General Properties of Rational
Stock-Market Fluctuations

- Unabridged version -

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Abstract

Which pricing kernel restrictions are needed to make low dimensional Markov models consistent with given sets of predictions on aggregate stock-market fluctuations ? This paper develops theoretical test conditions addressing this and related reverse engineering issues arising within a fairly general class of long-lived asset pricing models. These conditions solely affect the first primitives of the economy (probabilistic descriptions of the world, information structures, and preferences). They thus remove some of the arbitrariness related to the specification of theoretical models involving unobserved variables, state-dependent preferences, and incomplete markets.

Understanding the properties of aggregate stock-market behavior has long been the subject of both theoretical and empirical research in financial economics. While the statistical properties of the aggregate stock-market seem now to be well-understood, we still have a variety of theoretical models which compete at rationalizing the empirical findings. Perhaps surprisingly, the general properties of these theoretical models are poorly understood. As an example, we do not have a theory able to answer such questions as: When are price-dividend ratios procyclical ? When is stock-market volatility countercyclical ? When are expected returns positively related to expected dividend growth ? This paper introduces a theory which explicitly addresses these and related questions.

In the class of models covered by the theory of this paper, agents have fully rational expectations. The only additional assumptions that I make are that the state variables of the economy are Markov processes with continuous sample paths (i.e. diffusion processes) satisfying some basic regularity conditions, and that asset prices are arbitrage-free. The first assumption has been widely used in related asset pricing fields because it facilitates the kind of investigations that are undertaken in this paper [See, e.g., Bergman, Grundy and Wiener (1996), Romano and Touzi (1997), and Mele (2003)].[1] The second assumption is used to produce the general statements of the theory. To illustrate this theory, I provide examples of infinite horizon, general equilibrium models. However, I emphasize that the theory only requires absence of arbitrage.

Based only on the previous assumptions, I develop sets of theoretical test conditions on the primitives of the economy (laws of motion of the primitive state variables and the pricing kernel). These conditions restrict the primitives so as to make the resulting asset price processes consistent with a variety of patterns of aggregate stock-market behavior that are given in advance. As an example, I provide precise conditions for price-dividend ratios to be strictly increasing and concave in the variables tracking the business cycle conditions. In many cases of interest, these conditions guarantee that stock market volatility and Sharpe ratios display the same qualitative countercyclical behavior that we commonly observe in the data. In the same cases, these conditions guarantee the internal consistency of many existing general equilibrium models. Indeed, a presumption of all these models is that asset prices volatility is strictly positive. (This presumption guarantees that intertemporal optimization programs of infinitely-lived agents are

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[1] Alternatively, future research may consider discrete time models. In his celebrated article, for example, Lucas (1972) considered a discrete time model. He was able to study slope and convexity of rational pricing functions with respect to the state variables of the economy that he was considering. In continuous time models, these tasks are easier because the study of the solution (and its partial derivatives) to certain dynamic programming equations collapses to the study of the solution (and its partial derivatives) to partial differential equations.

well-defined.) But how can this condition be checked when volatility is endogenously determined ? As a by-product, the theoretical test conditions of this paper explicitly address this issue.

The perspective taken in this article differs from previous approaches in some fundamental respects. As is well-known, the majority of long-lived asset pricing models are inherently nonlinear and analytically intractable. Consequently, three well-known remedies have been hitherto devised. The first one removes nonlinearities through a series of simplifying assumptions [e.g., Mehra and Prescott (1985), Abel (1994, 1999), or Cecchetti, Lam and Mark (1993)]. The second one neglects nonlinearities through a first-order approximation of the models under study [e.g., Campbell and Shiller (1988)].[2] Finally, a third approach consists in solving the models numerically [e.g., Campbell and Cochrane (1999), Veronesi (1999), or Chan and Kogan (2002)]. The first two remedies have the clear advantage to isolate some important economic phenomena in a simple and understandable way. [An example of analysis based on these principles is the survey of Campbell (1999).] The third approach allows one to explicitly work out the consequences of nonlinearities. This article combines the relative strengths of the previous three approaches. First, I produce predictions which do not rely on any ad-hoc assumption. Second, these predictions do not hinge upon any closed-form solution or any numerical analysis of any particular model. At a very least, the results of this article should thus constitute the basis of a new method of investigation that complements previous approaches.

To illustrate one example of predictions of the theory developed in this article, consider the standard models with external habit formation. Among other things, these models may predict stock-market volatility to be countercyclical. As an example, Campbell and Cochrane (1999) found this property by relying on the numerical solution of their model. The theory in this article unveils the precise theoretical mechanism underlying this and related discoveries. It predicts that stock-market volatility is countercyclical whenever Sharpe ratios are "sufficiently" convex in the state variable tracking the business cycle conditions (see proposition 4 in section 5).

As another example of predictions of the theory, consider the learning models introduced by Veronesi (1999). This model predicts that long-lived asset prices are increasing and convex in the agents' posterior probability of the economy being in a good state. Veronesi offered many insights on such a *rational* "excess sensitivity" of price reaction to state variables. The theoretical test conditions of this article provide further precise insights on this and related learning models [such as the Brennan and Xia (2001) model]. They point to two main conclusions. First, the overreaction property observed by Veronesi is a robust property shared by many other learning

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[2]Kogan and Uppal (2001) have developed a refined approximation approach based on asymptotic analysis ideas.

models. Second, the same property is the manifestation of a more general characteristic of any long-lived asset pricing model. Precisely, I find that long-lived asset prices are *always* convex in *any given* state variable affecting the expected payoff under a bound on convexity of the risk-neutralized drift of the given variable (see propositions 2 and 3 in section 5). As I will show in section 4, such a bound arises naturally in many models with incomplete information and learning (and in simpler models with time-varying dividend growth rate). The robustness of such a "convexity" property suggests that theoretical explanations of large price movements do not necessarily have to rely on non-fundamental factors of asset price formation.

The previous predictions of the theory are part of a more elaborated, multidimensional framework of analysis (see proposition 5 in section 6). This framework encompasses two categories of multidimensional models each having its own economic motivation. Both categories extend the standard Lucas (1978) model of the (single) Markov consumption good process (the "payoff"). The extensions operate along the two most natural dimensions.

In the first one (considered in section 4), one state variable affects the expected consumption growth. Such a state variable may be observed or not. If it is not observed - and if agents attempt to learn its value through observation of the past - nonlinearities may arise. It is precisely the presence of such nonlinearities which makes the resulting pricing problem so complex. The theoretical test conditions of this paper address this problem in great generality. However, I stress that these conditions do not depend on assumptions such as partial observability of the state.

In the second one (considered in section 5), one state variable affects all sets of admissible (i.e. no arbitrage) Sharpe ratios on long-lived assets. Special cases of the resulting economies are the "habit" economies mentioned earlier, or certain incomplete markets economies. Again, I emphasize that the theoretical test conditions I obtain do not depend on assumptions regarding preferences or the market structure.

Finally, the theory in this article is related to the "integrability" problem studied by He and Leland (1993), Wang (1993), Cuoco and Zapatero (2000), and others. The integrability problem consists in recovering preferences (and beliefs) from the knowledge of a given equilibrium asset price process. In this article, I also derive restrictions on price kernels which make them consistent with given rational asset price processes. One distinctive feature of this article is that it is not confined to settings with complete markets and/or standard expected utility functions. Furthermore, I consider multidimensional settings and I *do* provide accurate descriptions of *both implied* kernel properties *and implied* primitive processes. On the other hand, the theoretical test conditions of this article only impose sufficient restrictions on kernels and other primitives

of models.

The article is organized in the following manner. The next section describes the primitives of the model. Section 2 outlines how the motivational issues of this introduction are addressed in this paper. Section 3 develops a simplified version of the theory. Sections 4 and 5 are the main core of the paper; section 4 examines models including learning mechanisms and, more generally, stochastic consumption growth; and section 5 analyzes models with time-varying risk-aversion. Section 6 extends the theory to four-factor models. Section 7 concludes. Five appendices gather proofs, examples, and results omitted in the main text.

1   The model

I consider a pure exchange economy endowed with a flow of a (single) consumption good. Let $Z = \{z(\tau)\}_{>0}$ be the process of instantaneous rate of consumption endowment. With the exception of section 6, I assume that consumption equals the dividends paid by a long-lived asset (see below). Accordingly, I use the terms "consumption" and "dividends" interchangeably. Let $Y = \{y(\tau)\}_{>0}$ be an additional multidimensional state vector. I assume that (Z, Y) constitutes a multidimensional diffusion process, with $z(0) = z$ and $y(0) = y$ (say), where $(z, y) \in \mathbb{Z} \times \mathbb{Y}$, $\mathbb{Z} \subset \mathbb{R}_{++}$ and $\mathbb{Y} \subset \mathbb{R}^{d-1}$ ($d \geq 2$). Consequently, I fix a probability space (Ω, F, P) and a family $\{F(\tau) : \tau \geq 0\}$ of sigma-algebras that is the augmented filtration of a standard Brownian motion in $\mathbb{R}^d$. To keep the analysis as simple as possible, I consider the case in which $d = 2$. As I will show in sections 3, 4 and 5, this case is general enough to include many existing models. Extensions to higher dimensions are considered in section 6 and appendix E.

A long-lived asset is an asset that promises to pay Z. Let $Q = \{q(\tau)\}_{\geq 0}$ be the corresponding asset price process. As is well-known, absence of arbitrage opportunities implies that there exists a positive pricing kernel $M = \{\mu(\tau)\}_{\geq 0}$ such that

$$q(\tau)\mu(\tau) = E\left[\int_\tau^\infty \mu(s)z(s)ds\right], \quad \tau \geq 0, \tag{1}$$

where E is the expectation operator taken under probability measure P.[3]

Given the previous assumptions on the information structure of the economy, the triple

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[3] "Bubbles" are not considered in this paper.

(Z, Y, M) necessarily satisfies:

$$
\begin{cases}
dz(\tau) = m_0\left(z(\tau), y(\tau)\right) d\tau + \sigma\left(z(\tau)\right) dW_1(\tau) \\[2mm]
d\mu(\tau) = -\mu(\tau)\left[R(\tau)d\tau + \lambda_1(\tau)dW_1(\tau) + \lambda_2(\tau)dW_2(\tau)\right] \\[2mm]
dy(\tau) = \varphi_0(z(\tau), y(\tau))d\tau + \xi_1\left(y(\tau)\right)dW_1(\tau) + \xi_2\left(y(\tau)\right)dW_2(\tau)
\end{cases}
\tag{2}
$$

where W_1 and W_2 are independent standard Brownian motions; $m_0, \sigma, \varphi_0, \xi_1$ and ξ_2 are given functions guaranteeing a strong solution[4] to (Z, Y); the assumptions that consumption volatility σ only depends on z and that the state variable volatility does not depend on z are made to keep the presentation simple. Finally, R, λ_1 and λ_2 are some F-adapted processes satisfying all the regularity conditions needed for the representation in (1) to exist. As is also well-known, R represents the instantaneous (or short-term) rate process, and λ_i $(i = 1, 2)$ are the unit prices of risk associated with the two sources of risk W_1 and W_2.

In this paper, I consider classes of models predicting that the asset price process Q in (1) satisfies the Markov property:

$$
q(\tau) \equiv q(z(\tau), y(\tau)),
$$

where function $q(z, y) \in \mathcal{C}^{2,2}(\mathbb{Z} \times \mathbb{Y})$ (the space of continuous and twice continuously differentiable functions on $\mathbb{Z} \times \mathbb{Y}$). A simple condition ensuring the existence of such a pricing function is that

$$
\mu(\tau) \equiv \mu(z(\tau), y(\tau), \tau) = e^{-\int_0^\tau \,(z(s), y(s))ds}\, p(z(\tau), y(\tau)),
\tag{3}
$$

for some bounded positive function δ, and some positive function $p(z, y) \in \mathcal{C}^{2,2}(\mathbb{Z} \times \mathbb{Y})$. Indeed, let us define the (undiscounted) "Arrow-Debreu adjusted" asset price process as:

$$
w(z, y) \equiv p(z, y) \cdot q(z, y).
$$

By the assumed functional form of μ, and Itô's lemma, $R(\tau) \equiv R(z(\tau), y(\tau))$ and $\lambda_i(\tau) \equiv \lambda_i(z(\tau), y(\tau))$ $(i = 1, 2)$, where functions R and λ_i are given in appendix A [see eqs. (A2)]. Under usual regularity conditions, eq. (1) can then be understood as the unique Feynman-Kac stochastic representation of the solution to the following partial differential equation

$$
Lw(z, y) + f(z, y) = \delta(z, y)w(z, y), \quad \forall(z, y) \in \mathbb{Z} \times \mathbb{Y},
\tag{4}
$$

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[4]See definition 2.1 in Karatzas and Shreve (1991) (p. 285).

where $f \equiv pz$, Lw is the usual infinitesimal generator of (2): $Lw \equiv \frac{1}{2}\sigma^2 w_{zz} + m_0 w_z + \frac{1}{2}(\xi_1^2 + \xi_2^2)w_{yy} + \varphi_0 w_y + \sigma\xi_1 w_{zy}$, and subscripts denote partial derivatives. [See, for example, Huang and Pagès (1992) (thm. 3, p. 53) or Wang (1993) (lemma 1, p. 202), for a series of regularity conditions underlying the Feynman-Kac theorem in infinite horizon settings arising in typical financial applications.]

Eq. (4) can be further elaborated so as to emphasize a more familiar characterization of no-arbitrage asset prices. By the definition of R and λ_i ($i = 1, 2$) given in appendix A [eqs. (A2)], and $Lw(\tau) \equiv \frac{d}{ds}E[pq]\big|_{s=}$, one has that q is solution to:

$$Lq + z = Rq + (q_z\sigma + q_y\xi_1)\lambda_1 + q_y\xi_2\lambda_2, \quad \forall(z,y) \in \mathbb{Z} \times \mathbb{Y}. \tag{5}$$

Under regularity conditions, the Feynman-Kac representation of the solution to eq. (5) is:

$$q(z,y) = \int_0^\infty C(z,y,\tau)d\tau, \tag{6}$$

where

$$C(z,y,\tau) \equiv \mathbb{E}\left[\exp\left(-\int_0 R(z(t),y(t))dt\right) \cdot z(\tau)\bigg| z, y\right],$$

and $\mathbb{E}$ is the expectation operator taken under the risk-neutral probability P^0 (say). Finally, (z, y) are solution to

$$\begin{cases} dz(\tau) = m\left(z(\tau), y(\tau)\right)d\tau + \sigma\left(z(\tau)\right)d\widehat{W}_1(\tau) \\[2mm] dy(\tau) = \varphi(z(\tau), y(\tau))d\tau + \xi_1\left(y(\tau)\right)d\widehat{W}_1(\tau) + \xi_2\left(y(\tau)\right)d\widehat{W}_2(\tau) \end{cases} \tag{7}$$

where $\widehat{W}_1$ and $\widehat{W}_2$ are two independent P^0-Brownian motions, and m and φ are risk-adjusted drift functions defined as $m(z, y) \equiv m_0(z, y) - \sigma(z)\lambda_1(z, y)$ and $\varphi(z, y) \equiv \varphi_0(z, y) - \xi_1(y)\lambda_1(z, y) - \xi_2(y)\lambda_2(z, y)$. [See, for example, Huang and Pagès (1992) (prop. 1, p. 41) for mild regularity conditions ensuring that Girsanov's theorem holds in infinite horizon settings.]

The objective of the article is to provide general qualitative properties of the rational pricing mapping $(z, y) \mapsto q(z, y)$ under the kernel assumption (3) and the additional technical condition that q and its partial derivatives may be represented through the Feynman-Kac theorem. [Mele (2002) (appendices A, B, C) develops regularity conditions ensuring the feasibility of such a representation for a technically related problem.] In the next section, I highlight the main issues motivating such a level of analysis. In section 3, I address a feasibility question: How is it possible to pursue the objectives of this article without any knowledge of analytical solutions ? To gain insight into this feasibility issue, I will then illustrate how the theory works through a series of simple examples related to the recent literature.

2 Issues

This article singles out general properties of long-lived asset prices that can be streamlined into three categories: "monotonicity properties", "convexity properties", and "dynamic stochastic dominance properties". I now produce examples illustrating the economic content of such a categorization.

- *Monotonicity.* Consider a model predicting that $q(z, y) = z \cdot v(y)$, for some positive function $v \in \mathcal{C}^2(\mathbb{Y})$. (The remainder of this article contains many examples of this kind of models.) By Itô's lemma, asset return volatility is $\mathrm{vol}(z) + \frac{v'(y)}{v(y)}\mathrm{vol}(y)$, where $\mathrm{vol}(z) > 0$ is consumption growth volatility and $\mathrm{vol}(y)$ has a similar interpretation. As is well-known, empirical evidence suggests that actual returns volatility is too high to be explained by consumption volatility [see, e.g., Campbell (1999) for a survey]. Naturally, additional state variables may increase the overall returns volatility. In this simple example, state variable y inflates returns volatility whenever the price-dividend ratio v is increasing in y. At the same time, such a monotonicity property would ensure that asset returns volatility be strictly positive. Eventually, strictly positive volatility is one crucial condition guaranteeing that dynamic constraints of optimizing agents are well-defined.

- *Convexity*: I. Next, suppose that y is some state variable related to the business cycle conditions. Another robust stylized fact is that stock-market volatility is countercyclical [see, e.g., Schwert (1989)]. If $q(z, y) = z \cdot v(y)$ and $\mathrm{vol}(y)$ is constant, returns volatility is countercyclical whenever v is a *concave* function of y. Even in this simple example, second-order properties (or "nonlinearities") of the price-dividend ratio are critical to the understanding of time variation in returns volatility.

- *Convexity*: II. Alternatively, suppose that expected dividend growth is positively affected by a state variable g. If v is increasing and *convex* in $y \equiv g$, price-dividend ratios would typically display "overreaction" to small changes in g. The empirical relevance of this point was first recognized by Barsky and De Long (1990, 1993).[5] More recently, Veronesi (1999)

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[5]In their empirical work, Barsky and De Long considered feeding a variant of the Gordon's model (1962) with a (time-varying) estimate of the long-term dividend growth rate. Naturally, the Gordon's model is based on the assumption that the dividend's growth is constant. Nevertheless, the Barsky and De Long procedure is of great interest. It highlights the role played by a convex function in vehicling small changes in the dividend growth rate to large changes of the price-dividend ratios.

addressed similar convexity issues by means of a fully articulated equilibrium model of learning.

