) do not adjust for improvements in output quality. Nonetheless, the correlations between the two series are high: 91% for the full series, and 60% if one excludes Computers and Office Equipment, which is an outlier in both cases. The Spearman rank correlation is a remarkable 62%.

Values of ϕ_i in Table 2 are below unity for all sectors. The data are consistent with a wide distribution of values of ϕ_i across different types of capital good when compared to the value of ϕ_x for the "aggregate" capital sector in Table 1. Interestingly, all capital goods industries appear to be more "fertile" than non-durables when compared to ϕ_c in Table 1.[15]

[13] In particular, Jorgenson et al (2005) account for intermediate goods in their measurement strategy, whereas the current model lacks intermediate goods.

[14] TFP estimates are based on relative prices of capital goods (adjusted for quality following Gordon (1990)) and the benchmark value of γ_s. See Cummins and Violante (2002) for details on the construction of the price indices. In Table 2, industries are aggregated to yield a consistent partition between the classifications used by the two sources – using the BEA investment expenditure shares for the price data, and Domar weights for the JHS data. We are grateful to Gianluca Violante for providing us with relative price data.

[15] In a more disaggregated industry classification we find that only one category – Agricultural Machinery excluding Tractors – is less fertile than the consumption sector.

Capital good sector	γ_i^{MODEL}	ϕ_i	γ_i^{JHS}
Computers and office equipment	20.15	0.84	10.40
Communication equipment	9.66	0.69	1.24
Aircraft	8.82	0.66	0.74
Instruments and photocopiers	6.29	0.53	1.17
Autos and trucks	3.50	0.17	0.29
Fabricated metal products	3.32	0.13	0.56
Electrical transm. distrib. and industrial appl.	3.22	0.10	0.55
Other	2.98	0.03	0.51
Ships and boats	2.67	-0.08	0.45
Machinery	2.58	-0.12	0.19
Electrical equipment, n.e.c.	2.51	-0.15	0.83
Mining and oilfield machinery	1.89	-0.52	-0.16
Furniture and fixtures	1.66	-0.73	0.89
Structures	1.34	-1.14	-1.05

Table (2) – Total factor productivity growth γ_i in the model, based on the quality-adjusted price of capital (Cummins and Violante (2002)). We also report TFP values from Jorgenson et al (2005) (JHS). The Spearman rank correlation is 62%.

4.2 On R&D intensities

We now turn to the quantitative implications of the model for R&D intensity in the 2-sector case. According to (31), this requires calibrating three additional parameters: δ_T, σ_x and σ_c. In the model, R&D is any activity that increases TFP. Thus, the empirical counterpart of L_i in the data includes not just research scientists but everyone whose function is to complement and support their research activities rather than the production of final goods and services, e.g. their laboratory assistants, administrative staff, sanitation and security service providers, etc. It could also include the labor of people not occupied in research institutions if their activities add to the accumulation of knowledge, such as certain management, marketing, and other activities.[16] Thus, the empirical counterpart of model R&D employment probably involves "much more" than the numbers reported by the NSF. Still, although direct

[16] For example, in quality-adjusted terms, an activity that identifies that a certain scent or color of soap is more valued by customers than another might be considered TFP-enhancing, *caeteris paribus*.

comparison may be difficult, our results are suggestive as to the relative importance of different parameters.

$\delta_T =$	0.0	0.01	0.05	0.1	0.2	0.3
R&D intensity, c	7.6	11.4	13.9	14.6	14.9	15.1
R&D intensity, x	32.7	35.9	42.3	45.5	48.1	49.2

$\sigma_x =$	0.0	0.2	0.4	0.6	0.8	1.0
R&D intensity, c	7.6	7.6	7.6	7.6	7.6	7.6
R&D intensity, x	32.7	32.5	32.3	32.2	32.0	31.9

$\sigma_c =$	0.0	0.2	0.4	0.6	0.8	1.0
R&D intensity, c	7.6	8.1	8.8	9.5	10.4	11.5
R&D intensity, x	32.7	32.7	32.7	32.7	32.7	32.7

Table (3) – R&D intensity: Sensitivity to δ_T, σ_x and σ_c. Results assume that all paremeters but the one of interest are set to their benchmark values – however, this is without loss of generality since R&D intensity in any given sector does not depend on spillovers in other sectors.

We will study several different values. Nonetheless, a benchmark is required. Nadiri and Prucha (1996) estimate depreciation rates for knowledge of 12%, and Pakes and Schankerman (1984) find values up to 25%. However, these are all measures of the *economic* depreciation of ideas and, as with capital, this is distinct from "physical" depreciation. Consequently, the value of δ_T in the model is likely to be considerably smaller. Indeed, while the notion of "physical" depreciation is easy to interpret for the case of capital, the same is not the case for knowledge. We use $\delta_T = 0$ as a benchmark. Estimates of cross-industry R&D spillovers that might be useful for calibrating σ_i exist only for the manufacturing sector, and mostly for a subset of industries that are responsible for the bulk of reported R&D. Nonetheless, Schultz (2006) finds that statistically significant spillovers do not exist for most manufacturing industries. Adams and Jaffe (1996) also argue that spillovers are unlikely to be a significant source of returns to R&D. We select $\sigma_x = 0$ and $\sigma_c = 0$ as a benchmark.

The sensitivity of R&D intensity to δ_T, σ_x and σ_c is displayed in Table (3). Raising δ_T increases the increases the R&D intensity of both sectors as larger R&D investments are required for given rates of productivity growth. The spillover parameters have far less impact on R&D intensity. Their impact is consistent with Proposition

5: however, the primary determinant of R&D intensity appears to be ϕ_i, not σ_i. Thus, in the model economy, TFP growth rates and R&D intensity may be highly correlated in cross-section, because they are both primarily determined by ϕ_i and not because there is any causal relationship between them. Indeed, Wilson (2002) finds that R&D intensity is indeed related to the decline in the quality-adjusted price of capital goods in industry cross-section – which in the current model is indicative of relative TFP growth differences.

