Jon Danielsson and Jean-Pierre Zigrand

Equilibrium asset pricing with systemic risk

Article (Accepted version)
(Refereed)

Original citation:
Danielsson, Jon and Zigrand, Jean-Pierre (2008) *Equilibrium asset pricing with systemic risk.* Economic Theory, 35 (2). pp. 293-319. ISSN 1432-0479 DOI: 10.1007/s00199-007-0238-3

EQUILIBRIUM ASSET PRICING WITH SYSTEMIC RISK*

Jón Daníelsson and Jean–Pierre Zigrand
London School of Economics and FMG
Houghton Street, London WC2A 2AE, United Kingdom
email: j.p.zigrand@lse.ac.uk
Tel: 020 79556201

February 15, 2007

Received: /revised version:

Abstract

We provide an equilibrium multi-asset pricing model with micro-founded systemic risk and heterogeneous investors. Systemic risk arises due to excessive leverage and risk taking induced by free-riding externalities. Global risk-sensitive financial regulations are introduced with a view of tackling systemic risk, with Value-at-Risk a key component. The model suggests that risk-sensitive regulation can lower systemic risk in equilibrium, at the expense of poor risk-sharing, an increase in risk premia, higher and asymmetric asset volatility, lower liquidity, more comovement in prices, and the chance that markets may not clear.

Journal of Economic Literature classification numbers: G12, G18, G20, D50.
Keywords: systemic risk, value-at-risk, risk sensitive regulation, general equilibrium

*An earlier version was presented at the Financial Stability Seminar at the Bank of England, CERGE-EI, Lancaster University, Lehman Brothers, London School of Economics, University of Maastricht, University of Würzburg and at the European Finance Association Meetings. We thank Michel Habib, José Scheinkman, Hyun Shin and two anonymous referees for their helpful comments. Jean-Pierre Zigrand is a lecturer in Finance at the LSE, and is the corresponding author. The authors would like to acknowledge financial support under the EPSRC Grant GR/S83975/01 at the Financial Markets Group, London School of Economics.

1 Introduction

There is still a sizable gap between the literature on equilibrium asset pricing models and the real world. We would like to point out two major contributing factors for this lack of realism. First, markets are neither complete nor frictionless. A large body of literature has studied asset pricing under incomplete markets as well as under various frictions, such as portfolio constraints. Much less work has been done when financial institutions are subjected to risk-sensitive constraints. Risk sensitive regulation, where statistical risk models are used to determine allowable levels of risk and of bank capital, has recently become the cornerstone of international financial regulations. Given the compulsory nature of the Basel-II prescriptions, asset prices, allocations and welfare will all be strongly affected by risk sensitive regulation. As a result, standard frictionless asset pricing methodologies may no longer be appropriate.

Second, very few, if any, equilibrium models provide micro-foundations for the risk underlying said risk-sensitive regulation. While some papers, for instance see Basak and Shapiro (2001)[1] and Cuoco and Liu (2005), model asset pricing in a risk-regulated world, that world is by assumption first-best and would therefore not warrant any regulation in the first place. The rationale for regulating risk must lie in the fear of what has been called a *systemic event*. Empirically, such an event seems to be priced in the markets. For instance, the implied distribution of post-'87 out-of-the money put options is substantially negatively skewed (consult Bates (2000), Pan (2002) and Carr and Wu (2003) who argue that this is due to the fear of substantial negative return jumps). While modelling a systemic crash as an exogenous shock may be useful in practice, in the absence of any market failure it is nevertheless not clear why this would require regulation. It is also not clear what a systemic event is in the first place (see De Bandt and Hartmann, 2000, for a survey). For concreteness we provide a formal definition of a systemic event.

Our task in this simple model with a continuum of agents (with varying coefficients of risk aversion and heterogeneous in their regulatory status: while some are regulated others may not be) is to formalize our intuition about systemic events that are due to an externality-induced free-riding market failure. Investors face random asset endowments. Being individually negligible they disregard the effects of their actions on aggregate outcomes. As a result in equilibrium an excessive fraction of total risk is concentrated on a small but significant number of highly leveraged investors, that is investors who borrow large sums by shorting the riskless asset and investing those sums in the risky assets. In turn, it suffices that an unanticipated event transforms this imbalance between those agents who own the

[1]Basak and Shapiro are foremost interested in modelling the optimal dynamic portfolio process of a regulated investor in complete Brownian markets and under various forms of constraints. They find for instance that in the worst states, regulated investors may take on more risk than non-regulated investors and consequently increase the stock market volatility in an economy with two log-utility agents, one of which is regulated. We find a similar result driven by agent heterogeneity.

shares and those agents who have spare investable capital into a systemic crisis.

More concretely, we model assets as rights to the output stream of firms, as in (a static version of) Lucas (1978). Following Holmstrom and Tirole (1998), we introduce an intermediate date at which firms may face a sudden (perfectly correlated across firms, i.e. aggregate) liquidity need. In order to keep the production process going, the existing shareholders are asked to provide liquidity to the firms by lending them an amount of riskless assets proportional to their shares in the firms. This sudden demand for additional working capital conveys no information as to the worth of the firm. We can view this stage either as pre-bankruptcy deliberations or as a stylized rights issue. The liquidity is reimbursed to them at date 1 with interest, provided that the productive sector was able to raise sufficient liquidity. Refinancing may fail since the holders of large equity positions may not themselves have the required incentives to accumulate enough liquidity to lend to the firms. In this paper the frictions consist of the assumptions that a) the productive sector must be refinanced as a whole (the outputs of the various firms are also inputs into each other, say), and b) that markets are closed at the intermediate date. This assumption is both theoretically and empirically reasonable. The initial investors, much like venture capitalists, gather private information about the projects they are investing in, or are at least perceived as doing so. This asymmetry might make fire-selling the project and/or attracting short-term liquidity from third parties impossible. The probability of a systemic crash increases along with imbalances in agents' leverage and risk taking. We measure systemic risk by the degree of imbalance of risk taking and leverage among agents.

Systemic risk therefore arises due to externalities and does warrant regulation. How successful are risk-sensitive regulations of the VaR type? Our model demonstrates that regulatory risk constraints lower the risk of a systemic event in equilibrium by preventing some regulated investors from accumulating excessively levered risky positions. Even though in equilibrium this means that more risk is held by risk-tolerant unregulated investors, equilibrium prices adjust in a way as to guarantee that even the unregulated investors, while holding more risk, also hold commensurately more of the safe asset.

But risk-sensitive regulations do impose social costs as well. First, risk-sensitive regulations may prevent market clearing in some circumstances if all financial institutions are regulated. The probability of markets not clearing increases with the tightness of the risk constraint. The basic intuition is that in periods of stress, such as with large fire-sales, the risk that would have to be taken on by the buyers could violate all potential buyers' regulatory constraints. Since non-diversifiable aggregate risk needs to be held at an equilibrium, no equilibrium can exist. No matter how cheap the risky assets are, regulated financial institutions face binding risk constraints and cannot buy more risky assets at any price. We show that there is a set of states for which there is no market clearing equilibrium. We also exhibit a set of states where equilibria exist but where prices and real allocations are indeterminate (nominal and real indeterminacy).

Second, the feedback-effects of regulation on the behaviour of prices are also important in and by themselves. We demonstrate that the equilibrium pricing function in a regulated economy exhibits, as regulation becomes tighter, less depth and more volatility (the covariance matrix is more positive definite, meaning that any portfolio of assets is more volatile). The fundamental intuition behind these results rests in the endogenous equilibrium level of risk-aversion in the market as a result of agent heterogeneity. The regulatory constraint causes the pricing function to become more concave for typical trades, since risk will have to be transferred from the more risk-tolerant to the more risk-averse. In order for the more risk-averse to take on the additional risk, the discount will have to be bigger the more risk-averse the marginal buyers are. Hence for a given change in demand, prices move more with regulation than without, implying higher (local) volatility and lower liquidity post regulation.

A well known source of major financial losses is the fact that correlations or comovements of assets are amplified in times of stress. While margin calls and wealth effects have been among the proposed explanations, as in Kyle and Xiong (2001), we are not aware of any models that are able to generate increased comovements in periods of stress from the regulatory constraints. Our model suggests that one of the explanations for the observed state-dependent comovement may be the impact of risk constraints on portfolio optimization, especially in times of stress. Even in the absence of wealth effects and even if assets have independent payoffs and independent demand innovations, sufficiently strict regulations will cause some agents to adjust their risk position by scaling down their holdings in the risky assets, thereby introducing comovements. This effect will be most pronounced during financial crises. As a result, a Basel style regulation introduces the potential for an endogenous increase in correlation, thereby decreasing the agents ability to diversify and increasing the severity of financial crises. Financial institutions therefore require equilibrium risk premia that are higher than predicted by frictionless models.

2 The Model

Our economy is based upon a standard two-dates constant absolute risk–aversion model without asymmetric information and with a stochastic asset supply. There are two families of agents: regulated financial institutions (RFI) that are subjected to regulatory risk constraints (e.g. banks) and unregulated institutions (UFI) (e.g. hedge funds). The standard two-dates model is extended by adding an intermediate date, date one, to it. At date zero the UFIs and RFIs invest their (random) endowments in both risky and "riskless" (a zero-coupon bond) assets. Consumption occurs at date two. We follow common modelling practice by endowing financial institutions with their own utility functions (such as in Basak and Shapiro, 2001). At the intermediate date one, as further explained below, a refinancing need may arise, which we refer to as a *liquidity event*.

There are N nonredundant risky assets that promise, in the absence of any liquidity event, normally distributed payoffs $\boldsymbol{d} \sim N\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right)$ at date two, independent of the random endowments of assets.[2] The hats indicate payoffs, returns $(\ldots, d_i/q_i, \ldots)$ are distributed as $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Asset 0 is the "riskless asset" and promises to pay off the deterministic amount d_0, except in a *systemic event*, defined as a liquidity event at which refinancing fails due to the actions taken by investors, when the payoff is zero. Asset payoffs and returns are accordingly conditionally, but not unconditionally, normally distributed. The event tree may be represented schematically as:

Insert figure 1 here

Each FI is characterized by its type h, which determines risk-aversion and endowments, and by its regulation status t, which is either $t = \{r\}$ if the FI is regulated, or $t = \{u\}$ if it is unregulated. Each type of financial institution $h \in [\underline{\ell}, \overline{\ell}]$ is characterized by a constant coefficient of absolute risk aversion (CARA) α^h as well as an initial endowment of the riskless asset θ_0^h and of the risky assets $\tilde{\boldsymbol{\theta}}^h := \boldsymbol{\theta}^h - \boldsymbol{\epsilon}^h$, where $\boldsymbol{\epsilon}^h$ represents the random component of the endowments in the risky assets, with $\boldsymbol{\epsilon} := \int_{\underline{\ell}}^{\overline{\ell}} \boldsymbol{\epsilon}^h dh$. For simplicity, we assume that $\boldsymbol{\epsilon}^h = \frac{\boldsymbol{\epsilon}}{\overline{\ell}-\underline{\ell}}$, that $\alpha^h = h$ (but for clarity we still label agent h's coefficient by α^h rather than by h only) and that all institutions are risk-averse, $\underline{\ell} > 0$. A fraction $\eta(h)$ of agents of each type h are regulated, the remaining fraction is unregulated. We impose no restrictions on η, and none of the propositions in the paper depend qualitatively on η, provided that a non-null subset of agents are regulated. A FI (h, t) invests its initial wealth W_0^h in a portfolio comprising both riskless and risky assets, $(y_0^{h,t}, \boldsymbol{y}^{h,t})$. The time–zero wealth of an agent of type h (regulated or unregulated) comprises initial endowments in the riskless asset, θ_0^h, as well in risky assets, $\tilde{\boldsymbol{\theta}}^h$, so that $W_0^h \equiv q_0\theta_0^h + \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^h$. The price vector of risky assets is denoted by $\boldsymbol{q}$. Since the time-zero budget constraint $q_0\theta_0^h + \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^h \geq q_0 y_0^{h,t} + \boldsymbol{q}'\boldsymbol{y}^{h,t}$ is homogeneous of degree zero in prices, we can normalize, without loss of generality, the price of the riskless asset to $q_0 \equiv 1$, i.e. the riskless asset is used as the time–zero numéraire. We can write $R_f := d_0/q_0 = d_0$ for the return on the riskless asset in the absence of a systemic event. At date 2, the consumption commodity plays the role of the numéraire.

[2] Independence simplifies our analysis. If supply shocks were not independent of payoffs, then asset prices would convey payoff-relevant information. The information extraction problem is easy to solve for some parametric distributions, such as the normal distribution, but contributes little to the issues at hand and does require additional parametric assumptions. Normality of payoffs may be at odds with option-like derivative securities and should be viewed as an approximation over shorter periods in the presence of such option-like derivatives. In any Brownian model, returns on derivatives are normal over short horizons.

