On Modelling Endogenous Default*

Dimitrios P. Tsomocos† Lea Zicchino‡

September 30, 2005

Abstract

Not only in the classic Arrow-Debreu model, but also in many mainstream macro models, an implicit assumption is that all agents honour their obligations, and thus there is no possibility of default. That leads to well-known problems in providing an essential role for either money or for financial intermediaries. So, in more realistic models, the introduction of minimal financial institutions, for example default and banks, becomes a logical necessity. But if default involved no penalties, everyone would do so. Hence there must be default penalties to

*This paper represents the views and analysis of the authors and should not be thought to represent those of the Bank of England. The paper would not have been written without the continuous encouragement, support, and helpful comments by Charles Goodhart, who first raised the economice relevance of the interconnection among endogenous default and exogenous probabilities of default. We are also grateful to C. Mayer and H. Polemarchakis for their valuable comments.

†Bank of England, Said Business School and St. Edmund Hall, University of Oxford, and Financial Markets Group. Email: dimitrios.tsomocos@sbs.ox.ac.uk

‡Bank of England. Email: lea.zicchino@bankofengland.co.uk

allow for an equilibrium with partial default. What we show here is that there is an equivalence between a general equilibrium model with incomplete markets (GEI) and endogeneous default, and a model with exogenous probabilities of default (PD). The practical, policy implications are that a key function of regulators (via bankruptcy codes and default legislation), or the markets (through default premia) are broadly substitutable. The balance between these alternatives depends, however, on many institutional details, which are not modelled here, but should be a subject for future research.

1 Introduction

One of the key features of recent papers aiming to analyze financial fragility (Goodhart, Sunirand and Tsomocos (2006) and Tsomocos (2003a, 2003b)) is the modelling of endogenous default. The idea of including the possibility of default in general equilibrium models can be traced back at least to Shubik and Wilson (1977). Subsequently Dubey and Geanakoplos (1992) and Dubey, Geanakoplos, Shubik (2005) formally analyzed default in models with and without uncertainty.[1] In the classic Arrow-Debreu model, an implicit assumption is that all agents honour their obligations, and thus there is no possibility of default. However, when one uses models, such as strategic market games *á la* Shapley and Shubik (1977), then the introduction of minimal institutions, for example money, credit and default, becomes a logical necessity. In particular, Shubik and Wilson (1977) allow agents to choose their repayment rates. Thus equilibrium becomes compatible with partial or complete abrogation of agents' contractual obligations. If agents are not accountable for their repayments, they will rationally choose not to repay any of their debts. Thus, we are naturally led to introduce default penalties[2]

[1] Hart and Moore (1994) have also analyzed endogenous default based on the idea of the inalienability of labour. Since future labour contracts can not be binding, default may arise endogenously.

[2] Default penalties may be either pecuniary or non-pecuniary and they are modelled by subtracting a linear term from the utility function which is proportional to the level

that constrain agents' choices of repayment. In sum, if these default penalties are infinite then the model reduces to the standard Arrow-Debreu model - plus the added constraint, under uncertainty and incomplete markets, that none will borrow - whereas if these penalties are zero no equilibrium can be established, since there will be unbounded credit demand and zero credit supply. In conclusion, Shubik and Wilson treat default continuously (i.e. they allow for partial default in equilibrium), thus providing a useful framework to analyze financial fragility as we encounter it in reality (and not only extreme phenomena such as complete disruption of credit markets as, for example, in Diamond and Dybvig (1983)).

Goodhart, Sunirand and Tsomocos (GST) incorporate a number of commercial banks in a general equilibrium model with incomplete markets. Each bank raises funds in the interbank market and from depositors, and extends loans to households. Both banks and households may default on their debt contracts for strategic reasons or due to ill fortune. If they choose not to honour their contract obligations fully, they incur penalties in proportion to the amount by which they default. One of the features of the model, and generally of the models with endogenous default mentioned above, is that assets are characterized as pools. Different sellers of the same asset typically

---

of debt. See Dubey and Geanakoplos (1992) for the general treatment of default in these classes of models.

default in different events, and in different proportions. However, the buyers of the asset receive a *pro rata* share of all the sellers' deliveries. When the pools are large, a buyer can reasonably assume that both the price of the asset and the pool delivery rate are unaffected by the number of shares he buys. These features maintain anonymity in the market and price taking behaviour.

