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Optimizing Randomized Patrols

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Abstract

A key operational problem for those charged with the security of vulnerable facilities (such as airports or art galleries) is the scheduling and deployment of patrols. Motivated by the problem of optimizing randomized, and thus unpredictable, patrols, we present a class of patrolling games on graphs. The facility can be thought of as a graph Q of interconnected nodes (e.g. rooms, terminals) and the Attacker can choose to attack any node of Q within a given time T. He requires m consecutive periods there, uninterrupted by the Patroller, to commit his nefarious act (and win). The Patroller can follow any path on the graph. Thus the patrolling game is a win-lose game, where the Value is the probability that the Patroller successfully intercepts an attack, given best play on both sides. We determine analytically optimal (minimax) patrolling strategies for various classes of graphs, and discuss how our results could support decisions about hardening facilities or changing the topology of the terrain to be patrolled.

Subject classifications: Games, noncooperative; Military, search/ surveillance; Decision Analysis, risk

Area of review: Military and Homeland Security

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1 Introduction

A key operational problem for those charged with the security of vulnerable facilities is the scheduling and deployment of patrols. This problem is encountered by, for example:

- security guards patrolling a museum or art gallery
- antiterrorist officers patrolling an airport or shopping mall
- police forces patrolling a city containing a number of potential targets for theft such as jewelry stores
- soldiers patrolling an occupied city or territory
- air marshals patrolling an airline network
- inspectors patrolling a container yard or cargo warehouse

Such problems have been studied in diverse literatures. For example, a well-known problem in computational geometry deals with the position of security guards in art galleries (Urrutia, 2000) and a classical Operations Research literature exists on the scheduling of police patrols (see e.g. Larson (1972) and references therein). The importance of randomized patrols has been recognized in law enforcement for some time, but not the nature of the randomization (e.g. Sherman and Eck (2002, p. 297)). Much of the optimization literature on this subject (e.g. Chelst, 1978) concentrates on the important problem of how to deploy randomized patrols to maximize the probability of intercepting a crime in progress, *when the crime frequency of different locations is taken as given* (often a realistic assumption, at least in the short term). Such models however are not game theoretic and do not capture the idea of a patrolling schedule as a strategy selected in the face of an intelligent and malign adversary, for example an art thief or terrorist, which is a distinctive feature of the class of models we study in this paper. Although there do exist differential game formulations of the relationship between police and criminal

(Isaacs, 1999, Feichtinger, 1983) these tend to focus on a dynamic (and often strategic) process of mutual adjustment rather than confronting the problem confronted by the scheduler who sits down to determine the path which the patrol will take.

Game theoretic analyses have recently featured prominently in OR studies of homeland security and counterterrorism (e.g. Brown et al, 2006, Bier and Azaiez, 2009, Lindelauf et al., 2009). An attractive and unique feature of game theoretic formulations in the context of patrolling is that they provide insight into how a Patroller should randomize her patrols. There is a clear common-sense rationale for randomization: a predictable Patroller is an ineffective one. Yet a naive "maximum entropy" heuristic (Fox et al, 2005) may be not fare much better: faced with n targets it may not make sense to spend $1/n$ of the available patrolling time with each of them. This dilemma has attracted considerable attention recently amongst practitioners and the research community has responded to this challenge: in particular, the work of Paruchuri and colleagues (2007) provides a number of heuristic models which illustrate how equilibrium randomized strategies can be approximated when the problem is formulated as a Stackelberg (leader-follower) game, and such models have found use in real security situations (Gordon, 2007; Newsweek, 2007).

Our work on this problem is inspired by the theory of search games, on which an extensive mathematical literature developed over the last few decades. This theory captures situations in which a Searcher aims minimize the time taken to find a stationary or mobile Hider who does not want to be found (Alpern and Gal, 2003). There are also related literatures on Inspection games (Avenhaus, von Stengel and Zamir, 2002), in which an Inspector who seeks to catch an Inspectee red-handed, and Infiltration games (Auger, 1991; Garnaev, Garnaeva Goutal, 1996; Garnaev, 2000 and Alpern, 1992) in which a Guard seeks to prevent an Infiltrator from penetrating some sensitive facility. Similar attack/ defence games have been studied in military operations research (Washburn, 2003), dating as far back as Morse and Kimball (1950). Many such games are of independent mathematical interest and have been studied in a purely mathematical settings (e.g. Baston, Bostock and Ferguson, 1989). Various results are available for how the Searcher/ Inspector/ Guard/ Defender should proceed, depending on the assumptions about the structure of the mathematical space which she inhabits. A particularly productive line of research in the search game literature has been to explore the case where the search space can be thought of as a graph, as we do here.

In this paper we formulate a game which we call the *Patrolling Game*. Unlike the work of Paruchuri and colleagues, our problem is a zero-sum game, and provides for a defender who is mobile, being able to travel between locations in the course of his shift (a "Patroller"). Unlike search games, our "Attacker" (the equivalent of the search game "Hider") may commence his attack at any time and has to be detected within a given time-window in order to forestall the performance of some misdeed. Our game is *win-lose* - a game of *type* rather than *degree* in the terminology of Isaacs (1999). Our problem is sufficiently idealized that it is possible to obtain insightful analytic results, but sufficiently realistic that it is recognizable as a practical problem faced by practitioners in various domains.

In this paper we present some analytic results for this game, and demonstrate that it yields patrolling (and attacking) strategies which are natural and intuitive. Moreover, we show how the game can be used to guide decisions about investment in hardening vulnerable sites or in adding additional passageways to enable the Patroller to shorten the time required to go between different sites that might be attacked. We are in this paper unable to present general analytic results for all games of this type, and it seems unlikely that such solutions exist. Indeed, even computing optimal strategies may be quite challenging, because of the combinatorial explosion in the Patroller's strategy space; in a companion paper (Alpern, Morton and Papadaki in preparation a), we present some algorithms for efficiently computing the value of this game for more complex (and realistic) examples.

This paper is organized as follows. We present in Section 2 a rigorous formulation of patrolling games, together with some elementary observations on properties of the Value. As the number of pure strategies for the players can be very large, we give in Section 3 three methods for reducing the number that we have to consider: symmetrization, dominance and decomposition. Section 4 discusses certain classes of strategies that the players can use on any graph, and which are optimal on certain classes of graphs. Section 5 solves patrolling games on certain classes of graphs: Hamiltonian, bipartite and line graphs. Section 6 considers how the game can be used to guide decisions in investment about hardening nodes or adding edges. Section 7 presents extensions of the model and concludes.

2 The Patrolling Game

In this section we give a formal description of the Patrolling Game $G = G(Q, T, m)$, where Q is the graph whose nodes are under attack, T is the total number of time units the game is played over, and m $(\leq T)$ is the number of (consecutive) periods required to successfully carry out an attack on a node. Roughly speaking, the Attacker picks a node i to attack and chooses some time interval $I = \{\tau, \tau + 1, \ldots, \tau + m - 1\}$ of length m in which to attack it. The Patroller follows a walk $w(t)$ on Q, that is, he chooses nodes $w(0), \ldots, w(T-1)$ with consecutive nodes the same or adjacent in Q. The Patroller wins if his walk is at the node i in some period t in which it is being attacked, that is, if $w(t) = i$ for some $t \in I$. Otherwise the Attacker wins. We also demonstrate some simple monotonicity results and some bounds on the value, which will be useful later on.