5 More on the Knowledge Production Function

5.1 Fertilization across sectors

A central result of the paper is that, in the model economy, long run differences in TFP growth persist only to the extent to which industries differ in terms of knowledge fertility ϕ_i. The model economy, however, assumes that there are no knowledge spillovers across sectors. Can differences in TFP growth rates persist even when such "cross fertilization" is allowed for, and might other parameters have a role to play when such spillovers are possible?

Suppose for simplicity that there are two sectors i and j in the model economy. The production function for knowledge is:

$$F_{iht} = A_i T_{jt}^{\phi_{ij}} \left(T_{iht}^{1-\sigma_i} T_{it}^{\sigma_i}\right)^{\phi_i} Q_{iht}^{\alpha} L_{iht}^{1-\alpha} \tag{32}$$

This is the same as (8), except that now it is possible for knowledge in sector i to influence sector j, and vice-versa. ϕ_{ij} is the extent to which sector i benefits from knowledge produced in sector j. In this case, deriving the properties of a balanced growth path yields the following results:

Proposition 6 *Proposition 1 holds when Φ is replaced by*

$$\Phi_1 = \left[1 - \phi_x - \phi_{xc}\left(\frac{1-\phi_x+\phi_{cx}}{1-\phi_c+\phi_{xc}}\right) - \frac{\alpha}{1-\alpha}\right]^{-1} \tag{33}$$

Proposition 7 *Along the balanced growth path, sectoral TFP growth rates satisfy the condition*

$$\gamma_c^{1-\phi_c+\phi_{xc}} = \gamma_x^{1-\phi_x+\phi_{cx}} \tag{34}$$

Thus, if $\gamma_c \geq 1$, then $\gamma_x \leq \gamma_c$ if and only if $\phi_x + \phi_{xc} \geq \phi_c + \phi_{cx}$.

It remains the case that differences in TFP growth rates are related only to the fertility parameters. What matters is no longer simply the "self-fertilization" parameter, but the sum of all the fertility parameters that benefit a given sector, regardless of which industry generates the knowledge. When industry cross-pollination is allowed for, the aggregate growth rate may depend upon fertility parameters other than those in the durables sector, but only to the extent to which durable goods benefit from knowledge produced in other industries. This is because there is an extra channel whereby the production of knowledge pertaining to durables may benefit that sector, via feedback through knowledge production in non-durables.

5.2 Duplication in research

We next consider the possibility of duplication in research, i.e. the knowledge production function may not be constant returns to rivalrous inputs:

$$F_{iht} = A_i \left(T_{iht}^{1-\sigma_i} T_{it}^{\sigma_i}\right)^{\phi_i} \left[Q_{iht}^{\alpha} L_{iht}^{1-\alpha}\right]^{\psi_i} \tag{35}$$

where ψ_i is the returns to rivalrous inputs in sector i. This departs from the empirical literature in that the stock of R&D spending no longer measures the stock of knowledge.

Proposition 8 *Proposition 1 holds when Φ is replaced by*

$$\Phi_2 = \left(\frac{1-\phi_x}{\psi_x} - \frac{\alpha}{1-\alpha}\right)^{-1}, \tag{36}$$

where $\psi_x \equiv \sum_{j=m}^{z} \frac{\psi_j \kappa_j (1-\phi_x)}{1-\phi_j}$ is a weighted average of ψ_j in the capital-producing sectors.

Proposition 8 is identical to Proposition 1 if we replace $(1-\phi_x)$ by $(1-\phi_x)/\psi_x$. As in Proposition 1, the only restriction on fertility is on the capital goods sector. The value of ϕ_x is still bounded away from unity due to the role of capital in the knowledge production function, but this bound is less strict when there is duplication in research in the "aggregate" capital sector ($\psi_x < 1$). Note that Jones (1995) also allows for $\psi < 1$, in fact our relationship $\gamma_x = g_N^{\Phi_2}$ is identical to that of Jones (1995) if $\alpha \to 0$. However, by assuming away capital in the knowledge production function, Jones' restriction is independent of ψ. Therefore allowing capital to be an input into the production of ideas not only implies that Jones' restriction is not sufficient (as we discuss in Proposition 1), but it also implies that the degree of diminishing returns also matters for the restrictions required by constant aggregate growth.

Proposition 9 *Along the balanced growth path, sectoral TFP growth rates satisfy the condition*

$$\gamma_i^{\frac{1-\phi_i}{\psi_i}} = \gamma_j^{\frac{1-\phi_j}{\psi_j}} \tag{37}$$

Thus, for any two sectors i, j with $\gamma_i, \gamma_j \geq 1$, we have

$$\gamma_i \geq \gamma_j \textit{ if and only if } \frac{\psi_i}{1-\phi_i} \geq \frac{\psi_j}{1-\phi_j}. \tag{38}$$

Proposition 8 is identical to Proposition 1 if we replace $\frac{1-\phi_i}{\psi_i}$ with $1-\phi_i$. The derminant of long run differences in TFP growth rates is still the fertility parameter, now adjusted by the degree of returns to rivalrous inputs. Observe that, if ψ_i takes similar values across sectors, the range of fertility values that are consistent with the data does not depend on the value of ψ_i itself.