- *Dynamic stochastic dominance.* An old issue in financial economics is about the relation between long-lived asset prices and volatility of fundamentals [see, e.g., Malkiel (1979), Pindyck (1984), Poterba and Summers (1985), Abel (1988) and Barsky (1989)]. The traditional focus of the literature has been the link between dividend (or consumption) volatility and stock prices. Another interesting question is the relationship between the volatility of additional state variables (such as the dividend growth rate) and stock prices. In some models, volatility of these additional state variables is endogenously determined. For example, it may be inversely related to the quality of signals about the state of the economy [see, e.g., David (1997) and Veronesi (1999, 2000)]. In many other circumstances, producing a probabilistic description of y is as arbitrary as specifying the preferences of a representative agent. [In fact, y is in many cases related to the dynamic specification of agents' preferences (see section 5).] The issue is then to uncover stochastic dominance properties of dynamic pricing models where state variables are possible nontradable.

In the next section, I provide a simple characterization of the previous properties. To achieve this task, I utilize (and extend) some general ideas in the recent option pricing literature. This literature attempts to explain the qualitative behavior of a contingent claim price function $C(z, y, \tau)$ [such as the one in eq. (6)] with as few assumptions as possible on z and y. Unfortunately, some of the conceptual foundations in this literature are not well-suited to pursue the purposes of this article. As an example, many available results are based on the assumption that at least one state variable is tradable. This is not the case of the "European-type option" pricing problem (6). In section 3.1, I introduce an abstract asset pricing problem which is appropriate to our purposes. In section 3.2, I apply this framework of analysis to study basic model examples of long-lived asset prices. Finally, sections 4, 5 and 6 provide systematic extensions of the results contained in section 3.

3   A simplified version of the theory

This section provides a derivation of the theory under a series of simplifying assumptions. These assumptions are made to illustrate the salient aspects of the theory in the easiest possible way, and will be relaxed in sections 4, 5 and 6. The reader willing to access directly to both more general results and in-depth discussion can proceed to section 4 without loss of continuity.

The major insights of this section are related to the price representation in eq. (6). According to eq. (6), a long-lived asset price $q(z, y)$ is a linear functional of European-type option prices $\{C(z, y, \ell)\}_{t \geq 0}$. The main idea in this section is to analyze simple situations where general properties of long-lived asset prices can be understood through the corresponding properties of European-type option prices. In this section, I develop results addressing monotonicity and convexity properties of asset price functions. To save space, results on dynamic stochastic properties are only succinctly presented in Appendix B (see proposition B1).

3.1 A canonical pricing problem

Consider a risk-neutral environment in which a cash premium ψ is paid off at some future date T. The cash premium is a given function of $\widetilde{x} \equiv x(T)$, where $X = \{x(\tau)\}_{\in [0,T]}$ $(x(0) = x)$ is some underlying state process. If the yield curve is flat at zero, $c(x) \equiv \mathbb{E}[\psi(\widetilde{x})|x]$ is the price of the right to receive ψ. The question is: Which joint restrictions on ψ and X are needed to make c concave/convex ? Furthermore: what is the relationship between volatility of $\widetilde{x}$ and c ?

When X is a proportional process (one for which the risk-neutral distribution of $\widetilde{x}/x$ is independent of x), there are simple answers to the previous questions. Consider for example the second question. The price c is:

$$ c(x) = \mathbb{E}\left[\psi(x \cdot C)\right], \quad C \equiv \frac{\widetilde{x}}{x}, \quad x > 0. $$

As this simple formula reveals, classical second-order stochastic dominance properties [see Rothschild and Stiglitz (1970)] apply when X is proportional: c decreases (increases) after a mean-preserving spread in C whenever ψ is concave (convex) [consistently for example with the prediction of the Black and Scholes (1973) formula]. This point was first made by Jagannathan (1984) (p. 429-430). In two independent papers, Bergman, Grundy and Wiener (1996) (BGW) and El Karoui, Jeanblanc-Picqué and Shreve (1998) (EJS) generalized these results to *any* diffusion process (i.e., not necessarily a proportional process).[6,7] But one crucial assumption of these exten-

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[6] The proofs in these two articles are markedly distinct but are both based on *price function* convexity. An alternate proof directly based on *payoff function* convexity can be obtained through a direct application ofconnae

sions is that X must be the price of a traded asset that does not pay dividends. This assumption is crucial because it makes the risk-neutralized drift function of X proportional to x. As a consequence of this fact, c inherits convexity properties of ψ, as in the proportional process case. As I demonstrate below, the presence of nontradable state variables makes interesting nonlinearities emerge. As an example, proposition 1 reveals that convexity of ψ is neither a necessary nor a sufficient condition for convexity of c.[8] Furthermore, "dynamic" stochastic dominance properties are more intricate than in the classical second order stochastic dominance theory (see proposition B1 in appendix B).

To substantiate these claims, I now introduce a simple, abstract pricing problem (taken to satisfy the technical regularity conditions maintained in section 1). Once again, I emphasize that the main purpose of this problem is to address in a simple way the issues of the previous section through a simple characterization of the long-lived asset pricing problem (6) (see section 3.2).

Auxiliary pricing problem. *Let X be the (strong) solution to*:

$$dx(\tau) = b\left(x(\tau)\right) d\tau + a\left(x(\tau)\right) d\widehat{W}(\tau),$$

where $\widehat{W}$ is a P^0-Brownian motion and b, a are some given functions. Let ψ and ρ be two twice continuously differentiable positive functions, and define

$$c(x, T) \equiv \mathbb{E}\left[\exp\left(-\int_0^{\mathsf{T}} \rho(x(t))dt\right) \cdot \psi(x(T)) \,\middle|\, x\right] \tag{8}$$

to be the price of an asset which promises to pay $\psi(x(T))$ at time T.

In this pricing problem, X can be the price of a traded asset. In this case $b(x) = x\rho(x)$. If in addition, $\rho' = 0$, the problem collapses to the classical European option pricing case with constant discount rates. If instead, X is not a traded risk, $b(x) = b_0(x) - a(x)\lambda(x)$, where b_0 is the physical drift function of X and λ is a risk-premium. The previous framework then encompasses a number of additional cases. As an example, set $\psi(x) = x$. Then, one may 1) interpret X as consumption process; 2) set $c(x, \tau) = C(x, y, \tau)$ in (6); and 3) restrict the *long-lived* asset price q to be driven by consumption only. As another example, set $\psi(x) = 1$ and $\rho(x) = x$. Then,

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[8]Kijima (2002) produced a counterexample in which option price convexity may break down in the presence of convex payoff functions. His counterexample was based on an extension of the Black-Scholes model in which the underlying asset price had a concave drift function. (The source of this concavity was due to the presence of dividend issues.) Among other things, the proof of proposition 1 reveals the origins of this counterexample.

c is a zero-coupon bond price as predicted by a simple univariate short-term rate model. The importance of these specific cases will be clarified in section 3.2 and appendix B. I now turn to characterize qualitative properties of c.

Proposition 1. *The following statements are true*:

a) If $\psi' > 0$, then c is increasing in x whenever $\rho' \leq 0$. Furthermore, if $\psi' = 0$, then c is decreasing (resp. increasing) whenever $\rho' > 0$ (resp. < 0).

b) If $\psi'' \leq 0$ (resp. $\psi'' \geq 0$) and c is increasing (resp. decreasing) in x, then c is concave (resp. convex) in x whenever $b'' < 2\rho'$ (resp. $b'' > 2\rho'$) and $\rho'' \geq 0$ (resp. $\rho'' \leq 0$). Finally, if $b'' = 2\rho'$, c is concave (resp. convex) whenever $\psi'' < 0$ (resp. > 0) and $\rho'' \geq 0$ (resp. ≤ 0).

Proposition 1-a) generalizes previous monotonicity results obtained by Bergman, Grundy and Wiener (1996). By the so-called "no-crossing property" of a diffusion, X is not decreasing in its initial condition x. Therefore, c inherits the same monotonicity features of ψ if discounting does not operate adversely. While this observation is relatively simple, it explicitly allows to address monotonicity properties of long-lived asset prices (see section 3.2).

Proposition 1-b) generalizes a number of existing results on price convexity. First, assume that ρ is constant and that X is the price of a traded asset. In this case, $\rho' = b'' = 0$. The last part of proposition 1-b) then says that convexity of ψ propagates to convexity of c. This result reproduces the findings in the literature that I surveyed earlier. Proposition 1-b) characterizes option price convexity within more general contingent claims models. As an example, suppose that $\psi'' = \rho' = 0$ and that X is not a traded risk. Then, proposition 1-b) reveals that c inherits the same convexity properties of the instantaneous drift of X. As a final example, proposition 1-b) extends one (scalar) bond pricing result in Mele (2003). Precisely, let $\psi(x) = 1$ and $\rho(x) = x$; accordingly, c is the price of a zero-coupon bond as predicted by a standard short-term rate model. By proposition 1-b), c is convex in x whenever $b''(x) < 2$ for all x. This corresponds to eq. (8) (p. 688) in Mele (2003).[9] In analyzing properties of long-lived asset prices, both discounting and drift nonlinearities play a prominent role. For the purpose of this paper, I therefore need the more general statements contained in proposition 1-b).

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[9] In appendix B, I have developed further intuition on this bounding number.

3.2 Applications to long-lived assets

Models in which long-lived asset prices are driven by only one state variable fail to explain the actual characteristics of aggregate stock-market behavior. The simplest multidimensional extensions consist in randomizing 1) the average consumption growth rate and 2) the Sharpe ratio. In section 3.2.1, I explore theoretical properties of models addressing the first extension. Properties of models with time varying Sharpe ratios are investigated in section 3.2.2. Sections 4 and 5 contain general versions of the simple predictions in sections 3.2.1 and 3.2.2.

3.2.1 Stochastic profitability growth

The first model of this section is a simple extension of the basic geometric Brownian motion model. Precisely, consider an economy in which the instantaneous rate of consumption Z satisfies

$$
\begin{cases}
\dfrac{dz(\tau)}{z(\tau)} = [g(\tau) - \sigma_0 \lambda]\, d\tau + \sigma_0 d\widehat{W}_1(\tau) \\[2em]
dg(\tau) = \varphi\left(g(\tau)\right) d\tau + \xi_1\left(g(\tau)\right) d\widehat{W}_1(\tau) + \xi_2\left(g(\tau)\right) d\widehat{W}_2
\end{cases}
\tag{9}
$$

where $\widehat{W}_i$ $(i = 1, 2)$ are two independent P^0-Brownian motions, and σ_0, λ are positive constants. This model is a special case of system (7) [notably, for $m(z, g) = z(g - \sigma_0 \lambda)$]. Accordingly, I interpret λ as a risk-premium coefficient and $G = \{g(\tau)\}_{\geq 0}$ as a stochastic consumption growth rate. In this model, agents may be unable to observe G. But I initially assume that G is measurable with respect to the agents' information set. To simplify the exposition, I assume that the short-term rate $R = r$, a constant.

To compute the long-lived price function $q(z, g)$, I utilize the representation in eq. (6). The result is that the price-dividend ratio $v(g) \equiv q(z, g)/z$ satisfies:

$$
v(g) = \int_0^\infty B(g, \tau) d\tau,
\tag{10}
$$

where

$$
\begin{aligned}
B(g, \tau) &\equiv \mathbb{E}\left[\beta(\tau) \cdot \exp\left(-\int_0^\tau (r - g(u) + \sigma_0 \lambda) du\right)\,\middle|\, g\right] \\[1em]
&= \overline{\mathbb{E}}\left[\exp\left(-\int_0^\tau (r - g(u) + \sigma_0 \lambda) du\right)\,\middle|\, g\right].
\end{aligned}
\tag{11}
$$

13

Here $\beta(\tau) \equiv \exp(-\frac{1}{2}\sigma_0^2\tau + \sigma_0\widehat{W}_1(\tau))$, $\overline{\mathbb{E}}$ is the expectation operator taken with respect to a new probability measure $\overline{P}$ (say), and g is solution to:

$$dg(\tau) = [\varphi(g(\tau)) + \sigma_0\xi_1(g(\tau))]\,d\tau + \xi_1(g(\tau))\,d\overline{W}_1(\tau) + \xi_2(g(\tau))\,d\overline{W}_2(\tau),$$

where $\overline{W}_1(\tau) = \widehat{W}_1(\tau) - \sigma_0\tau$ is a Brownian motion under $\overline{P}$, and $\overline{W}_2 = \widehat{W}_2$. Put another way, this model predicts that function C in eq. (6) is given by $C(z, g, \tau) = z \cdot B(g, \tau)$. Properties of v can therefore be understood through the corresponding properties of B in eq. (11).

First, consider the simple case in which G is constant. In this case, eq. (10) reduces to Gordon's (1962) formula. This formula predicts that the price-dividend ratio v is increasing and convex in g. Does a similar property hold when G is a random process ? This question is of fundamental importance as it is related to the overreaction issue highlighted by Barsky and De Long (1990, 1993) and overviewed in section 2.

Surprisingly, the answer to the previous question is neat. The price-dividend ratio v is always increasing in g; furthermore,

$$v''(g) > 0 \quad \text{whenever} \quad \varphi_0''(g) + (\sigma_0 - \lambda)\,\xi_1''(g) > -2 \quad \text{for all } g \in \mathbb{G}. \tag{12}$$

This result is a special case of propositions 2 and 3 in section 4. To demonstrate it here, I recognize B as a special case of the canonical pricing problem introduced in section 3.1 (precisely, B is a bond pricing function). The previous theoretical conditions then follow by a direct application of proposition 1. Specifically, monotonicity properties ($v' > 0$) follow by the "no-crossing" property of a diffusion. Convexity properties follow by proposition 1-b). As we will see in section 4, both properties may fail to hold if R is a function of g (see proposition 3 and example 1).

The previous theoretical test condition imposes a joint restriction on both the law of motion of the state variable g (φ_0 and ξ_1) and degrees of risk-aversion (λ). Suppose for example that φ_0 and ξ_1 are both linear functions. Then, eq. (12) implies that the price-dividend ratio v is *always* convex (i.e. independently of risk-aversion). As a second example, suppose that $\varphi_0'' = 0$. Then, eq. (12) tells us that v is convex whenever ξ_1 is concave *and* risk-aversion is sufficiently high. As it turns out, ξ_1 is nonconvex in many economies with partially observed state variables and learning mechanisms [see, e.g., Brennan and Xia (2001) and Veronesi (1999)]. Eq. (12) then formally describes how the effects of such learning mechanisms impinge upon the equilibrium price process. This is the major insight of the present subsection. For completeness, in appendix B I have illustrated the mechanism through which learning leads to nonconvexities of ξ_1 in a simple example (see example B2).

14

3.2.2   Time-varying discount rates

I now analyze the main mechanism linking asset prices variations to random fluctuations in Sharpe ratios. I consider a simple model in which (risk-neutralized) consumption Z solves

$$
\begin{cases}
\dfrac{dz(\tau)}{z(\tau)} = [g_0 - \sigma_0 \lambda(s(\tau))]\, d\tau + \sigma_0 d\widehat{W}(\tau) \\[2ex]
ds(\tau) = [\phi\left(s(\tau)\right) - \xi\left(s(\tau)\right)\lambda(s(\tau))]\, d\tau + \xi\left(s(\tau)\right) d\widehat{W}(\tau)
\end{cases}
\tag{13}
$$

where $\widehat{W}$ is a P^0-Brownian motion; g_0, σ_0 are positive constants; ϕ, ξ are given functions; and λ is the Sharpe ratio (or unit risk-premium).

Time-varying Sharpe ratios arise naturally in economies where agents have preferences non-separable in time [see, e.g., Campbell and Cochrane (1999)]. They also arise in certain incomplete markets economies [see Basak and Cuoco (1998)]. Section 5 develops theoretical test conditions that can be used to predict the behavior of all models arising within such economies. In these economies, the short-term rate R is a function of the state (z, s). To simplify the exposition of this section, I assume that R is a constant r. More general results are established in section 5.

In system (13), state variable $S = \{s(\tau)\}_{\geq 0}$ drives variations in the Sharpe ratio λ. In many cases of interest, it represents a state variable tracking the business cycle conditions (see section 5, example 3 and 4). In the same cases, the functional form of λ is easily deduced from first principles (see, e.g., example A1 in appendix A) - as an example, all models examples in section 5 predict that λ is decreasing in s. On the other hand, the functional form of both ϕ and ξ is typically "variation-free", i.e. it is *not restricted* by standard asset pricing theories.

In this article, I develop joint restrictions on ϕ, ξ and λ that are consistent with properties of the pricing function $q(z, s)$ that are given in advance. As an example, it is well-known that stock-market volatility is countercyclical (see section 2). By Itô's lemma, volatility of q is "countercyclical" whenever ξ is constant and q is a concave function of s. But how can we ensure that q is concave in s in this and more complex situations (with possible non constant ξ) ? The conditions in this section explicitly address this issue.

Similarly as in section 3.2.1, the starting point is to compute $q(z, s)$ through the evaluation formula in eq. (6). If Z is solution to (13), then $q(z, s) = z \cdot v(s)$ w

In the previous formula, $\overline{\mathbb{E}}$ is the expectation operator taken under a new measure $\overline{P}$, and S is solution to

$$ds(\tau) = \{\phi\left(s(\tau)\right) - [\lambda(s(\tau)) - \sigma_0] \cdot \xi(s(\tau))\}\, d\tau + \xi\left(s(\tau)\right) d\overline{W}(\tau),$$

where $\overline{W}$ is a $\overline{P}$-Brownian motion.[10]

According to eq. (14), the price-dividend ratio v is a linear functional of bond prices $\{B(s,\ell)\}_{t \geq 0}$ in a fictitious economy where the short-term rate is given by $\rho(s) \equiv r - g_0 + \sigma_0 \lambda(s)$. Furthermore, function B in (14) is a special case of the canonical pricing problem in section 3.1 (namely for $X \equiv S$ and $\psi \equiv 1$). Therefore, general properties of v in (14) may be deduced through an application of proposition 1 to function B.

Monotonicity properties are straightforward. By proposition 1-a), B is increasing in s whenever λ is decreasing in s. Convexity properties of v can be deduced through an application of proposition 1-b). Precisely, B is concave in s whenever $[\phi\left(s\right) + \sigma_0\xi(s) - \xi\left(s\right)\lambda(s)]'' < 2\rho'(s)$ and $\rho''(s) > 0$, all $s \in \mathbb{S}$. By using the definition of ρ, and by rearranging terms, I then arrive at the following theoretical test condition. Suppose that $\lambda' < 0$. Then, v is *concave* if

$$\forall s \in \mathbb{S}, \quad \lambda''(s) > 0 \quad \text{and} \quad [\phi\left(s\right) + \sigma_0\xi(s) - \xi\left(s\right)\lambda(s)]'' - 2\sigma_0\lambda'(s) < 0. \tag{15}$$

The previous condition is a special case of proposition 4 in section 5. It imposes a natural lower bound on convexity of the Sharpe ratio λ. This lower bound can be understood heuristically as follows. Suppose that $\phi = \xi = 0$. The price-dividend ratio is then as predicted by the standard Gordon's (1962) model, viz $v(s) = (r - g + \sigma_0\lambda(s))^{-1}$, where s is the (constant) value of S. In this case, the theoretical test condition (15) collapses to $\lambda' > 0$ and $\lambda'' > 0$. That is, convexity of λ translates to concavity of v in a natural way whenever $\lambda' > 0$. The condition that $\lambda' > 0$ is of course very tight. As condition (15) reveals, randomizing S makes the model gain in increased flexibility through the additional (nonzero) terms ϕ and ξ. Section 5 contains additional details on the link between convexity of Sharpe ratios and concavity of price-dividend ratios.