2.1 Formulation

More formally, the Patrolling Game $G = G(Q, T, m)$ is a win-lose (and hence zero-sum) game between a maximizing Patroller (female) and a minimizing Attacker (male). It comes in two forms, the *one-off game* $G^o = G^o(Q, T, m)$ and the *periodic game* $G^p = G^p(Q, T, m)$. The one-off game is played out over a given time interval $\mathcal{T} = \{0, 1, 2, \ldots, T-1\}$ of length T on a graph Q with n nodes $\mathcal{N}$ and edges $\mathcal{E}$. We will tend to assume that Q is connected unless stated otherwise. A pure strategy for the Attacker is a pair $[i, I]$, where $i \in \mathcal{N}$ is called the *attack node* and $I \subset \mathcal{T}$ is an m-interval called the *attack interval.* A pure strategy for the Patroller is a walk $w : \mathcal{T} \rightarrow Q$ called a *patrol.* If $i \in w(I)$ we say that the patrol intercepts the attack, in which case the Patroller wins and the payoff is $P = 1$; otherwise we have $P = 0$. Thus the payoff is given by

$$P(w, [i, I]) = \begin{cases} 1 \text{ (Patroller wins)}, & \text{if } i \in w(I) \text{ (Patroller intercepts attack)}, \\ 0, \text{ (Attacker wins)}, & \text{if } i \notin w(I) \text{ (attack is successful)}, \end{cases}$$

The Value V^o of this game G^o is thus the probability that the attack is successfully intercepted. Except in trivial cases, optimal strategies must be mixed.

The periodic game G^p is similar except that the patrols (Patroller pure strategies) are now walks of period T (satisfying $w(t + T) = w(t)$ for all T). Attack intervals are now m-intervals in the *time circle* $\mathcal{T}^* = \mathcal{T} \bmod (T)$, so for example if T is 24 and m is 5, the attack could be carried out overnight, during the interval $\{22, 23, 0, 1, 2\}$ (10 o'clock to 2 in the morning). We can also view the patrols as walks $w : \mathcal{T}^* \rightarrow \mathcal{N}$. The periodic game is simpler to analyze because the attack can be assumed to take place equiprobably in any time interval, which simplifies the analysis (see Subsection 3.1). When the values of the games differ, we will use the superscripts V^p and V^o to distinguish between the Values, using V when the result applies to both cases. V (V^p, V^o) can be considered as parameterized by Q, T, and m just as G is, but most of the time writing $V(Q, T, m)$ is distracting and confusing and we will tend to suppress some or all of these arguments. We denote by $d(i, i')$ the distance function on the node set $\mathcal{N}$, the minimum number of edges between i and $i' \in \mathcal{N}$.

This formulation makes a number of assumptions which are not in fact as restrictive as they might appear. The first is the assumption an attack will take place. An immediate response to this is that even though attacks occur very rarely, one should patrol on the assumption that an attack will happen - otherwise what is the point of patrolling at all? A more sophisticated response is that the parties are really engaged in a non-zero sum deterrence game and the Patroller only has to reduce the probability of attack to a level where the expected value of the attack is less than the value to the Attacker of engaging in an attack elsewhere (another airport, another art museum). As it turns it, however, the game studied in this paper can be seen as being embedded in a larger non-zero sum deterrence game in the manner of Avenhaus, von Stengel and Zamir (2002). In this case the key to the analysis of the larger non-zero sum game is precisely the analysis of the game discussed in the current paper. The second and third assumptions are that the node values are equal (all paintings are worth the same amount of money; the damage inflicted by an attack at some airport terminal will be the same as at any other airport terminal), and that the distances between nodes are equal, respectively. Obviously this may well not hold in an application setting. However, it is not hard to modify the modelling framework to include these features and although the resulting games are not analytically tractable, they can be analyzed computationally with the mechanisms discussed in the companion paper Alpern, Morton and Papadaki (in preparation a).

2.2 General Properties of the Value V

We now make some observations about the Value V, which apply to both versions of the game. We start with a monotonicity result (Lemma 1), the last part of which involves the well known notion of *identifying* nodes of a graph. Formally this can be defined by a projection map $\pi : \mathcal{N} \to \mathcal{N}'$, where $\mathcal{N}'$ is the node set of the new graph Q', and $\pi^{-1}(j')$ represents a set of nodes of Q that have been identified. For example, we can easily obtain the line graph with n nodes L_n by vertically identifying nodes of the cycle with $2(n-1)$ nodes $C_{2(n-1)}$ (see Figure 1 for the case $n = 5$).

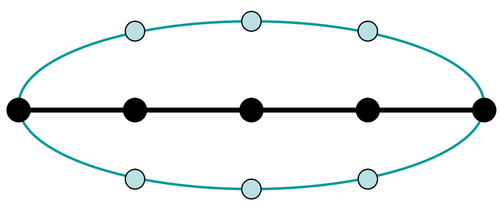

Figure 1. L_5 as a projection of C_8.

Lemma 1

1. $V(Q,T,m)$ is nondecreasing in m.

2. $V(Q,T,m)$ cannot decrease if an additional edge is added between two nonadjacent nodes of the graph Q. That is, V is nondecreasing in $\mathcal{E}$ (with the ordering on the latter understood in the sense of set inclusion).

3. $V^p(Q,T,m) \leq V^o(Q,T,m)$

4. If Q' is obtained from Q by node identification, $V(Q') \geq V(Q)$.

Proof. The first part follows from the observation that a patrol that intercepts an attack $[i,I]$ also intercepts $[i,I']$ if $I' \supset I$. The next two are based on the fact that in a zero sum game a player cannot do worse if he gets additional strategies. The last is based on the following observation: If a patrol w intercepts an attack on a node i of Q then the patrol $\pi(w)$ intercepts the associated attack on the node $\pi(i)$ of Q'. So the Patroller can ensure that the expected payoff is at least $V(Q)$ by choosing patrols w for Q according to some optimal mixed strategy, and then playing the projected patrol $\pi(w)$. ■

The next result gives easy general bound on the Value.

Lemma 2 $\frac{1}{n} \leq V \leq \frac{m}{n}$, *for* V *equal to* V^p *or* V^o *and any parameters* Q, T *and* m. *More generally,* $V \leq \omega/n$, *where* ω *is the maximum number of nodes that any patrol can cover (*ω *depends on whether the one-off or periodic version is being played).*

Proof. The Patroller can obtain the left inequality by randomly picking a node and waiting there. The Attacker can obtain the right inequality by attacking a random node during some fixed time interval I. Of these n pure strategies, the Patroller can intercept at most $|w(I)| \leq |I| = m$ of them, giving the bound m/n, or more generally the bound ω/n, since $|w(I)| \leq \omega$ by definition. ■

It is worth observing that ω is bounded above by the node size of the largest component of Q (if it is not connected), with ω/n equalling $1/n$ for the completely disconnected graph. Also note that for the one-off game with $m = 1$, where Q is the complete graph K_n, this is a special case of Ruckle's "Simple Search Game" (Ruckle, 1983). Since we thus have (from Part 2 of Lemma 1) $V = 1/n$ whenever $m = 1$, we will assume for the remainder of the paper that $m \geq 2$.

