Multi-Agent Production Equilibrium Models with Expansion

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it). The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgment is made. This thesis may not be reproduced without my prior written consent. I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

Statement of co-authored work

I confirm that a version of Chapters 2, 3 were jointly co-authored with Prof. Costantinos Kardaras and Prof. Mihail Zervos. I confirm that Chapter 4 was jointly co-authored with Prof. Umut Çetin.

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Abstract

This thesis is concerned with a multi-agent equilibrium expansion model where agents are faced with an exogenous stochastic constant elasticity demand function. Producers simultaneously decide their production schedule, via a sequential equilibrium market clearing condition, as well as their optimal expansion schedule which is formulated as the solution of a singular stochastic control problem. In particular, agents take into account both the fact that their expansion has an adverse effect to the price and also the effect of their actions on the rest of the agents. For every agent, the value function and the optimal control process is determined and a Nash equilibrium for the market is established. The problem is divided into two sections, the monopolist case, where a single agent dominates the market and the competitive case in which all agents form a price-taking continuum, and the problem takes the form of a mean-field stochastic differential game. In both cases the value function as well as the control is calculated in closed form.

In a different topic using an implicit numerical scheme and under mild conditions we recover, in a compact way, the optimal weak convergence rate for a Cox–Ingersoll–Ross (CIR) process despite the fact that the coefficients of the underlying Stochastic differential equation are not Lipschitz.

Contents

Chapter 1

Introduction

1.1 Description of this thesis

Capacity expansion under uncertainty involves planning for future growth and resource allocation despite unpredictable factors such as fluctuating demand, technology shifts, and economic changes. The idea, part of a field called industrial organization[Tir88] started as the necessity to provide a theoretical framework for investment in industries like heavy process industries, communication networks, electrical power services, water resource systems, oil and gas sector, pharmaceuticals, high-tech industries and real estate. Additionally, it has provided valuable assistance to government policy, especially in areas like infrastructure investment, environmental regulation, and Research and Development (R&D) subsidies, where timing and uncertainty play critical roles.

Early on, apart from the pioneering work of Manne [Man61] many models assumed perfect forecasting of the economic conditions[Lus82]. This approach is very valuable but has a significant disadvantage, in a constantly transforming world with constant shocks in the supply/demand side[Coc94, FVGQKRR15], unexpected structural shifts on a sector[She98] or even political uncertainty [JY12] perfect forecasting becomes meaningless. To this end, much of the work shifted in a stochastic environment where the demand has been described as a Brownian motion with a drift[Man61] or as mixed diffusion/pure jump process model to accommodate for unexpected structural demand shifts [Tap79, BT79] as a reference, a small number of papers of the same period are cited for the interested reader[GM77, Nic77, ESO81, MP87a, PKR91]. Another crucial element to the necessity of a mathematical framework is the fact that most major investments are almost irreversible and even if the de-investment process is possible the cost associated with it might be detrimental for future growth. As highlighted by Pindyck [Pin88] industry investment can be quite specific which means that it is not transferable to other sectors and thus if the demand for the produced product falls significantly it renders the investment to become a sunken cost, a reasoning which doesn't apply only to the production industry but also to technology equipment and software. Nevertheless, let us mention that Abel & Eberly[AE96] argue that in some cases partial reversibility is plausible under the assumption that the cost of investment is higher than the de-investment profit. A simple approach to investment is to follow a Net present value(NPV)-strategy, i.e. invest when the value of the capital is at least as large as the corresponding costs which works in a deterministic environment but fails in an dynamic uncertain one. Instead, under uncertainty one should aim to minimize the probability of the occurrence of a bad scenario which is to have low or even negative return to investment. Therefore, a plausible strategy would be to aim to invest whenever the net present value is sufficiently positive and do nothing otherwise or in the extreme case where returns are expected to be low to choose to exit the market. Such type of strategies are described by Dixit & Pindyck in their book *Investment under uncertainty* [DP94] and are known as the real options theory which incorporates the *value* of waiting for additional information regarding market conditions or for the current market condition to reach a certain threshold before committing to investments. The aforementioned option is called the *value of waiting* and it has to be compared against the *value of expanding* as well as if permitted the *value of exiting* and the *value to reduce investment.*