The aggregate amount of outstanding risky assets owned by investors is $\tilde{\boldsymbol{\theta}}^a := \int_{\underline{\ell}}^{\overline{\ell}} \tilde{\boldsymbol{\theta}}^h \mathrm{d}h$. The random component $\boldsymbol{\epsilon}$ is assumed to be distributed on $\mathbf{E} \subset \mathbb{R}^N$ according to the law $\mathbb{P}^{\epsilon}$, for simplicity assumed to be independent of the law governing asset payoffs, $\mathbb{P}^d$. In this paper, we do *not impose any* assumptions upon the distribution of $\boldsymbol{\epsilon}$ other than to assume that its support $\mathbf{E}$ is open and convex, in order to occasionally apply differential calculus. Instead of interpreting $\boldsymbol{\epsilon}$ as noisy asset endowments, with the appropriate adjustments one could interpret $\boldsymbol{\epsilon}$ as noise trader supplies. Because the total endowment of risky assets has to be absorbed by the UFIs and RFIs in equilibrium, prices depend upon $\boldsymbol{\epsilon}$. This is the only role of stochastic asset endowments. In a dynamic version of our model where dividends or news about the value of firms govern the resolution of uncertainty, they can be dropped entirely, as in Danielsson et al. (2004).

The aim of the regulations for risk-taking is to control extreme risk-taking by individual financial institutions. In theory, a large number of possible regulatory environments exist for this purpose. In practice, we are not aware of any published research into the welfare properties of alternative market risk regulatory methodologies,[3] and as a result, we adopt the standard market risk methodology, i.e., Value-at-Risk (VaR). The constraint takes the form:[4]

$$\mathbb{P}\left[(E[W^h] - W^h) \geq VaR\right] \leq \bar{p},$$

i.e. the probability of a *loss* larger than the uniform regulatory number VaR is no larger than $\bar{p}$. Each RFI maximizes the expected utility subject to both the budget constraint and the VaR constraint by choosing the optimal asset holdings. In the next section, we go into the details of how the liquidity events play out.

[3]Among those methodologies one could enumerate various schemes to explicitly limit risk-taking or leverage, lending-of-last-resort practices, regulation of the admissible financial contracting practices with a view of overcoming agency or free-riding problems, and so forth. Of course, we know from Artzner et al. (1999) that VaR is not a desirable measure from a *purely statistical* point of view because it fails to be subadditive. Furthermore, Ahn et al. (1999) show that the VaR measure may not be reliable because it is easy for a financial institution to legitimately manage reported VaR through options. Alexander and Baptista (2002) caution against using mean-VaR portfolio allocation as opposed to the standard mean-variance analysis.

[4]We follow standard practice (as advocated by the Basel Committee on Banking Supervision (1996) and by Jorion (2001) for instance) and use the relative VaR, i.e. the dollar loss relative to the mean (the unexpected loss), rather than the absolute VaR, i.e. the dollar loss relative to the initial value. Over short horizons the two coincide, but over longer horizons the relative VaR has proved more useful as it appropriately accounts for the time value of money. Indeed, over large horizons, with many data-generating processes calibrated to past data, the absolute VaR number would be swamped by the drift term.

3 Modelling Systemic Crises

While many authors attribute systemic fragility to an excessive piling-on of debt (e.g. Kindleberger (1978), Feldstein (1991)), those theories have relied explicitly or implicitly on irrationality.[5] In our model no such irrationality is required to generate excessive leverage, which arises solely by the fact that the less risk-averse FIs are not bearing the full social costs of their actions. Shares are rights to the output stream of firms. Firms may face a sudden aggregate liquidity need (assumed to be independently distributed of payoffs and demand innovations) which can only be satisfied by a further injection of capital (cash) from the shareholders in proportion to the size of their existing share holdings. This liquidity is reimbursed to them at date 1 with interest R_f, provided that the productive sector as a whole is able to raise sufficient liquidity. Since each investor believes he is too small to affect the aggregate allocation, and therefore whether the refinancing is successful or not, he may have an incentive to disregard the social cost of his actions and accumulate an excessively risky and leveraged position. What is an "excessive" level for a FI is specified within the model, and depends on the actions of all other FIs.

Formally, assume that a liquidity event $\mathfrak{L}$ occurred, and that each shareholder is asked during the emergency meetings with the firms' stakeholders to contribute to firm i K_i units of the riskless asset per unit of asset i held, with $\mathbf{K} := (\dots, K_i, \dots)$. This is similar to the fixed costs assumption in Marshall (1998). While shareholders do not have to come to the rescue of the productive sector by contributing working capital, it is a weakly dominant strategy to do so. The total amount of riskless assets lent by (h, t) to the productive sector is therefore the full amount $\mathbf{K}'\boldsymbol{y}^{h,t}$ if h has the required liquidity, $y_0^{h,t} \geq \mathbf{K}'\boldsymbol{y}^{h,t}$, otherwise $y_0^{h,t}$ only:

$$L^{h,t} := \min\{y_0^{h,t}, \mathbf{K}'\boldsymbol{y}^{h,t}\}$$

Defining $S^{h,t} := \mathbf{K}^\top y^{h,t} - L^{h,t}$, The financing shortfall stemming from investor (h, t) is:

$$S_+^{h,t} := \max\{0, S^{h,t}\}$$

Aggregate shortfall $S(\boldsymbol{\epsilon}, \bar{v})$ is defined as

$$S(\boldsymbol{\epsilon}, \bar{v}) := \int_{\underline{\ell}}^{\overline{\ell}} [\eta(h) S_+^{h,r} + (1 - \eta(h)) S_+^{h,u}] dh$$

[5]Some authors have studied financial stability purely from a banking perspective. The bank-run literature (e.g. Bryant (1980) and Diamond and Dybvig (1983), and subsequent work based thereon, e.g. Gorton and Huang (2003)) exploits coordination problems between multiple equilibria, while the interbank literature studies how shocks can be amplified through balance-sheet networks, such as Allen and Gale (2000) and Cifuentes et al. (2005). The sunspots features in some of the coordination models have been replaced by a unique equilibrium in the recent literature based on the global games concept (e.g. Carlsson and Van Damme (1993) and Morris and Shin (1998)). Our model in contrast is market based rather than banking based.

Aggregate output collapses if refinancing fails, i.e. if the proceeds are too low, and each investor's consumption is at the survival level. The event "refinancing fails" is the event that in aggregate $S(\boldsymbol{\epsilon}, \bar{v}) > \bar{S}$ for some $\bar{S}$. Implicit in this definition is the idea that even if working capital can be reallocated across firms at date one, there just is not enough to sustain all firms' production plans. The event (viewed as a measurable subset of $\mathbf{E}$) whereby a latent refinancing imbalance $S(\boldsymbol{\epsilon}, \bar{v}) > \bar{S}$ exists at equilibrium is denoted by $\mathfrak{F}_{\bar{v}}$. When no ambiguity arises, we simply denote it by $\mathfrak{F}$.

The assumption that output completely collapses is made for simplicity only and reflects a strongly interdependent production sector. While none of our main results depend on a precise micro foundation for such an interdependent sector, for the sake of concreteness we outline one such economy. Before nature chooses whether there will be a liquidity shock or not, each firm i is in the process of producing a heterogeneous intermediate output. We say that the production sector is *strongly interdependent* if the intermediate input-output matrix is symmetric, indecomposable and if every intermediate input of any firm i is crucial (meaning that the output of firm i is zero in case there is some intermediate input in the input list I_i of firm i that is no longer supplied to i). Indecomposability is a standing assumption in standard input-output analysis, see for instance Nikaido (1968). Indecomposability and cruciality are equivalent here to requiring that for any two firms i and j, there is a sequence of distinct firms $\{k_1 = i, k_2, \ldots, k_{n-1}, k_n = j\}$ such that a minimal amount of intermediate output by j is required as an intermediate input in the production of intermediate output k_{n-1}, and a minimal amount of of intermediate output k_{n-1} in turn is required as an intermediate input in the production of k_{n-2} and so forth all the way up to i. Any two sectors are directly or indirectly linked in this way. If I_i is the list of firms whose intermediate inputs are required in the production of intermediate output i, then the output of intermediate output i is (here $\boldsymbol{d} = (\ldots, d_i, \ldots)$ is the normally distributed random payoff variable introduced above, $\prod$ is the product operator and $\mathbf{1}_{\text{event}}$ stands for the indicator function which equals 1 if the event is true and zero otherwise):

$$\text{intermediate output}_i = e^{d_i}\mathbf{1}_{\left\{\prod_{j\in I_i} \text{intermediate output}_j > 0\right\}}$$

Before the intermediate outputs can be shipped, nature determines whether a liquidity shock occurs or not. If the liquidity shock occurs and refinancing fails for at least some firm j, then intermediate output$_j = 0$, and the intermediate outputs of all firms collapse. We might interpret the aggregate nature of the liquidity shock as consisting of necessary investments into the transportation network for inputs between firms. A shortfall larger than $\bar{S}$ represents the event whereby at least one link of the transportation network is no longer operational due to underinvestment. If refinancing succeeds, then intermediate output$_i = e^{d_i}$, all firms i. The intermediate output is then in a second production phase transformed into the homogeneous consumable final output via the production function

$$\text{final output}_i = \ln(\text{intermediate output}_i)$$

If some firm fails to refinance itself (or if some transportation link fails to receive adequate investments), then by strong interdependence all final outputs are $-\infty$ and each investor's consumption is normalized to be equal to some arbitrarily small survival amount $x^{h,t} = \underline{x} > -\infty$, all h, $t \in \{u, r\}$.[6] To summarize,

$$\mathfrak{L} \text{ and } \mathfrak{F} \quad \Rightarrow \quad x^{h,t} = \underline{x} \quad \text{a.s., all } h, \quad t \in \{u, r\}$$

Definition 1 (Systemic Crash, Normal Market Conditions) *We define a* systemic crash *(or a* systemic event *or* collapse*) as the event* $\mathfrak{L} \cap \mathfrak{F}$*. The ex-ante probability of a systemic event is* $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F}) = \mathbb{P}(\mathfrak{L})\mathbb{P}(\mathfrak{F})$*.*

Normal market conditions *are defined as the event* $\mathfrak{N} := (\mathfrak{L} \cap \mathfrak{F})^c$ *which obtains with ex-ante probability* $\mathbb{P}(\mathfrak{N}) = 1 - \mathbb{P}(\mathfrak{L} \cap \mathfrak{F})$*.*

Notice that probabilities depend on $\bar{v}$ as well as on the chosen distribution of risk among the agents.

4 The Decision Problem of the Financial Institutions

The RFI's program consists in choosing demand schedules to solve the following program.

Problem 1 (Risk-Constrained Program)

$$\max_{\{(\boldsymbol{y}^h, y_0^h)\}} \mathbb{P}(\mathfrak{L} \cap \mathfrak{F}|\boldsymbol{\epsilon})u^h(\underline{x}) + (1 - \mathbb{P}(\mathfrak{L} \cap \mathfrak{F}|\boldsymbol{\epsilon}))E[u^h(x^h)\|\mathfrak{N}, \boldsymbol{\epsilon}]$$

$$\begin{aligned} \text{subject to} \quad & y_0^h + \boldsymbol{q}'\boldsymbol{y}^h \leq \theta_0^h + \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^h \\ & x^h = W^h := \boldsymbol{d}'\boldsymbol{y}^h + R_f L^h + R_f(y_0^h - L^h) = \boldsymbol{d}'\boldsymbol{y}^h + R_f y_0^h \\ & \mathbb{P}\left[(E[W^h|\boldsymbol{\epsilon}] - W^h) \geq VaR\|\boldsymbol{\epsilon}\right] \leq \bar{p} \end{aligned}$$

Since individual institutions are negligible, this formulation gives rise to the free-riding externality mentioned above. Each financial institution chooses to neglect the effect of their actions on $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F}|\boldsymbol{\epsilon})$. Rationality on behalf of the investor requires that he correctly learns from $\boldsymbol{\epsilon}^h$ and $\boldsymbol{q}$ in equilibrium. Of course, since $\boldsymbol{\epsilon}^h$ fully reveals $\boldsymbol{\epsilon}$, $\boldsymbol{q}$ is uninformative given $\boldsymbol{\epsilon}^h$. The investor therefore knows whether a critical latent imbalance is built up or not. What he does not know is whether a liquidity event obtains that would turn the known latent imbalance into a systemic crisis.

[6] As usual in the CARA-normal setting, the consumption set is unbounded below and equals $\mathbb{R} \cup \{-\infty, +\infty\}$. The least desirable bundle –"the collapse of the productive sector"– is therefore effectively $-\infty$, and $\underline{x}$ is an arbitrary small number, i.e. negative with $|\underline{x}|$ arbitrarily large.