4 Stochastic consumption growth

This section develops general properties of the rational pricing function $q(z, y)$ introduced in section 1. These properties isolate the effects of random changes in average profitability. To

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[10]Such an additional change of measure arises because Z and S are correlated, and it is justified by the same arguments leading to eq. (11) in section 3.2.1.

emphasize this fact, I set $G \equiv Y$ in system (2). I then consider the following restrictions:

$$\forall (z,g) \in \mathbb{Z} \times \mathbb{G}, \quad \frac{\partial m_0(z,g)}{\partial g} \neq 0 \quad \text{and} \quad \frac{\partial \lambda_i(z,g)}{\partial g} = 0, \quad i = 1,2.$$

Models in which Sharpe ratios are driven by additional state variables are analyzed in section 5. Section 6 considers higher dimensional extensions (with fully interacting state variables) encompassing both models of this and the next section. In this and the next section, I disentangle the effects of random changes in average profitability from the effects of random changes in Sharpe ratios. This helps to develop intuition on the functioning of the more complex model in section 6. I now provide examples of models that are special cases of the framework covered in this section.

Example 1. [Veronesi (1999, 2000)]. Consider an infinite horizon economy in which a representative agent observes realizations of Z generated by:

$$dz(\tau) = \theta d\tau + \sigma_0 dw_1(\tau), \tag{16}$$

where w_1 is a Brownian motion, and θ is a two-states $(\overline{\theta}, \underline{\theta})$ Markov chain. θ is unobserved, and the agent implements a Bayesian procedure to learn whether she lives in the "good" state $\overline{\theta} > \underline{\theta}$. The equilibrium price of this economy is isomorphic to the equilibrium price of an economy in which (Z, G) are solution to:

$$\begin{cases} dz(\tau) = \left[g(\tau) - \gamma \sigma_0^2\right] d\tau + \sigma_0 d\widehat{W}_1(\tau) \\ \\ dg(\tau) = \left[k(\overline{g} - g(\tau)) - \gamma \sigma_0 \xi_1\left(g(\tau)\right)\right] d\tau + \xi_1\left(g(\tau)\right) d\widehat{W}_1(\tau) \end{cases}$$

where $\widehat{W}_1$ is a P^0-Brownian motion, $\xi_1(g) = (\overline{\theta} - g)(g - \underline{\theta})/\sigma_0$, $k, \overline{g}$ are some positive constants, and γ is the agent's CARA. (See example B2 in appendix B for heuristic details on such an isomorphism and filtering results for a simpler problem.) A related model is one in which Z is solution to:

$$\frac{dz(\tau)}{z(\tau)} = \theta d\tau + \sigma_0 dw_1(\tau), \tag{17}$$

and the agent receives additional signals $A = \{a(\tau)\}_{>0}$ about θ satisfying:

$$da(\tau) = \theta d\tau + \sigma_1 dw_2(\tau),$$

where w_2 is a Brownian motion independent of w_1. Similarly as for model (16), the nonarbitrage price of this economy is isomorphic to the nonarbitrage price of an economy in which

(Z, G) are solution to eq. (2), with $m_0(z, g) = gz$, $\sigma(z) = \sigma_0 z$, $\varphi_0(z, g) = p(\overline{g} - g)$, $\xi_1(z, g) = (\overline{\theta} - g)(g - \underline{\theta})/\sigma_0$, $\xi_2(z, g) = -\frac{\sigma_0}{\sigma_1}\xi_1(z, g)$ and $p, \overline{g}$ are some positive constants.[11]

Example 2. [Brennan and Xia (2001)]. A single infinitely lived agent observes Z, where Z is solution to:

$$\frac{dz(\tau)}{z(\tau)} = \widehat{g}(\tau)d\tau + \sigma_0 dw_1(\tau).$$

Similarly as in example 1, $\widehat{G} = \{\widehat{g}(\tau)\}_{>0}$ is unobserved. Unlike example 1, $\widehat{G}$ does not evolve on a countable number of states. Rather, it follows an Ornstein-Uhlenbeck process:

$$d\widehat{g}(\tau) = k(\overline{g} - \widehat{g}(\tau))d\tau + \sigma_1 dw_1(\tau) + \sigma_2 dw_2(\tau)$$

where $\overline{g}$, σ_1 and σ_2 are positive constants. The agent implements a learning procedure similar as in example 1. If she has a Gaussian prior on $\widehat{g}(0)$ with variance γ_*^2 (defined below), the nonarbitrage price takes the form $q(z, g)$, where (Z, G) are now solution to eq. (2), with $m_0(z, g) = gz$, $\sigma(z) = \sigma_0 z$, $\varphi_0(z, g) = k(\overline{g} - g)$, $\xi_2 = 0$, and $\xi_1 \equiv \xi_1(\gamma_*) = (\sigma_1 + \frac{1}{\sigma_0}\gamma_*)^2$, where γ_* is the positive solution to $\xi_1(\gamma) = \sigma_1^2 + \sigma_2^2 - 2k\gamma$.[12]

The models in the previous examples share the same basic economic motivation. Yet they make different assumptions on the probabilistic structure of the unobserved consumption growth rate. Do these two assumptions entail similar asset pricing implications ? More generally, which minimal assumptions must *any* two "stochastic consumption growth" models share in order to display comparable pricing properties ? Clearly, examples 1 and 2 only contain two possible

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[11] The formal structure of the Markov chain in the two models is slightly different. In Veronesi's (1999) model (16), θ switches from the good state $\overline{\theta}$ to the bad state $\underline{\theta}$ with probability $p_1 d\tau$ (resp. θ switches from the bad state $\underline{\theta}$ to the good state $\overline{\theta}$ with probability $p_2 d\tau$) over any infinitesimal amount of time, and $k = p_1 + p_2$, $\overline{g} = \pi\overline{\theta} + (1 - \pi)\underline{\theta}$, $\pi = p_2/(p_1 + p_2)$. In a simplified version of Veronesi's (2000) model, there is a probability $p d\tau$ that over any infinitesimal amount of time $d\tau$, new values of θ in (17) are drawn ($\overline{\theta}$ with probability f and $\underline{\theta}$ with probability $1 - f$, and $\underline{\theta} < \overline{\theta}$), and $\overline{g} = f\overline{\theta} + (1 - f)\underline{\theta}$.

[12] In their article, Brennan and Xia considered a slightly more general model in which consumption and dividends differ. They obtain a reduced-form model which is identical to the one in this example. In the calibrated model, Brennan and Xia found that the variance of the filtered $\widehat{g}$ is higher than the variance of the expected dividend growth in an economy with complete information. The results on γ^* in this example can be obtained through an application of theorem 12.1 in Liptser and Shiryaev (2001) (Vol. II, p. 22). They generalize results in Gennotte (1986) and are a special case of results in Detemple (1986). Both Gennotte and Detemple did not emphasize the impact of learning on the pricing function.

kinds of models with incomplete information and learning mechanisms.[13] Furthermore, models making expected consumption another *observed* diffusion may have an interest in their own [see Campbell (1999); and examples C1 and C2 in appendix C]. In this case, there might be no practical guidance as to how to choose a dynamic model of expected consumption changes. The theory of this section provides coverage to all such models, and allows one to gauge the implications of primitive assumptions on the form of the asset price function.

In describing the theory, I will make two simplifications. The first simplification is achieved by assuming that the short-term interest rate R is constant. Such an assumption isolates interesting phenomena in a neat way, and is relaxed in appendix A (see lemma A1). The resulting predictions of the theory are contained in proposition 2. Proposition 3 relaxes the assumption that R is constant, but restricts the general theory to situations where price-dividend ratios are independent of z. The reader interested in the general theory is referred to lemma A1 in appendix A.

The most basic properties of q that one may wish to isolate regard monotonicity (with respect to both g and z) and "overreaction" to changes in g (i.e. convexity of the price function with respect to g). Monotonicity properties are ensured by relatively simple restrictions. Overreaction is a more complex phenomenon. In sections 2 and 3.2.1, I provided a heuristic introduction to this topic. I now develop more technical details. Precisely, in the appendix I show that the second partial q_{gg} is solution to the following partial differential equation

$$0 = (L - k(z,g))q_{\mathsf{gg}}(z,g) + h(z,g), \quad \forall (z,g) \in \mathbb{Z} \times \mathbb{G},$$

where L is a partial differential operator defined in appendix A (see lemma A1), k is also given in appendix A, and finally,

$$h(z,g) \equiv m_{22}(z,g)q_{\mathsf{z}}(z,g) + \varphi_{22}(z,g)q_{\mathsf{g}}(z,g) + \left[2m_2(z,g) + \frac{\partial^2}{\partial g^2}((\sigma\xi_1)(z,g))\right]q_{\mathsf{zg}}(z,g), \quad (18)$$

with $m(z,g) \equiv m_0(z,g) - \lambda_1(z,g)\sigma(z)$, $\varphi(z,g) \equiv \varphi_0(z,g) - \lambda_1(z,g)\xi_1(g) - \lambda_2(z,g)\xi_2(g)$. By an application of the Feynman-Kac representation theorem, we have that the sign of q_{gg} is inherited by the sign of h. This insight justifies the last statement in the following proposition.

---

[13]The literature on continuous time models with incomplete information and Bayesian learning mechanisms is vast. It was initiated by Detemple (1986) and Gennotte (1986). David (1997) proposed the first model with unobservable processes living on a countable number of states. Veronesi (1999, 2000) and Brennan and Xia (2001) developed the first models analyzing the pricing function implications of learning phenomena. These last papers contain additional references to this topic. Models with incomplete information are quite distinct from models with asymmetric information such as the one developed by Wang (1993). Models with asymmetric information are so complex that they can only be treated at the cost of simplifying assumptions on the primitives. In turn, these simplifications often imply that the resulting price functions are only linear in the state variables.

Proposition 2. *Assume that for all $(z, g) \in \mathbb{Z} \times \mathbb{G}$, $\varphi_1(z, g) = 0$ and that R is constant; then, the price function $q(z, g)$ is increasing in z. If in addition $\partial^2((\sigma\xi_1)(z, g))/\partial z^2 = 0$, $q(z, g)$ is concave (resp. convex) in z whenever $m_{11}(z, g) < 0$ (resp. > 0) for all $(z, g) \in \mathbb{Z} \times \mathbb{G}$. Furthermore, if $q(z, g)$ is increasing in z, it is increasing (resp. decreasing) in g whenever $m_2(z, g) > 0$ (resp. < 0) for all $(z, g) \in \mathbb{Z} \times \mathbb{G}$. Finally, $q(z, g)$ is convex (resp. concave) in g whenever $h(z, g) > 0$ (resp. < 0) in (18) for all $(z, g) \in \mathbb{Z} \times \mathbb{G}$.*

I shall henceforth assume that state variable G positively affects average profitability; that is, $m_2 > 0$. In this case, monotonicity of q with respect to *both* z and g holds whenever $\varphi_1 = 0$. [In appendix C, I have developed a less stringent technical condition ensuring that $q_z > 0$; see eq. (C1).] Examples of models predicting that $\varphi_1 = 0$ naturally arise within infinite horizon economies with complete markets. Assume, for instance, that $\xi_2 = 0$ and that

$$\varphi(z, g) \equiv \varphi_0(g) - \xi_1(g)\lambda(z).$$

In this case, $\varphi_1 = 0$ whenever λ is independent of z.[14]

To fix ideas, I now assume that $\varphi_1 = 0$. Proposition 2 may then be used to understand second order properties of the pricing function $q(z, g)$ in many interesting cases. Consider, first, convexity properties *with respect to z*. Proposition 2 predicts that if the covariance between Z and G is at most linear in z, the price function q inherits the same qualitative features of the risk-neutralized drift function of Z. This is a generalization of a result found and discussed at length in section 3.1 (see, also, corollary B2 in appendix B for further insights). It implies that $q(z, g) \equiv v_c(g) + v(g)z$ whenever

$$\left| \frac{\partial^2 m(z, g)}{\partial z^2} \right| + \left| \frac{\partial^2 \sigma(z)}{\partial z^2} \right| = 0, \quad \forall (z, g) \in \mathbb{Z} \times \mathbb{G}. \tag{19}$$

Finally, proposition 2 contains general predictions on convexity properties of the price function q *with respect to g*. As an example, consider an economy in which consumption is a proportional process. In this case, condition (19) is satisfied. Since the interest rate is constant, $v_c = 0$ by eq. (6). Therefore, function h in (18) is: $h(z, g) = \left[(\sigma_0 - \lambda)\xi_1''(g) + \varphi_0''(g) + 2 \right] v'(g)z$. By proposition 2, "overreaction" of asset prices then occurs whenever the Sharpe ratio λ and the other primitives of the economy satisfy the following joint restriction:

$$\text{for all } g \in \mathbb{G}, \quad (\sigma_0 - \lambda)\xi_1''(g) + \varphi_0''(g) + 2 > 0.$$

---

[14] The risk-premium λ is independent of z in all complete markets economies in which either $\sigma(z) = \sigma_0 z$ and a representative agent has CRRA [as in example 1, model (17)]; or $\sigma(z) = \sigma_0$ and a representative agent has CARA [as in example 1, model (16)].

[See eq. (12) in section 3.2.1 for an alternative derivation.] If risk-premia are sufficiently high and ξ is nonconvex, the previous inequality holds whenever $\varphi_0'' > -2$ (see example B2 in appendix B for an illustration of the origins of nonconvexities of ξ_1 in models with learning mechanisms).[15]

How do predictions change when the assumption of constant interest rates is dropped ? As an example, in appendix A (lemma A1), I have shown that $q(z, g)$ is convex in g if

$$
\begin{aligned}
&\text{for all } (z, g) \in \mathbb{Z} \times \mathbb{G}, \quad \overline{h}(z, g) \equiv m_{22}(z, g)q_z(z, g) - R_{22}(z, g)q(z, g) \\
&+ \left[\varphi_{22}(z, g) - 2R_2(z, g)\right]q_g(z, g) + \left[2m_2(z, g) + \tfrac{2}{g^2}((\sigma\xi_1)(z, g))\right]q_{zg}(z, g) \quad > 0.
\end{aligned}
\tag{20}
$$

To simplify the exposition, I now develop qualitative properties of price-dividend ratios in models predicting them to be independent of z.

We have:

Proposition 3. *Suppose that $q(z, g) = z \cdot v(g)$ for some positive function v. Then $v' > 0$ (resp. < 0) if $zR_2(z, g) < m_2(z, g)$ (resp. $zR_2(z, g) > m_2(z, g)$). Furthermore, suppose that $v' > 0$; then v is convex (resp. concave) if both $m_{22}(z, g) - R_{22}(z, g)z > 0$ (resp. < 0) and $\varphi_{22}(z, g)z + 2\left[m_2(z, g) - zR_2(z, g)\right] + \tfrac{2}{g^2}(\sigma\xi_1((z, g))) > 0$ (resp. < 0).*

The pricing function $q(z, g)$ takes the form assumed in proposition 3 whenever condition (19) holds and both the short-term rate R and the coefficients of G are independent of z. The proof of this statement follows heuristically from eq. (6) - and it can be made rigorous through an elaboration of lemma A1 in appendix A. In this case, function $\overline{h}(z, g)$ in eq. (20) collapses to:

$$
\begin{aligned}
\overline{h}(z, g) =\ & \left\{m_{22}(z, g) - R_{22}(z, g)z\right\}v(g) \\
& + \left\{\varphi_{22}(z, g)z + 2\left[m_2(z, g) - R_2(z, g)z\right] + \frac{\partial^2}{\partial g^2}((\sigma\xi_1)(z, g))\right\}v'(g).
\end{aligned}
$$

The second part of proposition 3 immediately follows. As proposition 2 revealed, the pricing function partially inherits convexity properties of the risk-neutralized drift function of the state variables. Proposition 3 now reveals that the same convexity effects may be compensated by

---

[15] For instance, let η denote the representative agent's CRRA in example 1 [model (17)]; then, $h(z, g) = 2\eta v'(g)z$ in this example. Furthermore, it is easily seen that $q_{zg} = 0$ in model (16) (see, e.g., example B2 in appendix B; or apply lemma A1 in appendix A). It follows that $h(z, g) = -\gamma\sigma_0\xi_1''(g)q_g(z, g) = 2\gamma\sigma_0 q_g(z, g)$. In both models of example 1, risk aversion and concavity of ξ_1 come exactly as needed to make prices convex in g. Finally, $h(z, g) = 2v'(g)z$ in example 2; that is, prices are *always* convex in g in this example - i.e. independently of risk-aversion.

second-order properties of the short-term rate. Even when $m_{22} = R_{22} = 0$, the short-term rate can destroy the convexity properties in proposition 2, and make asset prices linear in g. As an example, this phenomenon occurs with model (17) and in appendix C, I show that a similar phenomenon may take place with model (16). Additionally, time-varying interest rates may induce price-dividend ratios to be decreasing in g! According to proposition 2, this happens whenever $zR_2 > m_2$. For example, in both model (17) and example 2, $v' < 0$ whenever $\eta > 1$ [see, also, Veronesi (2000, lemma 3(a)) for a related result]. In appendix C, I have provided further examples illustrating the theoretical test conditions in proposition 3.

Propositions 2 and 3 impose restrictions on the joint dynamics of expected returns, returns volatility and changes in g. Consider, for example, proposition 3, and set $\sigma(z) \equiv \sigma_0 z$, where σ_0 is some constant. To save space, I only consider the case $\xi_2 = 0$, and set $\xi \equiv \xi_1$ and $\lambda \equiv \lambda_1$. Excess returns volatility is then

$$\mathcal{V}(g) \equiv \sigma_0 + \frac{v'(g)}{v(g)} \xi(g),$$

and expected (percentage) returns are given by $\lambda(z)\mathcal{V}(g)$. Returns volatility is negatively related to g whenever function $\omega(z,g) \equiv v'(g)\xi(g)$ is positive and decreasing in g. In all models predicting that $v' > 0$, ω is decreasing in g for sufficiently high levels of g whenever ξ is $\cap$-shaped (as in the learning models in example 1). If on the contrary ξ is nondecreasing, time-varying expected dividend growth may now induce a *positive* relation between expected (non-percentage) returns and price-dividend ratios whenever price-dividend ratios are convex in g.[16] Menzly, Santos and Veronesi (2004) have recently demonstrated that should such a property occur in multidimensional settings, price-dividend ratios would then be weak predictors of future dividend growth - a well-known empirical feature [see, e.g., Campbell and Shiller (1988)]. The theory in this section isolates precise conditions under which price-dividend ratios are convex. In section 6, I develop its multidimensional extensions in which Sharpe ratios and interest rates may be driven by additional state variables.