6 Concluding remarks

We develop a multi-sector endogenous growth model in which technical progress is driven by the production of new knowledge. The model is specified so that the production functions for goods and for knowledge are close to those estimated in the empirical literature. In the model, we find that the only long-run determinant of productivity growth differences across sectors is the parameter which dictates the extent of decreasing returns to pre-existing knowledge in the production of new ideas – the extent to which knowledge is *fertile*. Although this parameter has not been identified as a potentially important source of cross-industry differences, it turns out to play a pivotal role in general equilibrium, and a contribution of our paper is to study the behavior of an economy in which this parameter may differ across industries. Moreover, the restriction on this parameter that determines the existence of a balanced growth path applies only to the fertility of knowledge in the *capital goods sector*, so that very large or very small intertemporal returns to knowledge "on average" may be consistent with balanced growth.

Our growth model attempts to take into account a number of factors that growth theorists and researchers of R&D have independently thought to be of potential importance to productivity change and R&D intensity. Interestingly, we find that inter-sectoral differences in productivity and R&D activity cannot be driven by most of those factors. Whether or not improvements in the quality of measures of knowledge eventually allow empirical work to confirm or reject this prediction, we suspect that any apparent tension between general equilibrium theories of growth and theories of R&D may lie in the nature of models of knowledge that underlie these theories. The rigorous formulation and testing of different theories of the way economically useful knowledge is produced, stored and disseminated remains a fertile area for future research.

Given the centrality of the fertility for knowledge, ϕ_i, one may ask what are its determinants. Nelson and Winter (1977) argue that innovations often follow "natural trajectories" that have a technological or scientific rationale rather than being driven by movements in demand. Similarly, Rosenberg (1969) talks of innovation following a "compulsive sequence." In a general equilibrium framework, demand plays an essential role in providing incentives to innovate: however, in our model the primary determinant of long run growth rates is the usefulness or relevance of past knowledge for the generation of new ideas, something that is reminiscent of these "natural trajectories" since it is leads to parametrically determined long-run TFP growth rates. Mokyr (2002) suggests a nuanced view of knowledge, whereby there is a distinction between the set of techniques that are known and the epistemic knowledge that underlies them. Perhaps ϕ_i reflects differences in the structure of the feedback between the techniques suggested by new epistemic knowledge and the epistemic knowledge that is generated by the application or refinement of new techniques. Furman and Stern (2006) identify the notion of "cumulativeness," which is related to the institutions that allow scientists access to preexisting knowledge. Although it is unclear a priori whether cumulativeness is related to ϕ_i, A_i, or both, this suggests that studying the networks that underlie the storage and transition of knowledge may be useful, and that there may be institutional dimensions to the value of ϕ_i. This raises potentially interesting policy questions.[17] In all, a theory of ϕ_i could be an interesting extension of the paper.

[17] The Jones (1995) result that decreasing returns to knowledge were required for a BGP in the presence of population growth was originally interpreted as ruling out the possibility of policy impacting growth. However, if this is itself an institutional parameter, then policies that, for example, enhance the density of the network of scientists may have a role to play in determining long run growth rates.

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A Derivations

A.1 Household maximization

We first determine the optimal spending across different goods taken as given the total per capita spending on consumption s_c and total spending on investment s_x. Omitting time subscript, the maximization problems across goods are

$$\max_{\{c_{ih}\}} c \qquad s.t. \qquad s_c = \sum_{i=1}^{m-1} \int_0^1 p_{ih} c_{ih} dh, \qquad \text{and}$$

$$\max_{\{x_{jh}\}} x \qquad s.t. \qquad s_x = \sum_{j=m}^{z} \int p_{jh} x_{jh} dh$$

where c and x are defined in the household problem. The optimal spending within sectors $i = 1, ..m-1$, across different varieties h, is

$$\left(\frac{c_{ih}}{c_{ih'}}\right)^{\frac{-1}{\mu_i}} = \frac{p_{ih}}{p_{ih'}} \Longrightarrow c_{ih'} = c_{ih} \left(\frac{p_{ih}}{p_{ih'}}\right)^{\mu_i} \tag{39}$$

which implies

$$c_i = \left(\int_0^1 c_{ih'}^{\frac{\mu_i - 1}{\mu_i}} dh'\right)^{\frac{\mu_i}{\mu_i - 1}} = c_{ih} \left[\int \left(\frac{p_{ih}}{p_{ih'}}\right)^{\mu_i - 1} dh'\right]^{\frac{\mu_i}{\mu_i - 1}} \tag{40}$$

Using (40), define price index, $p_i \equiv \frac{\int p_{ih} c_{ih} dh}{c_i} = \left[\int p_{ih}^{1-\mu_i} dh\right]^{1/(1-\mu_i)}$. We can now rewrite (40) as $c_i = c_{ih} \left(\frac{p_{ih}}{p_i}\right)^{\mu_i}$. Across good i, Cobb-Douglas utility yields $\frac{p_i c_i}{p_j c_j} = \frac{\omega_i}{\omega_j}$, so $p_i c_i = \omega_i s_c$, together with utility function,

$$p_c \equiv \frac{s_c}{c} = \prod_{i=1}^{m-1} p_i^{\omega_i} \tag{41}$$

and the demand for good ih is

$$c_{ih} = s_c \left(\frac{p_i}{p_{ih}}\right)^{\mu} \frac{\omega_i}{p_i} \tag{42}$$

The result follows analogously for investment,

$$x_{jh} = s_x \left(\frac{p_j}{p_{jh}}\right)^{\mu_j} \left(\frac{\kappa_j}{p_j}\right) \text{ and } x_j = s_x \left(\frac{\kappa_j}{p_j}\right), \tag{43}$$

where p_j is analogously defined as before, and

$$p_x \equiv \frac{s_x}{x} = \prod_{j=m}^{z} p_j^{\kappa_j} \tag{44}$$

Given the solution of the static maximization, the dynamic problem is

$$\begin{aligned}
& \max_{\{c_t, x_t\}} \sum_{t=0}^{\infty} (\beta g_N)^t u(c_t) \\
& s.t. \\
p_{ct} c_t + p_{xt} x_t &= w_t + R_t k_t + \pi_t \\
g_N k_{t+1} &= x_t + (1-\delta_k) k_t
\end{aligned}$$