Consider the auxiliary program where it is known that a systemic crash is impossible. Payoffs and returns are then normally distributed, and a sufficient statistic for portfolio risk is the volatility of W^h. The VaR constraint can therefore be stated as an exogenous upper bound $\bar{v}$ on portfolio variance,[7]

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h \leq \bar{v} \tag{1}$$

The auxiliary program can be written:

$$\max_{\{(\boldsymbol{y}^h, y_0^h)\}} E[u^h(x^h)|\mathfrak{N}, \boldsymbol{\epsilon}]$$

$$\begin{aligned} \text{subject to} \quad & y_0^h + \boldsymbol{q}'\boldsymbol{y}^h \leq \theta_0^h + \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^h \\ & x^h = W^h \equiv \boldsymbol{d}'\boldsymbol{y}^h + R_f L^h + R_f(y_0^h - L^h) = \boldsymbol{d}'\boldsymbol{y}^h + R_f y_0^h \\ & \boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h \leq \bar{v} \end{aligned}$$

Suppose that at the equilibrium with the original program the investor knows that there is no global imbalance, $\mathfrak{F}^c$. Then whether or not a liquidity event obtains, no systemic crash can occur, and the solution to Program 1 coincides with the solution to the auxiliary program. Next assume that the investor knows that $\mathfrak{F}$ obtains. Since neither $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F}|\boldsymbol{\epsilon})$ nor $u^h(\underline{x})$ are affected by the actions of investor h, his objective function again coincides with the one in the auxiliary program. The same is true for the VaR constraint:

$$\mathbb{P}\left[(E[W^h|\boldsymbol{\epsilon}] - W^h) \geq VaR\|\boldsymbol{\epsilon}\right] = \mathbb{P}(\mathfrak{L})\cdot 0 + (1-\mathbb{P}(\mathfrak{L}))\mathbb{P}^d\left[(E^d[W^h|\boldsymbol{\epsilon}] - W^h) \geq VaR\|\boldsymbol{\epsilon}\right] \leq \bar{p}$$

This is the main role played by the assumption that crisis consumption $\underline{x}$ is exogenous as it allows both two-fund separation and the given VaR constraint to hold even with the possibility of a non-null systemic event. The solution to Program 1 coincides with the solution to the auxiliary program, with $\bar{p}$ replaced by $\frac{\bar{p}}{(1-\mathbb{P}(\mathfrak{L}))}$. In the analysis that follows we simply write $\bar{p}$ with the understanding that it should be $\bar{p}/(1-\mathbb{P}(\mathfrak{L}))$ in case investors know there is a latent imbalance. The solution to the investor's problem is given by the following lemma:

Lemma 1 (Optimal Portfolio) *The optimal portfolio of risky assets for RFI (h,t) has the mean-variance form*

$$\boldsymbol{y}^{h,t} = \frac{1}{\alpha^h + \phi^{h,t}}\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q}) \tag{2}$$

[7]Indeed, denoting the cumulative standard normal distribution function by $N(\cdot)$, the VaR constraint can be reduced to a volatility constraint: $\mathbb{P}^d\left[(E^d[W^h|\boldsymbol{\epsilon}] - W^h) \geq VaR\|\boldsymbol{\epsilon}\right] \leq \bar{p}$ iff $N\left(\frac{-VaR}{\mathrm{Std}^d(W^h|\boldsymbol{\epsilon})}\right) \leq \bar{p}$ iff $\mathrm{Std}^d(W^h\|\boldsymbol{\epsilon}) \leq \frac{VaR}{-N^{-1}(\bar{p})}$ iff $\mathrm{Var}^d(W^h|\boldsymbol{\epsilon}) \leq \bar{v} := \left(\frac{VaR}{-N^{-1}(\bar{p})}\right)^2$.

where $\phi^{h,u} := 0$ *and* $\phi^{h,r} := \frac{2\lambda^{h,r}}{E^d[u^{h\prime}(W^h)|\boldsymbol{\epsilon}]} \geq 0$*, with* $\lambda^{h,r}$ *being the Lagrange multiplier of the VaR constraint. The effective degree of risk-aversion,* $\alpha^h + \phi^{h,r}$*, is independent of the initial wealth* W_0^h *and only depends on* α^h*,* $\boldsymbol{q}$ *and* $\bar{v}$*.*

A binding risk-regulation affects the portfolio through the effective degree of risk-aversion, $\alpha^h + \phi^{h,r}$. Whereas the coefficient of absolute risk-aversion is constant for unrestricted FIs, it is endogenous for the FIs subjected to the VaR regulations and is larger than their utility-based coefficient during volatile events, $\alpha^h + \phi^{h,r} \geq \alpha^h$. In volatile events RFIs shift wealth out of risky assets into the safe haven provided by the riskless asset. This is one way of capturing the often-heard expression among practitioners that "risk-aversion went up," or that there is a "flight to quality." This is reminiscent of the effect of portfolio insurance on optimal asset holdings found in Grossman and Zhou (1996). Also see Gennotte and Leland (1990) and Basak (1995). As a matter of convention, we reserve the term *risk-aversion* to the CARA coefficients α^h. We call $\alpha^h + \phi^{h,r}$ the coefficient of *effective risk-aversion*, and we call its inverse *risk appetite.* From here it can be easily shown that the FIs with risk aversions close to $\underline{\ell}$ are highly levered in that they borrow from the more risk averse and invest that borrowed money in risky projects, thereby effectively acting as banks.

Market clearing prices require that the total excess demand by regulated and unregulated institutions, $\int_{\underline{\ell}}^{\overline{\ell}}[\eta(h)\boldsymbol{y}^{h,r} + (1-\eta(h))\boldsymbol{y}^{h,u}]\mathrm{d}h - \tilde{\boldsymbol{\theta}}^a$ must equal zero. Equivalently they satisfy the relation:

$$\boldsymbol{q} = \frac{1}{R_f}\left[\hat{\boldsymbol{\mu}} - \Psi\hat{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\theta}}^a\right] \tag{3}$$

where

$$\Psi^{-1} := \int_{\underline{\ell}}^{\overline{\ell}}\left[\eta(h)\frac{1}{\alpha^h + \phi^{h,r}} + (1-\eta(h))\frac{1}{\alpha^h}\right]\mathrm{d}h \tag{4}$$

is the aggregate effective risk-tolerance. Prices are equal to risk-neutral prices minus a risk adjustment. Ψ can also be viewed as the reward-to-variability ratio (or a market-price of risk scalar) of the market $\tilde{\boldsymbol{\theta}}^a$, $\Psi = \frac{\hat{\mu}_{\mathrm{M}} - R_f q_{\mathrm{M}}}{\hat{\sigma}^2_{\mathrm{M}}}$. Compared to an economy without any VaR constraints where the market-price of risk scalar is $\gamma := \left(\int_{\underline{\ell}}^{\overline{\ell}}\frac{1}{\alpha^h}\mathrm{d}h\right)^{-1}$, we have $\Psi \geq \gamma$. But the market price of risk is not only higher in a constrained economy than in an unconstrained one, it also is endogenous and random through the additional risk aversion ϕ^h imposed by the regulations.[8] The sole pricing factor being the market portfolio

[8]Equations (2, 3 and 4) remain valid if utility functions are not of the constant absolute risk-aversion class. The only difference would be that $\alpha^h = \frac{-\mathrm{E}^d[u^{h\prime\prime}]}{\mathrm{E}^d[u^{h\prime}]}$, and therefore endogenous. While no closed-form solutions exist in this more general case, $\Psi \geq \gamma$ would still hold and the rationale underlying our results would survive with reasonable income effects. Since most results in the sequel are driven mainly by the

$\tilde{\boldsymbol{\theta}}^a$, it becomes apparent that assuming noise traders is equivalent to assuming a random aggregate endowment in risky assets of $\tilde{\boldsymbol{\theta}}^a$.

5 On Market Clearing

Our definition of a rational expectations competitive equilibrium as a pricing function Q mapping noise trades $\boldsymbol{\epsilon}$ to market clearing prices is entirely standard (see Radner (1979)):

Definition 2 *A* **competitive equilibrium** *is a pricing function Q together with its domain, $Q : \mathbf{E} \times \mathbb{R}_+ \to \mathbb{R}^N$, an asset allocation $(h \in [\underline{\ell}, \overline{\ell}], t \in \{r, u\}, \boldsymbol{\epsilon} \in \mathbf{E}) \mapsto (\boldsymbol{y}^{h,t}, y_0^{h,t})(\boldsymbol{\epsilon})$ and a consumption allocation $(h \in [\underline{\ell}, \overline{\ell}], t \in \{r, u\}, \boldsymbol{\epsilon} \in \mathbf{E}) \mapsto x^{h,t}(\boldsymbol{\epsilon})$ such that*

(i) Given any $(\boldsymbol{\epsilon}, \bar{v}) \in \mathbf{E} \times \mathbb{R}_+$ and $\boldsymbol{q} \in Q(\boldsymbol{\epsilon}, \bar{v})$, $(\boldsymbol{y}^{h,t}, y_0^{h,t}, x^{h,t})$ solve FI (h,t)'s optimization problem, and this is true for all FIs $(h,t) \in [\underline{\ell}, \overline{\ell}] \times \{r, u\}$.

(ii) Markets for risky assets clear, $\int_{\underline{\ell}}^{\overline{\ell}} \left[\eta(h)\boldsymbol{y}^{h,r} + (1-\eta(h))\boldsymbol{y}^{h,u}\right] dh = \tilde{\boldsymbol{\theta}}^a$, for each $\boldsymbol{\epsilon} \in \mathbf{E}$.

(iii) Expectations are confirmed: the pricing function under which investors optimize coincides with the equilibrium pricing function.

Proposition 1 solves for the equilibrium Ψ (see Equations (15, 16, 17) in the Appendix for an exact expression) and prices. Most proofs are contained in the Appendix, and all figures are at the end of the paper.

Proposition 1 (Existence) *If $\eta(h) < 1$ over a set of positive Lebesgue measure, there exists a unique competitive equilibrium for any $(\boldsymbol{\epsilon}, \bar{v}, \underline{\ell}) \in \mathbf{E} \times [0, \infty) \times (0, \overline{\ell}]$.*

If $\eta = 1$ almost everywhere, there exists an equilibrium for $\underline{\ell} \in [0, \overline{\ell}]$ and for $(\bar{v}, \boldsymbol{\epsilon})$ satisfying $\boldsymbol{\epsilon} \in \mathcal{E}(\bar{v}, \underline{\ell}) := \left\{\boldsymbol{\epsilon} \in \mathbf{E} : [(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]^{1/2} \leq (\overline{\ell} - \underline{\ell})\sqrt{\bar{v}}\right\}$. For $(\bar{v}, \boldsymbol{\epsilon})$ such that $\boldsymbol{\epsilon} \in \operatorname{int} \mathcal{E}(\bar{v}, \underline{\ell})$, the equilibrium is unique, while for $(\bar{v}, \boldsymbol{\epsilon})$ such that $\boldsymbol{\epsilon} \in \partial\mathcal{E}(\bar{v}, \underline{\ell})$ asset prices and consumption allocations are indeterminate (within a certain range of prices) but the allocation of risky assets is not. No equilibrium exists for $\boldsymbol{\epsilon} \in \mathfrak{E} := \mathbf{E} \setminus \mathcal{E}(\bar{v}, \underline{\ell})$.

---

fact that risk-constraints effectively lower aggregate risk-tolerance, we feel comfortable as to the robustness of the results derived here. This is strengthened by the fact that for small risks (such as in a continuous-time framework) the CARA-normal model is essentially true without loss of generality, even if neither preferences are of the CARA type nor returns are normal. In the event of "normal market conditions" we can think of random payoffs as being a "small" risk. This strengthens the case for the CARA-normal model since the events that may lead to non-normal distributions ex-ante are embodied in the systemic event. We do not assume that payoffs or returns are ex-ante normally distributed, only that payoffs are conditionally normally distributed.

Equilibria always exist if there are unregulated financial institutions ($\eta < 1$ over a set of positive Lebesgue measure). If almost all institutions are regulated ($\eta = 1$), then there are combinations of regulatory levels $\bar{v}$ and asset endowment innovations $-\boldsymbol{\epsilon}$ in which markets cannot clear. This happens precisely if the endowment $\tilde{\boldsymbol{\theta}}^a$ that has to be absorbed by the regulated financial institutions is sufficiently different from 0 so that the number of agents over which the risk needs to be evenly spread, $\kappa(\boldsymbol{\epsilon}; \bar{v}) := \sqrt{\frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})' \hat{\boldsymbol{\Sigma}} (\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{\bar{v}}}$, is larger than the population: $\kappa > \overline{\ell} - \underline{\ell}$. This defines the non-existence event $\mathfrak{E}$. This feature is not a shortcoming of our model. In fact, *any* model would exhibit such a result as it relies solely on the universality of VaR constraints. Figure 2 illustrates this phenomenon in an economy with two assets and different levels of tightness $\bar{v}$. Each level of tightness determines an ellipsoid set of noise supplies that can be supported by a competitive equilibrium. For $\boldsymbol{\epsilon}$ outside of this ellipsoid, FIs cannot absorb the supply as described earlier, and markets break down. And for a tighter regulatory level $\bar{v}_2 < \bar{v}_1$, the set of supportable supplies shrinks even further, $\mathcal{E}(\bar{v}_2) \subset \mathcal{E}(\bar{v}_1)$. This suggests the policy implication that if the supervisory authorities impose stringent risk limits (in the sense that $\bar{v}$ is small enough to lead to $\mathcal{E}(\bar{v}) \not\ni \mathbf{0}$, i.e. $\bar{v} < \frac{\boldsymbol{\theta}^{a\prime} \hat{\boldsymbol{\Sigma}} \boldsymbol{\theta}^a}{(\overline{\ell} - \underline{\ell})^2}$), some agents need to be exempted from those constraints for markets to clear. For derivatives, however, $\mathbf{0} \in \mathbf{E}$, and no exemptions are required as long as regulations are not too strict.

