Strategic Financial Innovation in Segmented Markets*

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Abstract

We study an equilibrium model with restricted investor participation in which strategic arbitrageurs reap profits by exploiting mispricings across different trading locations. We endogenize the asset structure as the outcome of a security design game played by the arbitrageurs. The equilibrium asset structure depends realistically upon considerations such as depth, liquidity and gains from trade. It is not socially optimal in general; the degree of inefficiency depends upon the heterogeneity of investors. Finally, we use this framework to formally analyze Shiller's conjecture of the optimality of "macro markets."

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1 Introduction

The optimal design of securities has been the subject of a growing body of research. But in analyzing whether it is in the interest of economic agents to provide trading in such securities, the literature has typically abstracted from a number of market facts.

First, a common assumption has been that innovations are carried out by agents who do not themselves trade the securities they design, such as options or futures exchanges, or entrepreneurs who sell equity stakes in their firms. In reality, agents involved in financial innovation are often profit-seeking institutions that actively make markets and trade the new securities across markets, for arbitrage or hedging purposes. A large chunk of their profits comes from proprietary trading, and not simply from transaction costs received from investors (such as brokerage fees). The innovating agents often have stakes in more than one exchange or market. Their profits arise from endogenous bid-ask spreads within an exchange, as well as from price differentials across exchanges, say between derivatives exchanges and the underlying markets. Moreover, financial innovators are typically not price-takers but large strategic institutions who know how their actions affect prices. In this paper we propose a model that captures these features.

Second, the literature has focused on frictionless environments in which there is no incentive to introduce redundant assets. In actual fact, the majority of financial innovations are redundant, in the sense that they can be replicated via a self-financing portfolio of assets. This raises the question why investors who buy such an innovation, say a barrier option sold at a markup by an investment bank, do not replicate the derivative themselves. There are many reasons that come to mind—limited knowledge regarding the right hedging strategy, high transaction costs, high setup costs involved in buying a seat on an exchange or obtaining access to real-time data and trading as required by delta-hedging, etc. In this paper, such impediments to perfect and costless replication are captured in the assumption that various investor clientèles have restricted access to capital markets.

More precisely, there are several trading locations or "exchanges." With each exchange is associated a group of investors. Investors may only trade the assets available on their "local" exchange. In addition, there are agents who are "global" players—they are able to trade on all exchanges simultaneously. These agents profit by arbitraging away price differentials across exchanges; they have zero initial wealth. We refer to them as "arbitrageurs." Any

transfer of resources across exchanges is intermediated by the arbitrageurs. Moreover, they determine the asset structure, i.e. the assets available for trade on each exchange. The arbitrageurs can thus be viewed as intermediaries who can target their clients according to their needs and supply them with securities that were hitherto unavailable to them but may be globally redundant. The financial innovations together with the inter-market traders can be viewed as a means of integrating the various markets. We are interested in characterizing the asset structure that is optimal for arbitrageurs. In particular, we would like to know how the equilibrium structure that arises at a Nash equilibrium of the innovation game relates to the liquidity, depth and gains from trade of the various market segments. A natural further question is to ask if the equilibrium asset structure is socially optimal.

We study a two-period model with asset trading at date zero and uncertainty resolved at date one. Investors conform to a version of the CAPM in that they have quadratic preferences and random endowments of the single consumption good (the latter need not be spanned by the local assets). They behave competitively. For the moment let us fix the asset structure; we will shortly describe how it is determined endogenously by arbitrageurs. This asset structure may be completely arbitrary, with the assets trading on one exchange bearing no specific relationship to those trading on another. In the absence of arbitrageur activity, each exchange is essentially a standard incomplete markets economy (complete markets being a special case).

Having fixed the asset structure, we can describe the ensuing equilibrium roughly as follows. For any vector of asset supplies by arbitrageurs to the exchanges, there is a corresponding Walrasian equilibrium on each exchange. Equilibrium supplies are then determined in a Cournot game played by the arbitrageurs. The result is a Cournot-Walras equilibrium associated with each asset structure.

The asset structure itself is the outcome of a security design game among the arbitrageurs before any trade takes place. We show that there is a unique equilibrium of the security design game in which there is a single asset on each exchange. In the case in which there are only two exchanges, this asset is the difference between the autarky (absent arbitrageur activity) state-price deflators of these exchanges. In the case of multiple exchanges, the equilibrium asset on an exchange is the difference between its state-price deflator and a weighted sum of the state-price deflators of all exchanges. This weighted sum is in fact the complete-markets Walrasian state-price deflator of the entire integrated economy.

We show that this equilibrium is optimal for arbitrageurs in the sense that no other asset structure yields higher arbitrageur profits. Furthermore, if investors on each exchange are identical, but possibly heterogeneous across exchanges, the equilibrium asset structure is actually Pareto optimal (though the equilibrium allocation is not, since arbitrageurs are imperfectly competitive). Relative to an arbitrary asset structure, however, optimal innovation by arbitrageurs may hurt some agents. We characterize who wins and who loses, and provide sufficient conditions for all investors to gain. Finally, we note that if there are heterogeneous agents within exchanges, the equilibrium security design fails to be Pareto optimal, since arbitrageurs profit only from trade between exchanges and not from trade within exchanges.

One contribution of our paper is to endogenously derive an asset structure which is incomplete, without imposing a bound on the number of assets that may be introduced. Moreover, the assets that arbitrageurs innovate in our model may be redundant from the economy-wide perspective, an aspect of actual financial innovation that has often been remarked on in the literature.

Related literature: This paper lies at the intersection of two distinct literatures—the literature on security design and that on segmented markets. The security design literature has addressed the incentives for the design of financial instruments by different kinds of agents such as futures and options exchanges, investment banks and corporations, motivated by considerations such as risk sharing, taxes, transaction costs, or the need to raise capital. Our focus in the present paper is on the design of assets that serve a risk sharing role. Prior research that has addressed this aspect of security design is surveyed in Allen and Gale (1994) and Duffie and Rahi (1995). The segmented market framework that we employ in this paper is adapted from Zigrand (2003) where the asset structure is taken to be exogenous. Other papers in this line of research are Polemarchakis and Siconolfi (1997), Basak and Croitoru (2000), Cass et al. (2001) and Gromb and Vayanos (2002) on the theory side, and Luttmer (1999), Brav et al. (2002), Vissing-Jorgensen (2002) and Chen et al. (2003) on the empirical side. A recent paper also in this intersection of security design and restricted participation, with an emphasis on endogenous entry, is Calvet et al. (2002).

2 The Setup

We consider a two-period economy with uncertainty parametrized by the state space $S := \{1, \ldots, S\}$. Assets are traded in several locations or "exchanges." They are in zero net supply. We do not impose complete markets or the existence of a riskless asset.

Investor $i \in I^k := \{1, \ldots, I^k\}$ on exchange $k \in K := \{0, \ldots, K\}$ has endowments $(\omega_0^{k,i}, \omega^{k,i}) \in \mathbb{R} \times \mathbb{R}^S$, and preferences which allow a quasilinear quadratic representation,

$$U^{k,i}(x_0^{k,i}, x^{k,i}) = x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{1}{2} \beta^{k,i} (x_s^{k,i})^2 \right],$$

where $x_0^{k,i} \in \mathbb{R}$ is consumption at date 0, $x^{k,i} \in \mathbb{R}^S$ is consumption at date 1, and π_s is the probability (common across agents) of state s. The coefficient $\beta^{k,i}$ is positive. Investors are price-taking and can trade only on their own exchange.

In addition there is a set of arbitrageurs $N := \{1, \ldots, N\}$ who possess the trading technology which allows them to also trade across exchanges. For simplicity, we assume that arbitrageurs only care about time zero consumption. Arbitrageurs are imperfectly competitive.

Asset payoffs on exchange k are given by a full column rank payoff matrix R^k of dimension $S \times J^k$. The asset span on exchange k is the column space of R^k, which we denote by $\langle R^k \rangle$. We assume that all assets are arbitraged.[1]

3 Cournot-Walras Equilibria

We begin by solving for equilibrium for exogenously given asset payoffs. Let $y^{k,n}$ be the supply of assets on exchange k by arbitrageur n, and $y^k := \sum_{n \in N} y^{k,n}$ the aggregate arbitrageur supply on exchange k.

Definition 1 *Given an asset structure $\{R^k\}_{k \in K}$, a Cournot-Walras equilibrium (CWE) of the economy is an array of asset price functions, asset*

[1] This is an innocuous assumption. It is straightforward to extend our analysis to the case where, on a given exchange, some assets are not arbitraged, i.e. traded only by investors on the exchange, while other assets are arbitraged, i.e. traded by both investors and arbitrageurs. It turns out, however, that equilibrium prices of arbitraged assets are not affected by the payoffs of non-arbitraged assets. Thus the characteristics of non-arbitraged assets have no bearing on arbitrage trades or on security design by arbitrageurs.

demand functions, and arbitrageur supplies, $\{q^k : \mathbb{R}^{J^k} \to \mathbb{R}^{J^k}, \theta^{k,i} : \mathbb{R}^{J^k} \to \mathbb{R}^{J^k}, y^{k,n} \in \mathbb{R}^{J^k}\}_{k\in K, i\in I^k, n\in N}$*, such that*

1. *Investor optimization: For given* q^k*,* $\theta^{k,i}(q^k)$ *solves*

$$\max_{\theta^{k,i}\in\mathbb{R}^{J^k}} x_0^{k,i} + \sum_{s\in S}\pi_s\Big[x_s^{k,i} - \frac{\beta^{k,i}}{2}(x_s^{k,i})^2\Big]$$
$$s.t.\ \ x_0^{k,i} = \omega_0^{k,i} - q^k\cdot\theta^{k,i}$$
$$x^{k,i} = \omega^{k,i} + R^k\theta^{k,i}.$$

2. *Arbitrageur optimization: For given* $\{q^k(y^k), \{y^{k,n'}\}_{n'\neq n}\}_{k\in K}$*,* $y^{k,n}$ *solves*

$$\max_{y^{k,n}\in\mathbb{R}^{J^k}} \sum_{k\in K} y^{k,n\top} q^k\Big(y^{k,n} + \sum_{n'\neq n} y^{k,n'}\Big)$$
$$s.t.\ \ \sum_{k\in K} R^k y^{k,n} \leq 0.$$

3. *Market clearing:* $\big\{q^k(y^k)\big\}_{k\in K}$ *solves*

$$\sum_{i\in I^k}\theta^{k,i}(q^k(y^k)) = y^k, \qquad \forall k\in K.$$

Note that investors take asset prices as given, while arbitrageurs compete Cournot-style. Arbitrageurs maximize time zero consumption, i.e. profits from their arbitrage trades, but subject to the restriction that they are not allowed to default in any state at date 1. Equivalently, arbitrageurs need to be completely collateralized.