5 Time-varying Sharpe ratios

This section develops a theory analyzing the joint behavior of time-varying discount rates, asset returns and volatility. I consider models in which Sharpe ratios are driven by state variables that

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[16] While the previous statement is always true when applied to expected returns, in the appendix I conjecture that expected percentage returns are increasing in g whenever the curvature of the stochastic flow of G under a certain probability measure is positive (see appendix C, conjecture C1).

are only *indirectly* related to total consumption. To isolate the effects of time-varying Sharpe ratios on asset prices, I assume that total consumption Z is generated by a simple geometric Brownian motion

$$\frac{dz(\tau)}{z(\tau)} = g_0 d\tau + \sigma_0 dW_1(\tau), \tag{21}$$

where g_0 and σ_0 are positive constants. The unit-risk premia λ_i are then taken to satisfy the following conditions:

$$\forall (z,y) \in \mathbb{Z} \times \mathbb{Y}, \quad \frac{\partial \lambda_i(z,y)}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \lambda_i(z,y)}{\partial y} \neq 0, \quad i = 1, 2.$$

Further, I simplify the presentation and I set $\xi_2 = 0$ (and hence, $\lambda_2 = 0$) and define $W \equiv W_1$, $\xi \equiv \xi_1$ and $\lambda \equiv \lambda_1$. General results are in appendix D. In many models satisfying the previous restrictions, Y is some state variable tracking the business cycle conditions and $\partial \lambda / \partial y < 0$ (see examples 3 and 4 below). To distinguish the class of models studied in this section from the one analyzed in section 4, I set $S \equiv Y$. Therefore, S is assumed to be the (strong) solution to:

$$ds(\tau) = \phi(s(\tau))d\tau + \xi(s(\tau))dW(\tau),$$

where ϕ and ξ are functions guaranteeing the existence of a strong solution. The problem analyzed in this section is: Which general restrictions do we have to impose to λ, R, ϕ and ξ to make the rational price function $q(z,s)$ exhibit some general properties given in advance ? I now provide examples of models covered by the framework of this section.

Example 3. [Campbell and Cochrane (1999)]. Consider an infinite horizon, complete markets economy in which the representative agent has (undiscounted) instantaneous utility given by $u(c,x) = [(c-x)^{1-} - 1]/(1 - \eta)$, where c is consumption and x is a (time-varying) habit, or (exogenous) "subsistence level". In equilibrium $C = Z$. Let $s \equiv (z-x)/z$ (the "surplus consumption ratio"). By assumption, $S = \{s(\tau)\}_{\geq 0}$ is solution to:

$$ds(\tau) = s(\tau)\left[(1-\phi)(\overline{s} - \log s(\tau)) + \frac{1}{2}\sigma_0^2 l(s(\tau))^2\right]d\tau + \sigma_0 s(\tau)l(s(\tau))dW(\tau), \tag{22}$$

where l is a positive function given in appendix D. The Sharpe ratio predicted by the model is:

$$\lambda(s) = \eta \sigma_0 \left[1 + l(s)\right]$$

(see appendix D for additional details).[17]

---

[17] Chan and Kogan (2002) have proposed an alternative external habit model with "catching up with the Joneses". In their model, the "standard of living of others" is a process with *bounded* variation and the Sharpe ratio is driven by a procyclical state variable through nonlinearities induced by agents heterogeneity.

Time variation in the Sharpe ratio may also arise in economies where agents have classical preferences but may face an incomplete market structure. In these cases, Sharpe ratios are typically driven by state variables positively related to the utility of market participants.

Example 4. [Basak and Cuoco (1998)]. Two infinitely lived agents a and b have instantaneous (undiscounted) utility $u_{\mathsf{a}}(c) = (c^{1-} - 1)/(1 - \eta)$ and $u_{\mathsf{b}}(c) = \log c$. Only agent a invests in the stock market. While the competitive equilibrium is generically Pareto inefficient, agents' aggregation is still possible in this model. Let $\{\widehat{c}_{\mathsf{i}}(\tau)\}_{\geq 0}$ be the general equilibrium allocation process of agent i $(i = a, b)$. The agents' first order conditions are $u'_{\mathsf{a}}(\widehat{c}_{\mathsf{a}}(\tau)) = w_{\mathsf{a}} e^{\mu(\tau)}$ and $\widehat{c}_{\mathsf{b}}(\tau)^{-1} = w_{\mathsf{b}} e^{-\int_0^\tau \mathsf{R}(\mathsf{s})\mathsf{ds}}$, where μ is the pricing kernel process defined in section 1, and $w_{\mathsf{a}}, w_{\mathsf{b}}$ are two constants. Let $u(z, x) \equiv \max_{c_a + c_b = z}[u_{\mathsf{a}}(c_{\mathsf{a}}) + x \cdot u_{\mathsf{b}}(c_{\mathsf{b}})]$, where $x \equiv u'_{\mathsf{a}}(\widehat{c}_{\mathsf{a}})/u'_{\mathsf{b}}(\widehat{c}_{\mathsf{b}}) = u'_{\mathsf{a}}(\widehat{c}_{\mathsf{a}})\widehat{c}_{\mathsf{b}}$ is a *stochastic* social weight. By the definition of μ (see section 1), X is solution to

$$dx(\tau) = -x(\tau)\lambda(\tau)dW(\tau),$$

where λ is the Sharpe ratio. Then, the equilibrium price system in this economy is supported by a fictitious representative agent with utility $u(z, x)$.[18] The Sharpe ratio takes the following form:

$$\lambda(s) = \eta \sigma_0 s^{-1},$$

where now $s \equiv \widehat{c}_{\mathsf{a}}/z$ (see appendix D). Appendix D also provides the functional form of drift and diffusions of state variable S and interest rates in this example.

Qualitative properties of the models in the previous examples should depend critically on the assumptions made as regards the primitives of the economy. For example, Campbell and Cochrane assumed that function l in (22) is positive, decreasing and convex over the relevant range of variation of S. Remarkably, their model makes the intriguing predictions that price-dividend ratio are concave in s, and that expected returns and stock-market volatility are both countercyclical. Yet what is the precise mechanism linking convexity of Sharpe ratios, concavity of price-dividend ratios and countercyclical risk-premia and volatility ? The following proposition provides a theory addressing this question in great generality.

---

[18]Theorem 1 in Basak and Cuoco (1998) (p. 321) contains the rigorous statement of this result.

Proposition 4. *The rational price function $q(z,s)$ is given by $q(z,s) = z \cdot v(s)$, where v is a positive function satisfying the following properties*:

a) Suppose that $\forall s \in \mathbb{S}$, $R'(s) + \sigma_0 \lambda'(s) < 0$ (resp. > 0). Then, v is increasing (resp. decreasing).

b) Assume that v is increasing, and that $\forall s \in \mathbb{S}$, $R''(s) + \sigma_0 \lambda''(s) > 0$ (resp. ≤ 0) and $G(s) \equiv (\phi(s) - \lambda(s)\xi(s))'' + \sigma_0 (\xi''(s) - 2\lambda'(s)) - 2R'(s) < 0$ (resp. > 0). Then, v is concave (resp. convex).

It is useful (but not compulsory) to think of S as a state variable related to business cycle conditions that are relevant to stock-market participants - just as in the previous examples 3 and 4. Proposition 4-a) then formalizes a simple idea about discount rates $R + \sigma_0 \lambda$: If discount rates are countercyclical, price-dividend ratios are *automatically* procyclical. As is well-known, economic theory is ambiguous about the sign of R'. But as proposition 4-a) indicates, models making short-term rates R "too" procyclical may also entail counterfactual consequences (namely, countercyclical price-dividend ratios).

Proposition 4-b) contains a *second-order* analysis of the setting analyzed in this section. Similarly as in section 4, define expected (excess) returns ($\mathcal{E}$) and returns volatility ($\mathcal{V}$) as:

$$\mathcal{E}(s) \equiv \mathcal{V}(s) \cdot \lambda(s) \quad \text{and} \quad \mathcal{V}(s) \equiv \sigma_0 + \frac{v'(s)}{v(s)}\xi(s).$$

In these models, concavity of the price-dividend ratio v plays a critical role in explaining cyclical properties of both volatility and risk-premia. As an example, if ξ is constant, $\lambda' < 0$ and v is concave, then $\mathcal{V}$ and $\mathcal{E}$ are both countercyclical. The simple intuition behind this effect is that returns volatility increases on the downside when price-dividend ratios are concave in the variables related to business cycle conditions.

When is v concave then ? According to proposition 4-b), v is concave whenever discount rates $R + \sigma_0 \lambda$ are convex and λ has a curvature "sufficiently" high to make $G < 0$. Such a condition on the curvature on Sharpe ratios has a relatively simple explanation. Suppose that Sharpe ratios are decreasing and convex in s. In good times, Sharpe ratios are then relatively insensitive to small changes in the state-variables driving the business cycle conditions. Therefore, future dividends are discounted at approximately the same order of magnitude, and price-dividend ratios do not vary too much. As business-cycle conditions deteriorate, Sharpe ratios increase sharply (due to convexity), and future dividends are discounted at rapidly increasing orders of magnitude. Price-dividend ratios should now be highly responsive to news in bad times. If such an asymmetry

in discounting is sufficiently strong, price-dividend ratios are then concave in the state variables related to the business cycle. The condition that $G < 0$ in proposition 4-b) represents a precise prediction on how "sufficiently strong" such an asymmetry must be.[19]

6 Higher dimensional extensions

This section considers higher dimensional extensions of the theory. I take as primitive a general diffusion state process. I then restrict it to guarantee that all possibly resulting long-lived asset price processes are consistent with *given* sets of properties.

Consider the general formulation in section 1, and set $d = 4$. I assume that (Z, Y) satisfies:

$$
\begin{cases}
dz(\tau) = m_0\left(z(\tau), y(\tau)\right) d\tau + \sigma\left(z(\tau)\right) dW_1(\tau) \\[2mm]
d\mu(\tau) = -\mu(\tau)\left[R(z(\tau), y(\tau))d\tau + \sum_{j=1}^{4} \lambda_j\left(z(\tau), y(\tau)\right)dW_j(\tau)\right] \\[2mm]
dy^i(\tau) = \varphi_0^{(i)}(z(\tau), y(\tau))d\tau + \sum_{j=1}^{4} \xi_j^{(i)}\left(y(\tau)\right) dW_j(\tau) \qquad i = 1, 2, 3
\end{cases}
\tag{23}
$$

where $y = (y^1, y^2, y^3)^\top$ and $\{W_j\}_{j=1}^{4}$ are independent Brownian motions. Accordingly, the no-arbitrage price function is $q(z, y) \in \mathcal{C}^{2,2,2,2}(\mathbb{Z} \times \mathbb{Y})$, $\mathbb{Y} \subset \mathbb{R}^3$. Furthermore, functions m_0, σ, R, λ_j, $\varphi_0^{(i)}$, $\xi_j^{(i)}$ ($i = 1, 2, 3$ and $j = 1, \cdots, 4$) satisfy the same kind of conditions as those of eqs. (2) and (7) in section 1. To simplify notation, I set $\overline{\xi}_i^2 \equiv \left\|(\xi_1^{(i)}, \cdots, \xi_4^{(i)})\right\|^2$ and $\xi^{m,n} \equiv \sum_{j=1}^{4} \xi_j^{(m)} \xi_j^{(n)}$.

In this model, asset prices variations originate from the fluctuation of four factors: 1) dividends (Z); and 2) three state variables affecting expected dividend growth (m_0), risk-premia (λ_j) and the short-term interest rate (R). This formulation allows expected dividend growth, risk-premia and the short-term interest rate to be imperfectly correlated - even when risk-premia and the short-term rate do not depend on z. Brennan and Xia (2003) and Brennan, Wang and Xia (2003) have recently considered specific cases of system (23) allowing for closed-form solutions for the pricing function $q(z, y)$. Even the *individual* stock prices in the economies considered in Menzly, Santos and Veronesi (2004) can be thought of as being generated by a specific mechanism that

---

[19] Consider, e.g., the model in example 3. An application of proposition 4 to this model predicts that $v' > 0$. The proof of proposition 4 also reveals that a milder condition ensuring concavity of v in this model is that $-\sigma_0 \lambda'' v + G v' < 0$. Even where G is positive, the convexity effect induced by λ by the parameters reported by Campbell and Cochrane (1999) is so strong that $v'' \leq 0$. As regards the model in example 4, I found that proposition 4 predicts that $v' > 0$ and $v'' \leq 0$ in correspondence of sufficiently high levels of η (the analytical expressions of ϕ, ξ, R for this model are given in appendix D).

is similar to system (23). Here I aim at providing a general asset pricing characterization relying on as few assumptions as possible as regards the primitive dynamics.

In appendix E, I have developed a general theory [see eqs. (E2)]. To simplify its exposition, I illustrate it in the case in which R and λ_j are independent of z, and Z is a process with possibly time-varying expected growth, viz

$$\frac{dz(\tau)}{z(\tau)} = d_0(y_1(\tau), y_2(\tau), y_3(\tau))d\tau + \sigma_0 dW_1(\tau), \tag{24}$$

where σ_0 is a constant and function d_0 is twice differentiable in all its arguments. Similarly as in section 1, I set $d \equiv d_0 - \sigma_0 \lambda_1$. Accordingly, functions $\varphi^{(i)}$ denote risk-neutralized drift functions.

We have:

Proposition 5. *Assume that the pricing kernel is as in eqs. (23), that the short-term rate R and unit risk-premia λ_j are all independent of z and, finally, that the instantaneous dividend rate satisfies eq. (24). Then, the rational price function takes the form $q(z, y) = z \cdot v(y)$, where the price-dividend ratio v is positive, and satisfies the following properties*:

a) Suppose that $\overline{\xi}_i^2$, $\varphi^{(i)}$, $\xi_1^{(i)}$ $(i = 2, 3)$ and $\xi^{2,3}$ are independent of y_1. Then, v is increasing (resp. decreasing) in y_1 whenever $\frac{\partial}{\partial y_1}[d(y) - R(y)] > 0$ (resp. < 0) for all y.

b) Suppose that $\overline{\xi}_i^2$, $\varphi^{(i)}$, $\xi_1^{(i)}$ $(i = 2, 3)$, $\xi^{2,3}$ are independent of y_1 and that $\frac{\partial^2}{\partial y_1^2}\xi^{1,2}(y) = \frac{\partial^2}{\partial y_1^2}\xi^{1,3}(y) = 0$ for all y. Then, if v is increasing in y_1, it is also concave (resp. convex) in y_1 if $\frac{\partial^2}{\partial y_1^2}[d(y) - R(y)] \leq 0$ (resp. ≥ 0) and $2\frac{\partial}{\partial y_1}d(y) + \frac{\partial^2}{\partial y_1^2}\varphi^{(1)}(y) + \sigma_0\frac{\partial^2}{\partial y_1^2}\xi_1^{(1)}(y) - 2\frac{\partial}{\partial y_1}R(y) < 0$ (resp. > 0) for all y.

c) Suppose that for all y, v is increasing and (weakly) convex in y_1 (viz $\frac{\partial}{\partial y_1}v(y) > 0$ and $\frac{\partial^2}{\partial y_1^2}v(y) \geq 0$) and that for $i \neq j$, function $\overline{\xi}_i^2$ does not depend on factor y_j, $\frac{\partial}{\partial y_1}[d(y) - R(y)] > 0$, $\xi_1^{(1)} > 0$, $\frac{\partial}{\partial y_3}v(y) > 0$, $\frac{\partial^2}{\partial y_1 \partial y_3}[d(y) - R(y)] > 0$ and, finally, that $\varphi^{(2)}$ is independent of both y_1 and y_3. Then, for all y, $\frac{\partial^2}{\partial y_1 \partial y_3}v(y) > 0$ in all economies in which $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda \equiv \lambda_1(y_3)$, $\lambda' < 0$ and $\frac{\partial}{\partial y_3}[d(y) - R(y) - (\xi_1^{(1)}(y_1))'\lambda(y_3)] > 0$.

A simple example illustrating proposition 5 is one in which aggregate consumption $C = \{c(\tau)\}_{\geq 0}$ (say) is made up of one nonfinancial component (i.e. irrelevant for asset pricing) and dividends $Z = \{z(\tau)\}_{\geq 0}$; and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda \equiv \lambda_1$, where the risk-premium λ is driven by a hypothetical pro-cyclical "risk-preferences process" $S = \{s(\tau)\}_{\geq 0}$, and is such that $s \mapsto \lambda(s)$

is differentiable and decreasing. Let (Z, C, S) satisfy:

$$
\begin{cases}
\dfrac{dz(\tau)}{z(\tau)} & = \ d_0(g(\tau))d\tau \ \ + \ \ \sigma_0 dW_1(\tau) \\[2mm]
dg(\tau) & = \ \varphi_0^{(1)}(g(\tau))d\tau \ + \ \xi_1^{(1)}(g(\tau))dW_1(\tau) \ + \ \xi_2^{(1)}(g(\tau))dW_2(\tau) \\[2mm]
dc(\tau) & = \ \varphi_0^{(2)}(c(\tau))d\tau \ + \ \xi_1^{(2)}(c(\tau))dW_1(\tau) \hspace{3cm} + \ \xi_3^{(2)}(c(\tau))dW_3(\tau) \\[2mm]
ds(\tau) & = \ \varphi_0^{(3)}(s(\tau))d\tau \ + \ \xi_1^{(3)}(s(\tau))dW_1(\tau) \hspace{3cm} + \ \xi_3^{(3)}(s(\tau))dW_3(\tau)
\end{cases}
\tag{25}
$$

Suppose further that the interest rate R is a function of s only. By proposition 5-a,b), the price-dividend ratio is a function $v(g, s)$ 1) *increasing* and *convex* in the expected dividend growth whenever $d_0'(g) > 0$, $d_0''(g) \geq 0$ and $2d_0'(g) + \frac{2}{g^2}[\varphi_0^{(1)}(g) + (\sigma_0 - \lambda(s))\xi_1^{(1)}(g)] > 0$ all $(g, s) \in \mathbb{G} \times \mathbb{S}$; and 2) *increasing* in s if $-\sigma_0\lambda'(s) - R'(s) > 0$, and *concave* in s whenever $(\lambda(s) + \sigma_0 R(s))'' > 0$ and $-\frac{2}{s^2}[\varphi_0^{(3)}(g) + (\sigma_0 - \lambda(s))\xi_1^{(3)}(g)] - 2\sigma_0\lambda'(s) - 2R'(s) < 0$, all $(g, s) \in \mathbb{G} \times \mathbb{S}$. These properties represent multidimensional extensions of those found and discussed in sections 3, 4 and 5, and are not discussed further here.