6 Equilibrium Pricing Function

The imposition of the VaR constraints affects the equilibria directly, with interesting results on risk-taking, liquidity, and volatility. We present our main results about the equilibrium pricing function during normal market conditions in a series of Propositions, with all proofs relegated to the Appendix. We shall retain the following assumption in this section. Basically it requires that the set of possible stochastic asset supplies is such that for sufficiently strict regulations, some agents face binding VaR constraints. Evidently, the problem is not interesting otherwise.

Assumption [**A**]. For a given $\mathbf{E}$ assume that there is a $\bar{v}'$ such that there is a compact subset of $\mathbf{E}$, call it $\mathbf{E}'$, which is non-null, $\mathbb{P}^{\boldsymbol{\epsilon}}(\mathbf{E}') > \mathbf{0}$, and which is such that $\forall \boldsymbol{\epsilon} \in \mathbf{E}'$, $\kappa(\boldsymbol{\epsilon}) > \underline{\ell}(\ln \overline{\ell} - \ln \underline{\ell})$ for all $\bar{v} \leq \bar{v}'$. Also, assume that the covariance matrix of the stochastic asset supplies over $\mathbf{E}'$, $E[(\boldsymbol{\epsilon} - E[\boldsymbol{\epsilon}])(\boldsymbol{\epsilon} - E[\boldsymbol{\epsilon}])^\top \mathbf{1}_{\boldsymbol{\epsilon} \in \mathbf{E}'}]$, exists and is positive definite.

6.1 Prices and Risk Premia

From (3) and (4) we know that the equilibrium pricing function is

$$Q(\boldsymbol{\epsilon}, \bar{v}) = \frac{1}{R_f}\left[\hat{\boldsymbol{\mu}} - \Psi(\kappa(\boldsymbol{\epsilon}, \bar{v}))\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right] \tag{5}$$

with $\kappa(\boldsymbol{\epsilon}, \bar{v}) := \sqrt{\frac{(\boldsymbol{\theta}^a-\boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a-\boldsymbol{\epsilon})}{\bar{v}}}$. Since in economies where regulations are binding the reward-to-variability ratio is higher than in unregulated economies, $\Psi > \gamma$, it follows from (5) that at equilibrium, a binding risk-regulation induces lower prices for a risky asset j compared to the unconstrained economy iff the covariance of asset j's payoff with the payoff of the market portfolio $\tilde{\boldsymbol{\theta}}^a$, equivalently the beta, is positive, $(\hat{\boldsymbol{\Sigma}})_{j^{th}\text{row}}\tilde{\boldsymbol{\theta}}^a > 0$, and higher prices otherwise. Therefore equity risk premia are higher the more tightly regulated the economy is:

Proposition 2 (Equity Risk Premia) *Let $\bar{v}_2 < \bar{v}_1$. Then $\mu_i(\boldsymbol{\epsilon}, \bar{v}_2) - R_f > \mu_i(\boldsymbol{\epsilon}, \bar{v}_1) - R_f$.*

Here $\mu_i(\boldsymbol{\epsilon}, \bar{v}) := \hat{\mu}_i / Q_i(\boldsymbol{\epsilon}, \bar{v})$ is the conditional expected return on asset i. It is indeed easy to see that the CAPM with respect to the market portfolio holds. For instance, the excess return on asset i is $\mu_i - R_f = \beta_{M,i}(\mu_M - R_f)$, where $\mu_M - R_f = \frac{\Psi\hat{\sigma}_M^2}{q_M}$. The tighter the economy is regulated, the higher Ψ and the lower q_M, generating higher expected excess returns.[9]

Intuitively, a more tightly regulated economy transfers risk from the less risk-averse to the more risk-averse investors for markets to clear. But the latter need to be induced to buy into the risk by more advantageous prices, i.e. by higher expected returns. If the stylized coefficients of risk-aversion are too low to match asset returns when using frictionless models, the additional degree of effective risk-aversion $\Psi - \gamma$ due to risk-taking constraints, such as the ones imposed by the regulatory environment, may form the basis of a model that attempts to address the equity premium puzzle (as outlined by Mehra and Prescott, 1985; Weil, 1989). This is left as an avenue for future exploration.

6.2 Depth

The risk constraint affects the *depth* of the markets directly. In our context, depth is an appropriate measure of liquidity. The inverse of the depth of the entire market, shallowness $\mathfrak{s}(\boldsymbol{\epsilon}, \bar{v})$, is defined as the maximal extent to which an additional (unit-size) market order

[9] $\hat{\sigma}_M^2$ is the variance of the payoff of the residual market portfolio, and therefore exogenous. The price of the market portfolio, $q_M := \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^a$ is given by $R_f^{-1}\hat{\boldsymbol{\mu}}'\tilde{\boldsymbol{\theta}}^a - R_f^{-1}\Psi(\tilde{\boldsymbol{\theta}}^a)'\hat{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\theta}}^a$, decreasing unambiguously in Ψ.

for a portfolio impacts its price. Formally,

$$\mathfrak{s}(\boldsymbol{\epsilon}, \bar{v}) := \max_{\boldsymbol{\theta} \text{ subject to } \|\boldsymbol{\theta}\|=1} |\boldsymbol{\theta}' dQ| = \max_{\boldsymbol{\theta} \text{ subject to } \|\boldsymbol{\theta}\|=1} |\boldsymbol{\theta}'(\partial_{\boldsymbol{\epsilon}} Q)\boldsymbol{\theta}|$$

With this definition in mind, we can state:

Proposition 3 (Depth) *Depth is lower the tighter the constraint (i.e. the smaller $\bar{v}$), $\frac{\partial \mathfrak{s}(\boldsymbol{\epsilon},\bar{v})}{\partial \bar{v}} < 0$ for all $\boldsymbol{\epsilon} \in \mathbf{E}$. In particular, depth is lower in the regulated economy than in the unregulated economy for any $\boldsymbol{\epsilon} \in \mathbf{E}$.*

Refer to Figure 4 for an illustration. No RFI's risk taking constraint is binding for $\epsilon \in [\underline{\theta}^a(\bar{v}), \bar{\theta}^a(\bar{v})]$. We have not made any assumptions regarding the distribution of $\boldsymbol{\epsilon}$. However, in most cases we expect the market portfolio to be positive $\tilde{\boldsymbol{\theta}}^a > 0$. If we assume that $N = 1$, then the pricing function is concave over the relevant domain $\{\boldsymbol{\epsilon} : \tilde{\boldsymbol{\theta}}^a > 0\}$, and in most interesting cases –large positive shocks to the asset endowment that need to be absorbed, or restrictive regulations– the pricing function is strictly concave. The same can be shown for $N > 1$ given the proper restrictions on the domain of noise trades. If we assume that regulations are sufficiently strict so that some agents are hitting the regulatory constraint at $\boldsymbol{\epsilon} = 0$, $\bar{v} < \left(\frac{\sigma\theta^a}{\underline{\ell}(\ln \bar{\ell} - \ln \underline{\ell})}\right)^2$, and also that $\mathbb{P}^{\epsilon}([\theta^a, \infty)) = 0$, then an inflow raises prices less than the corresponding outflow lowers them. This is the widespread phenomenon dubbed by traders as "going up by the stairs and coming down by the elevator."

6.3 Volatility, Diversification and Comovements

In the single asset case, inspection of Figure 4 reveals that the time zero asset price becomes more volatile the stricter the VaR constraints are. In other words, uniform shallowness implies ex-ante volatility. The single-asset intuition can then be extended to the general case (a matrix M_1 is more positive definite than a matrix M_2 if $M_1 = M_2 + N$, with N positive definite):

Proposition 4 (Volatility) *Consider any two levels of regulation $\bar{v} < \bar{v}'$, at least one of them binding for some RFIs. The variance-covariance matrix of asset prices in the $\bar{v}$ economy is more positive definite than the one in the $\bar{v}'$ economy. It follows that the equilibrium price of* any portfolio *(and therefore of any security) becomes more volatile in the economy with tighter regulations, $\bar{v}$, than in economy $\bar{v}'$. In particular, there is more volatility in the constrained economy than in the unconstrained economy.*

The basic intuition behind these results is as follows. The endowment shocks which were absorbed by the more risk neutral RFIs in economy $\bar{v}'$ now have to be absorbed in the economy with $\bar{v} < \bar{v}'$ by the more risk averse. However, the more risk averse are less willing to absorb these (additional) units than the less risk averse. Hence the imposition of the risk constraint reduces market depth, and the market impact of a market order is larger. Since the arrival of market orders (asset endowments) is random, this generates more volatile asset prices.[10]

The fact that both individual assets and portfolios become necessarily more volatile suggests at the very least that diversification does not improve sufficiently to counteract the increases in the volatilities of the assets, since any portfolio, no matter how it is diversified, becomes more volatile. In fact, by the multi-asset nature of our model, we can naturally show that covariances between individual assets increase with stricter regulation:

Proposition 5 (Comovements) *Assets that are intrinsically statistically independent (i.e. the payoffs as well as the endowment shocks of the assets considered are statistically mutually independent) become positively correlated due to risk-regulations.*

Even if two asset classes are payoff-independent and hit by independent endowment or noise trader shocks, if the regulations are strict enough to bind over a set of positive measure, then a large liquidity shock hitting one asset class will induce the VaR regulation to bind for some RFIs. These RFIs will subsequently need to adjust their global risk position, thereby creating comovements in asset prices among classes that would seem to be unrelated. Furthermore, these comovements would be detectable mostly in crisis situations since the VaR constraints do not bind in subdued periods. It would therefore seem that the VaR constraints bear one further seed of instability by not only creating asset price volatility, but by inducing correlations during the exact periods where such correlations are most dangerous. This phenomenon is often referred to as "contagion" in the finance literature, e.g. in Kyle and Xiong (2001) and Kodres and Pritsker (2002). Our result provides a complementary contagion channel to the income effects channel, and clarifies why these comovements occur especially during crisis and what the impact of risk-regulations on contagion could be.

[10] In a certain sense the additional volatility is generated from the fact that some agents, while active in the market, are restricted from participating fully. In Allen and Gale (1994) it is shown that (endogenous) restricted participation (in the sense of either choosing to bear some fixed costs and fully participate, or not participate at all) also may lead to more volatile asset prices. In their two-asset setup the result is driven by the assumption that neither the riskless nor the long term risky asset can be shorted. This implies that if in equilibrium only the aggressive agents participate, then they will not hold much of a cash buffer to absorb liquidity trades. As a result prices are more volatile than in a full participation equilibrium where cash-rich backward investors also participate.

7 How Successful is the VaR Constraint?

To make the regulatory problem interesting and transparent, we make the following assumption in this section:

Assumption [B]. [**B1**]: $(\theta_0^{h,t}, \boldsymbol{\theta}^{h,t}) = \frac{1}{\bar{\ell}-\underline{\ell}}(\theta_0^a, \boldsymbol{\theta}^a)$, all (h,t). [**B2**]: $\mathbb{P}(\{\boldsymbol{\epsilon} \in \mathbf{E} : (\mathbf{K}+\boldsymbol{q})'\tilde{\boldsymbol{\theta}}^a \leq 0\}) = 0$. [**B3**]: $(\mathbf{E}, \mathbf{K}, \bar{v})$ are so that $0 < \mathbb{P}(\mathfrak{F}) < 1$.

Assumption [**B1**] insures a neutral distribution of endowments that is not biased in favour or against the success of the VaR regulations. Assumption [**B2**] prevents pathological cases whereby the value of the entire market is negative, and [**B3**] assumes that refinancing conditions are not so strict (so weak) as to lead to failure almost surely (almost never).

Proposition 6 implies that if [**B**] holds, the VaR regulations are effective in reducing the probability of a systemic crash.[11]

Proposition 6 *Assume that* [**B1**] *and* [**B2**] *hold.*

Lowering $\bar{v}$ *reduces the probability of a systemic event* $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F})$ *(but at the expense of a lower probability of market clearing* $\mathbb{P}(\mathfrak{E})$ *if* $\eta = 1$ *almost everywhere), strictly so under* [**B3**]. *Furthermore, if* $\eta = 1$ *almost everywhere and* $\mathbb{P}(\theta_0^a \geq \mathbf{K}'\tilde{\boldsymbol{\theta}}^a) = 1$*, then* $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F}) = 0$ *for small enough* $\bar{v}$.

The reason why this policy is effective is as follows. The less risk-averse FIs hold large amounts of the risky portfolio if regulation is weak, and therefore will have to borrow at the riskless rate to finance such a risky holding, unless they are endowed with large amounts of assets to start with, which we exclude by condition [**B**]. Stricter risk-limits curb both the amount of risky assets held by the less risk-averse RFIs as well as their required leverage, and therefore make it more likely that such institutions are able to take part in the refinancing of the firms. The more subtle point is, however, that stricter VaR limits reduce prices and thereby induce UFIs, and in particular the less risk-averse ones, to purchase the risky assets sold by the RFIs. This effect may indicate that systemic risk can increase with a tightening of regulations. But the non-trivial general equilibrium effect on prices and the differential elasticities if demand for risk-averse versus less risk-averse investors means that buyers (the UFIs and the more risk averse RFIs) can purchase their larger holdings in risky assets at lower prices. The net effect is that the less risk-averse leave in equilibrium more of their wealth invested in the riskless asset (sold to them by the more risk-averse agents), creating less of a systemic imbalance despite holding riskier portfolios. This benefit must be balanced by the loss of diversification and risk-sharing, by more shallow markets and the increased volatility of prices during normal market conditions. And if $\eta = 1$ almost

[11] If $\eta = 1$ almost everywhere and if $\boldsymbol{\epsilon}$ is so that there is no market clearing price vector, then we assume that markets shut and allocations coincide with endowments. In particular, $\mathfrak{E} \subset \mathfrak{F}^c$.

everywhere, the regulator faces a further cost in that markets may not clear. For very strict levels of $\bar{v}$ and very inclusive regulations, and provided the economy in aggregate does have enough of the riskless asset, then the probability of a systemic crash goes to zero, but the likelihood of market clearing is reduced as well.