The prospect of banks partially defaulting on deposit contracts (or in the interbank market) may seem unreasonable if suspension of convertibility is not a possibility. In other words, a bank's default on its demand deposits (or on its loan obligations towards other banks) triggers closure. Thus, a natural question arises as to whether the Shubik framework of endogenous default is compatible with the institutional features of the banking sector whereby, if there is any default on deposits (or interbank transactions), a bank is forced to shut. A related issue to the previous question is whether the modelling of endogenous default can serve as an appropriate microfoundation of the commonly used probabilities of default in applied analysis of credit risk, corporate default, etc., (see Merton (1974)). Our aim is to assess whether this class of models can address this issue.

We will argue that there exists an equivalence between a general equilibrium model with incomplete markets and endogenous default and one with exogenous probabilities of default. In particular, we will show that the Dubey, Geanakoplos and Shubik (DGS) model is equivalent to a model where assets are defined not only with respect to their returns but also with respect to their exogenously specified state dependent probabilities of default.

Put differently, we will show that the DGS model produces the same equilibrium allocation as a standard GEI economy where assets are defaultable. The upshot of our analysis is that endogenous default provides microfoundations to exogenous probabilities of default. Moreover, all these modelling techniques shed light on the general issue of partial fulfillment of contractual obligations in modern economies. Indeed, one would argue that a key function that the regulator (through default legislation), or the markets (through default premia) serve in our stylized model are broadly substitutable. Institutional factors, such as transaction costs or legislative and executive powers, influence which of the approaches are adopted in practice. However, at this level of simplicity we can not offer an analytical argument to determine which of the two will be implemented. There will be much adverse selection, with those expecting a high likelihood of defaulting trying to borrow. Thus, in the absence of common knowledge, default premia can not work very efficiently, collateral is scarce, and this provides the *raison d'être* for default penalties.

The rest of the paper is organized as follows. In section 2 we present the DGS model and show its equivalence with the defaultable asset economy. Section 3 concludes and offers some potential extensions to our analysis.

2 Endogenous default, probabilities of default and assets

We first consider the DGS model. It is the canonical general equilibrium model with incomplete markets where agents are allowed to default on their contractual obligations on asset sales. For the sake of exposition, we consider a simplified version in which time extends over two periods and there are only two possible states of nature in the second period. Thus, we allow only one asset to be traded in the first period to maintain market incompleteness. There also exists one commodity per period.[3] Without loss of generality, the asset is in zero net supply. Agents trade in the commodity market and also in the asset market in the first period. In the second period, the asset pays off and commodity trading occurs. In addition, agents are allowed to default in the asset market but they are penalized proportionally to the amount of

[3]The extension to a multicommodity GEI requires a more thorough analysis of different forms of default and of real versus nominal penalties.

their debt.[4]

Formally, the notation that will be used henceforth is as follows:

$t \in T = \{0, 1\}$ = time periods,

$s \in S = \{1, 2\}$ = set of states at $t = 1$,

$S^* = \{0\} \cup S$ = set of all states,

$h \in H = \{1, 2, ...H\}$ = set of economic agents (households),

$l \in L = \{1\}$ = set of commodities,

$R_+ \times R_+^S$ = commodity space,

$e^h \in R_+ \times R_+^S$ = endowments of households,

$u^h : R_+ \times R_+^S \to R$ = utility function of agent $h \in H$,

x_s^h = consumption of the commodity in state s by agent $h \in H$,

$A \in R_+^S$ = promise per unit of the asset of the commodity in each state $s \in S$,

$Q^h \in R_+^S$ = quantity restriction on sale of the asset for agent h,

$\lambda_s^h \in \bar{R}_+ = R_+ \cup \{\infty\}$ = real default penalty on agent h for the asset in state s.