A new interesting aspect of multidimensional models is related to the price reaction to *joint* movements in the underlying state variables. Proposition 5-c) provides a basic prediction on the sign of such a reaction in models in which the price is a monotonic function of any two state variables. As an example, suppose that in model (25), $\xi_1^{(1)} > 0$ and that $R'(s) < -\sigma_0\lambda'(s) - (\xi_1^{(1)}(g))'\lambda'(s)$, all $(g, s) \in \mathbb{G} \times \mathbb{S}$. Under these mild conditions, proposition 5-c) predicts that it is *always* true that $v_{sg} > 0$ in this model. In multidimensional models such as (25), changes in the expected dividend growth rate are very likely to have an even larger price impact than the one described in section 4. At the same time, such a magnifying effect induces in a natural way a positive relation between expected dividend growth and expected price variations (and hence returns) in this model. As proposition 5-c) reveals, this property is likely to emerge in much larger classes of economies under a set of rather mild conditions.

7 Conclusion

The basic *one-factor* Lucas (1978) asset pricing framework can considerably be enriched to allow for time-variation in both dividend growth and risk-adjusted discount rates. Such a research strategy has generated a new impetus in the literature. While the resulting models are making

a real progress towards our understanding of aggregate stock-market behavior, the same models are often based on new assumptions concerning the dynamics of unobservable processes (such as time-varying dividend growth, or habit formation). As for many other asset pricing problems, the choice of these assumptions is typically guided by economically sensible intuition, casual empirical evidence, or analytical convenience. Yet each particular assumption should carry a critical weight on to the overall general properties of the resulting pricing functions. This article adds a new perspective and explores such general properties in a framework relying only on three basic assumptions: 1) asset prices are arbitrage free; 2) agents have rational expectations; and 3) state variables follow low-dimensional diffusion processes.

The theoretical test conditions of this article enable one to understand qualitative properties of models directly from first principles. As a by-product, they explicitly investigate the robustness of well-known long-lived asset price properties to the modification of "typical" assumptions. In fact, I produced examples illustrating how to apply the theory of this article to shed new light on already existing models. Importantly, the theory developed in this article makes novel testable restrictions on the joint behavior of asset prices, risk-premia and the dynamics of consumption. Therefore, natural applications of this theory include the use of its predictions as a practical guidance to specification, estimation and testing of multidimensional long-lived asset prices models with rational expectations.

Appendix A: Preliminaries

This appendix contains preliminary results. First, I assume the regularity conditions in section 1, and derive the Feynman-Kac stochastic representation of the partial derivatives of long-lived asset prices for the models considered in sections 3 and 4 (see lemma A1). Second, I provide the analytical expression of interest rates and risk-premia corresponding to the price kernel in eq. (3) [see eqs. (A2)]. Finally, I illustrate how these analytical expressions fit into standard infinite horizon, general equilibrium models with complete markets (see example A1).

Lemma A1. *Let $w^1(z,g) \equiv q_{\mathsf{z}}(z,g)$, $w^2(z,g) \equiv q_{\mathsf{zz}}(z,g)$, $w^3(z,g) \equiv q_{\mathsf{g}}(z,g)$, $w^4(z,g) \equiv q_{\mathsf{gg}}(z,g)$ and $w^5(z,g) \equiv q_{\mathsf{zg}}(z,g)$. We have:*

$$w^{\mathsf{i}}(z,g) = \mathbb{E}\left[\int_0^\infty \kappa^{\mathsf{i}}(\tau) h^{\mathsf{i}}(\zeta_{\mathsf{i}}(\tau), \gamma_{\mathsf{i}}(\tau)) d\tau\right], \quad i = 1, \cdots, 5,$$

where κ^{i} are random, positive processes defined in the proof,

$$
\begin{aligned}
h^1(z,g) &= 1 + \varphi_1(z,g)q_{\mathsf{g}}(z,g) - R_1(z,g)q(z,g) \\[1em]
h^2(z,g) &= m_{11}(z,g)q_{\mathsf{z}}(z,g) + \varphi_{11}(z,g)q_{\mathsf{g}}(z,g) + \left[2\varphi_1(z,g) + \frac{2}{z^2}((\sigma\xi_1)(z,g))\right]q_{\mathsf{zg}}(z,g) \\
&\quad - R_{11}(z,g)q(z,g) - 2R_1(z,g)q_{\mathsf{z}}(z,g) \\[1em]
h^3(z,g) &= m_2(z,g)q_{\mathsf{z}}(z,g) - R_2(z,g)q(z,g) \\[1em]
h^4(z,g) &= m_{22}(z,g)q_{\mathsf{z}}(z,g) + \varphi_{22}(z,g)q_{\mathsf{g}}(z,g) + \left[2m_2(z,g) + \frac{2}{g^2}((\sigma\xi_1)(z,g))\right]q_{\mathsf{zg}}(z,g) \\
&\quad - R_{22}(z,g)q(z,g) - 2R_2(z,g)q_{\mathsf{g}}(z,g) \\[1em]
h^5(z,g) &= m_{12}(z,g)q_{\mathsf{z}}(z,g) + m_2(z,g)q_{\mathsf{zz}}(z,g) + \varphi_{12}(z,g)q_{\mathsf{g}}(z,g) + \varphi_1(z,g)q_{\mathsf{gg}}(z,g) \\
&\quad - R_1(z,g)q_{\mathsf{g}}(z,g) - R_{12}(z,g)q(z,g) - R_2(z,g)q_{\mathsf{z}}(z,g)
\end{aligned}
$$

and $\zeta_{\mathsf{i}}, \gamma_{\mathsf{i}}$ are solutions to some stochastic differential equations that are also given in the proof.

Proof. In the absence of arbitrage, the price function $q(z,g)$ is solution to eq. (5). By using the definition of m and φ in (7), eq. (5) is:

$$0 = \frac{1}{2}\sigma^2 q_{\mathsf{zz}} + mq_{\mathsf{z}} + \frac{1}{2}\xi^2 q_{\mathsf{gg}} + \varphi q_{\mathsf{g}} + \sigma\xi_1 q_{\mathsf{gz}} + z - Rq, \quad \forall (z,g) \in \mathbb{Z} \times \mathbb{G}, \tag{A1}$$

where $\xi^2 \equiv \xi_1^2 + \xi_2^2$. By differentiating eq. (A1) with respect to z and g an appropriate number of times, I find that w^{i} are solutions to the following partial differential equations:

$$0 = (L^{\mathsf{i}} - k^{\mathsf{i}})w^{\mathsf{i}}(z,g) + h^{\mathsf{i}}(z,g), \quad \forall(z,g) \in \mathbb{Z} \times \mathbb{G}, \quad i = 1, \cdots, 5,$$

where $L^{\mathsf{i}}w^{\mathsf{i}} = \frac{1}{2}\sigma^2 w_{zz}^{\mathsf{i}} + m^{\mathsf{i}}w_z^{\mathsf{i}} + \frac{1}{2}\xi^2 w_{gg}^{\mathsf{i}} + \varphi^{\mathsf{i}}w_g^{\mathsf{i}} + \sigma\xi_1 w_{zg}^{\mathsf{i}}$, and

$$k^1(z,g) = R(z,g) - m_1(z,g)$$

$$k^2(z,g) = R(z,g) - 2m_1(z,g) - \frac{1}{2}(\sigma(z)^2)''$$

$$k^3(z,g) = R(z,g) - \varphi_2(z,g)$$

$$k^4(z,g) = R(z,g) - 2\varphi_2(z,g) - \frac{1}{2}(\xi^2(g))''$$

$$k^5(z,g) = R(z,g) - m_1(z,g) - \varphi_2(z,g) - \frac{2}{z\,g}(\sigma\xi_1)(z,g)$$

where I have defined, $m^1 \equiv m + \frac{1}{2}(\sigma^2)'$, $\varphi^1 \equiv \varphi + \frac{}{z}(\sigma\xi_1)$, $m^2 \equiv m + (\sigma^2)'$, $\varphi^2 \equiv \varphi + 2\frac{}{z}(\sigma\xi_1)$, $m^3 \equiv m + \frac{}{g}(\sigma\xi_1)$, $\varphi^3 \equiv \varphi + \frac{1}{2}(\xi^2)'$, $m^4 \equiv m + 2\frac{}{g}(\sigma\xi_1)$, $\varphi^4 \equiv \varphi + (\xi^2)'$, $m^5 \equiv m + \frac{1}{2}(\sigma^2)' + \frac{}{g}(\sigma\xi_1)$, $\varphi^5 \equiv \varphi + \frac{1}{2}(\xi^2)' + \frac{}{z}(\sigma\xi_1)$. The result then follows by the Feynman-Kac probabilistic representation theorem: processes κ^{i} are given by $\kappa^{\mathsf{i}}(\tau) \equiv \exp(-\int_0 k^1(\zeta_{\mathsf{i}}(u), \gamma_{\mathsf{i}}(u))du)$, where ζ_{i} and γ_{i} are solutions to

$$\begin{cases} d\zeta_{\mathsf{i}}(\tau) &= m^{\mathsf{i}}(\zeta_{\mathsf{i}}(\tau), \gamma_{\mathsf{i}}(\tau))d\tau + \sigma(\zeta_{\mathsf{i}}(\tau))d\widehat{W}_1(\tau) \\[2mm] d\gamma_{\mathsf{i}}(\tau) &= \varphi^{\mathsf{i}}(\zeta_{\mathsf{i}}(\tau), \gamma_{\mathsf{i}}(\tau))d\tau + \xi_1(\gamma_{\mathsf{i}}(\tau))d\widehat{W}_1(\tau) + \xi_2(\gamma_{\mathsf{i}}(\tau))d\widehat{W}_2(\tau) \end{cases}$$

with $(\zeta_{\mathsf{i}}(0), \gamma_{\mathsf{i}}(0)) = (z,g)$, for $i = 1, \cdots, 5$. $\blacksquare$

Next, I characterize Sharpe ratios and interest rates in some classes of models considered in sections 3, 4 and 5.

Interest rates and risk premia in eq. (5). Let (Z, Y) be solution to the first and third equations in system (2). The exact expressions of R and λ in eq. (5) are obtained by an application of Itô's lemma to $\mu(\tau, z, y)$ in eq. (3), and by identifying drift and diffusion terms.

We have:

$$
\begin{cases}
R(z,y) & = \quad \delta(z,y) - \dfrac{Lp(z,y)}{p(z,y)} \\[2em]
\lambda_1(z,y) & = \quad -\sigma(z,y)\dfrac{\partial}{\partial z}\log p(z,y) - \xi_1(z,y)\dfrac{\partial}{\partial y}\log p(z,y) \\[2em]
\lambda_2(z,y) & = \quad -\xi_2(z,y)\dfrac{\partial}{\partial y}\log p(z,y)
\end{cases}
\tag{A2}
$$

Example A1 below is an important special case of this setting.

Example A1 (Infinite horizon, complete markets economy.) Consider an infinite horizon, complete markets economy in which total consumption Z is solution to eq. (2), with $\xi_2 \equiv 0$. Let a (single) agent's program be:

$$
\max E\left[\int_0^\infty e^{-} \ u(c(\tau), x(\tau))d\tau\right] \quad \text{s.t. } V_0 = E\left[\int_0^\infty \mu(\tau)c(\tau)d\tau\right], \quad V_0 > 0,
$$

where $\delta > 0$, the instantaneous utility u is continuous and thrice continuously differentiable in its arguments, and x is solution to

$$
dx(\tau) = \beta(z(\tau), g(\tau), x(\tau))d\tau + \gamma(z(\tau), g(\tau), x(\tau))dW_1(\tau).
$$

In equilibrium, $C = Z$, where C is optimal consumption. Provided the finiteness of this program's value [see Huang and Pagès (1992) (lemma 3, p. 42; and prop. 4, p. 47) for regularity conditions related to this kind of infinite horizon problems], we have that in terms of the representation in (A2), $\delta(z,x) = \delta$, and $p(z(\tau), x(\tau)) = u_1(z(\tau), x(\tau))/ u_1(z(0), x(0))$. Consequently, $\lambda_2 = 0$,

$$
\begin{aligned}
R(z,g,x) & = \quad \delta - \frac{u_{11}(z,x)}{u_1(z,x)}m_0(z,g) - \frac{u_{12}(z,x)}{u_1(z,x)}\beta(z,g,x) \\
& \quad -\frac{1}{2}\sigma(z,g)^2\frac{u_{111}(z,x)}{u_1(z,x)} - \frac{1}{2}\gamma(z,g,x)^2\frac{u_{122}(z,x)}{u_1(z,x)} - \gamma(z,g,x)\sigma(z,g)\frac{u_{112}(z,x)}{u_1(z,x)}
\end{aligned}
\tag{A3}
$$

and

$$
\lambda(z,g,x) = -\frac{u_{11}(z,x)}{u_1(z,x)}\sigma(z,g) - \frac{u_{12}(z,x)}{u_1(z,x)}\gamma(z,g,x).
\tag{A4}
$$

Appendix B: Proofs, examples and Dynamic Stochastic Dominance theory for section 3

Proof of proposition 1. Let $c(x, T-s) \equiv \mathbb{E}[\exp(-\int_s^T \rho(x(t))dt) \cdot \psi(x(T)) | x(s) = x]$. Function c is solution to the following partial differential equation:

$$\begin{cases} 0 = -c_2(x, T-s) + L^*c(x, T-s) - \rho(x)c(x, T-s), & \forall(x,s) \in \mathbb{R} \times [0,T) \\ \\ c(x,0) = \psi(x), & \forall x \in \mathbb{R} \end{cases} \quad \text{(B1)}$$

where $L^*c(x,u) = \frac{1}{2}a(x)^2 c_{\mathsf{xx}}(x,u) + b(x)c_{\mathsf{x}}(x,u)$. By differentiating twice eq. (B1) with respect to x, I find that $c^{(1)}(x,\tau) \equiv c_{\mathsf{x}}(x,\tau)$ and $c^{(2)}(x,\tau) \equiv c_{\mathsf{xx}}(x,\tau)$ are solutions to the following partial differential equations: $\forall(x,s) \in \mathbb{R}_{++} \times [0,T)$,

$$\begin{aligned} 0 &= -c_2^{(1)}(x, T-s) + \frac{1}{2}a(x)^2 c_{\mathsf{xx}}^{(1)}(x, T-s) + \left[b(x) + \frac{1}{2}(a(x)^2)'\right]c_{\mathsf{x}}^{(1)}(x, T-s) \\ &\quad - \left[\rho(x) - b'(x)\right]c^{(1)}(x, T-s) - \rho'(x)c(x, T-s), \end{aligned}$$

with $c_2^{(1)}(x,0) = \psi'(x)$ $\forall x \in \mathbb{R}$; and $\forall(x,s) \in \mathbb{R} \times [0,T)$,

$$\begin{aligned} 0 &= -c_2^{(2)}(x, T-s) + \frac{1}{2}a(x)^2 c_{\mathsf{xx}}^{(2)}(x, T-s) + \left[b(x) + (a(x)^2)'\right]c_{\mathsf{x}}^{(2)}(x, T-s) \\ &\quad - \left[\rho(x) - 2b'(x) - \frac{1}{2}(a(x)^2)''\right]c^{(2)}(x, T-s) \\ &\quad - \left[2\rho'(x) - b''(x)\right]c^{(1)}(x, T-s) - \rho''(x)c(x, T-s), \end{aligned}$$

with $c_2^{(2)}(x,0) = \psi''(x)$ $\forall x \in \mathbb{R}$ (in both equations, subscripts denote partial derivatives). By arguments similar to the ones used to prove lemma A1, we have that $c^{(1)}(x, T-s) > 0$ (resp. < 0) $\forall(x,s) \in \mathbb{R} \times [0,T)$ whenever $\psi'(x) > 0$ (resp. < 0) and $\rho'(x) < 0$ (resp. > 0) $\forall x \in \mathbb{R}$. This completes the proof of part a) of the proposition. The proof of part b) is obtained similarly. ∎

It is worth emphasizing that one consequence of proposition 1 is a general statement about conditional expectations of scalar diffusion processes. Precisely, we have:

Corollary B1. *A conditional expectation of a scalar diffusion is a concave (resp. convex) function of the initial condition whenever the drift function is concave (resp. convex).*

Proof. A conditional expectation of a scalar diffusion is function $c(x, \tau)$ with $\psi(x) = x$ and $\rho(x) = 0$ in the canonical pricing problem of section 3.1. The result immediately follows by plugging these ψ and ρ into the theoretical test conditions of proposition 1. ∎

Let $\psi(x) = x$, $b = m$, $a = \sigma$ and $X = Z$, the total consumption process. By combining eq. (6) with proposition 1, one obtains a general price characterization of scalar models:

Corollary B2 (Scalar long-lived asset price models.) *The rational price function q is positive and if $R' \leq 0$, it is increasing. Furthermore, suppose that q is increasing. Then, q is concave (resp. convex) whenever $m'' - 2R' < 0$ (resp. > 0) and $R'' \geq 0$ (resp. ≤ 0).*

Remark B1. In the economy of example A1, $R' \leq 0$ whenever $u(c) = \frac{c^{1-\eta}-1}{1-}$ and the elasticities of m_0 and σ are both bounded by one, as in example B1 below.

I now turn to an alternate proof of corollary B2. This proof is instructive. It provides intuition on the general strategy of proofs adopted to deal with the difficult multidimensional cases of sections 4, 5 and 6.

Alternate proof of corollary B2. In the scalar case, the stochastic representations of q_z and q_{zz} of lemma A1 reduce to:

$$q'(z) = \mathbb{E} \left\{ \int_0^\infty \kappa^1(\tau) \left[1 - R'(\zeta_1(\tau)) q(\zeta_1(\tau)) \right] d\tau \right\}, \tag{B2}$$

and

$$q''(z) = \mathbb{E} \left[\int_0^\infty \kappa^2(\tau) h^2(\zeta_2(\tau)) d\tau \right], \tag{B3}$$

where

$$h^2(z) \equiv \left[m''(z) - 2R'(z) \right] q'(z) - R''(z) q(z),$$

and κ^1, κ^2, ζ_1 and ζ_2 are as in lemma A1 (with $\xi_2 \equiv 0$ and $m(z, g) \equiv m(z)$). By eq. (B2), $q'(z) > 0$ for all z whenever $R' \leq 0$. Given this result, the second claim of the corollary immediately follows from the representation of q'' in (B3). ∎

Finally, I develop dynamic stochastic dominance theory related to the canonical pricing problem in section 3.1. We have:

Proposition B1 [Dynamic Stochastic Dominance (DSD) theory.] *Consider two economies A and B with volatilities a_A and a_B, and let c^i, $\pi_i(x) \equiv a_i(x) \cdot \lambda^i(x)$ and $\rho_i(x)$ ($i = A, B$) be the corresponding prices, risk-premia and discount rates in the canonical pricing problem of section 3. Let $a_A > a_B$. Then, $c^A < c^B$ whenever for all $(x, \tau) \in \mathbb{R} \times [0, T]$,*

$$V(x, \tau) \equiv -\left[\rho_A(x) - \rho_B(x)\right] c^B(x, \tau) - \left[\pi_A(x) - \pi_B(x)\right] c_x^B(x, \tau) + \frac{1}{2}\left[a_A^2(x) - a_B^2(x)\right] c_{xx}^B(x, \tau) < 0.$$

Proof. Clearly, c^A and c^B are both solutions to eq. (B1), but with different coefficients. Let $b_A(x) \equiv b_0(x) - \pi_A(x)$. The price difference $\Delta c(x, \tau) \equiv c^A(x, \tau) - c^B(x, \tau)$ is solution to the following partial differential equation: $\forall (x, s) \in \mathbb{R} \times [0, T]$,

$$0 = -\Delta c_2(x, T-s) + \frac{1}{2}\sigma^B(x)^2 \Delta c_{xx}(x, T-s) + b_A(x)\Delta c_x(x, T-s) - \rho_A(x)\Delta c(x, T-s) + V(x, T-s),$$

with $\Delta c(x, 0) = 0$ for all $x \in \mathbb{R}$, and V is as in the proposition. The result follows by the same reasoning produced in the proof of lemma A1 in appendix A. ∎

I now develop applications of DSD theory to illustrate properties of models with uncertain stochastic consumption growth rate [model (9) and example 3 in section 4] and models with time-varying Sharpe ratios [model (13)].