Let $\Pi := \text{diag}\,(\pi_1,\ldots,\pi_S)$ and $\mathbf{1} := (1\ldots1)^\top$. Investor (k,i)'s utility can be written as

$$U^{k,i} = \omega_0^{k,i} - q^k\cdot\theta^{k,i} + \mathbf{1}^\top\Pi(\omega^{k,i}+R^k\theta^{k,i}) - \frac{\beta^{k,i}}{2}(\omega^{k,i}+R^k\theta^{k,i})^\top\Pi(\omega^{k,i}+R^k\theta^{k,i}). \tag{1}$$

The first order condition for the investor's optimization problem gives us his asset demand function:

$$\theta^{k,i}(q^k) = \frac{1}{\beta^{k,i}}(R^{k\top}\Pi R^k)^{-1}[R^{k\top}\Pi p^{k,i} - q^k], \tag{2}$$

where $p^{k,i} := (\mathbf{1} - \beta^{k,i}\omega^{k,i})$ is agent (k,i)'s no-trade state-price deflator or pricing kernel. We can now use the market clearing condition to deduce the inverse demand mapping, i.e. the price vector on exchange k that sets aggregate demand, $\theta^k := \sum_i \theta^{k,i}$, equal to aggregate arbitrage supply, y^k:

$$q^k(y^k) = {R^k}^\top \Pi[p^k - \beta^k R^k y^k], \tag{3}$$

where $\beta^k := [\sum_i (\beta^{k,i})^{-1}]^{-1}$, $\omega^k := \sum_i \omega^{k,i}$, and $p^k := (\mathbf{1} - \beta^k \omega^k)$ is exchange k's autarky state-price deflator. The parameter $1/\beta^k$ represents the "depth" of exchange k, i.e. the price impact of a unit of arbitrageur trading. For instance, ceteris paribus, the market impact of a trade is smaller on exchanges with a larger population; it can be absorbed by more investors. Notice that we can interpret equilibrium prices as risk-neutral prices ${R^k}^\top \Pi \mathbf{1}$ from which a risk-aversion discount $\beta^k {R^k}^\top \Pi(\omega^k + R^k y^k)$ is subtracted.

Our assumptions on preferences, in conjunction with the absence of non-negativity constraints on consumption, guarantee that the equilibrium pricing function on an exchange does not depend on the initial distribution of endowments, but merely on the aggregate endowment of the local investors. The autarky state-price deflator p^k also does not depend on R^k, even though investors on exchange k do trade among themselves consumptions in the span of R^k.

We now solve the Cournot game among arbitrageurs, given the asset price function (3). It turns out that there is a unique CWE, and that this equilibrium is symmetric, i.e. $y^{k,n}$ is the same for all n. Let

$$P^k := R^k ({R^k}^\top \Pi R^k)^{-1} {R^k}^\top \Pi.$$

Since P^k is idempotent, it is a projection from $\mathbb{R}^S$ onto the asset span $\langle R^k \rangle$. Indeed, it is an orthogonal projection in $L^2(\Pi)$.[2] It is convenient to state arbitrageur supplies in terms of the supply of state-contingent consumption:

Lemma 1 (Equilibrium supplies) *Equilibrium arbitrageur supplies are unique and symmetric. For asset structure $\{R^k\}_{k \in K}$, they are given by*

$$R^k y^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P^k \left(p^k - p^A\right), \qquad k \in K \tag{4}$$

[2]For state-contingent consumption $x \in \mathbb{R}^S$, the $L^2(\Pi)$-norm of x is defined as follows: $\|x\|_2 := (x^\top \Pi x)^{\frac{1}{2}}$.

where $p^A \geq 0$ is a state-price deflator for the arbitrageurs, satisfying

$$\sum_k \frac{1}{\beta^k} P^k (p^k - p^A) \leq 0 \tag{5}$$

and

$$p^A_s \cdot \left[\sum_k \frac{1}{\beta^k} P^k (p^k - p^A) \right]_s = 0, \qquad \forall s. \tag{6}$$

Note that p^A can be chosen not to depend on N. We can clearly see (pretend $P^k = I$, all k, for the moment) from equation (4) that arbitrageurs supply consumption in state s to an exchange k when the price that agents on exchange k are willing to pay for a unit of state s consumption exceeds the arbitrageurs' shadow willingness to pay, p^A_s. This statement should be qualified, since while these are the "optimal" supplies in some sense (to be confirmed subsequently), they may not be in the span of the existing assets. Therefore, arbitrageurs will supply consumption if the excess willingness to pay, when projected onto the span of the permissible assets, is positive.[3] The factor of proportionality in (4) is determined by two considerations. First, the deeper is exchange k (i.e. the lower is β^k), the more arbitrageur n trades on this exchange, since he can afford to augment his supply without affecting margins as much. And second, the supply vector is scaled to zero as competition intensifies, because the whole pie shrinks and there are more players to share the smaller pie with (see also Lemma 3 below).

Lemma 1 gives us the equilibrium supply $y^{k,n}$ of arbitrageur n. The total equilibrium supply is then $y^k = N y^{k,n}$. Substituting into the pricing equation (3) determines the equilibrium prices. Let $\hat{p}^k$ be an equilibrium pricing kernel for exchange k, i.e. $q^k = {R^k}^\top \Pi \hat{p}^k$.

Lemma 2 (Equilibrium prices) *The following is an equilibrium state-price deflator for exchange k:*

$$\hat{p}^k = \frac{1}{1+N} p^k + \frac{N}{1+N} p^A. \tag{7}$$

[3] Of course, the projected state-price deflator $\overline{p}^k := P^k p^k$ is also a state-price deflator (${R^k}^\top \Pi (P^k p^k) = {R^k}^\top \Pi p^k = q^k$), and in fact it is the unique state-price deflator that is also marketed, i.e. in $\langle R^k \rangle$. We could therefore replace p^k and $P^k p^k$ by $\overline{p}^k$ throughout this paper and simplify some expressions. But since we shall determine spans endogenously in the sequel, we resist that temptation.

In particular, as N goes to infinity, the equilibrium valuation on each exchange converges to the arbitrageurs' valuation: $\lim_{N\to\infty} q^k = {R^k}^\top \Pi p^A$.

Note that, with incomplete markets on exchange k, there is a multiplicity of state-price deflators for a given R^k. However, the equilibrium valuation functional on $\mathbb{R}^S$, ${R^k}^\top \Pi \hat{p}^k$, is unique, and when $\hat{p}^k$ is viewed as a function of R^k, it is a valid state-price deflator for each R^k. From (2) and (7), we see that

$$\begin{aligned} R^k \theta^{k,i} &= \frac{1}{\beta^{k,i}} P^k (p^{k,i} - \hat{p}^k) \\ &= \frac{1}{\beta^{k,i}} P^k \left[(p^{k,i} - p^k) + \frac{N}{1+N} (p^k - p^A) \right]. \end{aligned} \tag{8}$$

Thus investor (k,i)'s net trade of state-contingent consumption is the sum of two components—an intra-exchange trade proportional to $P^k(p^{k,i} - p^k)$, and an inter-exchange trade proportional to $P^k(p^k - p^A)$.

Finally, we calculate the equilibrium profits of arbitrageur n (which do not depend on n since the CWE is symmetric), $\Phi := \sum_k q^k \cdot y^{k,n}$.

Lemma 3 (Equilibrium profits) *The equilibrium profits of an arbitrageur, for given asset structure $\{R^k\}$, are*

$$\Phi(\{R^k\}) = \frac{1}{(1+N)^2} \cdot \sum_k \frac{1}{\beta^k} \left\| P^k (p^k - p^A) \right\|_2^2. \tag{9}$$

As N goes to infinity, individual arbitrageur trades vanish, as do total arbitrageur profits $N\Phi$.

The following section provides an interpretation of the arbitrageurs' shadow values in terms of certain Walrasian pricing kernels.

4 A Walrasian Benchmark

It turns out that the equilibrium of the arbitraged economy that we have just computed bears a close relationship to an appropriately defined competitive equilibrium, with no arbitrageurs.