The approach of the previous discussion rings a bell regarding the appropriate mathematical formulation of these types of problems (actually most of the problems in economics). As probably the reader has guessed the stochastic control formulation of the problem has flourished in this field. In fact, many of the aforementioned works we have cited use such an approach, for example much of [DP94] is devoted to the dynamic programming approach. In addition, important work of the same period using a stochastic control approach is that of Davis, Dempster & Sethi[DDSV87]. In particular, the fact that we are mainly interested in comparing the *value of expanding*, *value of waiting* and the *value of exiting* makes the problem suitable to be usually formulated as a singular/optimal stopping/impulse control problem, and much work has been devoted to this approach. In this direction, interesting work has been conducted by Knudsen, Bernhard & Zervos[KMZ98] where they study the valuation of an investment producing a single commodity and provide the investor with the option to abandon the process, Duckworth & Zervos, where they consider an investment model that involves entry and exit decisions as well as decisions related to production scheduling [DZ00] and in a following paper where they assume that firms can enter and exit and determine the optimal production scheduling as well as the sequence of entry and exit decisions [DZ01]. Moreover, Riedel & Su and independently Pham studied a singular control model of irreversible capacity with capital depreciation and showed that the optimal policy is to retain an amount of capital above a certain base level [RS11, KLSP06]. Also an interesting approach was done by Guo Miao & Morellec where they consider the case of irreversible investment but with regime shifts in the demand shock, an approach which could simulate abrupt demand shocks during business cycles[GMM05]. Additionally, Mehri & Zervos considered a singular control model of reversible expansion [MZ07]. Moreover, Motairi & Zervos considered a model of irreversible capacity for which expansion also affects the price [AMZ17b] and De Angelis, Federico & Ferrari considered a model where the uncertainty is also extended to the expansion costs[DAFF17]. More recently, Dammann & Ferrari considered the investment problem for a firm that produces two different products[DF22]

A significant component that we have neglected is competition. In particular, firms compete for labor, investment and as participants of the market through prices. Quite a lot of effort has been put in addressing the effects of competition. For example, Baldursson addressed the problem of investment in oligopoly [BK96] , Baludrsson & Karatzas studied the problem for small firms organized by a central planner[Bal98] while Grenadier focused on the effect of increasing competition on the waiting option to delay investment [Gre02]. A similar approach was studied by Back & Paulsen[BP09] and Steg[Ste12]. In a different framework, Novy-Marx shows that in equilibrium it might be optimal for firms to delay investment even when the NPV is sufficiently positive [NM07]. Another interesting work includes the work by Huisman & Kort where they considered a leader/follower duopoly problem of capacity to study how investment of the leader deters the follower from investing[HK15].

In our model we consider an economy which produces a non-durable/non-storable good and with a sequential market clearing condition. The demand function is a constant price elasticity function where the base demand follows a geometric Brownian motion. Producers have to do a two-fold optimization. Firstly, with fixed capability (capacity) producers decide how much quantity they have to produce. The fact that the produced good is non-storable/non-durable along with the sequential market clearing condition makes the agents myopic. In this context capacity in this context could be anything from hiring more labour to installing new facilities or to an upgrade in technology. Secondly, producers decide the expansion of their capabilities taking into account the discounted lifetime reward they will receive. The capacity optimization problem is then formulated as a stochastic singular control problem in which additional investment has an adverse effect on the price process of the underlying product. We consider models of three different economies. Firstly, we consider the case of a monopolistic production economy for which the market is dominated by a single producer. In this case a closed form solution for the optimal schedule process as well as the value function is provided. In addition, we consider the case where the monopolist faces capital depreciation. Next, we consider the case where we have a fully competitive market in which all participants have heterogeneity in their initial capacity and discount rate as well as in production parameters. In this scenario firms have negligible influence to the price of the underlying commodity and thus each firm acts as a price taker, and the problem is formulated as a mean-field stochastic differential game. Every individual producer solves a singular stochastic control problem taking into account the initial price level, their initial capacity and the maximum of the price level. For each heterogeneous producer we obtain a closed form solution for the optimal schedule process as well as the value function. Moreover, a Nash equilibrium for the competitive market is established.