1. *Model (9)*. I assume throughout that ξ_2 does not depend on σ_0.

   1.1 Let $\partial[(\sigma_0 - \lambda)\xi_1]/\partial\sigma_0 = 0$ and $\partial\xi_1/\partial\sigma_0 < 0$.[20] If condition (12) holds, *q decreases with σ_0*. Indeed, B in eq. (11) decreases with σ_0 because: a) it decreases with $\sigma_0\lambda$; and b) condition (12) ensures that convexity effects are activated in proposition B1. (Due to these convexity effects, q is decreasing in σ_0 even when the risk-premium $\lambda = 0$.)

   1.2 Next, let $\partial\xi_1/\partial\sigma_0 = 0$, and set $\pi \equiv -\sigma_0\xi_1$ in the canonical pricing problem (CPP). By proposition B1, *q can now be increasing in σ_0 when $\lambda = 0$.*

   1.3 If condition (12) holds and $\lambda < \sigma_0$, *q increases with ξ_1.* This follows by setting $\pi \equiv -(\sigma_0 - \lambda)\xi_1$ in the CPP, and by an application of proposition B1.

   1.4 Finally, consider the price impact of ξ_2.[21] By proposition B2, *q increases with ξ_2* whenever condition (12) holds. Note that if the inequality in (12) is reversed, *q decreases with ξ_2.*

---

[20] These conditions emerge naturally in learning models such as the one in example 1 (section 4).

[21] As example 1 in section 4 reveals, ξ_2 is negatively related to the quality of additional sources of information in models of learning.

2. *Example 1 in section 4 [Model (16)].* This model can be analyzed with the simple tools of section 3. In terms of eq. (6), the model predicts that $q(z, g) = \int_0^\infty C(z, g, \tau)d\tau$, where

$$C(z, g, \tau) = e^{-r} \ (z - \sigma_0 \lambda \tau) + e^{-r} \int_0 \mathbb{E}\left[g(u)|\, g\right] du, \quad \tau \geq 0. \qquad (B4)$$

Then, *q is always decreasing in ξ if $\lambda > 0$.* Indeed, an application of proposition B2 reveals that there is a conflict between convexity effects and drift effects. I then perturb function ξ_1 with $\epsilon \cdot \xi_1$, $\epsilon > 0$. The expectation in eq. (B4) is:

$$\mathbb{E}\left[g(u)|\, g\right] = e^{-ku}g + \left(1 - e^{-ku}\right)\overline{g} - \lambda\epsilon \cdot \int_0^u e^{-k(s-t)}\mathbb{E}\left[\xi_1(g(s))|\, g\right] ds,$$

where ξ_1 is nonnegative by construction.

3. *Model (13).* Let $\xi > 0$, and let $\partial\xi/\partial\sigma_0 > 0$ and $\partial\lambda/\partial\sigma_0 > 0$.[22] Then, *q decreases with σ_0 whenever B is concave and $\partial(\lambda - \sigma_0)/\partial\sigma_0 > 0$. Finally, let $\lambda > \sigma_0$. Then q decreases with changes in ξ that are not related to σ_0.*

Finally, consider the scalar models analyzed in corollary B2. If R is not constant, proposition B2 can not be used to address stochastic dominance properties of q in great detail. Yet I claim that in the setting of example A1, q is decreasing in volatility whenever δ is constant and,

$$\text{for all } z \in \mathbb{Z}, \quad m_0''(z) \leq 0, \quad zA(z) < 1 \quad \text{and} \quad zP(z) < 2, \qquad (B5)$$

where $A \equiv -u''/u'$ and $P \equiv -u'''/u''$ are the absolute risk aversion and the absolute prudence coefficient. As the following proof reveals, *prices are increasing in volatility if $u'' = u''' = 0$ and $m_0'' > 0$.* In other terms, conditions (B5) make concavity effects dominate in proposition B1.[23]

Proof of sufficiency of eqs. (B5). Define $w(z, \tau) \equiv w(z, \tau; \sigma^2) \equiv c(z, \tau; \sigma^2)u'(z)$, where z is solution to the first equation in (2) with $m_0(z, y) \equiv m_0(z)$, and $c(z, \tau; \sigma^2) \equiv c(z, \tau)$, where $c(z, \tau)$ is as in eq. (8), with $\psi(z) = z$ for all $z \in \mathbb{Z}$. By definition, c is decreasing in σ^2 if and only if w is decreasing in σ^2. By eq. (6), $q(z) = \int_0^\infty c(z, \tau; \sigma^2)d\tau$. Therefore, q is decreasing

---

[22] These assumptions hold in all the model examples in section 5.

[23] Naturally, conditions (B5) are only sufficient.35(Y)76-2.1(a)1.t8(,)65.31(h)-2.6(ese)361(c(a)1.(co(h)-2.(ci)7.(pti(h)-2.ons)5965.3(;

in σ^2 whenever w is decreasing in σ^2. By assumption, and example A1, $\delta(z) = \delta > 0$ and $p(z(\tau)) = u'(z(\tau))/u'(z(0))$. By eq. (8), and the definition of the risk-neutral probability P^0, w satisfies:

$$e^{-\,\mathrm{t}}w(z(t), t) = e^{-\,\mathrm{s}}E\left[w(z(s), s)\right], \quad \tau > s > t > 0,$$

where $w(z, \tau) = zu'(z)$ for all $z \in \mathbb{Z}$. Therefore, w is solution to the following partial differential equation:

$$\begin{cases} 0 = w_2(z, s) + Lw(z, s) - \delta w(z, s), & \forall (z, s) \in \mathbb{Z} \times [0, \tau) \\ \\ w(z, \tau) = zu'(z), & \forall z \in \mathbb{Z} \end{cases}$$

where $Lw(z, s) = \frac{1}{2}\sigma(z)^2 w_{zz}(z, s) + m_0(z)w_z(z, s)$. The previous partial differential equation is in the same format as eq. (B1). In terms of eq. (B1), $\rho(z) = \delta$, $\psi(z) = zu'(z)$, $a(z) = \sigma(z)$ and $b(z) = m_0(z)$. Therefore, the theoretical test conditions of proposition 1 can be applied to the undiscounted Arrow-Debreu price w as well. Suppose then that $w' > 0$. Then, by proposition 1-b), $w'' < 0$ whenever

$$m_0''(z) \le 0, \quad z \in \mathbb{Z},$$

and the final payoff $zu'(z)$ is concave, viz.

$$\frac{d^2}{dz^2}[zu'(z)] < 0, \quad z \in \mathbb{Z}.$$

Finally, by proposition 1-a), $w' > 0$ whenever the final payoff $zu'(z)$ is increasing, or

$$\frac{d}{dz}[zu'(z)] > 0, \quad z \in \mathbb{Z}.$$

The prediction summarized by eqs. (B5) is obtained by explicitely developing the previous two conditions. As demonstrated above, $w'' < 0$ if eqs. (B5) hold. The result then follows by proposition B1. ∎

The following example illustrates corollary B2 and eqs. (B5).

Example B1. Consider the economy in example A1 (appendix A), and assume that δ is constant, $u(c) = (c^{1-} - 1)/(1 - \eta)$, and total consumption Z is lognormal:

$$dz(\tau) = z(\tau)(a - b\log z(\tau))d\tau + \sigma_0 z(\tau)dW(\tau),$$

where a, b, σ_0 are positive constants and W is a Brownian motion. Consequently, $R(z) = \delta + \eta(a - b \log z) - \frac{1}{2}\sigma_0^2 \eta(\eta + 1)$ and $\lambda(z) = \eta \sigma_0$. Hence $R'(z) < 0$ whenever $\eta > 0$ and by proposition 1, $q' > 0$ whenever $\eta > 0$. Furthermore, $m(z) = z(\bar{a} - b \log z)$, where $\bar{a} \equiv a - \eta \sigma_0^2$. Therefore,

$$m''(z) - 2R'(z) = \frac{b(2\eta - 1)}{z} < 0 \quad \Leftrightarrow \quad \eta < \frac{1}{2}.$$

Finally, $R'' > 0$. By corollary 1, $q'' < 0$ if $\eta \in (0, \frac{1}{2})$. More generally, the proof of corollary B2 reveals that $q'' < 0$ whenever,

$$\left[m''(z) - 2R'(z)\right] q'(z) - R''(z)q(z) < 0.$$

As eq. (B6) below reveals, there exist sufficiently high values of η reversing the previous inequality, thus making $q'' > 0$. But in all cases, eqs. (B5) can be used to conclude that q is decreasing in σ_0^2 whenever $\eta < 1$ (i.e. even when the representative agent is risk-neutral). To confirm these results analytically, consider the asset price solution in this economy:

$$q(z) = \int_0^\infty z^{[1-\exp(-b\tau)] + \exp(-b\tau)} e^{k(\tau)} d\tau, \tag{B6}$$

where

$$\begin{aligned} k(\tau) \quad &\equiv \quad -\delta\tau + \frac{1}{b}\left[1 - \exp(-b\tau)\right]\left[a(1-\eta) - \frac{1}{2}\sigma_0^2(\eta + 1)\right] \\ &\quad + \frac{\eta\sigma_0^2}{2b}\left[1 + \exp(-2b\tau) - 2\exp(-b\tau)\right] + \frac{\sigma_0^2}{4b}\left[1 - \exp(-2b\tau)\right](1 + \eta^2). \end{aligned}$$

Naturally, the (general equilibrium version of the) Gordon's (1962) model is obtained by sending $b \to 0$ in eq. (B6), leaving the well-known formula: $q(z)/z = (\delta - a(1-\eta) - \frac{1}{2}\sigma_0^2(\eta(\eta-1))^{-1}$. In the general case, $q'' < 0$ (resp. > 0) if and only if $\eta < 1$ (resp. > 1). And if $\eta > 0$, q is decreasing (resp. increasing) in σ_0^2 whenever $\eta < 1$ (resp. > 1). Note, however, that if $\eta > 1$, q is also decreasing in a - similarly as in the Gordon's model.

The next example is inspired from Veronesi (1999) (see example 1 in section 4), and deals with issues arising from learning mechanisms. Precisely, here I develop a new heuristic construction of nonlinear filters of partially observed processes. I then illustrate pricing implications through an application of corollary B1.

Example B2. Consider a static scenario in which Z is generated by $z = \theta + w$, where w has zero mean and unit variance, and it is continuous with bounded density function ϕ. For a given $A > 0$, let $p \equiv \Pr(\theta = A) = 1 - \Pr(\theta = -A)$, and $\pi(z)dz \equiv \Pr(\theta = A \,|\, z \in dz)$. We have:

$$\pi(z) - p = p(1-p)\frac{\phi(z-A) - \phi(z+A)}{p\phi(z-A) + (1-p)\phi(z+A)}.$$

The variance of the "probability changes" $\pi(z) - p$ is zero exactly where there is a degenerate prior on the state. More generally, it is a $\cap$-shaped function of the a priori probability p of the "good" state. Clearly, $g \equiv E(\theta = A \,|\, z)$ has the same property because it is linear in $\Pr(\theta = A \,|\, z)$.

Next, assume that w is Brownian motion and set $A \equiv A d\tau$. Let $z(0) = 0$ and $\pi \equiv \pi(z)$. By Itô's lemma,

$$d\pi(\tau) = 2A \cdot \pi(\tau)(1 - \pi(\tau))dW(\tau), \quad \pi(0) \equiv p,$$

where $dW(\tau) \equiv dz(\tau) - g(\tau)d\tau$ and $g(\tau) \equiv E(\theta = A \,|\, z) = A\pi(\tau) - A(1 - \pi(\tau))$. While this construction is heuristic, the result is correct [see, e.g., Liptser and Shiryaev (2001) (Vol. I, thm. 8.1 p. 318; and example 1 p. 371)]. W is then a Brownian motion with respect to $\sigma(z(t), t \leq \tau)$ [see Liptser and Shiryaev (2001) (Vol. 1, thm. 7.12 p. 273)]. The equilibrium in the economy with incomplete information is then isomorphic in its pricing implications to the equilibrium in a full information economy in which:

$$\begin{cases} dz(\tau) = [g(\tau) - \lambda]\,d\tau + d\widehat{W}(\tau) \\[2mm] dg(\tau) = -\lambda\xi(g(\tau))d\tau + \xi(g(\tau))d\widehat{W}(\tau) \end{cases} \tag{B7}$$

where $\widehat{W}$ is a P^0-Brownian motion, λ is a constant (say) risk-premium, and $\xi(g) \equiv (A - g)(g + A)$.

In this and related models, the instantaneous volatility of G is $\cap$-shaped. Under positive risk-aversion, the risk-neutralized drift of Z is thus convex in g. Finally, the pricing function is as in eq. (B4) (with $k = \overline{g} = 0$) and by corollary B1, $\mathbb{E}[g(u) \,|\, g]$ is convex in g whenever the drift of G in (B7) is convex, which as observed is always true.

On bond prices convexity. Consider a short-term rate process $\{r(\tau)\}_{\in[0,\mathsf{T}]}$ (say), and let $u(r_0, T)$ be the price of a bond expiring at time T when the current short-term rate is r_0:

$$u(r_0, T) = \mathbb{E}\left[\exp\left(-\int_0^{\mathsf{T}} r(\tau)d\tau\right)\,\Big|\,r_0\right].$$

As I pointed out in section 3.3.1, a *restricted version* of proposition 1-b) implies that in all scalar (diffusion) models of the short-term rate, $u_{11}(r_0, T) < 0$ whenever $b'' < 2$, where b is the risk-netraulized drift of r. I originally obtained this specific result in Mele (2003). Both the theory in Mele (2003) and the proof of proposition 1 rely on the Feynman-Kac representation of u_{11}. Here I provide a more intuitive derivation under a set of simplifying assumptions.

By Mele (2003) [eq. (6) p. 685],

$$u_{11}(r_0, T) = \mathbb{E}\left\{\left[\left(\int_0^\mathsf{T} \frac{\partial r}{\partial r_0}(\tau)d\tau\right)^2 - \int_0^\mathsf{T} \frac{\partial^2 r}{\partial r_0^2}(\tau)d\tau\right] \exp\left(-\int_0^\mathsf{T} r(\tau)d\tau\right)\right\}.$$

Hence $u_{11}(r_0, T) > 0$ whenever[24]

$$\int_0^\mathsf{T} \frac{\partial^2 r}{\partial r_0^2}(\tau)d\tau < \left(\int_0^\mathsf{T} \frac{\partial r}{\partial r_0}(\tau)d\tau\right)^2. \tag{B8}$$

To keep the presentation as simple as possible, I assume that r is solution to:

$$dr(\tau) = b(r(\tau))dt + a_0 r(\tau)dW(\tau),$$

where a_0 is a constant. We have,

$$\frac{\partial r}{\partial r_0}(\tau) = \exp\left[\int_0 b'(r(u))du - \frac{1}{2}a_0^2\tau + a_0 W(\tau)\right]$$

and

$$\frac{\partial^2 r}{\partial r_0^2}(\tau) = \frac{\partial r}{\partial r_0}(\tau) \cdot \left[\int_0 b''(r(u))\frac{\partial r(u)}{\partial r_0}du\right].$$

Therefore, if $b'' < 0$, then $\partial^2 r(\tau)/\partial r_0^2 < 0$, and by inequality (B8), $u_{11} > 0$. But this result can considerably be improved. Precisely, suppose that $b'' < 2$ (instead of simply assuming that $b'' < 0$). By the previous equality,

$$\frac{\partial^2 r}{\partial r_0^2}(\tau) < 2 \cdot \frac{\partial r}{\partial r_0}(\tau) \cdot \left(\int_0 \frac{\partial r(u)}{\partial r_0}du\right),$$

and consequently,

$$\int_0^\mathsf{T} \frac{\partial^2 r}{\partial r_0^2}(\tau)d\tau < 2\int_0^\mathsf{T} \frac{\partial r}{\partial r_0}(\tau) \cdot \left(\int_0 \frac{\partial r(u)}{\partial r_0}du\right)d\tau = \left(\int_0^\mathsf{T} \frac{\partial r(u)}{\partial r_0}du\right)^2,$$

which is inequality (B8).

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[24] All statements are to be understood to hold $P \otimes d\tau$ almost surely.