The effectiveness also depends on the distribution and relative mass of RFIs as encapsulated in $\eta(h)$. For instance, ceteris paribus, in an economy where η does not depend on h the regulations are more effective than in an economy where $\eta(h) = 0$ for $h \in [\underline{\ell}, \underline{\ell} + (\overline{\ell} - \underline{\ell})/2]$ and $\eta(h) = 1$ otherwise, for there the regulated institutions are not the ones imposing the externalities while the more risk-tolerant FIs that impose the majority of externalities are all unregulated.

We would like to conclude with a simple welfare analysis. While VaR based regulation is effective in the sense of lowering the probability of a systemic event, in light of the aforementioned welfare costs of regulation during normal market times it does not automatically follow though that risk sensitive regulation is socially beneficial. Whereas a general welfare study would take us too far afield, the following general observations illustrate the intuition that some regulation might be welfare enhancing while too strict a regulation would be detrimental. In a nutshell, provided subsistence consumption $\underline{x}$ is low enough, a strengthening of regulations will lower the probability of a systemic event and thereby increase expected utility. This is true up to a point beyond which further regulations are detrimental to welfare. The following argument formally proves this point.

Expected utility of FI h is given by the integral of $u^h(x^h)$ over the product measure $\mathbb{P}^{\mathfrak{L}\times\boldsymbol{\epsilon}\times d}$. Given a realisation of the liquidity and the endowment shocks $(\mathfrak{L}, \boldsymbol{\epsilon})$, write the expected utility over asset return realisations as $E^d[u^h(x^h)](\boldsymbol{\epsilon}, \mathfrak{L}) := w(\boldsymbol{\epsilon}, \mathfrak{L}; \bar{v})$. We then have the ex-ante utility

$$\begin{aligned}
E[u^h(x^h)] &= \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \int_{\boldsymbol{\epsilon}\in\mathfrak{F}} w(\boldsymbol{\epsilon}, \mathfrak{L}) d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) + \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \int_{\boldsymbol{\epsilon}\in\mathfrak{F}^c} w(\boldsymbol{\epsilon}, \mathfrak{L}^c) d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) \\
&\qquad + (1 - \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \int_{\boldsymbol{\epsilon}\in E} w(\boldsymbol{\epsilon}, \mathfrak{L}^c) d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) \\
&= \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \mathbb{P}^{\boldsymbol{\epsilon}}(\mathfrak{F}) u^h(\underline{x}) + \int_{\boldsymbol{\epsilon}\in E} w(\boldsymbol{\epsilon}, \mathfrak{L}^c) d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) - \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \int_{\boldsymbol{\epsilon}\in\mathfrak{F}} w(\boldsymbol{\epsilon}, \mathfrak{L}^c) d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) \qquad (6) \\
&= \mathbb{P}^{\mathfrak{L}}(\mathfrak{L}) \mathbb{P}^{\boldsymbol{\epsilon}}(\mathfrak{F}) \Big[u^h(\underline{x}) - E^{\boldsymbol{\epsilon}}[w(\boldsymbol{\epsilon}, \mathfrak{L}^c) \| \boldsymbol{\epsilon} \in \mathfrak{F}] \Big] + E^{\boldsymbol{\epsilon}}[w(\boldsymbol{\epsilon}, \mathfrak{L}^c)] \qquad (7)
\end{aligned}$$

Equation (7) illustrates the tradeoffs. As regulations become stricter, utility during normal market times decreases due to the various reasons mentioned before, which has to be balanced against the reduced probability of a systemic crisis. By Proposition 6, ex-ante utilility is increasing with a lower $\bar{v}$ if $u^h(\underline{x})$ is low enough, at least up to the point where $\mathbb{P}(\mathfrak{F}) = 0$. From that point onwards, ex-ante utility is decreasing with a lower $\bar{v}$. This can also be gleaned directly by differentiating the expression above to get that welfare increases with a smaller $\bar{v}$ iff

$$(1-\mathbb{P}^{\mathfrak{L}}(\mathfrak{L}))\int_E \frac{\partial w^h(\boldsymbol{\epsilon},\mathfrak{L}^c;\bar{v})}{\partial \bar{v}} d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) + \mathbb{P}^{\mathfrak{L}}(\mathfrak{L})\int_{\mathfrak{F}(\bar{v})^c} \frac{\partial w^h(\boldsymbol{\epsilon},\mathfrak{L}^c;\bar{v})}{\partial \bar{v}} d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})$$
$$< \mathbb{P}^{\mathfrak{L}}(\mathfrak{L})\int_{\partial_{\bar{v}}\mathfrak{F}(\bar{v})} [w^h(\boldsymbol{\epsilon},\mathfrak{L}^c;\bar{v}) - u^h(\underline{x})] d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})$$

The LHS represents the incremental welfare costs due to stricter regulation (the costs expected to be borne during normal market times) while the RHS represents the welfare gains due to the resulting lower probability of a systemic crash. By Proposition 6 we know that $\mathfrak{F}$ becomes smaller if $\bar{v}$ becomes smaller, and as long as that area $\partial_{\bar{v}}\mathfrak{F}(\bar{v})$ is in the support of $\mathbb{P}^{\boldsymbol{\epsilon}}$, which we assume, a low enough $\underline{x}$ guarantees that stricter regulations lead to a Pareto dominating allocation. When regulation is strict enough so that $\mathbb{P}(\mathfrak{F})$ is small enough, say zero, then the LHS is $\int_E \frac{\partial w^h(\boldsymbol{\epsilon},\mathfrak{L}^c;\bar{v})}{\partial \bar{v}} d\mathbb{P}^{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) > 0$ and the RHS is zero: the marginal costs outstrip the marginal gains. This suggests that the optimal level of regulation should be set at the interior point where the (possibly weighted) integral of the LHS over all h (the marginal social welfare cost) equals the (possibly weighted) integral of the RHS (the marginal social welfare gain).

8 Conclusion

The aim of this paper is two-fold. First, we are interested in modelling the underlying causes which generate systemic risk and lead to a rationale for regulating risk. This is in contrast with most models which impose risk regulation upon a first-best economy and where the conclusions may not be meaningful or realistic. We then study why and to what extent the current risk-regulation alleviates systemic risk. We show that risk-sensitive regulations of the VaR type do reduce the probability of a systemic event and therefore do alleviate some of the free-riding externalities. Such benefits do have to be balanced by the social costs imposed by the regulations. Pricing risk is shown to be endogenous. We demonstrate that the fact of regulating risk-taking changes the statistical properties of financial risk. Markets may not clear if regulations are too all-encompassing. We also derive equity premia which are larger than in standard models, going some way towards a resolution of the equity premium puzzle. We show that illiquidity, volatility and covariations are all larger than in an unregulated world. In particular they are especially large in periods of distress. This is so because in periods of distress, such as during panic sales, the market price of risk parameter Ψ, which also acts as the effective risk aversion parameter, is large. Markets act as if aggregate risk aversion went up, with less depth and more local volatility as a result, as illustrated on Figure 4. It is well-known that risk-modelling often fails in periods of stress due to the breakdown of established historical comovements. Our model exhibits some of the nonlinearities at the heart of this phenomenon and exhibits conditions under which there is an optimal interior amount of risk-sensitive regulation.

A Proofs

Proof of Lemma 1 The program consists in solving (the superscript d indicates that the expectation is computed with respect to the probability of the payoffs $\boldsymbol{d}$)

$$\max_{\{\boldsymbol{y}^h, y_0^h\}} E^d \left[u^h(d_0[\theta_0^h + \boldsymbol{q}'\tilde{\boldsymbol{\theta}}^h - \boldsymbol{q}'\boldsymbol{y}^h] + \boldsymbol{d}'\boldsymbol{y}^h)|\boldsymbol{\epsilon} \right] - \lambda^h \left[\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h - \bar{v} \right]$$

The FOCs (the program is strictly convex and constraint qualification holds), so the FOCs are both necessary and sufficient) are $E^d \left[u^{h\prime}(W^h)(\boldsymbol{d} - d_0\boldsymbol{q})|\boldsymbol{\epsilon} \right] = 2\lambda^h\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$, or equivalently

$$\mathrm{Cov}^d(u^{h\prime}(W^h), \boldsymbol{d}|\boldsymbol{\epsilon}) + E^d \left[u^{h\prime}(W^h)|\boldsymbol{\epsilon} \right] E[\boldsymbol{d}] - d_0 E^d \left[u^{h\prime}(W^h)|\boldsymbol{\epsilon} \right] \boldsymbol{q} = 2\lambda^h\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$$

Next, by Stein's Lemma [recall that Stein's Lemma asserts that if x and y are multivariate normal, if g is everywhere differentiable and if $E[g'(y)] < \infty$, then $\mathrm{Cov}(x, g(y)) = E[g'(y)]\mathrm{Cov}(x, y)$] and the fact that $\mathrm{Cov}^d(\boldsymbol{d}, W^h|\boldsymbol{\epsilon}) = \mathrm{Cov}^d(\boldsymbol{d}, \boldsymbol{d}'\boldsymbol{y}^h|\boldsymbol{\epsilon}) = \hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$ we get that:

$$\boldsymbol{y}^h = \frac{1}{\alpha^h + \phi^h}\hat{\boldsymbol{\Sigma}}^{-1}[\hat{\mu} - d_0\boldsymbol{q}]$$

where we also used the fact that in this CARA–Normal setup $\frac{-\mathrm{E}^d[u^{h\prime\prime}|\boldsymbol{\epsilon}]}{\mathrm{E}^d[u^{h\prime}|\boldsymbol{\epsilon}]} = \alpha^h$, and where we defined $\phi^h := \frac{2\lambda^h}{E^d[u^{h\prime}|\boldsymbol{\epsilon}]}$.

Finally, we'll derive the expression for $\alpha^h + \phi^h$ and show that it does not depend on the wealth of the institution. In order to accomplish this, we first need to find an expression for ϕ^h. To simplify expressions, define

$$\rho := (\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q})'\,\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q}) \tag{8}$$

It can easily be established that

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v} \quad (\text{and } \lambda^h \geq 0) \Rightarrow \alpha^h + \phi^h = \sqrt{\frac{\rho}{\bar{v}}} \tag{9}$$

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v} \quad (\text{so } \lambda^h = 0) \Rightarrow \alpha^h + \phi^h = \alpha^h \tag{10}$$

Indeed, assume that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v}$. Since $y^h = \frac{1}{\alpha^h+\phi^h}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q})$, this expression becomes $\left(\frac{1}{\alpha^h+\phi^h}\right)^2 \rho = \bar{v}$. Of course, if $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v}$ then $\lambda^h = 0$ and thus $\phi^h = 0$.

This implies that $\alpha^h + \phi^h$ is independent of W_0^h for given prices,

$$\alpha^h + \phi^h = \max\left\{\alpha^h, \sqrt{\frac{\rho}{\bar{v}}}\right\} \tag{11}$$

Indeed, assume first that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v}$. Then by (10) we have that $\alpha^h + \phi^h = \alpha^h$, so we need to show that $\alpha^h \geq \sqrt{\frac{\rho}{\bar{v}}}$. Now since $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = {\alpha^h}^{-2}\rho$, we know that ${\alpha^h}^{-2}\rho < \bar{v}$, so that indeed $\alpha^h > \sqrt{\frac{\rho}{\bar{v}}}$. Next, assume that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v}$. Then from (9) $\alpha^h + \phi^h = \sqrt{\frac{\rho}{\bar{v}}}$. So we need to establish that $\alpha^h \leq \sqrt{\frac{\rho}{\bar{v}}}$, which follows from $\phi^h \geq 0$. ■

Proof of Proposition 1 Before we proceed to the proof, notice that by Walras' Law, the markets for the riskless asset and for consumption clear if the market for risky assets clears. Indeed, denote aggregated FIs quantities by a superscript a: for any quantity x, $\int_{\underline{\ell}}^{\overline{\ell}}(\eta(h)x^{h,r} + (1-\eta(h))x^{h,u})dh = x^a$. Walras' Law at times 0 and 2 says that $(y_0^a - \theta_0^a) + \boldsymbol{q}'(\boldsymbol{y}^a - \boldsymbol{\theta}^a + \boldsymbol{\epsilon}) = 0$ [W0] and $x^a = d_0 y_0^a + \boldsymbol{d}'\boldsymbol{y}^a$ [W2]. So assume that $\boldsymbol{y}^a - \boldsymbol{\theta}^a + \boldsymbol{\epsilon} = 0$. Then by [W0] the market for the riskless asset clears as well, and by [W2] we immediately have clearing of the commodities market, $x^a = d_0\theta_0^a + \boldsymbol{d}'(\boldsymbol{\theta}^a - \epsilon)$, under "normal market conditions."