In the final chapter we tackle the problem of the weak convergence rate for the CIR process. Under mild conditions and with a mathematically clean and compact formulation we are able to prove that the weak convergence rate for a CIR proces is of the order $\mathcal{O}(1/N)$, where N is the number of steps. The weak convergence rate was studied by [Alf05] using additional hypotheses for the numerical scheme it was shown while recently in [MN21] using an appropriate but rather complicated stochastic discretization scheme to obtain a weak convergence rate. In the present chapter, inspired by [cH21] we used only elementary arguments and mild assumptions on the payoff function in order to find the optimal convergence rate.

1.2 Stochastic Singular control problems

In this section we give a short description to problems of stochastic singular control and to that of stochastic differential game theory. We do so that the reader will be more comfortable with the techniques and concepts we use throughout this thesis. Please note that the exposition of the topic is not intended to be detailed and we will refer the reader to the relevant literature for further details.

Apart from industry investment, singular stochastic control problems have been extensively used to describe many real world situations such as the so-called "satellite problems" [Jac99, FNV85, PV79] in which the problem is to keep an object fixed in some location or to move along a specific trajectory[BC67, Kar83, KOWZ00, Ban05, CDAGL22], optimal execution problems in algorithmic trading[GZ15], in queuing theory [Har88], in fiscal policy[FR20], sustainable exploitation of the ecosystem[LZ20, Alv00, H+22] and many other fields.

We now turn our attention to the mathematical formulation of the problem. Assume a d-dimensional Brownian motion W living on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ satisfying the usual conditions. In addition, let X denote the solution of the following stochastic differential equation (SDE)

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t; \quad X_{0-}^x = x \in \mathbb{R}^d \tag{1.2.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are deterministic functions such that a unique strong solution exists.

Let us assume that we want to control the d-dimensional SDE via a control process ξ and a direction $\eta \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1} := \left\{x \in \mathbb{R}^d : \|x\| = 1\right\}$. In this direction, each initial $x \in \mathbb{R}^d$ gives rise to $X^{\xi,x}$ which in turn obeys the following controlled SDE

$$dX_t^{\xi,x} = b(X_t^{\xi,x})dt + \sigma(X_t^{\xi,x})dW_t + \eta_t d\xi_t \tag{1.2.2}$$

We define the cumulative control process as

$$\zeta_t = \int_{[0,t)} \eta_s d\xi_s \tag{1.2.3}$$

One may notice that not only continuous behavior but also jumps might be optimal which implies that ζ might not be absolutely continuous controls of t. Therefore, we must enlarge the class of controls to admit ζ which may not be an absolutely continuous. However, we assume that each component of ζ is a process of bounded variation on every finite interval $[0, t]$ which means that every component of ζ can be represented as $\zeta = \zeta^+ - \zeta^-$, where $\xi^\pm$ are non-decreasing processes that are right continuous with left limits.

Hence, this lead us to define the set of admissible controls

$$\mathcal{A} = \{\zeta_t \in \mathbb{R} \mid \zeta_t \text{ is } \mathcal{F}_t \text{ adapted, cádlag with locally bounded variation}\} \tag{1.2.4}$$

Moreover, there is a one-to-one correspondence between left-continuous processes of bounded variation and signed measures. Therefore, by an application of Girsanov's theorem we have

$$\zeta_t = \int_{[0,t)} \eta_t d\xi_t \tag{1.2.5}$$

where, $\eta \in \mathbb{S}^{d-1}$ and ξ is the total variation of ζ.