The endogenous variables to be determined in equilibrium are the following:

$p \in R_+^{S^*}$ = commodity price vector,

[4] This model does not have a monetary sector but it can be easily incorporated along the lines of Dubey and Geanakoplos (1992) and Tsomocos (2003a, 2003b).

$q \in R_+ =$ asset price,

$K \in [0,1]^S =$ expected delivery rate on the asset,

$x^h \in R_+^{S^*} =$ consumption of the commodity by agent h,

$\theta^h \in R_+ =$ asset **purchase** of h,

$\varphi^h \in R_+ =$ asset **sale** of h,

$D^h \in R_+^S =$ delivery by agent h on the asset.

The quantity constraints on asset sales are finite and capture the idea that in any realistic model there exist some limits to credit. The parameters λ_s^h represent the marginal disutility of defaulting for each "real" dollar on assets in state s. Therefore, the payoff to households will be $\forall s \in S^*$:

$$\Pi_s^h \left(x_s^h, D_s^h, p_s \right) = u_s^h \left(x_s^h \right) - \frac{\lambda_s \max \left[0, \left(\varphi p_s A_s - p_s D_s^h \right) \right]}{p_s v_s},$$

where v_s is the base basket of goods which serves as a price deflator with respect to which the default penalty is measured.[5]

The standard assumptions hold:

(A1) $\forall s \in S^*$, $\sum_{h \in H} e_s^h > 0$ (i.e. the commodity is present in all states of the world).

(A2) $\forall s \in S^*$ and $h \in H$, $e_s^h > 0$ for some $s \in S^*$ (i.e. no household has

[5] In our case $L = 1$ so v_s can be set equal to 1, $\forall s \in S^*$. However, whenever $L > 1$ the specification of v_s becomes crucial.

the null endowment of the commodity in any state of the world).

(A3) Let A be the maximum amount of the commodity s that exists and let 1 denote the unit vector in R^S. Then $\exists Q > 0 \ni u^h(0, ..., Q, ...0) > u^h(A1)$ for Q in an ordinary component (i.e. strict monotonicity in every component). Also, continuity and concavity are assumed.

The economy is defined as a vector

$$E = \left\{ \left(u^h, e^h\right)_{h \in H}; A, \left((\lambda_s)_{s \in S}, Q^h\right)_{h \in H} \right\}.$$

The equilibrium of the economy is characterised by the vector $\left(x^h, \theta^h, \varphi^h, D^h, p, q, K\right)$, where $\left(x^h, \theta^h, \varphi^h, D^h\right)$ are arg max of the following problem:

$$\max_{x^h, \theta^h, \varphi^h, D^h} u^h\left(x_0^h\right) + \sum_s \pi_s \left(u^h\left(x_s^h\right) - \frac{\lambda_s^h \max\left[0, \left(\varphi p_s A_s - p_s D_s\right)\right]}{p_s v_s} \right) \tag{P1}$$

$$s.t. \quad p_0\left(x_0^h - e_0^h\right) + q\left(\theta^h - \varphi^h\right) \leq 0 \tag{1}$$

and $\forall s \in S$,

$$p_s\left(x_s^h - e_s^h\right) + p_s D_s \leq \theta K_s p_s A_s \tag{2}$$

The market clearing conditions are the following:

$$\sum_{h\in H}\left(x^h - e^h\right) = 0 \tag{3}$$

$$\sum_{h\in H}\left(\theta^h - \varphi^h\right) = 0 \tag{4}$$

$$K_s = \begin{cases} \frac{\sum_{h\in H} p_s D_s^h}{\sum_h \varphi p_s A_s^h}, & if \quad \sum_h \varphi p_s A_s^h > 0 \\ arbitrary, & if \quad \sum_h \varphi p_s A_s^h = 0 \end{cases} \tag{5}$$