The solution implies

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{p_{xt+1}/p_{ct+1}}{p_{xt}/p_{ct}} \left(1 - \delta_k + \frac{R_{t+1}}{p_{xt}}\right) \tag{45}$$

A.2 Firm's maximzation

The firm's maximization problem is

$$\begin{aligned}
& \max_{\{N_{it}, K_{it}, Q_{it}, L_{it}\}} \sum_{t=0}^{\infty} \lambda_t \frac{\Pi_{iht}}{p_{ct}} \\
& s.t. \\
T_{iht+1} &= f_{iht} + (1-\delta_T) T_{iht} \\
Y_{iht} &= N_t c_{iht} \qquad if \qquad i = 1, ..m-1 \\
Y_{iht} &= N_t x_{jht} \qquad if \qquad i = m, ...z
\end{aligned}$$

Given Π_{iht} in (10), the static efficiency conditions are standard:

$$\frac{\partial Y_{iht}/\partial N_{iht}}{\partial Y_{iht}/\partial K_{iht}} = \frac{w_t}{R_t} = \frac{\partial f_{iht}/\partial L_{iht}}{\partial f_{iht}/\partial Q_{iht}}$$

The assumptions on production functions imply so, for any two sectors i, j, $K_{iht}/N_{iht} = Q_{iht}/L_{iht}$, and $p_{iht}\left(1 + \frac{Y_{jht}}{p_{jht}} \frac{\partial p_{jht}}{\partial Y_{jht}}\right) T_{iht} = p_{jht}\left(1 + \frac{Y_{jht}}{p_{jht}} \frac{\partial p_{jht}}{\partial Y_{jht}}\right) T_{jht}$. Using the demand function, relative prices is

$$\frac{p_{iht}}{p_{jht}} = \frac{T_{jht}\left(1 - 1/\mu_j\right)}{T_{iht}\left(1 - 1/\mu_i\right)} \tag{46}$$

The dynamic efficiency condition involves the optimal R&D decision. The first order condition of the Lagragian with respect to T_{iht+1} is

$$\frac{\lambda_{t+1}}{p_{ct+1}}\frac{\partial \Pi_{iht+1}}{\partial T_{iht+1}} - \chi_{iht} + \chi_{iht+1}\left(\frac{\partial f_{iht+1}}{\partial T_{iht+1}} + 1 - \delta_T\right) = 0 \tag{47}$$

where

$$\chi_{iht} = \left(\frac{\lambda_t}{p_{ct}}\right)\frac{R_t}{\partial f_{iht}/\partial Q_{iht}} \tag{48}$$

is the shadow price for T_{iht+1}.

A.3 Market Equilibrium

The capital market clearing condition (11) and equal capital-labor ratios (49) imply

$$\frac{K_{iht}}{N_{iht}} = \frac{Q_{iht}}{L_{iht}} = \frac{K}{N} = k, \qquad \text{and} \tag{49}$$

$$R_t = \alpha p_{iht} T_{iht} k_t^{\alpha-1}\left(1 - \frac{1}{\mu_i}\right); \qquad w_t = (1-\alpha)\, p_{iht} T_{iht} k_t^{\alpha}\left(1 - \frac{1}{\mu_i}\right). \tag{50}$$

together with χ_{it} in (48), dynamic equation (47) becomes

$$\begin{aligned} & \frac{\lambda_t p_{iht}/p_{ct}}{\lambda_{t+1} p_{iht+1}/p_{ct+1}}\left(\frac{T_{iht}}{\left(T_{iht}^{1-\sigma_i} T_{it}^{\sigma_i}\right)^{\phi_i}}\right) \\ = \quad & A_i k_{t+1}^{\alpha} N_{iht+1} + \frac{T_{iht+1}}{\left(T_{iht+1}^{1-\sigma_i} T_{it+1}^{\sigma_i}\right)^{\phi_i}}\left(\phi_i (1-\sigma_i)\frac{F_{iht+1}}{T_{iht+1}} + 1 - \delta_T\right) \end{aligned} \tag{51}$$

We now focus on the symmetric equilibrium across h within i.

B Proofs

Proof of Lemma 1. See the derivation of (49) and (46) in the Firm's maximization. ■

Proof of Lemma 2. Define T_x such that:

$$N_t x \equiv T_x \left(\sum_{i=m}^{z} K_{it}\right)^{\alpha}\left(\sum_{i=m}^{z} N_{it}\right)^{1-\alpha} = T_x k^{\alpha} \sum_{i=m}^{z} n_{it}$$

where the equality follow from (49). To determine T_x, (43) implies $\frac{p_j T_j k^\alpha n_j}{p_i T_i k^\alpha n_i} = \frac{\kappa_j}{\kappa_i}$ and, by (46), $\frac{n_i}{\kappa_i} = \left(\frac{1-1/\mu_i}{1-1/\mu_j}\right)\frac{n_j}{\kappa_j}$, so

$$\sum_{i=m}^{z} n_{it} = \frac{n_j\left(1-1/\mu_x\right)}{\kappa_j\left(1-1/\mu_j\right)} \tag{52}$$

where we define μ_x such that

$$1-\frac{1}{\mu_x} = \sum_{i=m}^{z}\kappa_i\left(1-\frac{1}{\mu_i}\right) \Leftrightarrow \frac{1}{\mu_x} = \sum_{i=m}^{z}\frac{\kappa_i}{\mu_i} \tag{53}$$