3 Strategy Reduction Techniques

Even for small graphs, the number of pure strategies available to the players can be quite large. So for practical purposes, as well as in proofs, it is useful to have methods for reducing the number of strategies that must be considered. This section discusses three such methods: symmetrization, dominance and decomposition.

3.1 Symmetrization

Symmetry considerations can simplify both the placement and timing of attacks and patrols. First we consider the placement of attacks in terms of the spatial symmetry of Q. As an example, note that the nodes 2 and 3 are symmetrically placed in the Kite Graph KT of Figure 2. So it follows from well known arguments (discussed below) that there is an optimal mixed Attacker strategy with the property that, for any attack interval I, these two nodes are attacked with equal probability.

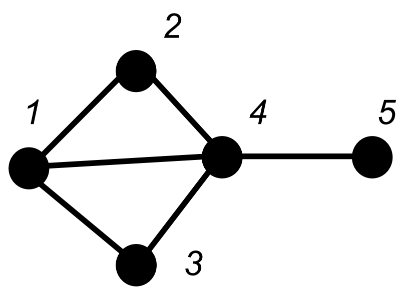

Figure 2. Kite Graph KT

This idea can be formalized by considering the automorphisms of Q, that is, the adjacency-preserving bijections of Q. (For the kite graph there are only the identity automorphism and the reflection about the vertical axis.) Calling nodes *equivalent* if some automorphism σ of Q takes one into the other, we need consider only attacks equiprobably distributed over the equivalence class of nodes. Similarly, two patrols w_1 and w_2 are equivalent if $w_2(t) = \sigma w_1(t)$ for some automorphism σ, and we can restrict our attention to the equiprobable mixture of such patrols.

A similar line of reasoning applies to time. In the periodic game all attack intervals are equivalent under some rotation of the time circle, so we need only consider the attack *node*. In the one-off game, attack intervals I_1 and I_2 are equivalent if $\gamma(I_1) = I_2$ where γ is the reflection automorphism of the time interval $\mathcal{T} = \{0, \ldots, T-1\}$ defined by $\gamma(t) = T - t$.

The fact that we need only consider symmetrical strategies, that is, mixed strategies which give equal probability to equivalent strategies, is demonstrated in Alpern and Asic (1985), and Zoroa and Zoroa (1993). Given a game G we call the modification of G where we restrict attention to attacker and patroller strategies which are equiprobable mixtures over the equivalence classes defined by the space and time automorphisms, the *symmetrization* of G; this symmetrized game has the same value as the original G, but has fewer strategies and so is easier to study.

3.2 Dominance

Since the Patrolling Game is a win-lose game, we can use the following weak notion of dominance. We say a pure strategy s_1 *dominates* a pure strategy s_2 of the same player if it wins against every opponent strategy that s_2 wins against, and against at least one more. The Value is unchanged if we successively eliminate dominated strategies.

As an example of how successive elimination of dominated strategies can be used, consider again the kite graph KT of Figure 2. The node 4 is what we call a *penultimate node*, that is, *a non-leaf node that is adjacent to a leaf node* (node 5 in KT). Our next result shows that there is an optimal strategy on KT which does not involve any attacks on a penultimate node.

Lemma 3 *Assume Q is connected and $T \geq 3$. For $m \geq 2$, patrols that stay on any node for three consecutive periods are dominated. For $m \geq 3$, attacks on penultimate nodes are dominated, and consequently the Attacker has an optimal strategy concentrated on nodes which are not penultimate.*

Proof. The proof is by iterated dominance. Suppose the patrol w_1 is at the same node i for the three consecutive periods $I = \{t-1, t, t+1\}$. Define w_2 to be the same as w_1 except that $w_2(t) = i'$, where i' is adjacent to i. For $m \geq 2$, the patrol w_2 intercepts every attack that w_1 intercepts, as well as the attack on i' during I, and hence dominates w_1. So we can now assume that the Patroller does not use patrols which stay at a node for three consecutive periods.

Next suppose that i' is a penultimate node adjacent to a leaf node i. We now show that any attack on node i' during an $m-$interval I is dominated by an attack on i during I. If w wins against the attack $[i, I]$, then $w(t) = i$, for some $t \in I$. Let the other two periods in I be called t' and t''. So by our earlier argument we know that either $w(t')$ or $w(t'')$ is i'. If $m \geq 3$, I must contain both t' and t'', and so $i' \in w(I)$, and w wins against $[i', I]$. Hence attacking at node i' is a dominated strategy. ■

3.3 The importance of timing: the line graph L_6 with $T = 5$ and $m = 3$

To illustrate the ideas of symmetry and dominance, we now analyze the line graph L_6 with nodes $i = 1, \ldots, 6$, for the case $T = 5$ and $m = 3$. The product of L_6 (drawn vertically) and the time space $\mathcal{T} = \{1, \ldots, 5\}$ (drawn horizontally) is shown three times in Figure 3. An attack with probability p at node i and time interval $\{t-1, t, t+1\}$ is represented by a p at the middle of the attack interval (i, t). Since there are three possible attack intervals $(\{1,2,3\}, \{2,3,4\}, \{3,4,5\})$ there are $6 \times 3 = 18$ possible attacks.

We first consider the one-off game $G^o(L_6, 5, 3)$, and *in a restricted form where the Attacker must use a time invariant strategy.* This is illustrated in the left drawing of Figure 3, where there are no attacks at penultimate nodes ($\{2, 4\}$) (using Lemma 3). Since nodes 1 and 6, and nodes 3 and 4 are equivalent under symmetry, we can assume they are attacked with equal probability. So the most general Attacker mixed strategy is shown, where $6x + 6y = 1$. The patrol $w_1 = (3, 2, 1, 2, 3)$ intercepts all attacks at node 1 and two attacks at node 3, so wins with probability $3x + 2y$; similarly $w_2 = (1, 2, 3, 4, \text{ any})$ intercepts one attack at node 1 and five attacks at nodes 3 and 4, and wins with probability $x + 5y$. These two patrols together dominate all others. So the Attacker minimaxes when $3x + 2y = x + 5y$. This occurs when $x = 1/10$ and $y = 1/15$, with minimax value $V^* = 13/30 = .43333\ldots$. (An easy calculation then shows that the Patroller should adopt w_1 and w_2 with probabilities 4/5 and 1/5.)