We now exhibit a solution to the fixed-point problem of existence. Fix some $\boldsymbol{\epsilon} \in \mathbf{E}$ and assume first that $\underline{\ell} > 0$. Recall from (4) that

$$\begin{aligned}\Psi^{-1} &= \int_{\underline{\ell}}^{\overline{\ell}} \eta(h)\frac{1}{\alpha^h + \phi^{h,r}}dh + \int_{\underline{\ell}}^{\overline{\ell}}(1-\eta(h))\frac{1}{\alpha^h}dh \\ &= \int_{I_1}\eta(h)\frac{1}{\alpha^h}dh + \int_{I_2}\eta(h)\sqrt{\frac{\bar{v}}{\rho}}dh + \int_{\underline{\ell}}^{\overline{\ell}}(1-\eta(h))\frac{1}{\alpha^h}dh \qquad (12)\end{aligned}$$

where $I_1 := \left\{h \in [\underline{\ell}, \overline{\ell}] : \alpha^h > \sqrt{\frac{\rho}{\bar{v}}}\right\}$ and $I_2 := \left\{h \in [\underline{\ell}, \overline{\ell}] : \alpha^h \leq \sqrt{\frac{\rho}{\bar{v}}}\right\}$.

In order to solve for the equilibrium, we use (3) to express $\boldsymbol{q}$ as a function of Ψ and then solve (12) for Ψ.

For convenience, we establish some preliminary calculations and notation. First, insert the pricing relation (3) into the definition of ρ from (8) to get the expression $\sqrt{\rho} = \Psi\sqrt{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}$. Second, define the relation

$$\kappa(\boldsymbol{\epsilon}) := \sqrt{\frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{\bar{v}}} \equiv \Psi^{-1}\sqrt{\frac{\rho}{\bar{v}}} \qquad (13)$$

$\kappa(\boldsymbol{\epsilon})$ represents the ratio of the standard deviation of the dividends of the residual market portfolio $\boldsymbol{\theta}^a - \boldsymbol{\epsilon}$ to the maximal allowable standard deviation of the payoffs of individual portfolios. Now we also use the assumption that $\alpha^h = h$. There are three cases to distinguish:

Assume first that $\kappa\Psi \leq \underline{\ell}$. Then we get $\Psi = \frac{1}{\ln\overline{\ell} - \ln\underline{\ell}}$.

Next, assume that $\kappa\Psi \in (\underline{\ell}, \overline{\ell})$. Then

$$\Psi^{-1} = (\ln \overline{\ell} - \ln \underline{\ell}) - \int_{\underline{\ell}}^{\Psi\kappa} [h^{-1} - (\Psi\kappa)^{-1}]\eta(h)dh \tag{14}$$

Finally, assume that $\kappa\Psi \geq \overline{\ell}$, in which case $\Psi = \frac{1-\kappa^{-1}\int_{\underline{\ell}}^{\overline{\ell}} \eta(h)dh}{\int_{\underline{\ell}}^{\overline{\ell}}(1-\eta(h))\frac{1}{h}dh}$.

This system is equivalent to the following, where we define $\underline{\kappa} := \underline{\ell}(\ln \overline{\ell} - \ln \underline{\ell})$ and $\overline{\kappa} := \overline{\ell} \int_{\underline{\ell}}^{\overline{\ell}}(1 - \eta(h))h^{-1}dh + \int_{\underline{\ell}}^{\overline{\ell}} \eta(h)dh$:

$$\Psi = \frac{1}{\ln \overline{\ell} - \ln \underline{\ell}} \qquad \text{if } \kappa \leq \underline{\kappa} \tag{15}$$

$$\Psi^{-1} = \ln \overline{\ell} - \ln \underline{\ell} - \int_{\underline{\ell}}^{\kappa\Psi} [h^{-1} - (\kappa\Psi)^{-1}]\eta(h)dh \qquad \text{if } \kappa \in (\underline{\kappa}, \overline{\kappa}) \tag{16}$$

$$\Psi = \frac{1 - \kappa^{-1}\int_{\underline{\ell}}^{\overline{\ell}} \eta(h)dh}{\int_{\underline{\ell}}^{\overline{\ell}}(1 - \eta(h))\frac{1}{h}dh} \qquad \text{if } \kappa \geq \overline{\kappa} \tag{17}$$

That the systems are equivalent can be verified as follows. First assume $\kappa \leq \underline{\kappa}$ and $\Psi = \frac{1}{\ln \overline{\ell} - \ln \underline{\ell}}$. Then $\Psi\kappa \leq \underline{\ell}$, which warrants the choice of Ψ. Now assume $\kappa \in (\underline{\kappa}, \overline{\kappa})$. Premultiply (16) by κ^{-1} and use the assumption that $\kappa > \underline{\ell}$ to get $\kappa\Psi > \underline{\ell}$. Now assume to the contrary that $\kappa\Psi \geq \overline{\ell}$. Using the fact that $\eta(h) = 0$ if $h > \overline{\ell}$, (16) can be rearranged by grouping the $\kappa\Psi$ terms to say $(\kappa\Psi)^{-1} = \frac{\ln \overline{\ell} - \ln \underline{\ell} - \int_{\underline{\ell}}^{\overline{\ell}} h^{-1}\eta(h)dh}{\kappa - \int_{\underline{\ell}}^{\overline{\ell}} \eta(h)dh} \leq \overline{\ell}^{-1}$ by assumption. Rewriting we get $\kappa \geq \overline{\kappa}$, a contradiction. So $\kappa\Psi < \overline{\ell}$. Finally, assume $\kappa \geq \overline{\kappa}$. Using (17) we get that $\kappa\Psi \geq \overline{\ell}$, which in turn justifies our choice of Ψ.

Recall that $\boldsymbol{\epsilon}$ affects Ψ only in as far as it affects κ, $\Psi(\kappa)$. For each κ, there is a unique Ψ solving this system. For the regions $\kappa \leq \underline{\kappa}$ and $\kappa \geq \overline{\kappa}$ this is obvious. So assume $\kappa \in (\underline{\kappa}, \overline{\kappa})$. Using x for $\kappa\Psi$, we rewrite (16) as $x = RHS(x) := \frac{\kappa + \int_{\underline{\ell}}^{x}[xh^{-1} - 1]\eta(h)dh}{\ln \overline{\ell} - \ln \underline{\ell}}$. It is easy to see that the mapping $RHS(x)$ is weakly increasing and convex, with slope strictly less than 1. Since $RHS(\underline{\ell}) = \lim_{x \searrow \underline{\ell}} RHS(x) = \frac{\kappa}{\ln \overline{\ell} - \ln \underline{\ell}}$. Assume that $\frac{\kappa}{\ln \overline{\ell} - \ln \underline{\ell}} \leq \underline{\ell}$, then $\kappa \leq \underline{\kappa}$, a contradiction, so $RHS(\underline{\ell}) > \underline{\ell}$. Similarly, $RHS(\overline{\ell}) < \overline{\ell}$ from the assumption $\kappa < \overline{\kappa}$. It follows that there is one and only one fixed point $x \in (\underline{\ell}, \overline{\ell})$.

Notice that if $\eta = 1$ almost everywhere, then

$$\Psi(\kappa) = \begin{cases} \frac{1}{\ln \overline{\ell} - \ln \underline{\ell}} & ; \kappa \in [0, \underline{\ell}(\ln \overline{\ell} - \ln \underline{\ell})] \\ -\frac{\kappa + \underline{\ell}}{\kappa W_{-1}\left(-(\kappa + \underline{\ell})\exp(-1 - \ln \overline{\ell})\right)} & ; \kappa \in (\underline{\ell}(\ln \overline{\ell} - \ln \underline{\ell}), \overline{\ell} - \underline{\ell}) \\ \text{any number} \geq \frac{\overline{\ell}}{\overline{\ell} - \underline{\ell}} & ; \kappa = \overline{\ell} - \underline{\ell} \\ \text{undefined} & ; \kappa > \overline{\ell} - \underline{\ell} \end{cases}$$

where $W_{-1}(\cdot)$ is the non-principal (lower) branch of the Lambert W-correspondence. Recall that the Lambert W-correspondence is defined as the multivariate inverse of the function $w \mapsto we^w$. In particular, the solution to $ax + b\ln x + c = 0$ is given by $x = \frac{b}{a}W_{-1}\left(\frac{a}{b}e^{-\frac{c}{b}}\right)$. Notice that the mapping Ψ is $\mathcal{C}^1$ (but not $\mathcal{C}^2$ since the second derivative to the left of $\underline{\ell}(\ln\overline{\ell} - \ln\underline{\ell})$ is zero while the second derivative to the right is $\frac{\eta}{\underline{\ell}^2(\ln(\overline{\ell})-\ln(\underline{\ell}))^4} > 0$) and that by construction the equilibrium Ψ satisfies $\Psi \geq \gamma$.

Over the entire domain the function $\Psi(\kappa)$ is illustrated in figure (3). In the case for $\eta = 1$ and $\kappa = 1 - \underline{\ell}$, which is equivalent to $\boldsymbol{\epsilon}$ being on the boundary of $\mathbf{E}$, the equilibrium can be shown to exhibit real indeterminacy. ■

Proof of Proposition 3 As preliminaries, let us record the following useful results:

J1 $\frac{\partial\kappa}{\partial\bar{v}} = -\frac{1}{2}\frac{\kappa}{\bar{v}}$, from the definition of κ, and $\partial_{\boldsymbol{\epsilon}}\kappa = \kappa^{-1}\bar{v}^{-1}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\epsilon} - \boldsymbol{\theta}^a)$.

J2 $\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi = -\frac{1}{2\bar{v}}\left[\kappa\frac{\partial^2\Psi}{\partial\kappa^2} + \frac{\partial\Psi}{\partial\kappa}\right]\partial_{\boldsymbol{\epsilon}}\kappa$. Indeed, since $\frac{d\Psi}{d\bar{v}} = \frac{\partial\Psi}{\partial\kappa}\frac{\partial\kappa}{\partial\bar{v}}$, we know from J1 that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi = \frac{d}{d\boldsymbol{\epsilon}}\left(\frac{\partial\Psi}{\partial\kappa}\frac{\partial\kappa}{\partial\bar{v}}\right) = \frac{\partial\kappa}{\partial\bar{v}}\frac{\partial^2\Psi}{\partial\kappa^2}\partial_{\boldsymbol{\epsilon}}\kappa + \frac{\partial\Psi}{\partial\kappa}\partial_{\boldsymbol{\epsilon}}\left(-\frac{1}{2}\frac{\kappa}{\bar{v}}\right)$.

J3 $\partial_{\boldsymbol{\epsilon}}Q$ is positive definite (downward-sloping equilibrium inverse demand). Indeed, $\partial_{\boldsymbol{\epsilon}}Q = R_f^{-1}\left[\Psi\hat{\boldsymbol{\Sigma}} - \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\partial_{\boldsymbol{\epsilon}}\Psi)'\right] = R_f^{-1}\left[\Psi\hat{\boldsymbol{\Sigma}} + \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\ (\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\kappa^{-1}\bar{v}^{-1}\frac{\partial\Psi}{\partial\kappa}\right]$, positive definite.

J4 For $\kappa \in (\underline{\kappa}, \overline{\kappa})$, $\frac{\partial\Psi}{\partial\kappa} = \frac{\kappa^{-2}\int_{\underline{\ell}}^{\kappa\Psi}\eta(h)dh}{\ln\overline{\ell} - \ln\underline{\ell} - \int_{\underline{\ell}}^{\kappa\Psi}h^{-1}\eta(h)dh} > 0$. Indeed, totally differentiate (16) and use (16) to sign.

The idea of the proof is to show that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q$ is negative definite. Intuitively, we want to show that the market impact of a trade goes up as the regulation is tightened, i.e. that $\frac{\partial}{\partial\bar{v}}|(d\boldsymbol{q})'(d\boldsymbol{\epsilon})| = \frac{\partial}{\partial\bar{v}}[(d\boldsymbol{q})'(d\boldsymbol{\epsilon})] < 0$ since $(d\boldsymbol{q})'(d\boldsymbol{\epsilon}) > 0$ as $\partial_{\boldsymbol{\epsilon}}Q$ is positive definite by J3. Now this expression equals $\frac{\partial}{\partial\bar{v}}\left[(d\boldsymbol{\epsilon})'\partial_{\boldsymbol{\epsilon}}Q(d\boldsymbol{\epsilon})\right] = (d\boldsymbol{\epsilon})'\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q(d\boldsymbol{\epsilon}) < 0$ for all $d\boldsymbol{\epsilon} \neq 0$, but that's the definition of negative definiteness.