So far we have defined an asset as a vector of payoffs in the various states of nature. We now extend the definition of an asset by associating each state contingent payoff with an exogenously specified repayment rate. Note that if this repayment value is equal to 100% in certain states and 0% in others, then we can calculate the probability of default of each asset.[6] Thus, assets are now defined as pairs (A, R), where $A = (A_1, ..., A_s, ..., A_S)$ is the vector of promises in the different states as before and $R = (R_1, ..., R_s, ..., R_S)$ is the vector of exogenous repayments. We can now define a single asset economy

[6] If we assume complete markets, we can calculate unique risk-neutral probabilities and thus unique probabilities of default. However, this is not true when markets are incomplete. In such a case, we may use the subjective probabilities over the states of nature of the issuer of an asset to obtain default probabilities. Of course, this implies that different issuers will have different probabilities of default.

$\tilde{E}$, where there is no endogenous default as follows:

$$\tilde{E} = \left\{ \left(u^h, e^h\right)_{h\in H}; (A, R)_{s\in S}, \left(Q^h\right)_{h\in H} \right\}.$$

The agents' maximisation problem can be written as follows:

$$\max_{x,\theta,\varphi} u(\tilde{x}_0) + \sum_s \pi_s u(\tilde{x}_s) \tag{P2}$$

$$s.t.: \tilde{p}_0(\tilde{x}_0 - \tilde{e}_0) + \tilde{q}\left(\tilde{\theta} - \tilde{\varphi}\right) \leq 0, \tag{6}$$

and $\forall s \in S$,

$$\tilde{p}_s(\tilde{x}_s - \tilde{e}_s) + \tilde{p}_s R_s A_s \tilde{\varphi} \leq \tilde{p}_s \tilde{\theta} R_s A_s, \tag{7}$$

where $\tilde{K}_s = \frac{\sum_h p_s R_s}{\sum_h \tilde{\varphi} p_s A_s}$.

The equilibrium is as before except that (P1) is replaced by (P2) and (5) no longer applies.

We are now ready to state our result, namely, that the two economies E and $\tilde{E}$ are equivalent for the appropriate selection of λ_s^h and R_s, $\forall s \in S$.

Proposition 1. (i) If $\left\{x^h, \theta^h, \varphi^h, D^h, p, q, K\right\}$ constitute an equilibrium for E, then there exist R_s, $\forall s \in S$, such that $\left\{x^h, \theta^h, \varphi^h, p, q\right\}$ constitute an equilibrium for $\tilde{E}$. Conversely, (ii) if $\left\{\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, \tilde{p}, \tilde{q}\right\}$ constitute an equilibrium for $\tilde{E}$, then there exist λ_s^h, D_s^h, K_s, $\forall s \in S$, and $h \in H$, such that $\left\{\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, D^h, \tilde{p}, \tilde{q}, K\right\}$ constitute an equilibrium for E.

Proof. In the case where there is no default, i.e. $D_s^h = \varphi^h A_s$, $\forall s \in S$, $h \in H$, set $K_s = R_s = 1$, $\forall s \in S$. Then the two problems (P1) and (P2) are identical and therefore produce the same optimal choices for the optimisation problem and markets clear at the same prices.