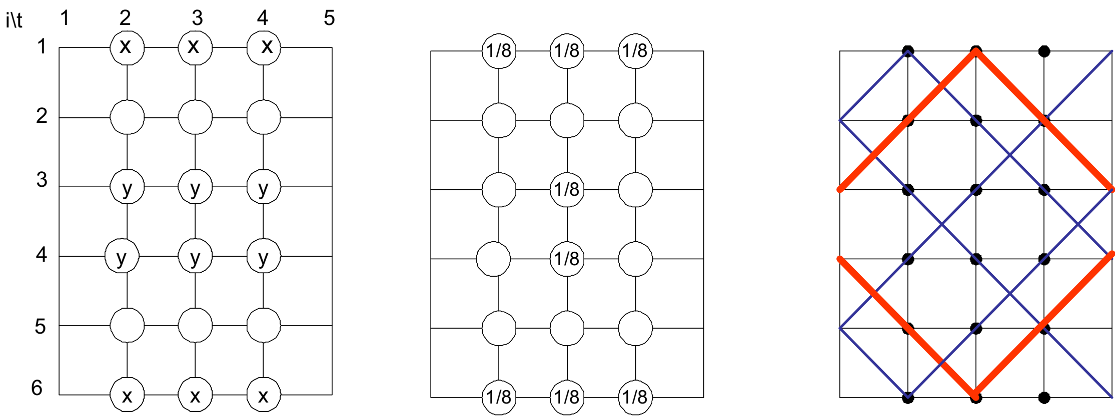

Figure 3. Optimal attacking and patrolling strategies for $G^o(L_6)$ with $T = 5$ and $m = 3$

In the (unrestricted) game $G^o(L_6, 5, 3)$ it is harder to derive the equilibrium strategy pair, but it is fairly easy to demonstrate that the Value is $V^o = 3/8 = 0.375$, which shows that no time invariant mixed Attacker strategy can be optimal. To see that $V^o \leq 3/8$, consider the Attack strategy shown in the middle drawing, and observe that no patrol can intercept more than three of the eight equiprobable attacks. An optimal Patroller mixed strategy is to adopt the four strategies $(2, 1, 2, 3, 4)$, $(2, 3, 4, 5, 6)$ and their reflections $(5, 6, 5, 4, 3)$, $(5, 4, 3, 2, 1)$ (drawn in thin blue slanted lines) with probability 1/8 each; and adopt the two equivalent strategies $(3, 2, 1, 2, 3)$ and $(4, 5, 6, 5, 4)$ (drawn in thick red lines) with probability 1/4 each. The assertion $V^o \geq 3/8$ follows from the observation that every one of the eighteen possible attacks is intercepted (hit, to the right of, to the left of) by at least three of the patrols, counting the thick red ones as two. It is also interesting to observe that all ten attacks which are not used at all in the middle drawing are intercepted by more than three of these patrols. Thus, the Value of the one-off game $G^o(L_6, 5, 3)$ is 3/8, but it requires the use of time dependent Attacker strategies: the middle node is only to be attacked in the middle time interval $\{3, 4, 5\}$.

Next we analyze the periodic version, the game $G^p(L_6, 5, 3)$. This is similar to the restricted version of the one-off game discussed above, except that the middle of the attack can be at any time, so comparing with the left drawing of Figure 3, the x's and y's would extend throughout the rows, and so we have $10x + 10y = 1$. The (periodic) patrol $w_3 = (3, 4, 3, 4, 3)$ intercepts all ten attacks at middle nodes 3 and 4, and wins with probability $10y$. The (periodic) patrol $w_4 = (1, 2, 3, 2, 1)$ intercepts four of the attacks at node 1 (all except the one during times 2,3,4) and three of the attacks at node 3, so wins with probability $4x + 3y$. These two (together with their symmetric translations) dominate all other patrols. So the Attacker minimaxes when $10y = 4x + 3y$. This has solution $x = 7/110$ and $y = 4/110$, with minimax $V^p = 4/11$. For the Patroller, w_3 (and its equivalents) should be used with probability $1/11$, w_4 with probability $10/11$.

To summarize, for line graph L_6, with $T = 5$ and $m = 3$, we have

$$V^p \approx 0.363\,64 < V^o = .375 < V^* \approx .43333\ .$$

Thus the Attacker does better in the one-off game, and thus the bound stated in Lemma 1 Part 3 need not be tight. Further, in this instance, the Attacker has to adopt a time dependent strategy in order to benefit fully.

3.4 Decomposition

Sometimes we can think of a graph Q as being made up of simpler graphs Q_1 and Q_2. We call this a *decomposition* of Q. The nodes of the original graph Q are the union of the nodes Q_1

and Q_2. All nodes which are adjacent in Q are also adjacent in any Q_i which contains both of them. See Figure 4 for an illustration. Q can of course be decomposed into multiple Q_i through repeated decomposition. If the nodes of Q_1 and Q_2 are disjoint and Q has no edges between nodes in distinct Q_i, then we say it is a *disjoint decomposition.*

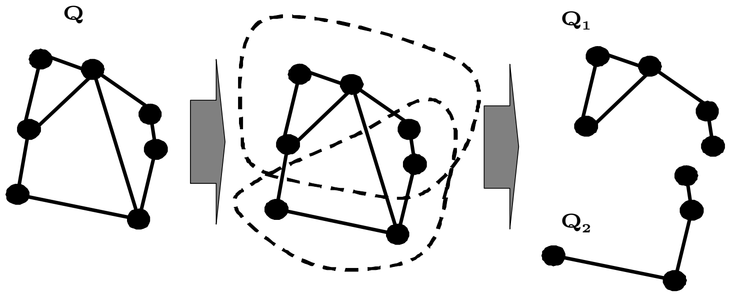

Figure 4. Decomposition of a network

Lemma 4 *Let* $V = V(Q, T, m)$ *and* $V_k = V(Q_k, T, m)$. *If the graphs* Q_k, $k = 1, \ldots, K$, *form a decomposition of* Q, *then*

$$V \geq \frac{1}{\sum_{k=1}^{K} 1/V_k},$$

with equality in the case of a disjoint decomposition.

Proof. Suppose the Patroller restricts himself to a family of mixed strategies S_k, where S_k is an optimal mixed strategy for the game $G(Q_k, T, m)$. Suppose he picks S_k with a probability q_k such that $q_k V_k = c$ is constant. In this case we have

$$1 = \sum_{k=1}^{K} q_k = c \sum_{k=1}^{K} 1/V_k, \text{ or } c = 1/\sum_{k=1}^{K} 1/V_k.$$

For any attack pair $[i, I]$, the node i belongs to the node set of some graph Q_k. So with probability q_k the Patroller will be optimally patrolling Q_k and in this case will intercept the Attacker with probability at least V_k. Hence the Patroller wins with probability at least $q_k V_k = c$. Hence the value of the game on graph Q is at least c, as claimed. If the Patroller is only allowed to search nodes from a single graph Q_k, the best he can do is win with probability c, so it follows that if the graphs Q_k have disjoint node sets and are disconnected, then $V = c$. ■

3.5 Example: the kite graph

To demonstrate the use of all of our strategy reduction techniques, we analyze the periodic game for the kite graph illustrated in Figure 5 with $T = m = 3$. The dominance argument of Lemma 3 showed that the Attacker would never attack node 4, as it is always better for him to attack the adjacent leaf node 5. Moreover, in the periodic case for $T = 3$, there is no feasible Patroller strategy which visits both node 5 and any one of 1,2, or 3. Therefore, we can remove node 4 and be confident that the periodic game on the resulting graph KT' has the same Value as the game on KT.