By definition, $x = \prod_{j=m}^{z}\left(\frac{x_j}{\kappa_j}\right)^{\kappa_j} = \prod_{j=m}^{z}\left(\frac{T_j k^\alpha n_j}{\kappa_j}\right)^{\kappa_j}$, so using (52) and (53), we obtain

$$x = k^\alpha\left(\sum_{i=m}^{z} n_{it}\right)\prod_{j=m}^{z}\left[T_j\left(\frac{1-1/\mu_j}{1-1/\mu_x}\right)\right]^{\kappa_j}$$

so the index of knowledge is

$$T_x = \left[\prod_{j=m}^{z}\left(\frac{1-1/\mu_j}{1-1/\mu_x}\right)\right]\prod_{j=m}^{z} T_j^{\kappa_j}. \tag{54}$$

■

Proof of Lemma 3. The Euler condition follows from (45) in the Consumer's Maximization. Using (44) and (46),

$$\begin{aligned}\frac{p_x}{p_i} &= \prod_{j=m}^{z}\left(\frac{p_j}{p_i}\right)^{\kappa_j} = \prod_{j=m}^{z}\left(\frac{T_i\left(1-1/\mu_i\right)}{T_j\left(1-1/\mu_j\right)}\right)^{\kappa_j} \\ &= T_i\left(1-1/\mu_i\right)\prod_{j=m}^{z}\left[T_j\left(1-1/\mu_j\right)\right]^{-\kappa_j} \quad \forall i,\end{aligned}$$

so by (54), we have

$$\frac{p_x}{p_i} = \frac{T_i\left(1-1/\mu_i\right)}{T_x\left(1-1/\mu_x\right)} \quad \forall i, \tag{55}$$

together with (50), we have

$$\frac{R}{p_x} = \frac{\alpha p_i T_i k^{\alpha-1}\left(1-\frac{1}{\mu_i}\right)}{p_x} = \alpha k^{\alpha-1} T_x\left(1-\frac{1}{\mu_x}\right),$$

so the expression for G follows from its definition. ■

Proof of Lemma 4. Euler equation (45) implies

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1+r_{t+1}} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{p_{xt}/p_{ct}}{G_{t+1}p_{xt+1}/p_{ct+1}} \tag{56}$$

Using (55), $\frac{p_{xt+1}/p_{it+1}}{p_{xt}/p_{it}} = \frac{\gamma_i}{\gamma_x}$, substitute into (51),result follows for (19). ■

Lemma 5 *The equilibrium employment shares for production are*

$$\begin{aligned} n_i &= \omega_i \frac{c/q}{T_x k^\alpha}\left(\frac{1-1/\mu_i}{1-1/\mu_x}\right) \qquad \forall i < m, &(57)\\ n_j &= \frac{1-\mu_j}{1-\mu_x}\kappa_j\left(\sum_{j=m}^{z} n_{jt}\right) \qquad \forall j \geq m. &(58)\end{aligned}$$

If positive, the R&D intensity of the firm satisfies

$$\frac{n_{it+1}}{l_{it+1}} = \frac{G_{t+1}\gamma_{it}^{\phi_i}/\gamma_{xt} - (1-\delta_T)}{\gamma_{it+1} - (1-\delta_T)} - \phi_i(1-\sigma_i) \qquad \forall i. \tag{59}$$

Proof. Combine (19) with (7) to derive n_i/l_i. To obtain n_i, use market clearing and the expenditure share of good $i = 1, ..m-1$, $p_i T_i k^\alpha n_i = p_i c_i = \omega_i p_c c$, and so $n_i = \omega_i \frac{c/q}{k^\alpha}\left(\frac{p_x}{T_i p_i}\right)$ where $q = p_x/p_c$. The result for n_i follows from (55). The proof for n_j, $j = m..z$ is analogous. ■

Proof of Proposition 1. Define $k_{et} = k_t T_{xt}^{-1/(1-\alpha)}$. Let $g_x \equiv x_{t+1}/x_t$ for all variables x. From the Euler condition (17), g_c is constant if g_q and G are constants. From (18), G is constant if and only if k_e is constant. Use Lemma 2 and (49) to rewrite the capital accumulation equation as $g_N g_k = k_{et}^{\alpha-1}\sum_{j=m}^{z} n_{jt} + 1 - \delta_k$, it follows that g_k is constant if and only if $\sum_{j=m}^{z} n_{jt}$ is constant. So (58) implies n_j are constants for $j = m, ..z$. By defintion, k_e is constant if and only if $g_k = \gamma_x^{-1/(1-\alpha)}$, which by (15) is constant if and only if γ_j are constants for all $j = m, ..z$, which is true by (19) if and only if

$$A_j k_{t+1}^\alpha N_{jt+1} T_{jt+1}^{\phi_j - 1} = A_j k_{et+1}^\alpha T_x^{\alpha/(1-\alpha)} n_{jt+1} N_{t+1} T_{jt+1}^{\phi_i - 1}$$

is constant. Given $\{n_j\}_{j=m,..z}$ are constants, we must have $\forall j \geq m$,

$$\gamma_x^{\alpha/(1-\alpha)} g_N \gamma_j^{\phi_i - 1} = 1 \tag{60}$$

which implies $\forall i, j \geq m$,

$$\gamma_i^{\phi_i - 1} = \gamma_j^{\phi_j - 1}, \tag{61}$$

so $\gamma_x = \prod_{j=m}^{z} \gamma_j^{\kappa_j} = \prod_{j=m}^{z} \left(\gamma_i^{\phi_i - 1}\right)^{\kappa_j / \left(\phi_j - 1\right)}$, so $\forall j \geq m$,

$$\gamma_x^{1-\phi_x} = \gamma_i^{1-\phi_i}, \tag{62}$$

where ϕ_x satisfies $(1-\phi_x)^{-1} = \sum_{j=m}^{z} \kappa_j \left(1-\phi_j\right)^{-1}$. Thus, (60) implies

$$\gamma_x^* = g_N^{\Phi}, \tag{63}$$

where $\Phi = (1 - \phi_x - \alpha / (1-\alpha))^{-1}$. Finally we need to show g_q is constant, using (41),

$$q^{-1} = \prod_{i=1}^{m-1} \left(\frac{p_x}{p_i}\right)^{\omega_i} = \prod_{i=1}^{m-1} \left(\frac{T_i\left(1-1/\mu_i\right)}{T_x\left(1-1/\mu_x\right)}\right)^{\omega_i}, \tag{64}$$