For any given initial condition $x \in \mathbb{R}^d$, we want to solve the problem of maximizing the following objective function

$$\mathcal{J}_x(\xi) = \mathbb{E}^x \left[\int_0^\infty e^{-rt} f(X_t^{\xi,x}) dt - \int_{[0,\infty)} e^{-rt} k(\eta_t) d\xi_t \right] \tag{1.2.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function, $k : \mathbb{S}^{d-1} \to \mathbb{R}^+$ is continuous, $r > 0$ is the discount factor.

The objective of the problem is to find $\eta \in \mathbb{S}^{d-1}, \xi^* \in \mathcal{A}$ such that

$$v(x) := \sup_{\eta \in \mathbb{S}^{d-1}, \xi \in \mathcal{A}} \mathcal{J}_x(\xi) \tag{1.2.7}$$

The Hamilton-Jacobi-Bellman equation of (1.2.7) is of (quasi)-variational nature and is given by:

$$\max\{\mathscr{L}v(x) + f(x) - \rho v(x),\ H_{\text{sing}}(\nabla v(x))\} = 0 \tag{1.2.8}$$

where

$$H_{\text{sing}}(p) := \sup_{\eta \in \mathbb{S}^{d-1}} (\eta p - k(\eta)) \tag{1.2.9}$$

and $\mathscr{L}$ is the infinitesimal generator of X defined by

$$\mathscr{L}f(x) := \lim_{h \to 0} \frac{\mathbb{E}^x[f(X_{t+h})] - f(X_t)}{h}; \quad f : \mathbb{R}^d \to \mathbb{R},$$

provided that the limit exists. in the case where the SDE is described by (1.2.1).

$$\mathscr{L} := \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \left(\sigma\sigma^\top\right)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.2.10}$$

Also in the special case where the cost is independent of direction η we get that

$$\max\{\mathscr{L}v(x) + f(x) - rv(x), |\nabla v(x)| - k\} = 0 \tag{1.2.11}$$

The solution of this problem can be characterized by the presence of the inaction region $\mathcal{W} = \{x \in \mathbb{R}^d \ : \ |\nabla v(x)| < k\}$. Typically, we can associate the solution of problem (1.2.7) with a Skorokhod reflection problem or a modified version of it. The classical definition of a Skorokhod problem is given by

Definition 1.2.1 *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. Let $x \in \mathbb{R}^d$ and unit vector $\hat{n} \in \mathbb{S}^{d-1}$ on $\partial\mathcal{O}$. We say that the process $\zeta \in \mathcal{A}$, given by*

$$\zeta_t = \int_{[0,t)} \eta_t d\xi_t, \tag{1.2.12}$$

$\eta \in \mathbb{S}^{d-1}$ and ξ is the total variation of ζ, is a solution to the classical Skorokhod problem for $X^{x,\zeta}$ starting in $x \in \overline{\mathcal{O}}$ reflected along the direction $\hat{n}$ if

(i) $|\eta| = 1$ and ξ is continuous and non-decreasing

(ii) the controlled process satisfies $\mathbb{P}(X_t \in \overline{\mathcal{O}},\ \forall t \in \mathbb{R}^+) = 1$

(iii) $\mathbb{P}$-a.s. it holds that for all $t \in \mathbb{R}^+$

$$\xi_t = \int_{[0,t)} 1_{\{X_s \in \partial\mathcal{O}, \eta_s = \hat{n}(X_s)\}} d\xi_s \tag{1.2.13}$$

Let us note that this definition is not enough to cover all cases. In particular, in order to control a Brownian motion in $d \geq 3$, Kruk used a generalized definition of the Skorokhod problem[Kru00]. Further work on the skorokhod problem has been conducted in [LS84, DF23]