Before we show that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q(d\boldsymbol{\epsilon})$ is negative definite, we want to relate this idea to the definition of shallowness, $\mathfrak{s}(\boldsymbol{\epsilon}, \bar{v}) := \max_{\boldsymbol{\theta}} |\boldsymbol{\theta}'(\partial_{\boldsymbol{\epsilon}}Q)\boldsymbol{\theta}|$ such that $\|\boldsymbol{\theta}\| = 1$, namely that $\frac{\partial\mathfrak{s}}{\partial\bar{v}} < 0$ iff $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q$ negative definite. Indeed, pick any $\boldsymbol{\theta}$ such that $\|\boldsymbol{\theta}\| = 1$, then it is immediate that $\frac{\partial(\boldsymbol{\theta}'\partial_{\boldsymbol{\epsilon}}Q\boldsymbol{\theta})}{\partial\bar{v}} = \boldsymbol{\theta}'\left(-\partial_{\boldsymbol{\epsilon},\bar{v}}Q\right)\boldsymbol{\theta}$, which proves the claim. A tighter $\bar{v}$ makes $\partial_{\boldsymbol{\epsilon}}Q$ more positive definite.

The pricing function is $Q(\boldsymbol{\epsilon}, \bar{v}) = R_f^{-1}\left[\hat{\boldsymbol{\mu}} - \Psi\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right]$, from which we can deduce that $\partial_{\bar{v}}Q = -R_f^{-1}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\frac{d\Psi}{d\bar{v}}$, and furthermore that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q = R_f^{-1}\hat{\boldsymbol{\Sigma}}\frac{d\Psi}{d\bar{v}} - R_f^{-1}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi$. This expression can be simplified, using J2, to

$$\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q = -\frac{1}{2}R_f^{-1}\frac{\partial\Psi}{\partial\kappa}\frac{\kappa}{\bar{v}}\hat{\boldsymbol{\Sigma}} - \frac{1}{2}R_f^{-1}\bar{v}^{-2}\left[\frac{\partial^2\Psi}{\partial\kappa^2}\kappa + \frac{\partial\Psi}{\partial\kappa}\right]\left[\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\right]\kappa^{-1}$$

The first term is negative definite by J4, while the second one is negative semidefinite. Indeed, it can be shown that the expression $\left[\frac{\partial^2\Psi}{\partial\kappa^2}\kappa + \frac{\partial\Psi}{\partial\kappa}\right]$ is strictly positive, while the term $\left[\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\right]$ is clearly positive semidefinite. This concludes the proof that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q$ is negative definite. ■

Proof of Proposition 4 The variance-covariance matrix of prices is given by

$$\begin{aligned}\Omega &:= E[(Q - E[Q])(Q - E[Q])^\top] \\ &= \frac{1}{R_f^2}\hat{\boldsymbol{\Sigma}}E[\Psi^2(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top]\hat{\boldsymbol{\Sigma}} - \frac{1}{R_f^2}\hat{\boldsymbol{\Sigma}}E[\Psi(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]E[\Psi(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top]\hat{\boldsymbol{\Sigma}}\end{aligned}$$

Differentiation this matrix with respect to $\bar{v}$ we get

$$\begin{aligned}\partial_{\bar{v}}\Omega = &\frac{1}{R_f^2}\hat{\boldsymbol{\Sigma}}E\left[2\Psi\frac{\partial\Psi}{\partial\bar{v}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top\right]\hat{\boldsymbol{\Sigma}} - \\ &\frac{1}{R_f^2}\hat{\boldsymbol{\Sigma}}\left[E[\Psi(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]E\left[\frac{\partial\Psi}{\partial\bar{v}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right]^\top + E\left[\frac{\partial\Psi}{\partial\bar{v}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right]E[\Psi(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]^\top\right]\hat{\boldsymbol{\Sigma}}\end{aligned}$$

In view of the nonpositive sign of $\frac{\partial\Psi}{\partial\bar{v}}$, both matrices are NSD. We show next that the first matrix is, in interesting economies at least where the VaR constraint does bind, in fact ND. Write $w := -\Psi\frac{\partial\Psi}{\partial\bar{v}}$, a positive random variable. By assumption [**A**], there is a strictly positive $\underline{w} := \inf_{\boldsymbol{\epsilon}\in\mathbf{E}'} w$ and furthermore

$$E[(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top\mathbf{1}_{\boldsymbol{\epsilon}\in\mathbf{E}'}] \qquad \text{is PD}$$

It follows that

$$\begin{aligned}&\det \underline{w}E[(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top\mathbf{1}_{\boldsymbol{\epsilon}\in\mathbf{E}'}] > 0 \\ \Rightarrow &\det E[w(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top\mathbf{1}_{\boldsymbol{\epsilon}\in\mathbf{E}'}] > 0 \\ \Rightarrow &\det E[w(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})^\top] > 0\end{aligned}$$

This shows that $\partial_{\bar{v}}\Omega$ is ND. It follows that stricter regulations (lower $\bar{v}$) make the variance-covariance matrix of prices more positive definite, and in particular each variance increases. Since the variance of a portfolio $\boldsymbol{\theta} \in \mathbb{R}^N$ is $\boldsymbol{\theta}^\top\Omega\boldsymbol{\theta}$, the variance of any portfolio increases as (binding) regulations become stricter. ■

Proof of Proposition 5 Consider any two assets, say assets 1 and 2. Intrinsic independence requires $\hat{\boldsymbol{\Sigma}}$ diagonal, ϵ_1 and ϵ_2 stochastically independent, and the absence of regulations so that $\Psi = \varphi$. Then $Q_1(\epsilon_1)$ and $Q_2(\epsilon_2)$, so Q_1 and Q_2 are stochastically independent.

Since $Q_i = \left[\hat{\mu}_i - \Psi(\boldsymbol{\epsilon})\hat{\boldsymbol{\Sigma}}_i(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right]/R_f$,

$$\text{Cov}(Q_1, Q_2) = \hat{\Sigma}_{11}\hat{\Sigma}_{22}\text{Cov}(\Psi\tilde{\theta}_1, \Psi\tilde{\theta}_2)\frac{1}{R_f^2} \geq 0$$

with $\Psi(\tilde{\theta}_1, \tilde{\theta}_2)$. The last inequality follows from the fact that independent rvs are associated, see Esary et al. (1967). Indeed, with $\tilde{\theta}_1$ stochastically independent of $\tilde{\theta}_2$, any increasing functions (ϕ_1, ϕ_2) satisfy $\text{Cov}(\phi_1(\tilde{\theta}_1, \tilde{\theta}_2), \phi_2(\tilde{\theta}_1, \tilde{\theta}_2)) \geq 0$, i.e. $(\tilde{\theta}_1, \tilde{\theta}_2)$ are associated. Since the VaR constraint is binding over a subset of states for some level of regulation, a strict inequality follows. ■

Proof of Proposition 6 Define by $v_*(\boldsymbol{\epsilon})$ the weakest level of regulation for which all RFIs hit their VaR constraints and by $v^*(\boldsymbol{\epsilon})$ the weakest level of regulation for which there is at least some RFI with a binding VaR constraint.[12] In order to ascertain the probability of failure in refinancing we need to study the mapping (we have used the fact that in equilibrium $y^{h,t} = \frac{\Psi}{h+\phi^{h,t}}\tilde{\boldsymbol{\theta}}^a$)

$$h \mapsto S^{h,t} := \frac{\Psi}{h + \phi^{h,t}}(K + \boldsymbol{q})'\tilde{\boldsymbol{\theta}}^a - \frac{1}{\overline{\ell} - \underline{\ell}}\left[\boldsymbol{q}'\tilde{\boldsymbol{\theta}}^a + \theta_0^a\right]$$

and $S^h := \eta(h)S^{h,r} + (1 - \eta(h))S^{h,u}$.

F1 $\int_{\underline{\ell}}^{\overline{\ell}} S^h dh = K'\tilde{\boldsymbol{\theta}}^a - \theta_0^a$, irrespective of $\bar{v}$ (this follows from (16)).

F2 S^h is continuous in h.

Assume [**B**] holds and that for a given $\boldsymbol{\epsilon}$, $\bar{v} \in (v_*(\boldsymbol{\epsilon}), v^*(\boldsymbol{\epsilon}))$, then $\Psi\kappa \in (\underline{\ell}, \overline{\ell})$ and S^h satisfies:

F3 $S^{h,r} = S^{h',r}$, all $h, h' \leq \Psi\kappa$. For such h and h' with a binding constraint, $\frac{\Psi}{h+\phi^h} = \kappa^{-1}$ (this is shown in the proof of Proposition 1), so S^h does not depend on h.

F4 $S^{h,r} < S^{h',r}$ for $h > h'$, $h > \Psi\kappa$, and $S^{h,u} < S^{h',u}$ for $h > h'$. Pick for instance $h > \Psi\kappa$ and $h' < \Psi\kappa$. Then $S^{h,r} - S^{h',r} = \left(\frac{\Psi}{h} - \kappa^{-1}\right)[(\mathbf{K} + \boldsymbol{q})'\tilde{\boldsymbol{\theta}}^a] < 0$ by [**B2**] and by the fact that $\frac{\Psi}{h} - \kappa^{-1} < 0$ due to the assumption $h > \Psi\kappa$.

F5 Consider either an arbitrary (h, u), or an (h, r) with $h > \kappa\Psi$. Then some algebra reveals that $\frac{\partial S^{h,t}}{\partial \bar{v}} < 0$ iff $h > \tilde{h}(\bar{v}) := \Psi(\overline{\ell} - \underline{\ell}) - \frac{(\overline{\ell}-\underline{\ell})R_f(\mathbf{K}+\mathbf{q})'\tilde{\boldsymbol{\theta}}^a}{\tilde{\boldsymbol{\theta}}^{a'}\boldsymbol{\Sigma}\tilde{\boldsymbol{\theta}}^a}$. For (h, r) with $h < \Psi\kappa$, $\frac{\partial S^{h,r}}{\partial \bar{v}} > 0$ always holds true.

[12] Set both terms equal to zero if $\eta \equiv 0$.

For a given $\bar{v}$, recall that the event $\mathfrak{F}_{\bar{v}}$ is defined as

$$\left\{ \boldsymbol{\epsilon} \in \mathbf{E} : S(\bar{v}) := \int_{\underline{\ell}}^{\overline{\ell}} \left[\eta(h) \max\{0, S^{h,r}(\boldsymbol{\epsilon}, \bar{v})\} + (1 - \eta(h)) \max\{0, S^{h,u}(\boldsymbol{\epsilon}, \bar{v})\} \right] dh > \bar{S} \right\}$$

Recall that for an $\boldsymbol{\epsilon} \in \mathfrak{E}$ (which can occur only if $\eta = 1$) the autarky allocation results, and by [**B1**] the autarky allocation can never lead to a critical imbalance, so we set $S^h(\boldsymbol{\epsilon}, \bar{v}) \equiv 0$ for $\boldsymbol{\epsilon} \in \mathfrak{E}$. We now show that $\mathfrak{F}_{\bar{v}} \subset \mathfrak{F}_{\bar{v}'}$ if $\bar{v} < \bar{v}'$. So pick $\boldsymbol{\epsilon} \in \mathfrak{F}_{\bar{v}}$. Then $S(\bar{v}) > \bar{S}$.

If $\boldsymbol{\epsilon}$ is so that S^h as a function of h is either uniformly nonnegative or uniformly nonpositive, then by F1 $S(\bar{v}) = S(\bar{v}')$. It follows that $\boldsymbol{\epsilon} \in \mathfrak{F}_{\bar{v}'}$.

Otherwise if S^h is neither nonnegative nor nonpositive for all h (and there must be a non-null set of such $\boldsymbol{\epsilon}$ by [**B3**]), then we have two cases. Consider first the case $\tilde{h} > \Psi\kappa$. If $S^{\tilde{h}}(\bar{v}) = S^{\tilde{h}}(\bar{v}') > 0$, then by F4 (the absolute value of) the integral of the negative part of $S^h(\bar{v}')$ is larger than the one of $S^h(\bar{v})$. Since by F1 the overall areas must coincide, the integral of the positive part of $S^h(\bar{v}')$ is larger than the one of $S^h(\bar{v})$, i.e. $S(\bar{v}') > S(\bar{v})$. If on the other hand $S^{\tilde{h}}(\bar{v}) = S^{\tilde{h}}(\bar{v}') < 0$, then we can focus on the positive parts of the two functions directly, since in that case $S^h(\bar{v}') > S^h(\bar{v})$ for h s.t. $S^h(\bar{v}) > 0$, from which again we can deduce that $S(\bar{v}') > S(\bar{v})$.

Consider now the case $\tilde{h} < \Psi\kappa$. By the assumption that $\tilde{h} < \Psi\kappa$, $S^h(\bar{v}') < S^h(\bar{v})$ for all $h > \Psi\kappa$ (by F5). The area of the negative part of $S^h(\bar{v}')$ is larger than the one of $S^h(\bar{v})$, so (again by F1) must be the positive areas, i.e. $S(\bar{v}') > S(\bar{v})$.

It follows that $\boldsymbol{\epsilon} \in \mathfrak{F}_{\bar{v}'}$ and that $\mathfrak{F}_{\bar{v}} \subset \mathfrak{F}_{\bar{v}'}$.