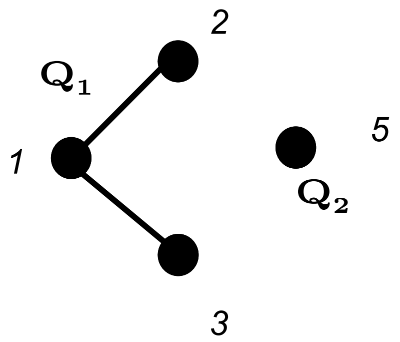

Figure 5. Decomposition of Kite graph KT into KT'

Lemma 4 shows that for Q_1 and Q_2 as in Figure 5, we have

$$V^p\left(KT'\right) = \frac{1}{1/V^p\left(Q_1\right) + 1/V^p\left(Q_2\right)} \tag{1}$$

Obviously $V^p\left(Q_2\right) = 1$, and it can be easily shown that $V^p\left(L_3\right) = 1/2$ for $T = m = 3$. Hence by (1) we have

$$V^p\left(KT\right) = V^p\left(KT'\right) = \frac{1}{1+2} = \frac{1}{3}.$$

This is an another example where the Patroller does strictly better in the one-off game, in which $V^o = 3/5$. To see this first note that by Lemma 2 the Attacker can ensure that $V^o \leq m/n = 3/5$ by attacking equiprobably at the five nodes. Then observe that by using the four patrols $(2,1,3)\,, (2,4,5)\,, (3,4,5)\,, (1,4,5)$ with respective probabilities $2/5, 1/5, 1/5, 1/5$ the

Patroller ensures any attack at any node will be intercepted with probability 3/5 and thus that $V^o \geq 3/5$. Note that if edge $(1, 4)$ is removed, the Value V^o goes down to 1/2; the Attacker chooses nodes 1 and 5 equiprobably and the Patroller chooses the first three of the above patrols with probabilities $1/2, 1/4, 1/4$.

4 Generic Strategies and Their Effectiveness

In general, the type of strategies available to the Patroller depends crucially on the path and circuit structure of the underlying graph Q. However, for purposes of analysis, it is possible to identify certain generic strategy types which are available on all graphs; or on all graphs in a class. For the Attacker, we define here the *uniform*, *independent and diametrical* strategies. For the Patroller, we define the *covering* strategy.

4.1 The uniform Attacker strategy

In zero sum games often the most random strategy is optimal. For the Attacker, this is the *uniform strategy*, in which the attack $[i, I]$ has i and I chosen equiprobably and independently over their domains. That is, *a random node is attacked at a random time.* In the periodic game this strategy is the equiprobable mixture of the nT possible attacks. For the purposes of the next result, we make use of a standard definition:

Definition 5 *A graph is **bipartite** if it has no odd cycles.*

The reader will note that (for example) all trees are bipartite.

We have already shown that $V \leq \frac{m}{n}$. For bipartite graphs we are able to tighten this bound.

Lemma 6 *If T is odd and Q is bipartite, the bound of Lemma 2 can be tightened to $V^p \leq \frac{(T-1)m+1}{nT}$. This bound is guaranteed by the Attacker adopting the uniform strategy in the periodic game.*

Proof. In the uniform strategy, all nT possible attacks are adopted with probability $1/(nT)$. If T is odd, and there are no odd cycles (because Q is bipartite), then for any w, $w(t) = w(t+1)$ for some t in the periodic game. In these two periods (that is, t and $t+1$), at most $m+1$ of the attacks can be intercepted, and as before at most m in each of the other $T-2$ periods. So at most $(T-2)m + (m+1) = (T-1)m + 1$ attacks can be intercepted altogether, giving the desired inequality. ■

4.2 The diametrical strategy

The diameter $\bar{d}$ of a graph Q is given by $\bar{d} = \max_{i,i' \in \mathcal{N}} d(i, i')$. A pair of nodes at distance $\bar{d}$ is called diametrical, and the Attacker's *diametrical strategy* is to attack these nodes equiprobably during a random time interval I. It is easy to show the following. If $\bar{d}$ is very large with respect to m and T then it is clear the best the Patroller can do against the diametrical strategy is to wait at one of the nodes and win half the time. On the other hand if m and T are large, the best the Patroller can do in the one-off game is go back and forth repeatedly on a geodesic between the diametrical points and win with probability $m/\left(2\bar{d}\right)$. Since he cannot do better in the periodic game, we have the following.

Theorem 7 $V \leq \max\left[m/\left(2\bar{d}\right), 1/2\right]$. *The diametrical strategy guarantees this payoff.*

4.3 Independent and covering strategies

The graph theoretic notion of independence and covering numbers has already been shown to be useful in related games of infiltration (Alpern, 1992). We give here modified versions of these concepts.

Definition 8 *A patrol w is called* ***intercepting*** *if it intercepts every attack on a node that it contains. That is, if a node i lies on a patrol w, then it appears in any subpath of w of length m. A set of intercepting patrols is called a* ***covering set*** *if every node of Q is contained in at* *least one of the patrols. The* ***covering number*** $\mathcal{J}$ *is the minimum cardinality of any covering set.*

Definition 9 *If, for any two nodes i and j, any patrol which intercepts an attack at node i in attack interval I, cannot also intercept an attack at j in attack interval I, then i and j will be said to be* ***independent****. In the one-off game G^o, this is equivalent to requiring any two nodes to satisfy $d(i, i') \geq m$; in the periodic game G^p, they must satisfy $d(i, i') \geq m$ or $T \leq 2\ d(i, i')$ (because the Patroller has to return to his starting point by the end of the period). The* ***independence number*** $\mathcal{I}$ *is the cardinality of a maximal independent set. Obviously $\mathcal{I} \leq \mathcal{J}$.*

Observe that both $\mathcal{I}$ and $\mathcal{J}$ depend on the parameters Q, T, m and on the version of the game that is played, G^o or G^p. For example, when $T = 3$ and $m = 3$, the node subset $\{1, 3\}$ of L_3 is independent for the periodic game but not for the one-off game.

For the Attacker, the *independent strategy* is to fix an attack interval and then choose the attack node equiprobably from some maximal independent set. For the Patroller, the *covering strategy* is to choose equiprobably from a minimal set of covering patrols.

Note that for $T = 2$, patrols can be identified with edges of Q, so these definitions reduce to the usual notion of an independent set not having adjacent nodes and a covering set consisting of edges.