Definition 2 *Given an asset structure $\{R^k\}_{k\in K}$, a Walrasian equilibrium with restricted consumption is a state-price deflator p^{RC}, and portfolios $\{\theta^{k,i}, \varphi^{k,i,\ell}\}_{k,\ell\in K, i\in I^k}$, such that*

1. *Investor optimization: For given* $q^k = {R^k}^\top \Pi p^{RC}$, $k \in K$, $\{\theta^{k,i}, \{\varphi^{k,i,\ell}\}_{\ell \in K}\}$ *solves*

$$\max_{\theta^{k,i} \in \mathbb{R}^{J^k},\, \varphi^{k,i,\ell} \in \mathbb{R}^{J^\ell}} x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{\beta^{k,i}}{2} (x_s^{k,i})^2 \right]$$

$$s.t. \quad x_0^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} - \sum_{\ell \in K} q^\ell \cdot \varphi^{k,i,\ell}$$

$$x^{k,i} = \omega^{k,i} + R^k \theta^{k,i}$$

$$0 \leq \sum_{\ell \in K} R^\ell \varphi^{k,i,\ell}.$$

2. *Market clearing:*

$$\sum_{i \in I^k,\, k \in K} R^k \theta^{k,i} + \sum_{i \in I^k,\, k \in K,\, \ell \in K} R^\ell \varphi^{k,i,\ell} = 0.$$

In a Walrasian equilibrium with restricted consumption agents can trade any asset on a centralized exchange, facing a common state-price deflator p^{RC}, but agents on exchange k can consume claims in $\langle R^k \rangle$ only. For agent (k,i), the portfolio that leads to future consumption is $\theta^{k,i}$. He can choose, in addition, an auxiliary portfolio $\{\varphi^{k,i,\ell}\}_{\ell \in K}$, provided the payoff is nonnegative.

Given asset payoffs $\{R^k\}$, we say that asset prices $\{q^k\}$ are globally weakly arbitrage-free if an agent with access to all the asset markets in the economy is unable to construct a weak arbitrage, i.e. a portfolio $\{z^k\}_{k \in K}$ such that $\sum_k R^k z^k \geq 0 \Rightarrow \sum_k q^k \cdot z^k \geq 0$. By the fundamental theorem of asset pricing, this is the case if and only if there exists $\psi \geq 0$ such that $q^k = {R^k}^\top \Pi \psi$, for all k. Clearly, due to the auxiliary portfolio, there cannot be a global weak arbitrage at a restricted-consumption Walrasian equilibrium; hence we can always choose $p^{RC} \geq 0$. This notion of equilibrium is of interest due to the following result:

Lemma 4 (Restricted-consumption Walrasian equilibrium) *There is a unique*[4] *Walrasian equilibrium with restricted consumption, with pricing*

[4]By uniqueness we mean that the equilibrium allocation and pricing functional are unique. There may, of course, be multiple pricing kernels that induce the equilibrium pricing functional.

kernel $p^{RC} \geq 0$ *satisfying* ${R^k}^\top \Pi(p^{RC} - p^A) = 0$. *Equilibrium net trades of state-contingent consumption are*

$$R^k \theta^{k,i} = \frac{1}{\beta^{k,i}} P^k (p^{k,i} - p^{RC}), \qquad i \in I^k, k \in K. \tag{10}$$

Thus arbitrageur valuations of assets at the CWE are the same as asset valuations at the restricted-consumption Walrasian equilibrium. Moreover, from Lemma 2, asset prices in the arbitraged economy converge to asset prices in the restricted-consumption Walrasian equilibrium, as the number of arbitrageurs goes to infinity. Comparing (8) and (10), we also see that the equilibrium allocation (for investors) in the arbitraged economy converges to the restricted-consumption Walrasian equilibrium allocation. It is in this sense that arbitrageurs serve to integrate markets.

A more natural notion of Walrasian equilibrium when markets are segmented, and one that has been widely studied in the general equilibrium literature, is Walrasian equilibrium with restricted participation. In this equilibrium agents trade on a centralized exchange, facing a common state-price deflator p^{RP}, but agents on exchange k can trade claims in $\langle R^k \rangle$ only.

Definition 3 *Given an asset structure* $\{R^k\}_{k \in K}$*, a Walrasian equilibrium with restricted participation is a state-price deflator* p^{RP}*, and portfolios* $\{\theta^{k,i}\}_{k \in K,\, i \in I^k}$*, such that*

1. *Investor optimization: For given* $q^k = {R^k}^\top \Pi p^{RP}$*,* $\theta^{k,i}$ *solves*

$$\begin{aligned} \max_{\theta^{k,i} \in \mathbb{R}^{J^k}} \; & x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{\beta^{k,i}}{2} (x_s^{k,i})^2 \right] \\ s.t. \;\; & x_0^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} \\ & x^{k,i} = \omega^{k,i} + R^k \theta^{k,i}. \end{aligned}$$

2. *Market clearing:*

$$\sum_{i \in I^k,\, k \in K} R^k \theta^{k,i} = 0.$$

One might think that a Walrasian equilibrium with restricted participation is no different from a Walrasian equilibrium with restricted consumption. This is not necessarily the case, however. The following example illustrates:

Example 1 Consider an economy consisting of two exchanges, 1 and 2, with a single agent on each exchange. The payoff matrices are

$$R^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad R^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The two exchanges are equally deep, with β^1 and β^2 both equal to $\bar{\beta}$, which satisfies

$$0 < \bar{\beta} < \frac{\pi_1}{1+\pi_1}. \tag{11}$$

Date one endowments are as follows: $\omega^1 = \mathbf{1}$ and $\omega^2 = (1/\bar{\beta} - 1)\mathbf{1}$. Autarky state-price deflators are, therefore, $p^1 = (1-\bar{\beta})\mathbf{1}$ and $p^2 = \bar{\beta}\mathbf{1}$, respectively. Exchange 1 values time one consumption more than exchange 2. In autarky, $q^1 = (1-\bar{\beta})\pi_1$, and $q^2 = \bar{\beta}$. The restriction (11) implies that $q^1 > q^2$, i.e. there exist profit opportunities for arbitrageurs, buying on exchange 2 and delivering to exchange 1.

Now consider the Walrasian equilibrium of this economy with restricted participation: agents face a common state-price deflator p^{RP}, but can only trade assets on their own exchange. However, since $\langle R^1 \rangle \cap \langle R^2 \rangle = \{0\}$, the two agents cannot trade with each other. Equilibrium asset prices are the same as in autarky. Since these prices allow for an arbitrage, albeit for a hypothetical agent with access to all markets, at least one of the state prices must be negative. The pricing kernel p^{RP} (which is unique since markets are complete in the integrated economy) solves $q^k = {R^k}^\top \Pi p^{RP}$, $k = 1, 2$:

$$p^{RP} = \begin{bmatrix} 1-\bar{\beta} \\ (1/\pi_2)[(1+\pi_1)\bar{\beta} - \pi_1] \end{bmatrix}.$$

It follows from (11) that $p_2^{RP} < 0$. Equilibrium consumption at date one (which is just the initial endowment for both agents) is below the bliss point $1/\bar{\beta}$.

In the restricted-participation Walrasian equilibrium, agents are unable to exploit the arbitrage opportunity, because doing so would take them outside their local asset span. Indeed, they would end up with excess consumption in state 2. In a restricted-consumption Walrasian equilibrium, on the other hand, agents can arbitrage away the mispricing. They simply dispose of some of the state 2 consumption good. Consequently $p_2^{RC} = 0$ (implying that $q^1 = q^2$). The desired net trade of state-contingent consumption is

$R^k\theta^k = \frac{1}{\beta^k} P^k(p^k - p^{RC})$, $k = 1, 2$. The projections P^1 and P^2 are:

$$P^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P^2 = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}. \tag{12}$$

Therefore, noting that $p_2^{RC} = 0$,

$$R^1\theta^1 = \frac{1}{\bar{\beta}} \begin{bmatrix} 1 - \bar{\beta} - p_1^{RC} \\ 0 \end{bmatrix}, \qquad R^2\theta^2 = \frac{1}{\bar{\beta}} \begin{bmatrix} \bar{\beta} - \pi_1 p_1^{RC} \\ \bar{\beta} - \pi_1 p_1^{RC} \end{bmatrix}.$$

Market clearing for state 1 gives us $p_1^{RC} = \frac{1}{1+\pi_1}$. It follows from (11) that the net trade of the agent on exchange 2 is negative in both states. Equilibrium consumptions at date one are respectively,

$$\omega^1 + R^1\theta^1 = \begin{bmatrix} \frac{1}{\bar{\beta}} \cdot \frac{\pi_1}{1+\pi_1} \\ 1 \end{bmatrix}, \qquad \omega^2 + R^2\theta^2 = \frac{1}{\bar{\beta}} \cdot \frac{1}{1+\pi_1} \cdot \mathbf{1},$$

both of which are below the bliss point $1/\bar{\beta}$. $\|$

The key difference between the two concepts of Walrasian equilibrium lies in the market clearing condition: at a restricted-consumption equilibrium, $\sum_k R^k\theta^k \leq 0$, while at a restricted-participation equilibrium, $\sum_k R^k\theta^k = 0$. The pricing kernel p^{RC} need not be strictly positive. This is due to the fact that investors are by construction behaving as if they are satiated in those directions of the consumption space that lie outside the imposed span. At equilibrium it can occur that investors collectively dispose of consumption in some states. These states are also the states in which, at a CWE, arbitrageurs dispose of consumption due to their inability to bring consumption back to time zero without disturbing their arbitrage portfolio. Arbitrageurs play a useful allocational role over and above allowing investors to trade their own claims abroad. They allow investors to trade *any* claim available in the economy. Investors can thereby exploit good deals in the global markets, which relaxes their time zero budget constraint. They are better off as a result, even if they have to discard consumption in some states at time one (via the arbitrageurs) to remain within their local asset span.[5]

[5] Arbitrageurs effectively dispose of consumption on behalf of investors, for whom disposal is not free in the arbitraged economy. This is reflected in the fact that state prices in the arbitraged economy are nonnegative in the limit, just as they are in the restricted-consumption economy where investors can dispose of excess consumption freely.