So g_q is constant if and only if γ_i are constants. From (57), n_i are constants for $i < m$. Using (19) for $i < m$, γ_i is constant if and only if (60) holds for $i < m$ as well. Therefore, (61) holds for $i < m$ as well. Use (49) to rewrite (7) as $\gamma_{it+1} = A_i T_i^{\phi_i - 1} N_{t+1} k_t^{\alpha} l_{it} + 1 - \delta_T$. This is constant from (60). l_i is constant from (59). ■

Proof of Proposition 2. The transversality conditions are: $\lim_{t\to\infty} \zeta_t k_{t+1} = \lim_{t\to\infty} \chi_{it} T_{it+1} = 0$, $\forall i$. χ_{it} and ζ_t are the corresponding shadow values. From the Firm's maximization and (55), we have

$$\frac{\chi_{it}}{\chi_{it-1}} = \frac{\lambda_t p_{xt}/p_{ct}}{\lambda_{t-1} p_{xt-1}/p_{ct-1}} \left(\frac{\gamma_x}{\gamma_i}\right) \gamma_i^{1-\phi_i} = \frac{1}{G}\left(\frac{\gamma_x}{\gamma_i}\right) \gamma_i^{1-\phi_i}$$

where the last equality follows from (56). Using (17), $1/G = \beta g_c^{1-\theta} \gamma_x^{-1/(1-\alpha)}$, together with (62),

$$\lim_{t\to\infty} \chi_{it} T_{it+1} = \chi_{i0} T_{i0} \lim_{t\to\infty} \left[\beta g_c^{1-\theta} \gamma_x^{-\alpha/(1-\alpha)} \gamma_x^{1-\phi_x}\right]^t = \chi_{i0} T_{i0} \lim_{t\to\infty} \left(\beta g_N g_c^{1-\theta}\right)^t$$

where the last equality follows from (63). Using (64) and (61),

$$g_q = \gamma_x^{(1-\phi_x)/(1-\phi_c) - 1} \tag{65}$$

where ϕ_c satisfies $1/(1-\phi_c) = \sum_i^{m-1} \omega_i / (1-\phi_i)$. So $g_c = g_q \gamma_x^{1/(1-\alpha)} = \gamma_x^{\Upsilon}$, where $\Upsilon \equiv \frac{1-\phi_x}{1-\phi_c} + \frac{\alpha}{1-\alpha}$. TVC for T_i holds if $\beta < \min\left(1/g_N, \bar{\beta}\right)$, where $\bar{\beta} \equiv (1/g_N)^{1+(1-\theta)\Phi\Upsilon}$.

The lagragian multiplier for k is the discounted marginal utility, $\zeta_t = (\beta g_N)^t \left(\frac{p_{xt}}{p_{ct}}\right) u'(c_t)$, so $\lim_{t\to\infty} \zeta_t k_t = \lim_{t\to\infty} (\beta g_N)^t \left(\frac{qk_t}{c_t}\right) c_t^{1-\theta}$. Since qk/c is constant, TVC for k holds if and only if $\beta < \min\left(1/g_N, \bar{\beta}\right)$. Finally, we derive conditions for positive l_i and n_i. From (7), $l_i > 0$ if and only if $\gamma_i > 1 - \delta_T$, using (62) and (63), we have $g_N^{\Phi/(1-\phi_i)} > (1-\delta_T)^{1/(1-\phi_x)}$. >From (19), $n_i > 0$ if and only if $\left[G_i^{\phi_i}/\gamma_x - (1-\delta_T)\right] - \phi_i (1-\sigma_i)\left[\gamma_i - (1-\delta_T)\right]$. Using (60) and (17)

$$\frac{G\gamma_i^{\phi_i - 1}}{\gamma_x} = \frac{G\gamma_x^{\phi_x - 1}}{\gamma_x} = \frac{\gamma_x^{\phi_x - 1}}{\beta g_c^{1-\theta} \gamma_x^{-1/(1-\alpha)} \gamma_x} = \left(\beta g_N g_c^{1-\theta}\right)^{-1} \tag{66}$$

where the last equality follows from (25). So $G_i^{\phi_i - 1}/\gamma_x > 1$ given $\beta < \bar{\beta}$. A sufficient condition for $n_i > 0$ is $\phi_i (1-\sigma_i) < 1$. ■

Proof of Proposition 3. Using (12) and (57),

$$\frac{y}{c} = \frac{p_x T_x k^\alpha}{c p_c} \sum_{i=1}^{z} \frac{1 - 1/\mu_x}{1 - 1/\mu_i} n_i = \frac{qk}{c} k_e^{\alpha - 1} \sum_{i=1}^{z} \frac{1 - 1/\mu_x}{1 - 1/\mu_i} n_i \Longrightarrow \frac{c}{y} = \frac{\sum_{i=1}^{m-1} n_i / (1 - 1/\mu_i)}{\sum_{i=1}^{z} n_i / (1 - 1/\mu_i)}$$

which is constant given n_i are constants. Using (50), the R&D expenditure share is

$$\frac{\sum_{i=1}^{z} (L_i w + Q_i R)}{\sum_{i=1}^{z} p_i T_i K_i^\alpha N_i^{1-\alpha}} = \frac{(1 - 1/\mu_x) p_x T_x k^\alpha \sum_{i=1}^{z} L_i}{p_x T_x k^\alpha \sum_{i=1}^{z} \frac{1-1/\mu_x}{1-1/\mu_i} N_i} = \frac{\sum_{i=1}^{z} l_i}{\sum_{i=1}^{z} n_i / (1 - 1/\mu_i)}.$$