Now we show that if $\eta = 1$ and $\mathbb{P}(\theta_0^a \geq \mathbf{K}'\tilde{\boldsymbol{\theta}}^a) = 1$, then $\mathbb{P}(\mathfrak{L} \cap \mathfrak{F}) = 0$ for small enough $\bar{v}$. As long as $\mathbf{E} \neq \{\theta^a\}$, there is a nonempty set $U := \{(\bar{v}, \boldsymbol{\epsilon}) \in \mathbb{R}_+ \times \mathbf{E} : \bar{v} = v_*(\boldsymbol{\epsilon})\}$. By definition, in each constellation in U all FIs face binding constraints, and $\kappa = \overline{\ell} - \underline{\ell}$ and $\Psi = \frac{\overline{\ell}}{\overline{\ell} - \underline{\ell}}$. We see that $S^h = \frac{1}{\overline{\ell} - \underline{\ell}}[K'\tilde{\boldsymbol{\theta}}^a - \theta_0^a] \leq 0$, irrespective of h, and $S = 0 < \bar{S}$. Therefore the result follows for all $\bar{v} \leq \inf_{\boldsymbol{\epsilon} \in \mathbf{E}} v_*(\boldsymbol{\epsilon})$. ■

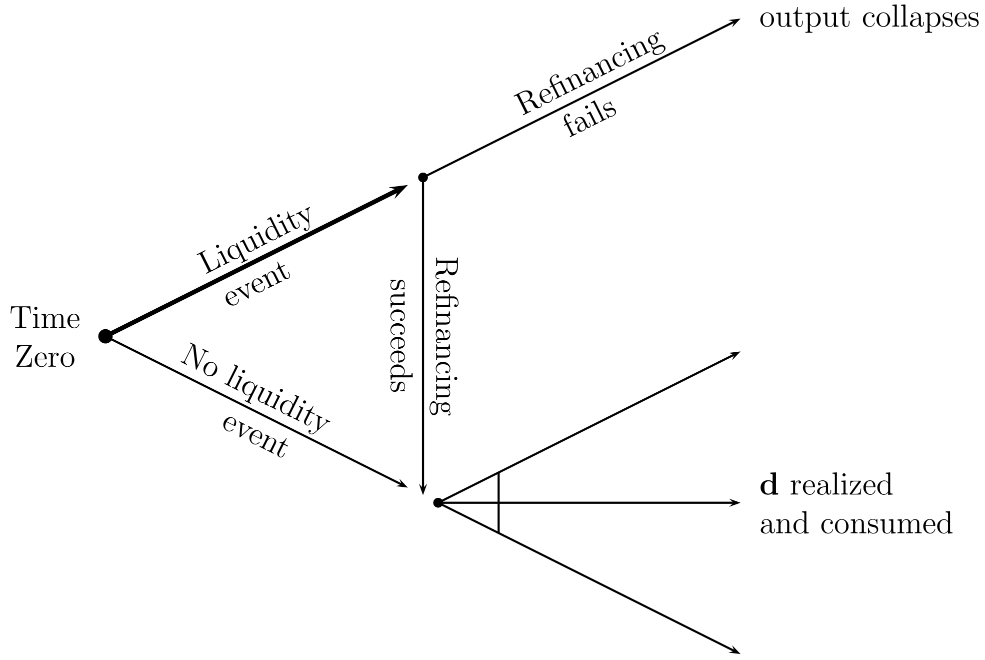

Figure 1: Event tree

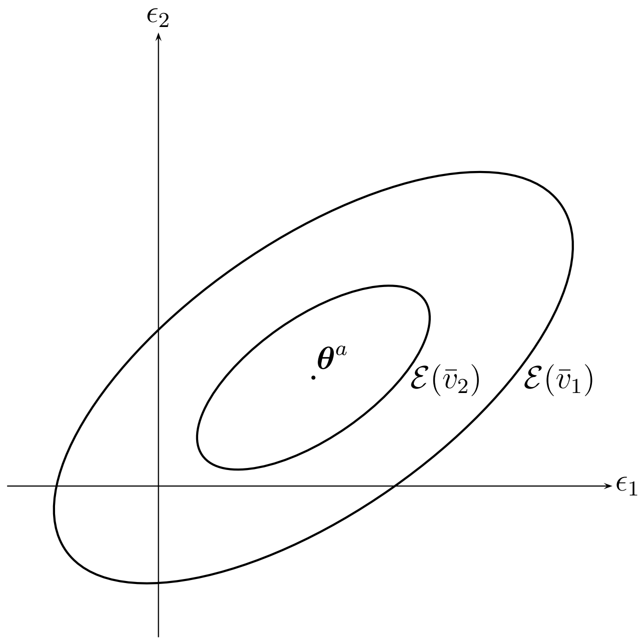

Figure 2: Equilibrium ellipsoids with increasingly restrictive risk constraints for $\eta = 1$

In this scenario there are two assets, and in the absence of any regulations, equilibria exist for $\boldsymbol{\epsilon} \in \mathbb{R}^2$. When the risk constraint is $\bar{v}_1$, the set of $\boldsymbol{\epsilon}$ that can be supported by an equilibrium is the larger ellipsoid, and includes zero noise trader demand. However a more restrictive constraint $\bar{v}_2$ does not include zero net demand, and hence equilibria do not exist if noise trades are zero.

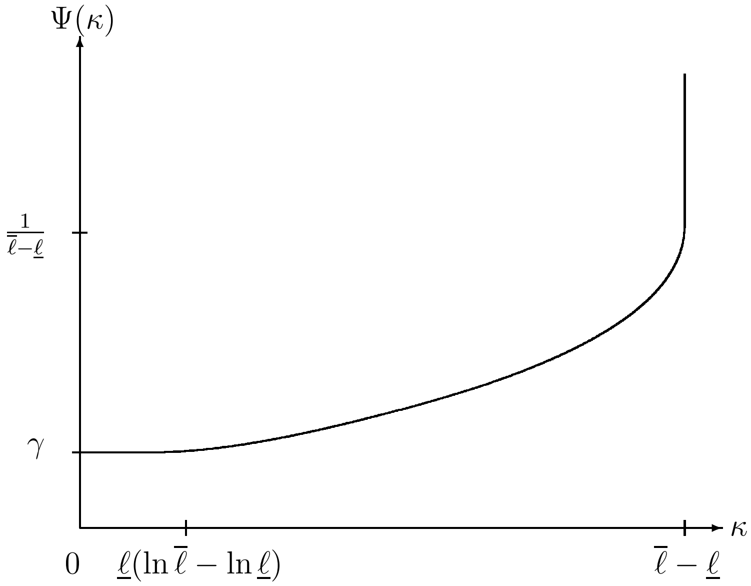

Figure 3: Illustration of the reward-to-risk function $\Psi(\kappa)$ when $\eta = 1$ and $\underline{\ell} > 0$.

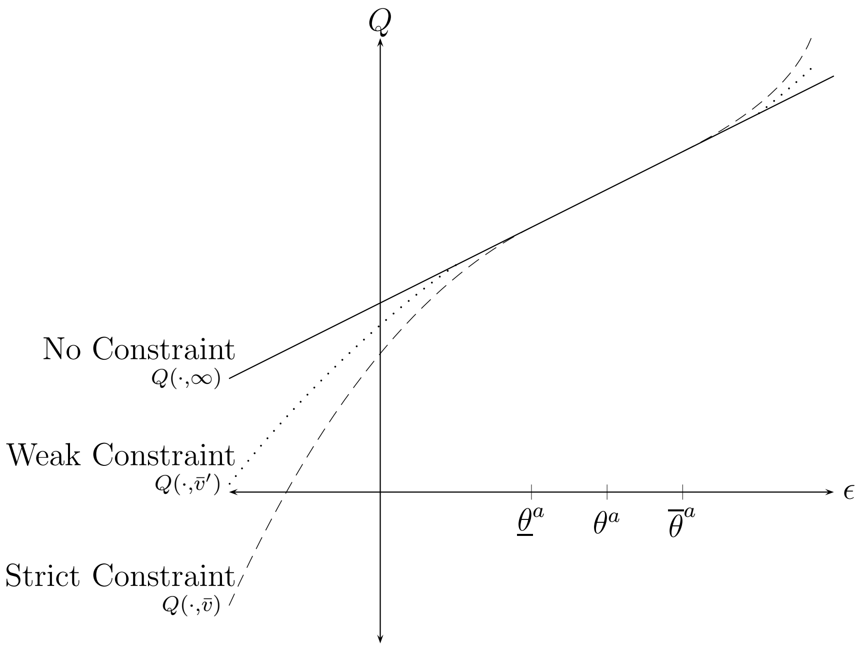

Figure 4: Pricing Function

The pricing function without constraints and with increasingly binding constraints, $\infty > \bar{v}' > \bar{v}$. The downside effects become more pronounced as the constraint becomes stricter.

References

Ahn, D., J. Boudukh, M. Richardson, and R. Whitelaw (1999). "Optimal Risk Management Using Options". *Journal of Finance 54*(1), 359–375.

Alexander, G. J. and A. M. Baptista (2002). "Economic Implications of Using a Mean-VaR model for Portfolio Selection: A Comparison with Mean-Variance Analysis". *Journal of Economic Dynamics and Control 26*, 1159–1193.

Allen, F. and D. Gale (1994). "Limited Market Participation and Volatility of Asset Prices". *The American Economic Review 84*(4), 933–955.

Allen, F. and D. Gale (2000). "Financial Contagion". *Journal of Political Economy 108*(1), 1–33.

Artzner, P., F. Delbaen, J. Eber, and D. Heath (1999). "Coherent Measures of Risk". *Mathematical Finance 9*(3), 203–228.

Basak, S. (1995). "A General Equilibrium Model of Portfolio Insurance". *Review of Financial Studies 8*, 1059–1090.

Basak, S. and A. Shapiro (2001). "Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices". *Review of Financial Studies 14*, 371–405.

Basel Committee on Banking Supervision (1996). *Overview of the Amendment to the Capital Accord to Incorporate Market Risk*.

Bates, D. (2000). "Post-'87 Crash Fears in the S&P 500 Futures Option market". *Journal of Econometrics 94*, 181–231.

Bryant, J. (1980). "A Model of Reserves, Bank Runs, and Deposit Insurance". *Journal of Banking and Finance 4*, 335–344.

Carlsson, H. and E. Van Damme (1993). "Global Games and Equilibrium Selection". *Econometrica 61*, 989–1018.

Carr, P. and L. Wu (2003). "What Type of Process Underlies Options? A Simple Robust Test". *Journal of Finance 68*, 2581–2610.

Cifuentes, R., G. Ferrucci, and H. S. Shin (2005). "Liquidity Risk and Contagion". *Journal of the European Economic Association 3*(2-3), 556–566.

Cuoco, D. and H. Liu (2005). "An Analysis of VaR-based Capital Requirements". Forthcoming, *Journal of Financial Intermediation*.

Danielsson, J., H. S. Shin, and J.-P. Zigrand (2004). "The Impact of Risk Regulation on Price Dynamics". *Journal of Banking and Finance 28*, 1069–1087.

De Bandt, O. and P. Hartmann (2000). "Systemic Risk: a Survey". Discussion paper series, no. 2634, CEPR.

Diamond, D. and P. Dybvig (1983). "Bank Runs, Deposit Insurance, and Liquidity". *Journal of Political Economy 91*, 401–419.

Esary, J. D., F. Proschan, and D. W. Walkup (1967). "Association of Random Variables, with Applications". *The Annals of Mathematical Statistics 38*(5), 1466–1474.

Feldstein, M. (1991). *The Risk of Economic Crisis.* NBER Conference Report. University of Chicago Press.

Gennotte, G. and H. Leland (1990). "Market Liquidity, Hedging, and Crashes". *American Economic Review 00*, 999–1021.

Gorton, G. and L. Huang (2003). "Banking panics and the origin of central banking". In D. Altig and B. Smith (Eds.), *Evolution and Procedures in Central Banking.* Cambridge University Press.

Grossman, S. and Z. Zhou (1996). "Equilibrium Analysis of Portfolio Insurance". *Journal of Finance 51*(4), 1379–1403.

Holmstrom, B. and J. Tirole (1998). "Private and Public Supply of Liquidity". *Journal of Political Economy 106*(1), 1–40.

Jorion, P. (2001). *Value at Risk, Second Edition.* McGraw Hill.

Kindleberger, C. (1978). *Manias, Panics, and Crashes.* Macmillan.

Kodres, L. and M. Pritsker (2002). "A Rational Expectations Model of Financial Contagion". *Journal of Finance 57*, 769–799.

Kyle, A. and W. Xiong (2001). "Contagion as a Wealth Effect". *Journal of Finance 56*, 1401–1440.

Lucas, R. E. J. (1978). "Asset Prices in an Exchange Economy". *Econometrica 46*, 1429–1445.

Marshall, D. (1998). "Understanding the Asian Crisis: Systemic Risk as Coordination Failure". *Economic Perspectives. Federal Reserve Bank of Chicago 22*(3), 13–28.

Mehra, R. and E. Prescott (1985). "The Equity Premium Puzzle". *Journal of Monetary Economics XV*, 145–161.

Morris, S. and H. S. Shin (1998). "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks". *American Economic Review 88*, 587–597.

Nikaido, H. (1968). *Convex Structures and Economic Theory*, Volume 51 of *Mathematics in Science and Engineering*. Academic Press.

Pan, J. (2002). "The jump-risk premia implicit in options: evidence from an integrated time-series study". *Journal of Financial Economics 63*(1), 3–50.

Radner, R. (1979). "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Price". *Econometrica 47*, 655–678.

Weil, P. (1989). "The Equity Premium Puzzle and the Riskfree Rate Puzzle". *Journal of Monetary Economics 24*, 401–422.