The two notions of Walrasian equilibrium coincide if and only if $p^{RP} \geq 0$, i.e. if and only if there is no global weak arbitrage at the restricted-participation Walrasian equilibrium. Then we have $p^{RC} = p^{RP} = p^A$. We consider some special cases in which we can explicitly solve for p^{RP}, and provide conditions under which the solution is nonnegative. Defining

$$\lambda^k := \frac{\frac{1}{\beta^k}}{\sum_{j=0}^{K} \frac{1}{\beta^j}},$$

we can write the market clearing condition for the restricted-participation Walrasian equilibrium as follows:

$$\sum_{k \in K} \lambda^k P^k (p^k - p^{RP}) = 0. \tag{13}$$

An explicit solution for p^{RP} can be obtained if there is an exchange (which we take to be exchange 0 without loss of generality) that satisfies the following maximal asset span condition:

(S1) $\langle R^0 \rangle$ contains $\langle R^k \rangle$, for all $k \in K$.

We will also need a nonsatiation condition:

(N1) $\mathbf{1} - \beta \cdot \sum_{k \in K} Q^k \omega^k \geq 0$,

where $\beta := \left[\sum_k (\beta^k)^{-1}\right]^{-1}$, and Q^k is defined as follows:

$$Q^0 = \left[\lambda^0 I + \sum_{k \neq 0} \lambda^k P^k\right]^{-1},$$

$$Q^k = \left[\lambda^0 I + \sum_{k \neq 0} \lambda^k P^k\right]^{-1} P^k, \qquad k \in \{1, \ldots, K\}.$$

Condition **N1** says that the representative agent with aggregate preference parameter β is nonsatiated at the aggregated projected endowment $\sum_{k \in K} Q^k \omega^k$.

Lemma 5 *Under* **S1**,

$$p^\Lambda := \sum_{k \in K} \lambda^k Q^k p^k \tag{14}$$

is a restricted-participation Walrasian pricing kernel. Furthermore, $p^\Lambda \geq 0$ *if and only if* **N1** *holds so that, under* **S1** *and* **N1**, *we can choose* $p^A = p^\Lambda$.

Moreover, under **S1**, $p^{RP} \geq 0$ if and only if there is no weak arbitrage on exchange 0. Therefore, condition **N1** is equivalent to assuming that there is an agent on exchange 0 who is nonsatiated at the restricted-participation Walrasian equilibrium. Note that p^{Λ} is the (unique) restricted-participation Walrasian pricing kernel in the economy in which asset payoffs are the same as in the original economy except that $R^0 = I$.

A further specialization of condition **S1** arises when all exchanges have the same payoff matrix R. Then from (13), we can choose p^{RP} to be equal to

$$p^{\lambda} := \sum_{k \in K} \lambda^k p^k.$$

The vector p^{λ} is the investors' economy-wide average willingness to pay, with the willingness to pay on each exchange weighted by its relative depth. It is apparent from (13) that p^{λ} is the pricing kernel for the complete-markets Walrasian equilibrium (with unrestricted participation).[6] The following regularity condition, a slight variation on **N1**, is equivalent to $p^{\lambda} \geq 0$:

(N2) $\mathbf{1} - \beta\omega \geq 0$,

where $\omega := \sum_k \omega^k$. It says that the representative investor with aggregate preference parameter β is weakly nonsatiated at the aggregate endowment ω.

There is another case in which $p^{RP} = p^{\lambda}$ solves (13): if $p^k - p^{\lambda} \in \langle R^k \rangle$, all k, for then $P^k(p^k - p^{\lambda}) = p^k - p^{\lambda}$. Accordingly, we will often need to refer to the following spanning condition:

(S2) Either (a) $R^k = R$, $k \in K$, or (b) $p^k - p^{\lambda} \in \langle R^k \rangle$, $k \in K$.

Lemma 6 *Under* **S2**, *the complete-markets Walrasian pricing kernel* p^{λ} *is a restricted-participation Walrasian pricing kernel. Furthermore,* $p^{\lambda} \geq 0$ *if and only if* **N2** *holds so that, under* **S2** *and* **N2**, *we can choose* $p^A = p^{\lambda}$.

In light of the foregoing discussion (Lemmas 2, 4, 5, and 6), we have:

[6] In the complete-markets case, p^{λ} is the unique state-price deflator. If markets are incomplete, so that the common payoff matrix R does not span $\mathbb{R}^S$, p^{λ} is still a state-price deflator, but it is not the only one.

Lemma 7 (Convergence to Walrasian equilibrium)

1. *As the number of arbitrageurs N goes to infinity, the equilibrium valuation on exchange k in the arbitraged economy converges to the restricted-consumption Walrasian equilibrium valuation, i.e.* $\lim_{N\to\infty} q^k = {R^k}^\top \Pi p^{RC}$. *Under* **S1** *and* **N1**, *this is also the restricted-participation Walrasian equilibrium valuation,* ${R^k}^\top \Pi p^\Lambda$. *Under* **S2** *and* **N2**, *it coincides with the complete-markets Walrasian equilibrium valuation,* ${R^k}^\top \Pi p^\lambda$.

2. *As the number of arbitrageurs N goes to infinity, the equilibrium allocation in the arbitraged economy converges to the restricted-consumption Walrasian equilibrium allocation. Under* **S1** *and* **N1**, *this is also the restricted-participation Walrasian equilibrium allocation. Under* **S2***(b)* *and* **N2**, *it coincides with the complete-markets Walrasian equilibrium allocation.*

We conclude this section by giving another interpretation of arbitrageur supplies for the case in which all exchanges have the same payoff matrix R, with corresponding projection matrix P. Assuming that **N2** holds, due to Lemma 6 we can rewrite equation (4) as follows:

$$Ry^{k,n} = \frac{1}{(1+N)\beta^k} \sum_{j\in K} \lambda^j P(p^k - p^j).$$

The difference $p^k - p^j$ reflects the overpricing of assets on exchange k compared to j. The net trade of k with all other exchanges can be arrived at by summing the trade of k with each particular exchange j, $P(p^k - p^j)$, weighted by its relative depth λ^j. When do arbitrageurs supply consumption to k in state s? The condition $p^k_s - p^{k'}_s > 0$, for some k', is clearly not sufficient, since there could be a third exchange j whose p^j_s is large enough to warrant that arbitrageurs take consumption away from both k and k' in order to transfer it to j. It must be the case that $p^k_s - p^\lambda_s \geq 0$, i.e. agents on k value consumption in state s more than the average depth-weighted valuation of all exchanges.

The nonsatiation condition **N2** will be a standing assumption in the remainder of the paper.

5 Optimal Security Design by Arbitrageurs

We have seen that there is a unique CWE associated with any asset structure $\{R^k\}_{k\in K}$. In this section we endogenize the security payoffs. Arbitrageurs play a security design game the outcome of which is an equilibrium asset structure. The payoffs of arbitrageurs are the profits they earn in the CWE associated with this asset structure. The asset structure $\{R^k\}$ is a Nash equilibrium of the security design game if no arbitrageur stands to gain by introducing additional assets that he may trade monopolistically (clearly this is also a Nash equilibrium of the associated game in which all arbitrageurs trade the additional securities). An asset structure is optimal for an arbitrageur if it yields the highest profits for the arbitrageur in the associated CWE, among all possible asset structures.

Proposition 1 *The following statements are equivalent:*

1. $p^k - p^\lambda \in \langle R^k \rangle$, *for all* $k \in K$;
2. *the asset structure* $\{R^k\}_{k\in K}$ *is optimal for arbitrageurs;*
3. *the asset structure* $\{R^k\}_{k\in K}$ *is a Nash equilibrium of the security design game.*

Thus the complete asset structure $R^k = I_{S\times S}$, for all k, is optimal for arbitrageurs, and a Nash equilibrium. Moreover, all optimal/equilibrium asset structures are payoff-equivalent for arbitrageurs. Among these, a minimal asset structure is one with the smallest number of assets. Such an asset structure would be the one chosen if each security issued bore a fixed cost c, no matter how small. In fact, such fixed costs are significant; see Tufano (1989) for an empirical assessment. The following result is immediate from Proposition 1 (when we say "unique", we mean "unique up to scaling").

Proposition 2 (Security design) *The asset structure*

$$R^k = p^k - p^\lambda, \qquad k \in K$$

is

1. *the unique minimal optimal asset structure for arbitrageurs; and*
2. *the unique minimal Nash equilibrium of the security design game.*

The optimal asset structure for arbitrageurs spans the net trades between exchanges in the complete-markets Walrasian equilibrium (from (10), these net trades are given by $R^k\theta^k = (1/\beta^k)(p^k - p^\lambda)$, $k \in K$). The minimal optimal security on exchange k is a swap, exchanging the autarky state-price deflator of exchange k for the complete-markets Walrasian state-price deflator of the entire integrated economy. If there are only two exchanges, say 0 and 1, then the optimal securities $p^0 - p^\lambda$ and $p^1 - p^\lambda$ are both collinear with $p^1 - p^0$, the difference of the autarky state-price deflators of the two exchanges.