Lemma 10 $\frac{1}{\mathcal{J}} \leq V \leq \frac{1}{\mathcal{I}}$ *(with $V = 1/\mathcal{I}$ when $\mathcal{I} = \mathcal{J}$).*

Proof. The Attacker's independent strategy gives the upper bound and the Patroller's covering strategy gives the lower bound. ■

The cases where $\mathcal{I} = \mathcal{J}$ deal with many patrolling games. For example, we can use this technique to give another solution to the kite graph KT of Figure 2 for the periodic game with $T = m = 3$. Here the nodes 2,3, and 5 form an independent set (because $2d(i, i') = 4 \geq 3 = T$) and intercepting patrols on the top left, top right and bottom edges (period 3 patrols of the form $(a, a, b, a, a, b \ldots))$ form a covering set. Thus Lemma 10 gives $V = 1/\mathcal{I} = 1/3$, as we demonstrated earlier by another method.

5 Patrolling on Special Classes of Graphs

We have no single form of analysis that is sufficiently robust to give the Value of an arbitrary patrolling game; in general the Value would have to be obtained computationally. However for certain classes of graphs we can determine the Value in terms of the parameters m and n (the number of nodes), at least for certain values of T. These classes are Hamiltonian graphs, bipartite graphs and line graphs.

5.1 Hamiltonian graphs

A *Hamiltonian graph* is a graph containing at least one cycle which visits each node exactly once (i.e. a Hamiltonian cycle). A special case of the Hamiltonian graphs is the simple cycle with n nodes C_n. (Another special case is the complete graph K_n.) The existence of a natural cycle in the underlying graph is a common feature of the problem faced in application settings, as often the area to be patrolled will be physically compact (consider, e.g. patrolling a campus). Note that if $m \geq n$ the Patroller can win by following the Hamiltonian cycle, so we assume that $m < n$. We define a *random Hamiltonian patrol* to be one which fixes some Hamiltonian cycle, starts at a random node i, and follows the cycle in a fixed direction, repeating as required. Such a patrol is always feasible in the one-off game G^o and is feasible in the periodic game G^p if T is a multiple of n. Using this mixed strategy, the Patroller can get the best possible interception probability V, namely the upper bound m/n of Proposition 2.

Theorem 11 *If Q is Hamiltonian then*

1. $V^o = \frac{m}{n}$;
2. $V^p \leq \frac{m}{n}$ *with equality if T is a multiple of n, and $V^p \to m/n$ as $T \to \infty$.*

Proof. First observe that in either case we have $V \leq m/n$ by Lemma 2. In the one-off game, suppose the Patroller adopts a random Hamiltonian patrol. Then for any attack interval I, $w(I)$ is a random m-arc of the Hamiltonian cycle, and as such contains the attack node i with probability m/n, as claimed. If T is a multiple of n, this strategy is also feasible in the periodic game. To obtain the limiting result, note that if $\sigma = T \mod n \neq 0$, the periodic Patroller can modify the random Hamiltonian patrol by waiting at a random node during a random σ-interval. This will not hurt him unless the attack interval I overlaps the waiting interval, which has probability $(\sigma + m - 1)/T$, so

$$\left(1 - \frac{\sigma + m - 1}{T}\right)\frac{m}{n} \leq V^p \leq V^o = \frac{m}{n}, \text{ and so } V^p \to \frac{m}{n}. \tag{2}$$

■

Since the above result applies to the cycle graph, we can use it to solve the game on some graphs which can be obtained from the cycle graph by identification of nodes. We now solve the periodic Patrolling Game for the eight node graph shown below on the left of Figure 6 in the case $T = 10$ and $m = 4$. First note that since the diameter is $\bar{d} = 5$ we have from Theorem 7 that the diametrical Attack strategy ensures that $V \leq m/\left(2\bar{d}\right) = 4/10$. By viewing the graph as a projection of C_{10} (with Value $m/10 = 4/10$ from Theorem 11) we conclude from Lemma 1 Part 4 that $V \geq 4/10$, so $V = 4/10$.

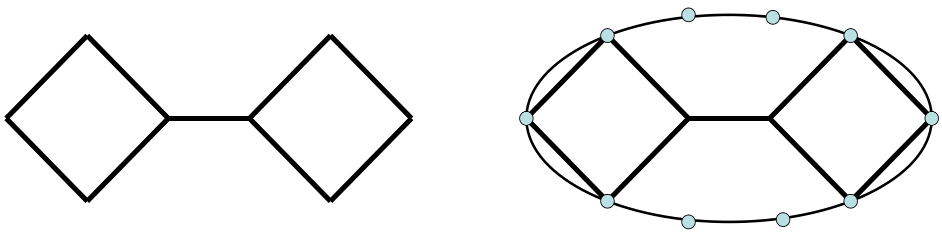

Figure 6. A graph shown as projection of C_{10}

5.2 Bipartite graphs

If Q is a *bipartite graph* (as defined earlier as having no odd cycles) we can partition its node set into *halfsets* $A = \{\alpha_1, \ldots, \alpha_a\}$ and $B = \{\beta_1, \ldots, \beta_b\}$, with $a \leq b$, such that its only edges are between nodes in A and nodes in B. If *all* such node pairs are edges then we say Q is the *complete bipartite graph* $K_{a,b}$.

If $m > 2b$ then the Patroller can win by using a patrol with period $2b$ which covers all the nodes, that is, the covering number $\mathcal{J}$ is 1. So we assume $m \leq 2b$.

Theorem 12 *If Q is bipartite with with halfsets of sizes $a \leq b$, then*

1. $V^o \leq m/(2b)$, *with equality if Q is complete bipartite ($K_{a,b}$);*
2. $V^p \leq m/(2b)$, *with equality if Q is complete bipartite ($K_{a,b}$) and T is a multiple of $2b$. If $Q = K_{a,b}$ then $V^p \to m/(2b)$ as $T \to \infty$.*

Proof. We first show that $V^o \leq m/(2b)$, which then gives the weaker inequality $V^p \leq m/(2b)$. Consider the Attacker mixed strategy of fixing an attack interval I and picking the node i equiprobably among the b elements of B. For any patrol w, the probability that a random i in B belongs to $w(I)$ equals $|w(I) \cap B|/b \leq (m/2)/b = m/(2b)$. If m is odd, the Attacker strategy must be modified to pick I and the shifted interval $I+1$ equiprobably. In this case for any w we have $|(w(I) \cap B)| + |w(I+1) \cap B| \leq m$ and the probability that the attack is intercepted by any w is given by

$$\frac{1}{2}\frac{|(w(I))|}{b} + \frac{1}{2}\frac{|(w(I+1) \cap B)|}{b} \leq \frac{m}{2b}.$$

The equality part follows for $a = b$ from Theorem 11, as $K_{b,b}$ is Hamiltonian. As in Theorem 11, we have separate results for the one-off and periodic cases, and specifically we are only able to show that our result applies for particular or limiting values of T in the periodic case. If $a < b$ then we can obtain $K_{a,b}$ from $K_{b,b}$ by identifying together a subset of $b - a$ nodes of one of the halfsets of $K_{b,b}$ and then applying Lemma 1 (part 4) to assert that $V^o(K_{a,b}) \geq V^o(K_{b,b}) = m/2b$

in general and with a similar limiting result for the periodic case. ■

In all these cases, informally speaking, an optimal strategy for the Attacker is to fix an attack interval and choose the attack node equiprobably from the larger halfset; an optimal strategy for the Patroller is to randomize over a collection of strategies which visit the larger half set every second time period. In the case of $m = 2$, Patroller chooses an edge joining the halfsets; the Attacker's and Patroller's strategies can be seen as a random choice from an independence and covering set respectively; in this case the Theorem can be understood as a version of König's Theorem (Harary, 1971, Theorem 10.2) in our context, since König's Theorem states that the independence and covering numbers of a bipartite graph are identical.