If the previous case does not obtain, then the first order conditions of E are:

$$\frac{\partial u^h\left(x_0^h\right)}{\partial x_0^h} - y_0 p_0 = 0 \tag{8}$$

$$\pi_s \frac{\partial u^h\left(x_s^h\right)}{\partial x_s^h} - y_s p_s = 0 \qquad \forall s \in S, \tag{9}$$

$$- y_0 q + y_1 p_1 K_1 A_1 + y_2 p_2 K_2 A_2 = 0 \tag{10}$$

$$- \pi_1 \lambda_1 A_1 - (1 - \pi_1)\lambda_2 A_2 + y_0 q = 0 \tag{11}$$

$$\pi_s \lambda_s - y_s p_s = 0 \qquad \forall s \in S, \tag{12}$$

$$y_0\left[p_0\left(x_0^h - e_0^h\right) + q\left(\theta^h - \varphi^h\right)\right] = 0 \tag{13}$$

$$y_s\left[p_s\left(x_s^h - e_s^h\right) + p_s D_s^h - \theta^h p_s K_s A_s\right] = 0 \qquad \forall s \in S, \tag{14}$$

where $y_s \in S^*$ are the Lagrange multipliers associated to the budget constraints.

Furthermore, the first order conditions of $\tilde{E}$ are:

$$\frac{\partial u^h\left(\tilde{x}_0^h\right)}{\partial \tilde{x}_0^h} - \tilde{y}_0\tilde{p}_0 = 0 \tag{15}$$

$$\pi_s\frac{\partial u^h\left(\tilde{x}_s^h\right)}{\partial \tilde{x}_s^h} - \tilde{y}_s\tilde{p}_s = 0 \qquad \forall s \in S, \tag{16}$$

$$-\tilde{y}_0\tilde{q} + \tilde{y}_1\tilde{p}_1R_1A_1 + \tilde{y}_2\tilde{p}_2R_2A_2 = 0 \tag{17}$$

$$\tilde{y}_0\tilde{q} - \tilde{y}_1\tilde{p}_1R_1A_1 - \tilde{y}_2\tilde{p}_2R_2A_2 = 0 \tag{18}$$

$$\tilde{y}_0\left[\tilde{p}_0\left(\tilde{x}_0 - \tilde{e}_0\right) + \tilde{q}\left(\tilde{\theta} - \tilde{\varphi}\right)\right] = 0 \tag{19}$$

$$\tilde{y}_s\left[\tilde{p}_s\tilde{\theta}^h R_sA_s - \tilde{p}_s\left(\tilde{x}_s - \tilde{e}_s\right) - \tilde{p}_s\tilde{\varphi}^h R_sA_s\right] = 0 \qquad \forall s \in S, \tag{20}$$

If we set $y_sp_s = \tilde{y}_s\tilde{p}_s$, $\forall s \in S^*$, then (15) and (16) produce the same consumption plan as (8) and (9). If we set $R_s = K_s$, $\forall s \in S$, then (10) and (17) are also identical. Note that (17) and (18) are identical as well. It remains to show that the budget constraints (19)-(20) are satisfied with the equilibrium values of E. (13) and (19) are identical. Finally, given that $D_s^h = R_sA_s\tilde{\varphi}^h$, (20) is also satisfied.

(ii) Let $K_s = R_s$, and $D_s^h = R_sA_s\tilde{\varphi}^h$, $\forall s \in S$, $h \in H$. By plugging (12) into (10) we obtain:

$$-y_0^h q + \pi_1 \lambda_1^h K_1 A_1 + (1 - \pi_1) \lambda_2^h K_2 A_2 = 0.$$

Given that $K_s = R_s$, $\forall s \in S$, and the optimal values $\tilde{y}_0^h, \tilde{q}$, we have:

$$-\tilde{y}_0^h \tilde{q} + \pi_1 \lambda_1^h K_1 A_1 + (1 - \pi_1) \lambda_2^h K_2 A_2 = 0.$$

Thus, we can find λ_s^h, $\forall s \in S$, that solve the previous equations. The rest of the proof is as in part (i).

Proposition 1 shows that an economy with endogenous default produces the same allocation and prices as long as the exogenous probabilities of default are set equal to the expected recovery rates of the economy with endogenous default. Conversely, an economy with exogenous probabilities of default can produce the same equilibrium allocations and prices, provided that we select the correct bankruptcy penalties. Then, the economy with endogenous default will produce deliveries and recovery rates equal to the exogenous probabilities of default that support the equilibrium allocation and prices of the economy with exogenous probabilities of default.