A reading of the optimal arbitrage supply (4) indeed suggests that the optimal security design for arbitrageurs should be as given in the proposition. Since the arbitrageurs' supply of state-contingent consumption is proportional to $p^k - p^A$, projected down onto the span of R^k, i.e. they choose the supply closest to $p^k - p^A$, it is clear that the optimal span should be exactly the one provided by $p^k - p^A$. Then $P^k(p^k - p^A) = p^k - p^A$, so that $p^A = p^\lambda$ from (13). It should be remarked that a single security on each exchange suffices for the arbitrageur to maximize his profits, and our result therefore generates incomplete markets endogenously, without any constraint on the number of securities. The reason is that, within any exchange k, asset prices are determined by an auction and arbitrageurs do not profit from those intra-exchange trades. Intuitively, an arbitrageur only profits from mispricings between the market price of the innovation and the replicating portfolio. He is therefore only concerned with the one-dimensional net trade he mediates between k and the rest of the economy, which can be accomplished via a single security collinear with the desired net trade. Relatedly, if $p^k = p^\lambda$, we claim that arbitrageurs do not find it profitable to introduce any assets on exchange k. But isn't it true that, with heterogeneous investors on exchange k, there must be some agent willing to pay or receive an amount different from the one on some other exchange, and therefore provide an incentive to innovate? The answer is no, since if such an asset is sold to investor (k, i), all other agents can trade that same asset as well, by nonexclusivity. The resulting price established on exchange k is such that no arbitrage opportunities arise across exchanges: $p^k = p^\lambda$.

Finally, it is interesting to realize that although net trades, and therefore equilibrium allocations, depend on the degree of competition N, the equilibrium asset structure does not depend on N. This is a feature of the linear-quadratic model in which demand functions are linear, and depth is a constant independent of trading volume. We discuss this further in the next section.

Our analysis readily extends to the more realistic case where arbitrageurs can innovate on all exchanges, but cannot affect the payoffs of the existing assets $\{R^k\}$. In other words, security design really represents incremental "innovation." For instance, if $R^k = I_{S\times S}$, for some k, one can interpret the result as the optimal design of redundant derivative securities on various exchanges. The following result is immediate from Proposition 1.

Proposition 3 (Innovation) *For given $\{R^k\}_{k\in K}$, the asset structure*

$$\begin{array}{lll} [R^k & (p^k - p^\lambda)] & \text{if} \quad p^k - p^\lambda \notin \langle R^k \rangle, \\ & R^k & \text{if} \quad p^k - p^\lambda \in \langle R^k \rangle, \end{array}$$

is

1. *a minimal optimal asset structure for arbitrageurs; and*
2. *a minimal Nash equilibrium of the security design game.*

Since for a given $\{R^k\}$ arbitrageurs find it optimal to supply state-contingent consumption proportional to $p^k - p^\lambda$ if allowed, they innovate on exchange k if and only if $p^k - p^\lambda \notin \langle R^k \rangle$, in which case they "further complete" the market by adding $p^k - p^\lambda$. Equivalently, they could add a security that makes $p^k - p^\lambda$ tradable in conjunction with R^k. For this reason, we can no longer say that $[R^k \quad (p^k - p^\lambda)]$ is the *unique* minimal asset structure.

6 Security Design and Social Welfare

Associated with an asset structure $\{R^k\}$, there is a unique CWE with the corresponding equilibrium payoffs for each arbitrageur and investor. Equilibrium arbitrageur profits are given by (9). We now turn to the equilibrium utilities of investors. Using investor (k,i)'s first order condition, we can write his utility (1) as:

$$U^{k,i} = \omega_0^{k,i} + \mathbf{1}^\top \Pi \omega^{k,i} - \frac{\beta^{k,i}}{2} \omega^{k,i^\top} \Pi \omega^{k,i} + \frac{\beta^{k,i}}{2} \| R^k \theta^{k,i} \|_2^2.$$

Note that $U^{k,i}$ depends on the asset structure only through the term $W^{k,i} := \beta^{k,i} \| R^k \theta^{k,i} \|_2^2$. We will find it convenient to refer to $W^{k,i}$ as the equilibrium utility of agent (k,i). Using (8), we have:

Lemma 8 (Equilibrium utilities) *The equilibrium utility of investor (k,i), for given asset structure $\{R^k\}$, is*

$$W^{k,i} = \frac{1}{\beta^{k,i}} \left\| P^k \left[(p^{k,i} - p^k) + \frac{N}{1+N}(p^k - p^A) \right] \right\|_2^2 .$$

Analogous to the optimality notion for arbitrageurs that we have considered before, an asset structure is optimal for an investor if it results in the highest equilibrium utility for the investor among all possible asset structures. An asset structure is Pareto optimal for a group of agents if there is no alternative asset structure that Pareto dominates it in equilibrium for this group. An asset structure is socially optimal if it is Pareto optimal for the set of all agents, arbitrageurs and investors.

We say that investors on exchange k are homogeneous if they have the same no-trade valuations, i.e. $p^{k,i} = p^k$, for all $i \in I^k$. We refer to an economy in which investors are homogeneous within each exchange as a clientèle economy. From the point of view of arbitrageurs, each clientèle $k \in K$ consists of agents with identical characteristics.

We will focus now on a clientèle economy, returning to the general heterogeneous agent case at the end of the section. Lemma 8 gives us the following welfare index for clientèle k:

$$W^k := \sum_{i \in I^k} W^{k,i} = \frac{1}{\beta^k} \left(\frac{N}{1+N} \right)^2 \| P^k (p^k - p^A) \|_2^2 . \tag{15}$$

Comparing this to (9), we see that $\sum_k W^k$ is proportional to arbitrageur profits Φ. Hence an asset structure that maximizes arbitrageur profits also maximizes the egalitarian social welfare function:

Proposition 4 (Optimality: clientèle economy) *In a clientèle economy, an optimal asset structure for arbitrageurs (which is also a Nash equilibrium of the security design game) is socially optimal.*

This is not surprising given that the optimal arbitrageur-chosen securities span the net trades between exchanges in the complete-markets Walrasian equilibrium of the integrated economy. However, while the equilibrium *securities* correspond to socially desirable ones, the *allocation* that results is not Pareto optimal. Arbitrageurs are strategic and restrict their asset supplies in order to benefit from the markup. This implies that not all gains from

trade are exhausted. It is only when the number of arbitrageurs N tends to infinity that equilibrium allocations converge to Walrasian allocations, and therefore to a Pareto optimum.

We can think of W^k, as given by (15) for an arbitrary asset structure $\{R^k\}$, as the inter-exchange gains from trade reaped by exchange k in moving from autarky to the arbitraged equilibrium. Using (4), $W^k = (1/\beta^k)\|R^k y^k\|_2^2$, so that the gains from trade are proportional to the magnitude of the state-contingent consumption trading volume. As N goes to infinity, the CWE converges to the Walrasian equilibrium with restricted consumption, and investors achieve the maximal gains from trade, given the asset structure $\{R^k\}$. At the optimal security design, the maximal gains from trade for exchange k are given by $(1/\beta^k)\|p^k - p^\lambda\|_2^2$. This is the welfare gain that exchange k can obtain by trading with other exchanges in a complete-markets Walrasian equilibrium.

Consider the minimal socially optimal security design $\{p^k - p^\lambda\}$. From Lemma 2, $p^k - p^\lambda$ is collinear with $\hat{p}^k - \hat{p}^\lambda$, where $\hat{p}^\lambda := \sum_{j\in K} \lambda^j \hat{p}^j$. In general, it is well-known (see Magill and Quinzii (1996)) that the state-price deflator evaluated at an equilibrium is locally the most valued security for an agent. In our case, this is $\hat{p}^k$ for clientèle k (if we take markets to be complete, this is the unique state-price deflator). The optimal security for clientèle k allows agents in this group to obtain the payoffs of their most valued security $\hat{p}^k$, while shorting the payoffs of the most valued securities of other clientèles $\{\hat{p}^j\}_{j\neq k}$. In equilibrium, agents are induced to hold the swap by prices and by the underlying motivation to diversify. Indeed, by buying $\hat{p}^k$ they get rid of their idiosyncratic risk and by shorting $\hat{p}^\lambda$ they acquire a position in the global market portfolio ($\hat{p}^k - \hat{p}^\lambda$ is collinear with $p^k - p^\lambda$, which is equal to $\beta\omega - \beta^k\omega^k$). It is a consequence of our linear-quadratic formulation that the optimal asset structure at the arbitraged equilibrium, namely $\{\hat{p}^k - \hat{p}^\lambda\}$, is the same as the optimal asset structure at the autarky equilibrium, $\{p^k - p^\lambda\}$. Thus the optimal security design in the arbitraged economy depends only on the autarky equilibrium and not on the amount supplied by arbitrageurs.

A socially optimal asset structure is not necessarily optimal for each clientèle. For example, starting from an initial asset structure $\{R^k\}$, if markets are completed for each clientèle, the resulting security design is socially optimal. However, some clientèles may be worse off since prices are typically affected by the introduction of new securities and lead to redistributions. This possibility has been discussed by Elul (1999), for instance. The following proposition addresses the important question as to *who* gains and who

loses as a result of an optimal financial innovation, and what the drivers are.

Proposition 5 (Welfare gains and losses) *In a clientèle economy with asset structure $\{R^k\}$, clientèle k is worse off at a socially optimal asset structure if and only if*

$$\|p^k - p^\lambda\|_2 < \|P^k(p^k - p^A)\|_2.$$

This follows directly from (15). Clientèle k is worse off if and only if the total marketable gains from trade are smaller after the innovation than before.

Example 2 Consider the economy in Example 1, with an additional exchange 0. There is a single agent on this exchange, and it has complete markets: $R^0 = I$. Choose $p^0 = \frac{1}{2}\mathbf{1}$. It is easy to check that $p^0 = p^\lambda$.