To illustrate the proof, consider the special case of the star graph $S_n = K_{1,n-1}$ consisting of a central node connected to $n-1$ extreme nodes. This models the situation where the Patroller has responsibility for the safekeeping of a building which has multiple wings, accessible through a common lobby area. We can view S_n as obtained from the even cycle graph $C_{2(n-1)}$ by identifying (say) all even numbered nodes, as in Figure 7.

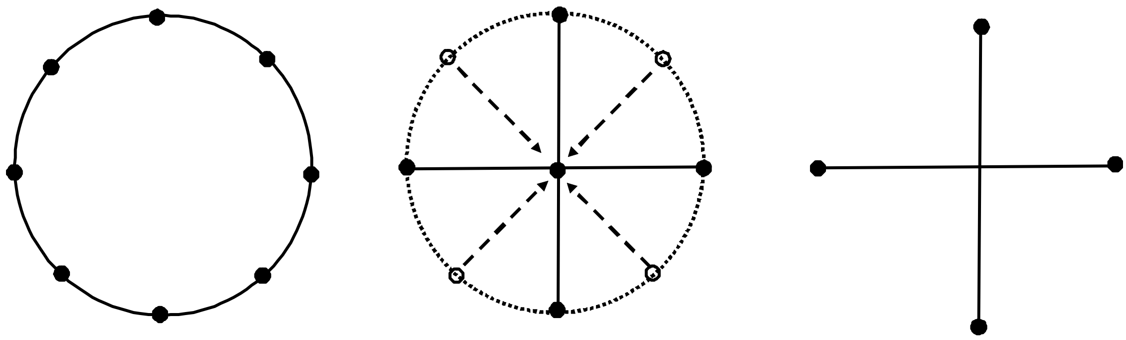

Figure 7. S_5 obtained from C_8 by node identification.

This mode of reasoning leads us to discover additional equilibrium pairs for the Hamiltonian case. Consider the cycle graph C_n for even n and T a multiple of n. We saw earlier that the uniform strategy was optimal for the Attacker. But since $C_n = K_{n/2,n/2}$ is bipartite Theorem 12 now gives the additional optimal strategy of attacking equiprobably on the odd (or even) nodes. In fact there is one more optimal Attacker strategy: since the diameter is $\bar{d} = n/2$, the diametrical strategy also gives $m/\left(2\bar{d}\right) = m/n$ by Theorem 7.

5.3 Line graphs

Line graphs L_n seem to be particularly complex to analyze and will be the main subject of a forthcoming article by the authors (Alpern, Morton and Papadaki, in preparation b), but we give here some special cases which illustrate the techniques we have developed earlier. We note that they are important in the patrolling context for their relation to the problem of patrolling a channel: preventing an agent from crossing a partially defended line between two regions.

Theorem 13 *If Q is a line and $n \leq m+1$, then*

1. $V^o = \frac{m}{2(n-1)}$;
2. $V^p \leq \frac{m}{2(n-1)}$ *with equality if T a multiple of $2(n-1)$; $V^p \to \frac{m}{2(n-1)}$ as $T \to \infty$.*

Proof. Since $\bar{d} = n-1$, it follows from Theorem 7 that $V^p \leq V^o \leq \frac{m}{2(n-1)}$. That this bound is tight follows from a use of Lemma 1 Part 4 and Theorem 11: we can arrive at L_n from $C_{2(n-1)}$ through node identification in the manner of Figure 1. Thus, $V(L_n) \geq V(C_{2(n-1)}) = \frac{m}{2(n-1)}$. ■

In most of the examples in this paper the optimal strategies have been highly random, in that the players used equiprobable mixtures of similar pure strategies. We did this mainly to keep things simple, but the reader should not be misled into thinking this is always the case.

For example, consider the game $G^p(L_5, 4, 3)$. We claim that the value of this game is $\frac{3}{7}$. Lemma 4 tells us that $V^p(L_5, 4, 3) \geq \frac{1}{\frac{1}{V^p(L_2,4,3)} + \frac{1}{V^p(L_3,4,3)}}$. Since $V^p(L_2, 4, 3) = 1$ and (from Theorem 13) $V^p(L_3, 4, 3) = \frac{3}{4}$, we have that $V^p(L_5, 4, 3) \geq \frac{3}{7}$. Now consider the game from the Attacker's point of view. There are four possible attack intervals I, and five possible attack nodes i. Suppose the Attacker randomizes equiprobably over the intervals and over the nodes with probabilities $\frac{3}{7}$ each for nodes 1 and 5, and $\frac{1}{7}$ for node 3. The Patroller could remain at node 1 or node 5 and intercept an attack with probability $\frac{3}{7}$ or randomly move between either 1 and 3 or 3 between and 5. She then has a $\frac{3}{4}$ chance of intercepting an attack which will take place with probability $\frac{3}{7}$ and a $\frac{3}{4}$ chance of intercepting an attack with will take place with probability $\frac{1}{7}$, for an overall probability of interception of $\frac{3}{4}\left(\frac{3}{7}+\frac{1}{7}\right)=\frac{3}{7}$. All other patrols yield a lower expected payoff and so $V^p\left(L_5,4,3\right)\leq\frac{3}{7}$.

Next consider $G^p\left(L_7,5,2\right)$. We claim that the value of this game is $\frac{1}{4}$. We have $\mathcal{I}=4$ and so from Theorem 10 we have $V^p\leq 1/4$. To ensure winning with this probability, the Patroller must use "biased oscillations" on edges (i,i') of the form (i,i,i',i,i'), which we denote as $i\longleftarrow i'$, with a random time rotation. Clearly $i\longleftarrow i'$ intercepts any attack on i and intercepts any attack on i' with probability 4/5 (that is unless the attack coincides with a repeated i). The optimal probabilities of the biased oscillations on consecutive nodes are shown below.

$$1\overset{4/16}{\leftarrow}2\overset{1/16}{\rightarrow}3\overset{3/16}{\leftarrow}4\overset{3/16}{\rightarrow}5\overset{1/16}{\leftarrow}6\overset{4/16}{\rightarrow}7$$

Attacks on any node are intercepted with probability at least 1/4, with equality except for the central node 4 (which should never be attacked). For example an attack on node 2 is intercepted with probability 4/5 if either $1\longleftarrow 2$ or $3\longleftarrow 2$ is adopted by the Patroller, that is with probability $(4/16+5/16)=5/16$. So it is intercepted with probability $(4/5)(5/16)=1/4$.