The upshot of our argument is that modelling endogenous default provides microfoundation to the models that treat default using exogenous default probabilities

3 Concluding remarks and extensions

One caveat that one should bear in mind in justifying our approach is that we live in a world where banking systems are usually concentrated. With a large number of agents, for example in a competitive equilibrium, conditions where everyone defaults on, say, 5% of their liabilities are equivalent to those where 5% of agents default on all their debts. This, however, is not the case when there are only a few agents in a concentrated field. If there are, say, only two agents in the field, and their failures are independent of each other, then in 0.25% of all cases there will be 100% default, in 9.5% of cases 50% default, and in 90.25% of cases no default, which is clearly vastly different from a 5% default rate amongst a large number of agents.

In most countries banking is a concentrated service industry. Moreover, reputational effects and cross-default clauses, amongst other things, mean that banks cannot default partially and remain open. If they cannot meet their payment obligations, (except under *force majeure* as in 9/11), they have to close their doors. Except when such closed banks are tiny, such closure does not however, in almost all cases, then turn into permanent liquidation. Effectively almost all banks are restructured, often via a 'bridge bank' arrangement, and shortly re-open, with the extent of short-fall of assets distributed amongst the various creditors, (the 'haircut' in the American phrase), the shareholders and taxpayers depending on the deposit insurance arrangements, bank bankruptcy laws and political pressures. In this latter sense, even though the banking system is concentrated, and banks have to close when they cannot meet due payments, it is perfectly valid to assess strategies as bringing about possible conditions in which a bank defaults by, say, 5% to all depositors, because that would be the effective loss of funds, or haircut, in the event of a bad state of the world.

Two possible main extensions of this paper focus on dealing with the coexistence of the institutional arrangements we examined and their relationship with collateral requirements as well as infinite horizon. First, the coexistence of alternative institutional arrangements should be established within the context of a single model. Also, we would like to investigate this equivalence in an infinite horizon model, where the possibility of Ponzi games arises (see Araújo *et al*, 2002).

References

Araújo, A., Páscoa, M. and Torres Martinez, J. P. (2002), "Collateral avoids Ponzi schemes in Incomplete markets", *Econometrica*, vol. 70, no. 4, pp. 1613-1638

Diamond, D., Dybvig, P. (1983), "Bank runs," *Journal of Political Economy* 91, 401-419.

Dubey, P., Geanakoplos, J. (1992), "The value of money in a finite-horizon economy: a role for banks. In: Dasgupta, P., Gale, D. et al. (Eds.), Economic Analysis of Market and Games. MIT Press, Cambridge, 407-444.

Dubey, P., Geanakoplos, J., Shubik, M. (2005), "Default and Punishment in General Equilibrium Model with Incomplete Markets," *Econometrica*, 73 (1), 1-37.

Goodhart, C.A.E., Pojanart, S. and Tsomocos, D. P. (2006), "A Model to Analyse Financial Fragility." Economic Theory 27, 107-142.

Hart, O., Moore, J. (1994), "A Theory of Debt Based on the Inalienability of Human Capital," *Quarterly Journal of Economics* 109 (4), 841-879.

Merton, R.C. (1974), "On the pricing of corporate debt: The risk structure of interest rates," *Journal of Finance* 29, 449-470.

Shapley, L.S., Shubik, M. (1977), "Trading using one commodity as a means of payment," *Journal of Political Economy* 85 (5), 937-968.

Shubik, M., Wilson, C. (1977), "The optimal bankruptcy rule in a trading economy using fiat money," *Journal of Economics* 37, 337-354.

Tsomocos, D.P. (2003a), "Equilibrium Analysis, banking and financial instability," *Journal of Mathematical Economics*, 39 (5-6), 619-655.

Tsomocos, D.P. (2003b), "Equilibrium Analysis, banking, contagion and financial fragility," Bank of England WP no. 175.