Under the optimal security design $R^k = p^k - p^\lambda$, or equivalently $R^k = I$, all k, the arbitrageurs' shadow valuation is p^λ, which coincides with exchange 0's valuation. There is no trade on 0. Arbitrageurs simply buy on 2 and deliver to 1. On the other hand, under the given asset structure, the arbitrageurs' shadow valuation is p^Λ. Using (14) and (12), we have

$$\begin{aligned} p^0 - p^\Lambda &= \left(\sum_{k\in K} \lambda^k P^k\right)^{-1} \sum_{k\neq 0} \lambda^k P^k (p^0 - p^k) \\ &= \frac{1/2 - \bar{\beta}}{\bar{\beta}} \left(\sum_{k\in K} \frac{1}{\beta^k} P^k\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

which is nonzero. Hence there is trade on exchange 0, and welfare is higher on this exchange than under the optimal security design. ‖

The example shows that not every exchange may benefit when all exchanges are completed because the completion of markets may erode the advantages an exchange may have had before the completion. Exchange 0 plays a valuable role in the arbitrage process at the initial asset structure, facilitating trade between the other two exchanges. At a socially optimal asset structure, however, it becomes entirely redundant. This is reminiscent of what was also found in Willen (2003). We can similarly show that when no clientèle initially enjoys a trading advantage, then all clientèles benefit from optimal security design.

Proposition 6 (Pareto-improving security design) *Consider a clientèle economy, with an initial asset structure that satisfies* **S2**. *Then no clientèle can be worse off at a socially optimal asset structure.*

Condition **S2**(b) means that the initial asset structure is already socially optimal. In the case of common payoff matrices, condition **S2**(a), no exchange is at a trading advantage at the initial equilibrium as trades that can be executed on one exchange can equally be carried out on some other exchange. Note that, in going from an initial asset structure to a socially optimal one, we allow for the possibility of removing some of the initial assets. When we restrict attention to innovation (not necessarily optimal) of additional assets, Proposition 6 extends to the general case where agents may be heterogeneous within exchanges.

Proposition 7 (Pareto-improving innovation) *Suppose arbitrageurs introduce new assets, and* **S2** *is satisfied at both the initial and the post-innovation asset structure. Then no investor can be worse off after the innovation.*

In particular, starting from a common asset structure $R^k = R$, if arbitrageurs (incrementally) innovate optimally, all agents are better off. Even though arbitrageurs are solely interested in their own profits, in this case maximal profit extraction means providing *each* investor with his favorite assets. The intuition is clearest in the limiting equilibrium as N goes to infinity. We know from Lemma 7 that the complete-markets Walrasian pricing kernel applies in this equilibrium. With quadratic preferences, adding assets leaves the pricing kernel unchanged. Innovation cannot hurt investors as it does not affect the prices of the initial assets. It should be noted, however, that we do require that both the pre- and post-innovation asset structures satisfy **S2**. This is the case, for example, when agents have access to the same asset markets before and after the innovation. In Example 2, **S2** was not satisfied at the initial equilibrium, and we saw that completing markets on all exchanges moved prices unfavorably for some agents.

Proposition 4 does not extend to the heterogeneous agent case. The minimal equilibrium security design $\{p^k - p^\lambda\}$ will in general be Pareto dominated by complete markets on every exchange. In a clientèle economy both these asset structures are socially optimal, and indeed payoff-equivalent for every investor and arbitrageur. This is not so with heterogeneous agents. If there is sufficient heterogeneity, with $I^k > S$ distinct investors on each exchange k, there will typically be S linearly independent optimal net trades on every exchange, so that the only socially optimal asset structure is the the one with complete markets on all exchanges.

The reason why the security design $\{p^k - p^\lambda\}$ fails to be socially optimal is the fact that arbitrageurs only care about the aggregate valuations on the various exchanges. They do not consider the effects of their security choice on the intra-exchange reallocation of resources that occur when investors on a given exchange trade the security among themselves. Such intra-exchange trades are absent in a clientèle economy ($p^{k,i} = p^k$, all k), which is why profit-maximization by arbitrageurs is socially efficient in that case.

7 Macro Markets

Robert Shiller (see Shiller (1993)) has argued that one of the most important categories of missing markets are markets for country GDP. Stock markets for instance allow investors to trade only the small component of national income that corresponds to corporate profits.

Since each trade needs a counterparty, macro markets can only be successful if there is both a demand and a supply for a claim on each country's GDP stream, not least because otherwise innovators would not choose to launch such products. Demand and supply must naturally be international. In that spirit, we proceed to study Shiller's conjecture within our formal model, reinterpreting exchanges as countries.

Assume initially that there are two countries, $k = 0, 1$. We have seen that arbitrageurs would find it optimal to introduce one single (further) asset in country k, $p^k - p^\lambda$, which in this simple example means a payoff collinear with $(\beta^0\omega^0 - \beta^1\omega^1)$. Since ω^k amounts in fact to GDP in country k, the equilibrium security corresponds to the depth-weighted difference of the GDPs in the two countries.

An asset structure $R^0 = R^1 = [\omega^0\ \omega^1]$ would also be an equilibrium of the design game, albeit not a minimal one. What does come out of this, though, is that introducing country k's GDP as a tradable asset in country k only is optimal neither for arbitrageurs nor for investors. Investors in country k want to be able to simultaneously sell a portion of their GDP and diversify their portfolios by buying a fraction of world GDP.

With K arbitrary, arbitrageurs could therefore introduce claims to the GDP of each and every country in all countries. But it is more reasonable and cheaper to introduce only the ideal security in country k, which is the difference of the GDP outcomes of country k with the rest of the world, properly weighted by depth.

With this caveat, Shiller's conjecture as to the importance of GDP markets is mirrored in our model: those are indeed the equilibrium assets designed and traded by profit-maximizing innovative institutions. Nevertheless, while macro GDP markets are optimal for arbitrageurs, Proposition 5 should caution us as to their *social* welfare properties. Shiller has not specified any particular social welfare function, and therefore claims must be evaluated carefully. Even with homogeneous investors within each country, some countries may lose out from the introduction of macro markets, especially those that provide substitute insurance. Realistically, these countries may also be the ones with the greatest political influence to prevent macro markets.

Finally, while Shiller's conjecture does yield a social optimum with homogeneous investors within each country, this is no longer true when they are heterogeneous. Even though macro markets are the ones that innovators will in fact establish, they are not socially efficient in general.

8 Conclusion

In this paper we analyze what happens if securities are designed not by a benevolent social planner, or a derivatives exchange, or by companies issuing financial assets based on the underlying hitherto nontraded cash-flows of closely-held real assets, but by large traders such as investment banks and hedge funds. We believe this corresponds closely to what happens in actual markets, with asset innovations completing markets for certain groups of investors ("exchanges") rather than for the economy as a whole. For instance, capital-protected investment vehicles have recently stood in the spotlight again, despite the fact that such assets are largely redundant. Price-fixing retail banks sell them at a markup to their clients, and investment banks sell them at a markup to the retail banks. Investment banks in turn have the ability to hedge them in the underlying markets.

We answer the question as to which assets we should expect to see in an economy with a variety of exchanges. Interestingly, we are able to provide an explicit and minimal answer: an exchange is offered to trade the difference between its own state-price deflator and the depth-weighted economy-wide state-price deflator. Only one asset is introduced per exchange.

Depth, or "liquidity," plays a major role in this paper. If introducing a new security was costly in our model, we would see that everything else equal, the only exchanges on which innovation occurs are the deep exchanges.

Shallow exchanges would be innovated upon if the gains from trade are large enough to compensate for their shallowness. One consequence is the adage, well-known to practitioners, that derivative securities can only be successful if there is sufficient demand for them from some clientèle, i.e. from the end-users. This is evident in our model: no matter how many intermediaries trade on an exchange, if the depth of the end-users tends to zero (for instance if the number of investors tends to zero), trading on the exchange vanishes as well.

We were not interested in just the equilibrium asset structure. The general equilibrium nature of our setup also allows us to study welfare properties. It is far from obvious if the securities introduced by arbitrageurs in order to extract the largest profits from wedges in investors' marginal willingness to pay lead to a socially optimal outcome. Arbitrageurs are driven by mispricings and depth considerations, not by socially beneficial gains from trade. Still, we can show that if investors within an exchange have identical valuations, so they form a homogeneous clientèle, then an equilibrium structure is socially optimal and independent of the degree of competition between arbitrageurs. However, equilibrium allocations are not socially optimal and do depend on the degree of competition. We also provide a necessary and sufficient criterion that characterizes those investors who gain and those who lose from said innovations. If investors within an exchange are heterogeneous, the equilibrium security design is no longer socially optimal. The reason is that arbitrageurs ignore the gains from trade within exchanges since they only profit from inter-exchange reallocations of resources. The equilibrium asset structure is therefore geared toward extracting the maximum inter-exchange gains from trade, at the expense of intra-exchange gains from trade.

Our paper provides a rich framework for studying a number of issues, some of which we have only barely touched on here. One is a more general concept of liquidity. Arbitrageurs provide liquidity by mediating trades, much in the same way as market-makers do. This suggests a measure of liquidity that ties together the seemingly disparate notions of depth, bid-ask spreads, trading volume, and gains from trade. We pursue this idea in Rahi and Zigrand (2004b). Second, the arbitraging scenario we have studied in this paper, wherein all arbitrageurs are simultaneously active on all exchanges, is but one possible description of intermediation in a segmented economy. In Rahi and Zigrand (2004a) we look at the case where arbitrageurs choose a subset of exchanges on which to trade, and analyze the resulting distribution of arbitrageur activity. We are able to derive simple rules as to the optimal

exchanges on which to innovate. Quite intuitively, arbitrageurs gravitate to those exchanges which, other things being equal, are deeper and stand to gain most from trading with other exchanges.