Appendix C: Proofs and examples for section 4

Proof of proposition 2. By assumption, $R_1(z,g) = \varphi_1(z,g) = 0$ for all $(z,g) \in \mathbb{Z} \times \mathbb{G}$. Therefore, the stochastic representation of q_z in lemma A1 (appendix A) takes the form:

$$q_z(z,g) = \mathbb{E}\left[\int_0^\infty e^{-r - \int_0^\tau m_1(\zeta_1(u), \gamma_1(u))du}\,d\tau\right],$$

where ζ_1 and γ_1 are as in lemma A1. Hence $q_z > 0$. As regards the sign of q_{zz}, the assumptions of the proposition imply that function h^2 in lemma A1 is $h^2(z,g) = m_{11}(z,g)q_z(z,g)$ and consequently,

$$q_{zz}(z,g) = \mathbb{E}\left\{\int_0^\infty e^{-r - \int_0^\tau [2m_1(\zeta_2(u), \gamma_2(u)) + \frac{1}{2}(\sigma(\zeta_2(u))^2)'']du} \left[m_{11}(\zeta_2(\tau), \gamma_2(\tau))q_z(\zeta_2(\tau), \gamma_2(\tau))\right]d\tau\right\},$$

where ζ_1 and γ_1 are as in lemma A1. Since $q_z > 0$, the claim of the proposition about the sign of q_{zz} immediately follows. Finally, all the assumptions of the proposition imply (in conjunction with lemma A1) that

$$q_g(z,g) = \mathbb{E}\left[\int_0^\infty e^{-r - \int_0^\tau \sigma_2(\zeta_3(u), \gamma_3(u))du}m_2(\zeta_3(\tau), \gamma_3(\tau))q_z(\zeta_3(\tau), \gamma_3(\tau))d\tau\right]$$

and

$$q_{gg}(z,g) = \mathbb{E}\left\{\int_0^\infty e^{-r - \int_0^\tau [2\sigma_2(\zeta_4(u), \gamma_4(u)) + \frac{1}{2}(\sigma(\zeta_4(u))^2)'']du}h(\zeta_4(\tau), \gamma_4(\tau))d\tau\right\}$$

where ζ_3, γ_3, ζ_4 and γ_4 are as in lemma A1, and $h(z,g)$ is as in eq. (18) in the main text. The stochastic representation for q_g reveals that $q_g > 0$ (resp. < 0) if $m_2(z,g)q_z(z,g) > 0$ resp. (< 0) for all $(z,g) \in \mathbb{Z} \times \mathbb{G}$. And by the stochastic representation for q_{gg}, we have that $q_{gg} > 0$ (resp. < 0) if $h(z,g) > 0$ (resp. < 0) for all $(z,g) \in \mathbb{Z} \times \mathbb{G}$. $\blacksquare$

On the sign of q_z. Let $y(\tau) \equiv \partial z(\tau)/\partial z$ and $x(\tau) \equiv \partial g(\tau)/\partial z$ to be the first partials of the stochastic flows $z(\tau)$ and $g(\tau)$ with respect to the initial condition z of the dividend rate. Consider the following condition: there exist two constants c_0 and c_1 such that $c_1 + r > 0$, $c_0 > -(c_1 + r)$ and,

$$\forall \tau > 0, \quad \mathbb{E}\left[m_1(z(\tau), g(\tau))y(\tau) + m_2(z(\tau), g(\tau))x(\tau)\right] > c_0 \cdot \exp(-c_1\tau). \tag{C1}$$

As shown below, $\mathbb{E}[y(\tau)] > 0$ for sufficiently small τ. But if the dividend rate is not Markov, there may be sets T such that $\mathbb{E}[y(\tau)] < 0$ for all $\tau \in T$. Condition (C1) then ensures that such sets do not contribute too much to the overall sign of q_z.

Proof that $q_z > 0$ under condition (C1). The Feynman-Kac representation of the solution to the partial differential equation (A1) is $q(z,g) = \mathbb{E}\left[\int_0^\infty e^{-r}\, z(\tau)d\tau\right]$. By differentiating it with respect to z leaves:

$$q_z(z,g) = \int_0^\infty e^{-r}\, \mathbb{E}\left[y(\tau)\right] d\tau, \qquad \text{(C2)}$$

where $\mathbf{s}(\tau) \equiv (y(\tau)\ \ x(\tau))^\top$ satisfies:

$$
\begin{cases}
d\mathbf{s}(\tau) = \mathbf{M}(z(\tau),g(\tau))\mathbf{s}(\tau)d\tau + \mathbf{\Sigma}(z(\tau),g(\tau))\mathbf{s}(\tau)dW(\tau) \\[2mm]
\mathbf{s}(0) = (1\ \ 0)^\top
\end{cases}
$$

and

$$
\mathbf{M}(z,g) = \begin{pmatrix} m_1(z,g) & m_2(z,g) \\ \varphi_1(z,g) & \varphi_2(z,g) \end{pmatrix}
$$

$$
\mathbf{\Sigma}(z,g) = \begin{pmatrix} \sigma'(z) & 0 \\ 0 & \xi'(g) \end{pmatrix}
$$

This shows that

$$\mathbb{E}\left[y(\tau)\right] = 1 + \int_0 \mathbb{E}\left[m_1(z(u),g(u))y(u) + m_2(z(u),g(u))x(u)\right] du.$$

In particular, function $\tau \mapsto \mathbb{E}\left(y(\tau)\right)$ is continuous with $\lim_{\downarrow 0}\mathbb{E}\left(y(\tau)\right) = 1$. Therefore, there exists a τ^* such that for all $\tau \leq \tau^*$, $\mathbb{E}\left(y(\tau)\right) > 0$, as I claimed before giving condition (C1). To show that condition (C1) guarantees that $q_z > 0$, replace the previous relation into (C2) to obtain

$$
q_z(z,g) = \frac{1}{r} + \int_0^\infty e^{-r} \int_0 \left\{\mathbb{E}\left[m_1(z(u),g(u))y(u) + m_2(z(u),g(u))x(u)\right] du\right\} d\tau
$$

$$
> \frac{c_1 + r + c_0}{r\,(c_1 + r)},
$$

where the second line is obtained through condition (C1). Therefore, for all z, g, $q_z(z,g) > 0$ whenever $c_1 + r > 0$ and $c_0 > -(c_1 + r)$. ∎

Proof of proposition 3. If $q(z,g) = z \cdot v(g)$, functions h^3 and h^4 in lemma A1 collapse to $\overline{h}^3(z,g) \equiv [m_2(z,g) - R_2(z,g)z] \cdot v(g)$ and $\overline{h}^4(z,g) \equiv [m_{22}(z,g) - R_{22}(z,g)z]\,v(g) + [\varphi_{22}(z,g)z +$

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$2(m_2(z,g) - zR_2(z,g)) + \frac{2}{g^2}((\sigma\xi_1)(z,g))]v'(g)$, and the stochastic representations for $q_g = zv'$ and $q_{gg} = zv''$ are then as follows:

$$zv'(g) = \mathbb{E}\left[\int_0^\infty \kappa^3(\tau)\overline{h}^3(\zeta_3(\tau), \gamma_3(\tau))d\tau\right]$$

and

$$zv''(g) = \mathbb{E}\left[\int_0^\infty \kappa^4(\tau)\overline{h}^4(\zeta_i(\tau), \gamma_i(\tau))d\tau\right],$$

where κ^i, ζ^i and γ^i $(i = 3, 4)$ are as in lemma A1. ∎

Example C1. Consider a representative agent economy in which $m_0(z, g) = zg$, $\sigma(z) = \sigma_0 z$, ξ_1 and ξ_2 are independent of z. The agent has impatience rate $\delta > 0$ and instantaneous utility function $u(c) = (c^{1-} - 1)/(1 - \eta)$ and acts as in example A1 in appendix A. It is easy to show that in this case, $q(z, g) = z \cdot \int_0^\infty C(g, \tau)d\tau$, where

$$
\begin{aligned}
C(g, \tau) &= \mathbb{E}\left[\exp\left(-\int_0 R(g(t))dt\right)\frac{z(\tau)}{z}\right] \\
&= \mathbb{E}\left\{\exp\left[-(\delta + \frac{1}{2}\sigma_0^2\eta(1-\eta))\tau + (1-\eta)\int_0 g(t)dt - \frac{1}{2}\sigma_0^2\tau + \sigma_0\widehat{W}_1(\tau)\right]\right\} \\
&= \overline{\mathbb{E}}\left\{\exp\left[-(\delta + \frac{1}{2}\sigma_0^2\eta(1-\eta))\tau + (1-\eta)\int_0 g(t)dt\right]\right\},
\end{aligned}
$$

and $\overline{\mathbb{E}}$ is the expectation taken under measure $\overline{P}$ defined through the Radon-Nikodym derivative $d\overline{P}/dP^0 = \exp(-\frac{1}{2}\sigma_0^2\tau + \sigma_0 d\widehat{W}_1(\tau))$. Next, I assume that g is normally distributed, with $\varphi(z, g) = a - \theta g$, ξ_1 and ξ_2 constants, and $a \equiv a_0 - \eta\sigma_0\xi_1$, where a_0 is a positive parameter under the physical probability measure. By Girsanov's theorem,

$$dg(\tau) = [\overline{a} - \theta g(\tau)]d\tau + \xi_1 d\overline{W}_1(\tau) + \xi_2 d\overline{W}_2(\tau),$$

where $\overline{W}_1$ is a $\overline{P}$-Brownian motion, $\overline{W}_2 = \widehat{W}_2$ is also a $\overline{P}$-Brownian motion, and $\overline{a} \equiv a_0 + \sigma_0\xi_1(1-\eta)$. Using the fact that $\int_0 g$ is a $\overline{P}$-Gaussian process, some computations lead to:

$$C(g, \tau) = e^{-[\ -\frac{(1-\eta)\overline{a}}{\theta} + \frac{1}{2}\ _0^2\ (1-\ ) - \frac{1}{2}(\frac{(1-\eta)\xi_0}{\theta})^2]\ + [g - \frac{\overline{a}}{\theta} - (1-\ )(\frac{\xi_0}{\theta})^2]\frac{1-\eta}{\theta}(1-e^{-\theta\tau}) + \frac{(1-\eta)^2\xi_0^2}{4\theta^3}(1-e^{-2\theta\tau})},$$

where $\xi_0^2 \equiv \xi_1^2 + \xi_2^2$. As is clear, v is always convex; and $v' > 0$ (resp. $v' < 0$) whenever $\eta < 1$ (resp. $\eta > 1$), consistently with the prediction of proposition 3. Note that this model differs

from the Brennan and Xia (2001) one because R is not being kept constant here. Also, a model previously developed by Goldstein and Zapatero (1996) is a special case of the present model (namely, for $\xi_2 = 0$).

Example C2. ("Three-halves"). Let us be given the economy in example C1, and assume that $\varphi_0(g) = \kappa(a - g)g$, $\xi_1(g) = \xi_0 g^{\frac{3}{2}}$ and $\xi_2(g) = 0$, where κ, a and ξ_0 are constants. This model was introduced by Ahn and Gao (1999) in the term structure literature,[25] and by Lewis (2000) in the stochastic volatility option pricing literature. By proposition 3, the price-dividend ratio $v(g)$ is *concave* (resp. convex) in g whenever $\kappa > 1 - \eta$ *and* $\xi_0 < 0$ (resp. $\kappa < 1 - \eta$ *and* $\xi_0 > 0$).

Convexity issues in model (16). Consider the economy in example 1 [model (16)]. If the representative agent has CARA $\gamma > 0$ and impatience rate $\delta > 0$, $R = \delta + \gamma g - \frac{1}{2}\sigma_0^2 \gamma^2$. By lemma A1, $h^1 = 1$ (implying that $q_z > 0$), $h^2 = 0$ (implying that $q_{zz} = 0$), $h^3 = q_z - \gamma q$, $h^4 = 0$ (implying that $q_{gg} = 0$) and $h^5 = -\gamma q_z$ (implying that $q_{zg} < 0$). Therefore,

$$q(z, g) = c_0 + c_1 \cdot g + c_2 \cdot z + c_3 \cdot g \cdot z,$$

where $\{c_j\}_{j=0}^3$ are some constants with $c_3 < 0$ and $c_2 > |c_3 \cdot \max_{g \in \mathbb{G}} g|$ ($|c_3 \cdot \max_{g \in \mathbb{G}} g| < \infty$). Prices are *not* convex in g. Furthermore, q is nondecreasing in g for sufficiently low levels of γ on any compact set $\mathcal{O}$ of $\mathbb{Z} \times \mathbb{G}$. Indeed, let $q_g(z, g; \gamma) \equiv q_g(z, g)$. By lemma A1, $\lim_{\to 0} q_g(z, g; \gamma) > 0$. Hence, for fixed (z, g), there exists a $\gamma_0(z, g) : q_g(z, g; \gamma) > 0$ for all $\gamma \leq \gamma_0(z, g)$. Now set $(z^0, g^0) \in \arg\min_{(z,g) \in \mathcal{O}} \gamma_0(z, g)$. Then, $q_g(z, g; \gamma) > 0$ for all $\gamma \leq \gamma_0(z^0, g^0)$ on $\mathcal{O}$.

Finally, I develop details leading to a conjecture on additional qualitative properties of the price-dividend ratio $v(g)$ in proposition 3.

Let the short-term rate $R \equiv r$, a constant, and set

$$\mathcal{R}(\tau) \equiv \exp\left[-(r + \lambda_0 \sigma_0)\tau + \int_0 g(u)du\right],$$

and

$$\mathcal{G}_j(\tau) \equiv \int_0 \frac{\partial^j g}{\partial g_0^j}(u)du, \quad j = 1, 2,$$

where

$$dg(\tau) = [\varphi(g(\tau)) + \sigma_0 \xi(g(\tau))]\, d\tau + \xi_1(g(\tau))d\overline{W}_1(\tau) + \xi_2(g(\tau))d\overline{W}_2(\tau)$$

---

[25] In the present example, the short-term rate $R(g)$ is as in example C1, and thus follows the same dynamics originally assumed by Ahn and Gao. (The pricing kernels in the two economies differ, however.)

(as in section 3.2.1), and $\partial g / \partial g_0$ is the sensitivity of the stochastic flow g with respect to the initial condition g_0. Accordingly, $\mathcal{G}_2$ tracks the (cumulative) *curvature* of the stochastic flow. Finally, define

$$A \equiv A(g) \equiv cov \left[\int_0^\infty \mathcal{R}(\tau) d\tau, \ \int_0^\infty \mathcal{R}(\tau) \mathcal{G}_1(\tau)^2 d\tau \right].$$

I assume that

$$A > 0, \quad \text{all } g \in \mathbb{G}.$$

Let $B(g, t) \equiv \overline{\mathbb{E}}[[\mathcal{R}(t)]$. Under regularity conditions similar to ones provided in Mele (2002, appendix A, lemma A4), the partials of B in eq. (11) satisfy

$$B_1(g, t) \ = \ \overline{\mathbb{E}}[\mathcal{R}(t)\mathcal{G}_1(t)], \quad \text{all } (g, t) \in \mathbb{G} \times [0, \infty).$$

$$B_{11}(g, t) \ = \ \overline{\mathbb{E}}[\mathcal{R}(t)\mathcal{G}_1(t)^2] + \overline{\mathbb{E}}[\mathcal{R}(t)\mathcal{G}_2(t)], \quad \text{all } (g, t) \in \mathbb{G} \times [0, \infty).$$

I also assume that the conditions of proposition 3 ensuring convexity of B are satisfied, and so:

$$B_{11}(g, t) > 0, \quad \text{all } (g, t) \in \mathbb{G} \times [0, \infty).$$

By the stochastic representation in eq. (6), the ratio,

$$\mathcal{Z}(g) \equiv \frac{v'(g)}{v(g)}$$

is non-concave whenever v is convex and

$$\left[\int_0^\infty B_1(g, \tau) d\tau \right]^2 \leq \left[\int_0^\infty B_{11}(g, \tau) d\tau \right] \cdot \left[\int_0^\infty B(g, \tau) d\tau \right], \quad \text{all } g \in \mathbb{G}. \tag{C3}$$

We have,

$$
\begin{aligned}
\left[\int_0^\infty B_1(g,\tau)d\tau\right]^2 &= \left\{\int_0^\infty \overline{\mathbb{E}}\left[\mathcal{R}(\tau)\mathcal{G}_1(\tau)\right]d\tau\right\}^2 \\[2mm]
&= \left\{\overline{\mathbb{E}}\left[\int_0^\infty \mathcal{R}(\tau)\mathcal{G}_1(\tau)d\tau\right]\right\}^2 \\[2mm]
&\leq \overline{\mathbb{E}}\left[\int_0^\infty \mathcal{R}(\tau)\mathcal{G}_1(\tau)d\tau\right]^2 \\[2mm]
&\leq \overline{\mathbb{E}}\left[\left(\int_0^\infty \mathcal{R}(\tau)d\tau\right)\cdot\left(\int_0^\infty \mathcal{R}(\tau)\mathcal{G}_1(\tau)^2 d\tau\right)\right] \\[2mm]
&\leq \left\{\int_0^\infty \overline{\mathbb{E}}\left[(\mathcal{R}(\tau)\right]d\tau\right\}\cdot\left\{\int_0^\infty \overline{\mathbb{E}}\left[\mathcal{R}(\tau)\mathcal{G}_1(\tau)^2\right]d\tau\right\} + A \\[2mm]
&< \left\{\int_0^\infty \overline{\mathbb{E}}\left[(\mathcal{R}(\tau)\right]d\tau\right\}\cdot\left\{\int_0^\infty \overline{\mathbb{E}}\left[\mathcal{R}(\tau)\mathcal{G}_1(\tau)^2\right]d\tau\right\} \\[2mm]
&= \left[\int_0^\infty B(g,\tau)d\tau\right]\cdot\left\{\int_0^\infty \left[B_{11}(g,\tau) - \overline{\mathbb{E}}\left(\mathcal{R}(\tau)\mathcal{G}_2(\tau)\right)\right]d\tau\right\} \\[2mm]
&\leq \left[\int_0^\infty B(g,\tau)d\tau\right]\cdot\left[\int_0^\infty B_{11}(g,\tau)d\tau\right], \quad \text{all } g \in \mathbb{G},
\end{aligned}
$$

where I assumed that I was allowed to apply both Fubini-Tonelli theorem and Cauchy-Schwartz inequality. If any, this presumption can only be validated through a series of integrability constraints on $\mathcal{R}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ that I have been unable to develop in full generality.

The previous heuristic arguments lead me to conjecture that under suitable integrability conditions, the following statement is true:

Conjecture C1 [Nonnegative stochastic flow curvature structures.] *Suppose that $A > 0$, $v'' > 0$ and that the stochastic solution flow G has nonnegative (cumulative) curvature, viz $\mathcal{G}_2 \geq 0$. Then, $\mathcal{Z}$ is strictly convex.*

An example not necessarily nested into the framework of the previous conjecture is the following one. Suppose that B is exponential affine, viz $B(t) \equiv B(g,t) = \exp(\mathcal{A}(t) + \mathcal{B}(t)g)$, for

some functions $\mathcal{A}$ and $\mathcal{B}$. Then, eq. (C3) holds whenever:

$$\left[\int_0^\infty B(\tau)\mathcal{B}(\tau)d\tau\right]^2 \leq \left[\int_0^\infty B(\tau)\mathcal{B}(\tau)^2 d\tau\right] \cdot \left[\int_0^\infty B(\tau)d\tau\right], \quad \text{all } g \in \mathbb{G},$$

which is Cauchy-Schwartz inequality. Therefore, $\mathcal{Z}$ is non-concave in affine models under the standard regularity conditions underlying Cauchy-Schwartz inequality.