Finally, the real interest rate is constant from (56). ■

Proof of Proposition 4. Condition (29) follows from (26), which implies $\gamma_i/\gamma_j = \gamma_j^{(\phi_i - \phi_j)/(1-\phi_i)}$. By (26) $\gamma_i \geq 1$ if $\phi_i \leq 1$ $\forall i$. Together they imply: if $\gamma_i, \gamma_j \geq 1$, then $\gamma_i \geq \gamma_j$ if and only if $\phi_i \geq \phi_j$. ■

Proof of Corollary 2. See derivation of (65) in proof for Proposition 1. ■

Proof of Proposition 5. From (31),

$$\frac{n_i}{l_i} - \frac{n_j}{l_j} = \left\{ \begin{array}{c} \frac{G\gamma_i^{\phi_i}/\gamma_x - (1-\delta_T)}{\gamma_i - (1-\delta_T)} - \frac{G\gamma_j^{\phi_j}/\gamma_x - (1-\delta_T)}{\gamma_j - (1-\delta_T)} \\ + \phi_j (1-\sigma_j) - \phi_i (1-\sigma_i) \end{array} \right\}. \tag{67}$$

By (60), the first part of the difference becomes

$$\frac{G\gamma_x^{\phi_x - 1}\left(\frac{\gamma_i}{\gamma_x}\right) - (1-\delta_T)}{\gamma_i - (1-\delta_T)} - \frac{G\gamma_x^{\phi_x - 1}\left(\frac{\gamma_j}{\gamma_x}\right) - (1-\delta_T)}{\gamma_j - (1-\delta_T)}$$

$$= \frac{G\gamma_x^{\phi_x-1}\left(\frac{\gamma_j}{\gamma_x}\right)+\gamma_i-G\gamma_x^{\phi_x-1}\left(\frac{\gamma_i}{\gamma_x}\right)-\gamma_x}{\left[\gamma_i-(1-\delta_T)\right]\left[\gamma_j-(1-\delta_T)\right]}(1-\delta_T)$$

$$= \frac{\left(G\gamma_x^{\phi_x-1}/\gamma_x-1\right)\left(\gamma_j-\gamma_i\right)}{\left[\gamma_i-(1-\delta_T)\right]\left[\gamma_j-(1-\delta_T)\right]}(1-\delta_T)$$

From (66) we have $G\gamma_x^{\phi_x-1}/\gamma_x > 1$, so the first part of (67) is positive if and only if $\gamma_j > \gamma_i$. When $\gamma_i, \gamma_j \geq 1$, condition (60) implies $\gamma_j \geq \gamma_i$ if and only if $\phi_j \geq \phi_i$. The second part of (67) is positve if and only if $\phi_j(1-\sigma_j) \geq \phi_i(1-\sigma_i)$. ■

Proof of Propositions 6 **and** 7. Since T_j is taken as given by firm ih, in terms of the firm dynamic optimization, we just need to replace previous A_i with the term $T_{jt}^{\phi_{ij}}A_i$, which now depends on t. Firm's optimal condition (51) becomes

$$\frac{T_{it}^{1-\phi_i}\lambda_t p_{it}/p_{ct}}{\lambda_{t+1}p_{it+1}/p_{ct+1}}$$
$$= T_{jt}^{\phi_{ij}}A_i k_{t+1}^{\alpha}N_{it+1}+\left(\frac{T_{jt}^{\phi_{ij}}}{T_{jt+1}^{\phi_{ij}}}\right)T_{it+1}^{1-\phi_i}\left[\phi_i(1-\sigma_i)\left(\gamma_{it+1}-1+\delta_T\right)+1-\delta_T\right]$$

As in Lemma 4 using (56) and (55), we have for sector x :

$$\begin{aligned} G_{t+1}\gamma_{xt}^{\phi_x-1} &= T_{ct}^{\phi_{xc}}A_x k_{et+1}^{\alpha}n_{xt+1}T_{xt+1}^{\alpha/(1-\alpha)+\phi_x-1}N_{t+1} \\ &\quad +\gamma_{ct}^{-\phi_{xc}}\left[\ \phi_x(1-\sigma_x)\left[\gamma_{xt+1}-(1-\delta_T)\right]+1-\delta_T\right] \end{aligned} \tag{68}$$

and for sector c :

$$\begin{aligned} G_{t+1}\frac{\gamma_{ct}^{\phi_c}}{\gamma_{xt}} &= T_{xt}^{\phi_{cx}}A_c k_{et+1}^{\alpha}n_{ct+1}T_{xt+1}^{\alpha/(1-\alpha)}T_{ct+1}^{\phi_c-1}N_{t+1} \\ &\quad +\gamma_{xt}^{-\phi_{cx}}\left[\phi_c(1-\sigma_c)\left[\gamma_{ct+1}-(1-\delta_T)\right]+1-\delta_T\ \right] \end{aligned} \tag{69}$$