A Appendix

Proof of Lemma 1 Using (3), we can write the Lagrangian for arbitrageur n as follows:

$$\mathcal{L} = \sum_k [p^k - \beta^k R^k y^{k,n} - \beta^k R^k y^{k,\backslash n}]^\top \Pi R^k y^{k,n} - p^{A,n\top} \Pi \sum_k R^k y^{k,n},$$

where $y^{k,\backslash n}$ is the aggregate supply of assets on exchange k of all arbitrageurs but n. $p^{A,n}$ is the Lagrange multiplier vector attached to the no-default constraints, and can be interpreted as a (shadow) state-price deflator of the arbitrageur. The first order conditions are:

$$R^{k\top} \Pi [p^k - \beta^k R^k y^{k,\backslash n} - 2\beta^k R^k y^{k,n} - p^{A,n}] = 0, \qquad k \in K \tag{16}$$

together with complementary slackness:

$$p^{A,n} \geq 0, \quad \sum_k R^k y^{k,n} \leq 0, \quad \text{and} \quad p^{A,n}_s \cdot \left[\sum_k R^k y^{k,n}\right]_s = 0, \quad \forall s. \tag{17}$$

The existence of the multipliers follows as usual from the linearity of the inequalities, as shown in Arrow et al. (1961) for instance.

We first demonstrate that a CWE is symmetric, i.e. $y^{k,n}$ does not depend on n, and we can choose $p^{A,n} = p^A$ for all n. The reaction correspondence of arbitrageur n, for given supply of the remaining arbitrageurs $\{y^{k,\backslash n}\}$, is single-valued due to the strict concavity of the program. From the first order conditions (16) and (17),

$$y^{k,n} = \frac{1}{\beta^k}(R^{k\top} \Pi R^k)^{-1} R^{k\top} \Pi [p^k - p^{A,n}] - y^k, \qquad k \in K \tag{18}$$

and

$$p^{A,n} \geq 0, \qquad \sum_k \frac{1}{\beta^k} P^k (p^k - p^{A,n}) - \sum_k R^k y^k \leq 0 \tag{19}$$

with complementary slackness state-by-state. In other words, there is some $p^{A,n} \geq 0$ satisfying the no-default and complementary slackness conditions so that the unique $y^{k,n}$ chosen if all others choose $y^{k,\backslash n}$ is given by (18). Many feasible $p^{A,n}$ may exist, but any $p^{A,n}$ and $\tilde{p}^{A,n}$ that represent the reaction function must satisfy $R^{k\top}\Pi p^{A,n} = R^{k\top}\Pi\tilde{p}^{A,n}$, otherwise single-valuedness is violated.

It follows that $y^{k,n}$ cannot depend on n at an equilibrium. Indeed, assume to the contrary that an equilibrium $\{y^{k,n}\}$ is such that $y^{k,n} \neq y^{k,n'}$ for some k and some pair (n, n'). Then (18) implies that $p^{A,n} \neq p^{A,n'}$. The inequalities (19) for arbitrageurs n and n' depend only on the aggregate quantities y^k, for all k. So given $\{y^{k,n}\}$, $p^{A,n'}$ is also a valid shadow price for arbitrageur n. But then $R^{k\top}\Pi p^{A,n} = R^{k\top}\Pi p^{A,n'}$, implying that $y^{k,n} = y^{k,n'}$.

Having established symmetry, we can easily solve (18) for $y^{k,n}$ and verify that (4) holds. This is in fact the unique solution. Given (4), (5) and (6) follow from (17). ∎

Proof of Lemma 2 Using (3) and (4),

$$\begin{aligned} q^k &= R^{k\top}\Pi\left[p^k - \frac{N}{1+N}P^k(p^k - p^A)\right] \\ &= R^{k\top}\Pi\left[p^k - \frac{N}{1+N}(p^k - p^A)\right] \\ &= R^{k\top}\Pi\left[\frac{1}{1+N}p^k + \frac{N}{1+N}p^A\right]. \end{aligned}$$

∎

Proof of Lemma 3 Using Lemma 1, Lemma 2 and (6), some straightforward algebra gives us the equilibrium profits of arbitrageur n, for asset

structure $\{R^k\}$:

$$\begin{aligned}\Phi(\{R^k\}) &= \sum_k q^k \cdot y^{k,n} \\ &= \frac{1}{(1+N)^2} \cdot \sum_k \frac{1}{\beta^k} {p^k}^\top \Pi P^k \left(p^k - p^A\right) \\ &= \frac{1}{(1+N)^2} \cdot \sum_k \frac{1}{\beta^k} \left(p^k - p^A\right)^\top \Pi P^k \left(p^k - p^A\right) \\ &= \frac{1}{(1+N)^2} \cdot \sum_k \frac{1}{\beta^k} \|P^k(p^k - p^A)\|_2^2.\end{aligned}$$

■

Proof of Lemma 4 The Lagrangian for agent (k,i)'s optimization problem is:

$$\begin{aligned}\mathcal{L} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} - \sum_{\ell \in K} q^\ell \cdot \varphi^{k,i,\ell} + \mathbf{1}^\top \Pi(\omega^{k,i} + R^k\theta^{k,i}) \\ - \frac{\beta^{k,i}}{2}(\omega^{k,i} + R^k\theta^{k,i})^\top \Pi(\omega^{k,i} + R^k\theta^{k,i}) + {\psi^{k,i}}^\top \Pi \sum_{\ell \in K} R^\ell \varphi^{k,i,\ell}.\end{aligned}$$

The first order conditions give us

$$\theta^{k,i} = \frac{1}{\beta^{k,i}}({R^k}^\top \Pi R^k)^{-1} {R^k}^\top \Pi(p^{k,i} - p^{RC}) \tag{20}$$

$${R^\ell}^\top \Pi \psi^{k,i} = q^\ell = {R^\ell}^\top \Pi p^{RC}, \qquad \forall \ell \in K \tag{21}$$

$$\psi^{k,i} \geq 0, \qquad \sum_{\ell \in K} R^\ell \varphi^{k,i,\ell} \geq 0 \tag{22}$$

$$\psi_s^{k,i} \cdot \left(\sum_{\ell \in K} R^\ell \varphi^{k,i,\ell}\right)_s = 0, \qquad \forall s \in S \tag{23}$$

Equation (20) gives us (10). Equations (21) and (22) are the usual no-arbitrage conditions. In particular, ${R^k}^\top \Pi \psi^{k,i}$ is independent of (k,i). Indeed, we can choose $\psi^{k,i}$ to be the same for all (k,i), and p^{RC} equal to this common value. Thus $p^{RC} \geq 0$.

We now show that $p^A = p^{RC}$ satisfies (5) and (6). From (20), we see that

$$\sum_{k,i} R^k \theta^{k,i} = \sum_k \frac{1}{\beta^k} P^k (p^k - p^{RC}).$$

From market clearing,

$$\sum_{k,i} R^k \theta^{k,i} = -\sum_{k,i,\ell} R^\ell \varphi^{k,i,\ell} \tag{24}$$

which is less than or equal to zero, by (22). Therefore, (5) is satisfied.

Next we need to establish that (6) holds with p^A replaced by p^{RC}. Assume that $p_s^{RC} > 0$, some s. Then $\psi_s^{k,i} > 0$, all (k,i). By (23) we know $\left(\sum_\ell R^\ell \varphi^{k,i,\ell}\right)_s = 0$, all (k,i). By (24), $\left(\sum_{k,i} R^k \theta^{k,i}\right)_s = 0$, i.e.

$$\left[\sum_k \frac{1}{\beta^k} P^k (p^k - p^{RC})\right]_s = 0$$

as desired.

On the other hand, assume now that $\left(\sum_{k,i} R^k \theta^{k,i}\right)_s < 0$, some s. By (24), $\left(\sum_{k,i,\ell} R^\ell \varphi^{k,i,\ell}\right)_s > 0$, which itself implies that for at least some (k,i), $\left(\sum_\ell R^\ell \varphi^{k,i,\ell}\right)_s > 0$. By (23), $\psi_s^{k,i}$ is zero, and hence so is p_s^{RC}.

We now prove uniqueness of restricted-consumption Walrasian equilibria. It suffices to show that if $x, y \in \mathbb{R}_+^S$ are equilibrium pricing kernels, then they induce the same pricing functional, i.e. $P^k(x-y) = 0$, for all $k \in K$. The equilibrium allocation is then uniquely determined by (10). In order to save on notation, we use the following shorthand:

$$A := \sum_k \frac{1}{\beta^k} \Pi P^k,$$
$$b := \sum_k \frac{1}{\beta^k} \Pi P^k p^k.$$

Then x and y satisfy the following system of inequalities:

$$Ax - b \geq 0, \qquad x^\top (Ax - b) = 0, \tag{25}$$
$$Ay - b \geq 0, \qquad y^\top (Ay - b) = 0. \tag{26}$$

We know that the system above does have a solution, for instance $x = y = p^A$. Since A is positive semidefinite, we have $(x-y)^\top A(x-y) \geq 0$, i.e.

$$y^\top Ax \leq \frac{1}{2}(x^\top Ax + y^\top Ay). \tag{27}$$

Furthermore, since $y \geq 0$, from (25) and (26) we have $y^\top Ax \geq y^\top b = y^\top Ay$, and similarly $y^\top Ax \geq x^\top b = x^\top Ax$. Therefore, (27) must hold with equality, i.e. $(x-y)^\top A(x-y) = 0$, or $\sum_k \frac{1}{\beta^k}(x-y)^\top \Pi P^k (x-y) = 0$. Now since ΠP^k is positive semidefinite for each k, this implies that $(x-y)^\top \Pi P^k (x-y) = 0$, or $\|P^k(x-y)\|_2^2 = 0$, for all k. Hence $P^k(x-y) = 0$ for all k. ∎

Proof of Lemma 5 From (14),

$$\left[\lambda^0 I + \sum_{k\neq 0} \lambda^k P^k\right] p^\Lambda = \lambda^0 p^0 + \sum_{j\neq 0} \lambda^j P^j p^j.$$

Premultiplying both sides by P^0, and noting that **S1** implies that $P^0 P^k = P^k$:

$$\left[\lambda^0 P^0 + \sum_{k\neq 0} \lambda^k P^k\right] p^\Lambda = \lambda^0 P^0 p^0 + \sum_{j\neq 0} \lambda^j P^j p^j,$$

i.e,

$$\sum_k \lambda^k P^k (p^k - p^\Lambda) = 0.$$

Therefore, $p^{RP} = p^\Lambda$ solves (13). It is easy to verify that $p^\Lambda = \mathbf{1} - \beta \cdot \sum_{k\in K} Q^k \omega^k$, so that $p^\Lambda \geq 0$ if and only if **N1** holds. Note that the inverse in the definition of Q^k exists since the matrix

$$\lambda^0 \Pi^{-1} + \sum_{k\neq 0} \lambda^k R^k (R^{k^\top} \Pi R^k)^{-1} R^{k^\top}$$

is positive definite, hence invertible. ∎

In order to prove Proposition 1, we need to establish the following result.