Appendix D: Proofs and examples for section 5

Proof of proposition 4. I provide second-order properties of price-dividend ratios for the general case. By eq. (5), $q(z,s)$ is solution to:

$$0 = \frac{1}{2}\sigma_0^2 z^2 q_{zz} + (g_0 - \sigma_0\lambda_1)z q_z + \frac{1}{2}\overline{\xi}^2 q_{ss} + (\phi - \xi \cdot \lambda)q_s + \sigma_0 z \xi_1 q_{zs} + z - Rq,$$

where $\overline{\xi}^2 \equiv \xi_1^2 + \xi_2^2$ and $\xi \cdot \lambda \equiv \xi_1\lambda_1 + \xi_2\lambda_2$. Next, suppose that $q(z,s) = z \cdot v(s)$. By replacing the proposed solution into the previous equation leaves:

$$0 = \frac{1}{2}\overline{\xi}^2 v'' + (\phi + \sigma_0\xi_1 - \lambda \cdot \xi)\,v' - (R - g_0 + \sigma_0\lambda_1)\,v + 1.$$

By the same arguments as in lemma A1, $v > 0$. Finally, define $v_1 \equiv v'$ and $v_2 \equiv v''$. By differentiating the previous differential equation, I find that v_1 and v_2 are solutions to:

$$0 = \frac{1}{2}\overline{\xi}^2 v_1'' + \left[\phi + \frac{1}{2}(\overline{\xi}^2)' + \sigma_0\xi_1 - \lambda \cdot \xi\right]v_1' - \left[R - g_0 + \sigma_0\lambda_1 - (\phi + \sigma_0\xi_1 - \lambda \cdot \xi)'\right]v_1 + \ell_1,$$

and

$$\begin{aligned}
0 &= \frac{1}{2}\overline{\xi}^2 v_2'' + \left[\phi + (\overline{\xi}^2)' + \sigma_0\xi_1 - \lambda \cdot \xi\right]v_2' \\
&\quad - \left[R - g_0 + \sigma_0\lambda_1 - 2\phi' - 2\sigma_0\xi_1' + 2(\lambda \cdot \xi)' - \frac{1}{2}(\overline{\xi}^2)''\right]v_2 + \ell_2,
\end{aligned}$$

where

$$\ell_1(s) \equiv -\left[R'(s) + \sigma_0\lambda_1'(s)\right]v(s),$$

$$\begin{aligned}
\ell_2(s) &\equiv -\left[R''(s) + \sigma_0\lambda_1''(s)\right]v(s) \\
&\quad + \left[(\phi(s) + \sigma_0\xi_1(s) - \lambda(s) \cdot \xi(s))'' - 2\left(R'(s) + \sigma_0\lambda_1'(s)\right)\right]v_1(s)
\end{aligned}$$

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The result follows by setting $\xi_2 = 0$, $\xi = \xi_1$, $\lambda = \lambda_1$, and by a direct application of the Feynman-Kac representation theorem, as in lemma A1 in appendix A. ∎

Additional details on example 3. Campbell and Cochrane (1999) originally considered a discrete-time model. The diffusion limit of their consumption process is simply eq. (21) given in the main text. By example A1 [eq. (A4)],

$$\lambda(z, x) = \frac{\eta}{s}\left[\sigma_0 - \frac{1}{z}\gamma(z, x)\right]. \tag{D1}$$

Next, $x = z(1 - s)$, where s is solution to eq. (22). By Itô's lemma, $\gamma = [1 - s - sl(s)]z\sigma_0$. By eq. (D1), $\lambda(s) = \eta\sigma_0[1 + l(s)]$, as claimed in the main text. (This result only *approximately* holds in the original discrete time framework.) Finally, by an application of formula (A3), $R(s) = \delta + \eta\left(g_0 - \frac{1}{2}\sigma_0^2\right) + \eta(1 - \phi)(\overline{s} - \log s) - \frac{1}{2}\eta^2\sigma_0^2[1 + l(s)]^2$. R is constant whenever

$$l(s) = \frac{1}{\overline{S}}\sqrt{1 + 2(\overline{s} - \log s)} - 1, \quad s \in \left(0, \overline{S} \cdot e^{\frac{1}{2}(1 - \overline{S}^2)}\right),$$

where $\overline{S} = \sigma_0\sqrt{\eta/(1 - \phi)} = \exp(\overline{s})$. This corresponds to the same original restriction as in Campbell and Cochrane.

Additional details on example 4. To apply the theoretical test conditions in proposition 4 to the model in example 4, one needs to know Sharpe ratios and interest rates - which Basak and Cuoco (1998) do not report in their article. The representative agent in this economy has (undiscounted) instantaneous utility $u(z, x)$ as in example A1. By eq. (A4), λ thus satisfies:

$$\lambda(z, x) = -\frac{u_{11}(z, x)}{u_1(z, x)}\sigma_0 z + \frac{u_{12}(z, x)x}{u_1(z, x)}\lambda(z, x).$$

This is:

$$\begin{aligned}
\lambda(z, x) &= -\frac{u_1(z, x)u_{11}(z, x)}{u_1(z, x) - u_{12}(z, x)x} \cdot \frac{\sigma_0 z}{u_1(z, x)} \\
&= -\frac{u_a''(\widehat{c}_a)}{u_1(z, x)}\sigma_0 z \\
&= -\frac{u_a''(\widehat{c}_a)\widehat{c}_a}{u_a'(\widehat{c}_a)}\sigma_0 s^{-1}.
\end{aligned}$$

where the second line follows by Basak and Cuoco [identity (33), p. 331] and the third line follows by the definition of $u(z, x)$ and s. The Sharpe ratio reported in the main text follows by

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the definition of u_a. The interest rate is also found using example A1. The final result is:

$$R(s) = \delta + \frac{\eta g_0}{\eta - (\eta - 1)s} - \frac{1}{2}\frac{\eta(\eta+1)\sigma_0^2}{s(\eta - (\eta-1)s)}.$$

Finally, by applying Itô's lemma to $s = c_a / z$, and using the optimality conditions for agent a, I find that drift and diffusion functions of s are given by:

$$\phi(s) = g_0\left[\frac{(1-\eta)(1-s)}{\eta + (1-\eta)s}\right]s - \frac{1}{2}\frac{(\eta+1)\sigma_0^2}{\eta + (1-\eta)s} + \frac{1}{2}\frac{(\eta+1)\sigma_0^2}{s} + \sigma_0(s-1),$$

and $\xi(s) = \sigma_0(1-s)$.

Appendix E: Proofs for section 6

Proof of proposition 5 [parts a and b]. By absence of arbitrage opportunities, $q(z, y_1, y_2, y_3)$ is solution to:

$$0 = (L_q - R(z, y_1, y_2, y_3))q(z, y_1, y_2, y_3) + z, \tag{E1}$$

where

$$\begin{aligned} L_q q &= \frac{1}{2}\sigma^2 q_{zz} + m q_z + \frac{1}{2}\sum_{i=1}^{3}\bar{\xi}_i^2 q_{y_iy_i} + \sum_{i=1}^{3}\varphi^{(i)}q_{y_i} + \sigma\sum_{i=1}^{3}\xi_1^{(i)}q_{zy_i} \\ &\quad + \xi^{1,2}q_{y_1y_2} + \xi^{1,3}q_{y_1,y_3} + \xi^{2,3}q_{y_2,y_3}, \end{aligned}$$

and $m \equiv m_0 - \lambda\sigma$.

Let $a \equiv q_{y_1}$ and $b \equiv q_{y_1y_1}$. Functions a and b are solutions to:

$$0 = (L_j - k_j)j + h_j, \quad j = a, b, \tag{E2}$$

where $k_a = R - \frac{(1)}{y_1}$, $k_b = R - 2\frac{(1)}{y_1} - \frac{1}{2}\frac{2}{y_1^2}\bar{\xi}_1^2$,

$$L_a a = L^3 a + L_{aa}a$$

$$L_b b = L^4 b + L_{bb}b$$

with

$$L_{\mathsf{aa}}[a] \;=\; \sum_{i=2}^{3}\left(\tfrac{1}{2}\overline{\xi}_i^2 a_{\mathsf{y}_i\mathsf{y}_i} + \varphi^{(i)}a_{\mathsf{y}_i} + \sigma\xi_1^{(i)}a_{\mathsf{z}\mathsf{y}_i}\right) + \frac{\partial\xi^{1,2}}{\partial y_1}a_{\mathsf{y}_2} + \xi^{1,2}a_{\mathsf{y}_1\mathsf{y}_2} + \frac{\partial\xi^{1,3}}{\partial y_1}a_{\mathsf{y}_3} + \xi^{1,3}a_{\mathsf{y}_1\mathsf{y}_3}$$
$$+\,\xi^{2,3}a_{\mathsf{y}_2\mathsf{y}_3}$$

$$L_{\mathsf{bb}}[b] \;=\; L_{\mathsf{aa}}[b] + \frac{\partial\xi^{1,2}}{\partial y_1}b_{\mathsf{y}_2} + \frac{\partial\xi^{1,3}}{\partial y_1}b_{\mathsf{y}_3}$$

and L^3 and L^4 are the operators defined in appendix A (lemma A1) (with $(y_1,\overline{\xi}_1^2,\varphi^{(1)},\xi_1^{(1)})$ replacing (g,ξ^2,φ,ξ_1)), and finally,

$$h_{\mathsf{a}}(z,y) \;=\; m_2(z,y_1,y_2,y_3)q_{\mathsf{z}}(z,y_1,y_2,y_3) - R_2(z,y_1,y_2,y_3)q(z,y_1,y_2,y_3) + M(z,y_1,y_2,y_3)$$

$$\begin{aligned}
h_{\mathsf{b}}(z,y) \;=\;& m_{22}(z,y_1,y_2,y_3)q_{\mathsf{z}}(z,y_1,y_2,y_3) + \varphi_{22}^{(1)}(z,y_1,y_2,y_3)q_{\mathsf{y}_1}(z,y_1,y_2,y_3)\\
&+\left[2m_2(z,y_1,y_2,y_3) + \tfrac{2}{y_1^2}(\sigma\xi_1^{(1)})(z,y_1,y_2,y_3)\right]q_{\mathsf{z}\mathsf{y}_1}(z,y_1,y_2,y_3)\\
&-R_{22}(z,y_1,y_2,y_3)q(z,y_1,y_2,y_3) - 2R_2(z,y_1,y_2,y_3)q_{\mathsf{y}_1}(z,y_1,y_2,y_3)\\
&+N(z,y_1,y_2,y_3)
\end{aligned}$$

where

$$M \;\equiv\; \sum_{i=2}^{3}\left(\frac{1}{2}\frac{\partial\overline{\xi}_i^2}{\partial y_1}q_{\mathsf{y}_i\mathsf{y}_i} + \frac{\partial\varphi^{(i)}}{\partial y_1}q_{\mathsf{y}_i} + \sigma\frac{\partial\xi_1^{(i)}}{\partial y_1}q_{\mathsf{z}\mathsf{y}_i}\right) + \frac{\partial\xi^{2,3}}{\partial y_1}q_{\mathsf{y}_2,\mathsf{y}_3}$$

$$\begin{aligned}
N \;\equiv\;& \sum_{i=2}^{3}\left[\frac{1}{2}\frac{\partial^2\overline{\xi}_i^2}{\partial y_1^2}q_{\mathsf{y}_i\mathsf{y}_i} + \frac{\partial\overline{\xi}_i^2}{\partial y_1}q_{\mathsf{y}_1\mathsf{y}_i\mathsf{y}_i} + \frac{\partial^2\varphi^{(i)}}{\partial y_1^2}q_{\mathsf{y}_i} + 2\frac{\partial\varphi^{(i)}}{\partial y_1}q_{\mathsf{y}_1\mathsf{y}_i} + \sigma\frac{\partial^2\xi_1^{(i)}}{\partial y_1^2}q_{\mathsf{z}\mathsf{y}_i} + 2\sigma\frac{\partial\xi_1^{(i)}}{\partial y_1}q_{\mathsf{y}_1\mathsf{z}\mathsf{y}_i}\right]\\
&+\frac{\partial^2\xi^{1,2}}{\partial y_1^2}q_{\mathsf{y}_1,\mathsf{y}_2} + \frac{\partial^2\xi^{1,3}}{\partial y_1^2}q_{\mathsf{y}_1,\mathsf{y}_3} + \frac{\partial^2\xi^{2,3}}{\partial y_1^2}q_{\mathsf{y}_2,\mathsf{y}_3} + 2\frac{\partial\xi^{2,3}}{\partial y_1}q_{\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_3}
\end{aligned}$$

By arguments nearly identical to the ones developed in appendix A, $j > 0$ (resp. < 0) whenever $h_j > 0$ (resp. < 0), $j = a,b$. In particular, let R be independent of z, and let $m(z,y) = z\cdot d(y)$, where $d(y) = d_0(y) - \sigma_0\lambda_1(y)$, as assumed in the proposition. Then, function $q(z,y) = z\cdot v(y)$ satisfies eq. (E1), and functions $h.$ in eq. (E2) are $h_{\mathsf{a}} = \overline{h}_{\mathsf{a}}$ and $h_{\mathsf{b}} = \overline{h}_{\mathsf{b}}$, where:

$$\overline{h}_{\mathsf{a}}(z,y) \;=\; z\cdot\left\{\left[d_{\mathsf{y}_1}(y) - R_{\mathsf{y}_1}(y)\right]v(y) + \overline{M}(y)\right\}$$

$$\begin{aligned}
\overline{h}_{\mathsf{b}}(z,y) \;=\;& z\cdot\left[d_{\mathsf{y}_1\mathsf{y}_1}(y) - R_{\mathsf{y}_1\mathsf{y}_1}(y)\right]v(y)\\
&+z\cdot\left[2d_{\mathsf{y}_1}(y) + \varphi_{\mathsf{y}_1\mathsf{y}_1}^{(1)}(y) + \sigma_0\tfrac{2}{y_1^2}\xi_1^{(1)}(y) - 2R_{\mathsf{y}_1}(y)\right]v_{\mathsf{y}_1}(y) + z\cdot\overline{N}(y)
\end{aligned}$$

and

$$\overline{M} \equiv \sum_{i=2}^{3}\left[\frac{1}{2}\frac{\partial \overline{\xi}_i^2}{\partial y_1}v_{y_i y_i} + \left(\frac{\partial \varphi^{(i)}}{\partial y_1} + \sigma_0\frac{\partial \xi_1^{(i)}}{\partial y_1}\right)v_{y_i}\right] + \frac{\partial \xi^{2,3}}{\partial y_1}v_{y_2,y_3}$$

$$\overline{N} \equiv \sum_{i=2}^{3}\left[\frac{1}{2}\frac{\partial^2 \overline{\xi}_i^2}{\partial y_1^2}v_{y_i y_i} + \frac{\partial \overline{\xi}_i^2}{\partial y_1}v_{y_1 y_i y_i} + \frac{\partial^2 \varphi^{(i)}}{\partial y_1^2}v_{y_i} + 2\frac{\partial \varphi^{(i)}}{\partial y_1}v_{y_1 y_i} + \sigma_0\frac{\partial^2 \xi_1^{(i)}}{\partial y_1^2}v_{y_i} + 2\sigma_0\frac{\partial \xi_1^{(i)}}{\partial y_1}q_{y_1 y_i}\right]$$
$$+ \frac{\partial^2 \xi^{1,2}}{\partial y_1^2}v_{y_1,y_2} + \frac{\partial^2 \xi^{1,3}}{\partial y_1^2}v_{y_1,y_3} + \frac{\partial^2 \xi^{2,3}}{\partial y_1^2}v_{y_2,y_3} + 2\frac{\partial \xi^{2,3}}{\partial y_1}v_{y_1,y_2,y_3}$$

Parts a) and b) in proposition 5 then follow by arguments similar to ones used to show lemma A1: $v_{y_1} > 0$ whenever $\overline{h}_a > 0$ and $v_{y_1 y_1} > 0$ (resp. < 0) whenever $\overline{h}_{bb} > 0$ (resp. < 0). ∎

Proof of proposition 5 [part c]. I only demonstrate that part c of proposition 5 holds for three-factor models. The general case is dealt with similar arguments. Let $g \equiv y_1$ and $s \equiv y_3$. The price-dividend ratio $v(g,s)$ satisfies:

$$0 = \frac{1}{2}\overline{\xi}_1^2(g)v_{gg}(g,s) + \frac{1}{2}\overline{\xi}_3^2(s)v_{ss}(g,s) + \xi^{1,3}(g,s)v_{gs}(g,s)$$

$$+ \left[\varphi_0^{(1)}(g) + (\sigma_0 - \lambda(s))\,\xi_1^{(1)}(g)\right]v_g(g,s) + \left[\varphi_0^{(3)}(s) + (\sigma_0 - \lambda(s))\,\xi_1^{(3)}(s)\right]v_s(g,s)$$

$$- [R(s,g) - d(s,g)]\,v(g,s) + 1.$$

By arguments similar to ones in the proof of proposition 5, I find that $w \equiv v_{\mathsf{gs}}$ is solution to:

$$
\begin{aligned}
0 &= \frac{1}{2}\bar{\xi}_1^2(g)w_{\mathsf{gg}}(g,s) + \frac{1}{2}\bar{\xi}_3^2(s)w_{\mathsf{ss}}(g,s) + \xi^{1,3}(g,s)w_{\mathsf{gs}}(g,s) \\
&\quad + \left[\frac{1}{2}(\bar{\xi}_1^2(g))' + \xi_{\mathsf{s}}^{1,3}(g,s) + \varphi_0^{(1)}(g) + (\sigma_0 - \lambda(s))\,\xi_1^{(1)}(g)\right] w_{\mathsf{g}}(g,s) \\
&\quad + \left[\frac{1}{2}(\bar{\xi}_3^2(s))' + \xi_{\mathsf{g}}^{1,3}(g,s) + \varphi_0^{(3)}(s) + (\sigma_0 - \lambda(s))\,\xi_1^{(3)}(s)\right] w_{\mathsf{s}}(g,s) \\
&\quad - \Big[R(g,s) - d(g,s) - \xi_{\mathsf{gs}}^{1,3}(g,s) - (\varphi_0^{(1)}(g))' + \xi_1^{(1)}(g)'\lambda(s) \\
&\qquad\quad -(\varphi_0^{(3)}(s) - \xi_1^{(3)}(s)\lambda(s))' + \sigma_0\left((\xi_1^{(1)}(g))' + (\xi_1^{(3)}(s))'\right)\Big] w(g,s) \\
&\quad + h^{\mathsf{w}}(g,s),
\end{aligned}
$$

where

$$
\begin{aligned}
h^{\mathsf{w}}(g,s) &\equiv [d_{\mathsf{gs}}(g,s) - R_{\mathsf{gs}}(g,s)]\,v(g,s) + \left[d_{\mathsf{s}}(g,s) - R_{\mathsf{s}}(g,s) - (\xi_1^{(1)}(g))'\lambda'(s)\right] v_{\mathsf{g}}(g,s) \\
&\quad + [d_{\mathsf{g}}(g,s) - R_{\mathsf{g}}(g,s)]\,v_{\mathsf{s}}(g,s) - \xi_1^{(1)}(g)\lambda'(s)v_{\mathsf{gg}}(g,s).
\end{aligned}
$$

The result follows by the Feynman-Kac representation theorem applied to function w. ∎

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