6 Hardening Nodes or Adding Edges

Up to now we have taken the network Q as given. But the agency that controls the Patroller (the "Defender") may be able to pay to either 'harden' a site so that it is immune to attack or to build a passageway to help the Patroller move more quickly between sites. In our formulation this corresponds either to reducing the Attacker's strategy space by removing an attack node, or adding an edge to the graph. In this section we give some very simple examples in the easy case where all sites (nodes) are equally expensive to harden and new passageways (edges) are equally expensive to build (more complex examples will be dealt with in our forthcoming paper on the computational aspects of this problem).

6.1 Hardening nodes

We can modify the Patrolling Game so that a certain set of nodes of Q cannot be attacked. For simplicity, we assume that only one node i can be hardened. We focus on the graph A shown in Figure 8, and consider $G^p(A,3,3)$. The analysis has been made easy by taking an example where all the problems satisfy $\mathcal{I} = \mathcal{J}$, so that the Value is given by the relevant value of $1/\mathcal{I}$, which in the case of A is $\frac{1}{3}$. Our arguments regarding independent and covering sets can still be used, by requiring independent sets to avoid i and requiring a collection of intercepting patrols to include all but i. When removing i from the nodes to be attacked (the node i is circled), we can also remove any edges such that the resulting distances between the other nodes do not increase. Since there are no 3−cycles, the intercepting patrols can all be identified with edges, so the calculation of $\mathcal{I}$ and $\mathcal{J}$ is particularly easy.

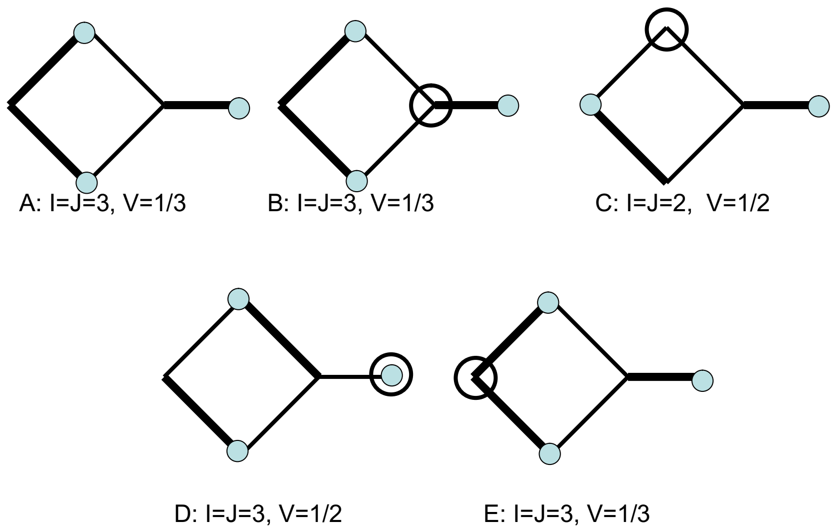

Figure 8. The graph A and four ways of hardening a node

From the analysis presented in Figure 9 it can be seen that hardening any but the top/bottom nodes or the leaf node is a waste of money, as there is no improvement in the interception probability V, but that hardening any of these nodes gives a $1/6 = 1/2 - 1/3$ probability improvement.

6.2 Adding edges

We observed earlier (Lemma 1) that adding edges can not hurt the Patroller, and used this to analyze certain networks. Here we take a different point of view and ask whether it pays for the Player that controls the Patroller to spend money to add an edge between two nodes in order to increase the efficacy of the Patroller. We consider again the network A and $G^p(A, 3, 3)$ and ask how the interception probability increases, if at all, by adding edges between nonadjacent nodes of A. The independent sets are indicated in all cases by disks at nodes in Figure 9. The covering sets are either identified with edges or 3−cycles, which are thickened.

We observe that adding edges as in B or E achieves no increase at all in the interception probability (the Value V), but as in C or D increases it to 1/2 from 1/3. The intuition is that these edges create 3−cycles, which are more efficient for patrols than oscillations on edges with a repeated node.

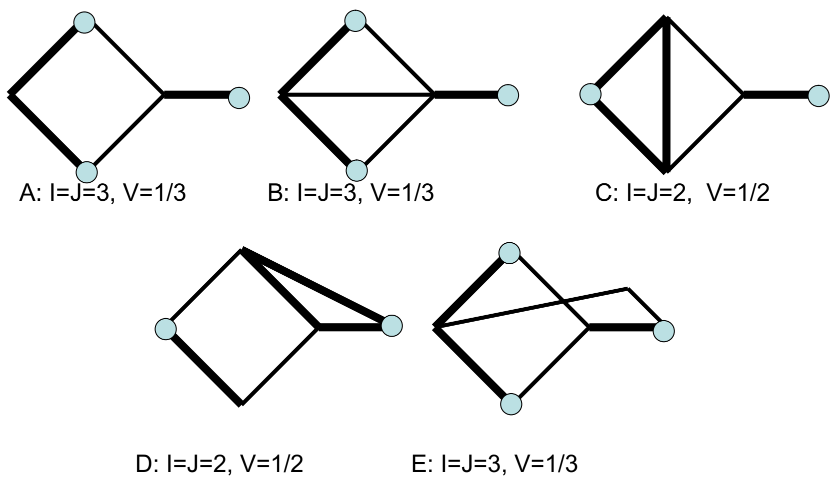

Figure 9. The graph A and four ways of adding an edge

7 Conclusion

In this paper we have described a simple, intuitive model which can serve as the basis for obtaining optimal randomized patrols. The assumptions of the present paper can be relaxed in the following extended models:

- The nodes can have different values (e.g. paintings of varying artistic merit in an art gallery). The simplest way to model this is to keep the same strategies but view the Patroller as wanting to minimize the value of a successfully attacked node. So this version comes with a cost vector c, where $c(i)$ denotes the cost of a successful attack on node i.

- Some nodes may be unequally hard, or even impossible, to attack. We thus replace the parameter m by a vector M, where $M(i)$ denotes the number of periods required to successfully attack node i. In a related way, edges may have lengths attached to them. (Two nodes with an intervening node which cannot be attacked are in some sense two time units apart.)

- There may be multiple patrollers and/ or attackers. Some of the results of the current paper transfer over easily to this situation - for example one can simply "multiply up" the numerator of Lemma 10 to handle the situation where there are several patrollers - but in general this seems to be rather more complex.

- Perhaps the Patroller must start at a known node $i = 0$ in the one-off game. Of course, in this case the diameter of Q cannot be large with respect to m, otherwise the Attacker will always win. A natural conjecture is that the Attacker would attack earlier in order to take advantage of his greater knowledge of where the Patroller will be, but is this always so?

- It may be natural to consider a continuous time formulation of this problem. An attack takes place at any point of the network (not necessarily a node) on a continuous time interval of fixed length. The Patroller uses a unit speed path and wins if he is at the attacked point at some time during the attack interval. This would model, for example, the defense of a pipeline system.

- The Patroller may be alerted (perhaps noisily and with some error) to the presence of an Attacker; and the Attacker may be alerted by a confederate who can identify when a

Patroller leaves a particular node (for example, if the Patroller is in a marked police car).

Many of these problems are not analytically tractable, and some of them will be discussed in our forthcoming paper on computational aspects of these games (Alpern, Morton and Papadaki in preparation a).

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