Fact 1 *$\|P^k v\|_2 \leq \|v\|_2$, for all $v \in \mathbb{R}^S$. Moreover, $\|P^k v\|_2 = \|v\|_2$ if and only if $P^k v = v$.*

Proof :

$$\begin{aligned}\|P^k v\|_2^2 &= v^\top \Pi R^k (R^{k^\top} \Pi R^k)^{-1} R^{k^\top} \Pi v \\ &= (\Pi^{1/2} v)^\top \Pi^{1/2} R^k (R^{k^\top} \Pi R^k)^{-1} R^{k^\top} \Pi^{1/2} (\Pi^{1/2} v)\end{aligned}$$

Defining $x := \Pi^{1/2} v$ and $y := \Pi^{1/2} P^k v = \Pi^{1/2} R^k ({R^k}^\top \Pi R^k)^{-1} {R^k}^\top \Pi^{1/2} x$, we see that $\|P^k v\|_2^2 = x \cdot y = \|x\| \|y\| \cos(\theta)$, where θ is the angle between x and y, and where $\|x\| := \sqrt{x \cdot x}$. Now $y \cdot y = x \cdot y \geq 0$. If $x \cdot y = 0$, the result follows. Otherwise we get $\|P^k v\|_2^2 = x \cdot y = \|x\| \sqrt{x \cdot y} \cos(\theta)$ which we can solve for $x \cdot y = \|x\|^2 \cos^2(\theta)$. We therefore find that $\|P^k v\|_2^2 = \|x\|^2 \cos^2(\theta) \leq \|x\|^2 = v^\top \Pi v = \|v\|_2^2$.

Now suppose that $\|P^k v\|_2 = \|v\|_2$. We want to show that $P^k v = v$. If $x \cdot y = 0$, then $x \cdot y = \|P^k v\|_2^2 = \|v\|_2^2 = 0$, so that $P^k v = v = 0$. If, on the other hand $x \cdot y > 0$, then $\|P^k v\|_2 = \|v\|_2$ implies that $\cos^2(\theta) = 1$, i.e. x and y are collinear. But $x \cdot y = y \cdot y \neq 0$. Hence $x = y$, or $P^k v = v$. ∎

Proof of Proposition 1 Proof of 1 ⇒ 2: Suppose that condition **S2**(b) holds, i.e. $p^k - p^\lambda \in \langle R^k \rangle$, $k \in K$. Then, from Lemma 6, we can choose $p^A = p^\lambda$. Using Lemma 3, and noting that $P^k(p^k - p^\lambda) = p^k - p^\lambda$, equilibrium arbitrageur profits are given by

$$\Phi = \frac{1}{(1+N)^2} \cdot \sum_k \frac{1}{\beta^k} \|p^k - p^\lambda\|_2^2. \tag{28}$$

In order to establish that the proposed asset structure is optimal for every arbitrageur, we need to show that

$$\sum_k \lambda^k \|p^k - p^\lambda\|_2^2 \geq \sum_k \lambda^k \|P^k(p^k - p^A)\|_2^2. \tag{29}$$

Fact 1 implies that

$$\sum_k \lambda^k \|p^k - p^\lambda\|_2^2 \geq \sum_k \lambda^k \|P^k(p^k - p^\lambda)\|_2^2. \tag{30}$$

Furthermore, noting that $x^\top A x - y^\top A y = (x-y)^\top A (x+y)$, for any vectors

x, y, and any square matrix A, we have

$$\begin{aligned}
&\sum_k \lambda^k \|P^k(p^k - p^\lambda)\|_2^2 - \sum_k \lambda^k \|P^k(p^k - p^A)\|_2^2 \\
&= \sum_k \lambda^k \left[(p^k - p^\lambda)^\top \Pi P^k (p^k - p^\lambda) - (p^k - p^A)^\top \Pi P^k (p^k - p^A)\right] \\
&= \sum_k \lambda^k (p^A - p^\lambda)^\top \Pi P^k (2p^k - p^\lambda - p^A) \\
&= \sum_k \lambda^k (p^A - p^\lambda)^\top \Pi P^k (p^A - p^\lambda) + 2(p^A - p^\lambda)^\top \Pi \sum_k \lambda^k P^k (p^k - p^A) \\
&= \sum_k \lambda^k \|P^k(p^A - p^\lambda)\|_2^2 - 2{p^\lambda}^\top \Pi \sum_k \lambda^k P^k (p^k - p^A) \qquad \text{(using (6))} \\
&\geq 0
\end{aligned}$$

where the last inequality follows from (5) and the fact that $p^\lambda \geq 0$. In conjunction with (30), this implies (29).

Proof of $2 \Rightarrow 1$: If **S2**(b) is violated, (30) holds with strict inequality due to Fact 1, implying that profits are strictly lower than (28).

Proof of $1 \Rightarrow 3$: We show that $\{R^k\}$ is a Nash equilibrium of the security design game, provided **S2**(b) holds. Suppose arbitrageur n deviates by introducing additional assets with payoff matrix D^k on exchange k, $k \in K$. Let $y_D^{k,n}$ be arbitrageur n's supply of the additional assets on exchange k. Note that we allow the deviating arbitrageur to have monopolistic access to these assets, so that he is the sole supplier. Using (3), we can write the Lagrangian for the optimization problem of arbitrageur n as

$$\begin{aligned}
\mathcal{L} = &\sum_k \left[p^k - \beta^k (R^k y^k + D^k y_D^{k,n})\right]^\top \Pi \left[R^k y^{k,n} + D^k y_D^{k,n}\right] \\
&- \mu^\top \Pi \sum_k (R^k y^{k,n} + D^k y_D^{k,n}),
\end{aligned}$$

where μ is the Lagrange multiplier vector on the no-default constraints. The first order conditions with respect to $y^{k,n}$ and $y_D^{k,n}$ are ${R^k}^\top \Pi Y = 0$, and ${D^k}^\top \Pi Y = 0$, respectively, where

$$Y := p^k - \beta^k R^k y^{k,\backslash n} - 2\beta^k R^k y^{k,n} - 2\beta^k D^k y_D^{k,n} - \mu.$$

It is easy to verify that a solution to the problem is $\mu = p^\lambda$, $y_D^{k,n} = 0$, and $y^{k,n}$ equal to arbitrageur n's supply on exchange k given the initial asset structure

$\{R^k\}$, namely $R^k y^{k,n} = \frac{1}{(1+N)\beta^k}(p^k - p^\lambda)$, $k \in K$. The no-default constraints hold with equality. Complementary slackness is satisfied since $p^\lambda \geq 0$. This solution is in fact unique since the program is globally concave. Thus the deviating arbitrageur has nothing to gain by deviating.

Proof of $3 \Rightarrow 1$: An asset structure $\{R^k\}$ that does not satisfy **S2**(b) is not a Nash equilibrium since introducing the asset $p^k - p^\lambda$ on each exchange k where it is not already available, (strictly) increases the profits of all arbitrageurs, provided they all trade the additional assets. Introducing these assets must therefore increase the payoff of an individual arbitrageur who has monopolistic access to them. ■

Proof of Proposition 6 Suppose the initial asset structure $\{R^k\}$ satisfies **S2**. Then, from Lemma 6, we can choose $p^A = p^\lambda$. Therefore, $\|P^k(p^k - p^A)\|_2 = \|P^k(p^k - p^\lambda)\|_2 \leq \|p^k - p^\lambda\|_2$, using Fact 1. The result now follows from Proposition 5. ■

Proof of Proposition 7 Let the pre- and post-innovation asset structures be, respectively, $\{R^k\}$ and $\{\bar{R}^k\}$, with projection matrices $\{P^k\}$ and $\{\bar{P}^k\}$. Since $\langle R^k \rangle \subset \langle \bar{R}^k \rangle$, we have $P^k \bar{P}^k = P^k$. Moreover, both asset structures satisfy **S2**, so that we can choose $p^A = p^\lambda$ in both cases, from Lemma 6. Let

$$\zeta^{k,i} := (p^{k,i} - p^k) + \frac{N}{1+N}(p^k - p^A).$$

We have $\|P^k \zeta^{k,i}\|_2 = \|P^k \bar{P}^k \zeta^{k,i}\|_2 \leq \|\bar{P}^k \zeta^{k,i}\|_2$, using Fact 1. The result now follows from Lemma 8. ■

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