Constant γ_x and γ_c requires the first term in both (68) and (69) to be constant, i.e. $\gamma_c^{\phi_{xc}}\gamma_x^{\alpha/(1-\alpha)+\phi_x-1}g_N = 1$ and $\gamma_x^{\phi_{cx}}\gamma_x^{\alpha/(1-\alpha)}\gamma_c^{\phi_c-1}g_N = 1$. Combining the two equations

$$\gamma_c^{\phi_{xc}}\gamma_x^{\phi_x-1} = \gamma_x^{\phi_{cx}}\gamma_c^{\phi_c-1} \Leftrightarrow \gamma_x^{1-\phi_x+\phi_{cx}} = \gamma_c^{1-\phi_c+\phi_{xc}}, \tag{70}$$

which implies $\gamma_x = g_N^{\Phi_1}$, where $\Phi_1 = \left(1-\phi_x-\phi_{xc}\left(\frac{1-\phi_x+\phi_{cx}}{1-\phi_c+\phi_{xc}}\right)-\frac{\alpha}{1-\alpha}\right)^{-1}$. ■

Proof of Proposition 8 and 9. Lemma $1-3$ hold as before. Firm's optimal condition (51) becomes

$$
\begin{aligned}
& G_{t+1}\frac{\gamma_{it}}{\gamma_{xt}}\left(\frac{T_{it}}{\left(T_{it}^{1-\sigma_i}T_{it}^{\sigma_i}\right)^{\phi_i}}\right)k_t^{(\alpha-1)(1-\psi_i)}Q_{it}^{1-\psi_i} \\
= \; & \psi_i A_i k_{t+1}^{\alpha}N_{it+1}+\frac{T_{it+1}k_{t+1}^{(\alpha-1)(1-\psi_i)}Q_{it+1}^{1-\psi_i}}{\left(T_{it+1}^{1-\sigma_i}T_{it+1}^{\sigma_i}\right)^{\phi_i}}\left(\phi_i\left(1-\sigma_i\right)\frac{F_{it+1}}{T_{it+1}}+1-\delta_T\right)
\end{aligned}
$$

Using Lemma 1, it simplifies to

$$
G_{t+1}\frac{\gamma_{it}^{\phi_i}}{\gamma_{xt}}=\left\{\begin{array}{c} A_i k_{t+1}^{\alpha}N_{it+1}T_{it+1}^{\phi_i-1}\psi_i k_t^{(1-\alpha)(1-\psi_i)}Q_{it}^{\psi_i-1}+ \\ \left(\frac{k_{t+1}}{k_t}\right)^{\alpha(\psi-1)}\left(\frac{l_{it+1}}{l_{it}}\right)^{1-\psi}g_N^{1-\psi_i}\left[\phi_i\left(1-\sigma_i\right)\left(\gamma_{it+1}-1+\delta_T\right)+1-\delta_T\right]\end{array}\right\}
$$

The restriction for BGP requires the first term to be constant,

$$
\begin{aligned}
& A_i k_{t+1}^{\alpha}N_{it+1}T_{it+1}^{\phi_i-1}\psi_i k_t^{(1-\alpha)(1-\psi_i)}Q_{it}^{\psi_i-1} \\
= \; & \psi A_i k_{et+1}^{\alpha}T_{xt+1}^{\alpha/(1-\alpha)}n_{it+1}N_{t+1}T_{it+1}^{\phi_i-1}k_{et}^{\alpha(\psi_i-1)}T_{xt}^{\alpha(\psi_i-1)/(1-\alpha)}l_{it}^{\psi_i-1}N_t^{\psi_i-1}
\end{aligned}
$$

which is constant if

$$
\gamma_x^{\alpha/(1-\alpha)}g_N\gamma_i^{\phi_i-1}\gamma_x^{\alpha(\psi_i-1)/(1-\alpha)}g_N^{\psi_i-1}=1\Leftrightarrow\left(\gamma_x^{\alpha/(1-\alpha)}g_N\right)^{\psi_i}\gamma_i^{\phi_i-1}=1 \qquad \forall i \tag{71}
$$

so across sectors we have

$$
\frac{\gamma_j^{\phi_j-1}}{\gamma_i^{\phi_i-1}}=\left(\gamma_x^{\alpha/(1-\alpha)}g_N\right)^{\psi_i-\psi_j} \qquad \forall i,j \tag{72}
$$

So $\gamma_x=\prod_{j=m}^{z}\gamma_j^{\kappa_j}=\prod_{j=m}^{z}\left[\left(\gamma_x^{\alpha/(1-\alpha)}g_N\right)^{\psi_i-\psi_j}\gamma_i^{\phi_i-1}\right]^{\kappa_j/\left(\phi_j-1\right)}$, which implies $\gamma_x^{\phi_x-1}=\gamma_i^{\phi_i-1}\left[\gamma_x^{\alpha/(1-\alpha)}g_N\right]^{\psi_i-\psi_x}$, for $i\geq m$, where $\psi_x=\sum_{j=m}^{z}\frac{\kappa_j(1-\phi_x)}{1-\phi_j}\psi_j$. Together with (72)

$$
\gamma_j^{\phi_j-1}=\gamma_x^{\phi_x-1}\left[\gamma_x^{\alpha/(1-\alpha)}g_N\right]^{\psi_x-\psi_j} \qquad \forall j \tag{73}
$$

But (71) implies $\left(\gamma_x^{\alpha/(1-\alpha)}g_N\right)^{\psi_j}=\gamma_j^{1-\phi_j}$, $\forall j$, substitute into (73),

$$
\gamma_j^{\frac{1-\phi_j}{\psi_j}}=\gamma_x^{\frac{1-\phi_x}{\psi_x}} \qquad \forall j \tag{74}
$$

Substitute back to (71), we have $\left(\gamma_x^{\alpha/(1-\alpha)}g_N\right)^{\psi_i}\gamma_x^{\phi_x-1}\left[\gamma_x^{\alpha/(1-\alpha)}g_N\right]^{\psi_x-\psi_i}=1$, which implies $\gamma_x=g_N^{\Phi_2}$, where $\Phi_2=\left(\frac{1-\phi_x}{\psi_x}-\frac{\alpha}{1-\alpha}\right)^{-1}$. ■

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