At this point we describe a very simple model of irreversible investment taken from Pham[Pha09]. Assume a one-dimensional Brownian motion B living on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ Let us assume that the production capacity process evolves as

$$dX_t^C = -\delta X_t^C dt + \sqrt{2}\sigma dB_t + dC_t \ ; \ X_{0-} = x \tag{1.2.14}$$

where δ is the depreciation rate, σ denotes the volatility constant of random fluctuations of capacity and C_t is the number of units of capacity that the firm purchases at cost kdC_t. The objective of the firm is to maximize the lifetime reward, i.e.

$$v(x) = \sup_{C\in\mathcal{A}} \mathbb{E}^x \left[\int_0^\infty e^{-rt} f(X_t^C) dt - \int_{[0,t)} e^{-rt} k dC_t \right], \tag{1.2.15}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a C^1 concave non-decreasing function.

In this case the (quasi)variational inequality is given

$$\max\{\sigma^2 v_{xx}(x) - \delta v_x - rv(x) + f(x), v_x(x) - k\} = 0. \tag{1.2.16}$$

In this case one can show that the value function v is a classical C^2 solution and the optimal control process is given by the solution of a Skorokhod problem (1.2.1).

In particular, let $x^* := \inf\{x \in \mathbb{R}^+ : v_x(x) < k\}$ then C^* is a right-continuous with left limits (RCLL) and non-decreasing process such that $X_t^* \in [x^*, \infty), \forall t \geq 0$ and $C_t^* = \int_{[0,t)} 1_{\{X_t^* = x^*\}} dC_s^*$. Moreover, if $x \geq x^*$ then C^* is continuous otherwise there is a jump at $t = 0-$ with $C_0^* = x^* - x$ and $X_0^* = x^*$.

The previous discussion was intended to be a brief description of the problem of singular stochastic control. The interested reader may refer to [Kru00, FS06], from which the discussion was mainly based, to see a rigorous approach and the assumptions that have to be made in order to have existence/uniqueness of classical or viscosity solutions. Finally, let us note that there is another class of singular stochastic control problems in which the Hamiltonian exhibits singularities[FS06, Car16, Pha09]

1.3 Stochastic differential games

In this section we give an informal description of the concept of stochastic differential game theory directed to the unfamiliar audience. Stochastic differential game theory is an extension of game theory to extend the concept of a game in a continuous time framework with uncertainty. Core applications can be found in economics, such as the study of wealth inequality, principal-agent games and firm competition [CL21, KMV18, AHL+22, Aiy94, San08, San07] as well as in engineering/computer science where it is widely used in communications networks and algorithms designed with the help of game theory to improve computational efficiency[Han12, ZYB21]

Of course the references are not exhaustive and represent only a minor fraction of the field.

To begin with, consider that N agents participate in a game G where each agent takes an action α_i taken from an admissible class $\mathcal{A}_i$ and for simplicity the total admissible space is given by $\mathcal{A} := \prod_{i=1}^{N} \times A_i$. In addition there is a an m-dimensional Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ and a state process X, being driven by W, which is given by

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha, \alpha_t)dW_t, \tag{1.3.1}$$

where $b : [0,T] \times \mathbb{R}^d \times \mathcal{A} \mapsto \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{A} \mapsto \mathbb{R}^{d\times m}$ are deterministic functions, where α^i refers to the control of agent i while $\alpha^{-i} = (\alpha_1, ..., \alpha^{i-1}, \alpha^{i+1}, ..., \alpha_N)$ refers to the controls of the rest of agents excluding i.

Additionally, assume that every agent tries to maximize her objective function

$$\mathcal{J}_x^i(\alpha^i, \alpha^{-i}) = \mathbb{E}^x \left[\int_0^T f_i(s, X_s, \alpha_s^i, \alpha_s^{-i}) ds + g_i(X_T) \right] \tag{1.3.2}$$

Definition 1.3.1 *(Nash Equilibrium) A set of admissible strategies $\alpha^* = (\alpha_1^*, ..., \alpha_N^*) \in \mathcal{A}$ is said to be a Nash equilibrium of the system if for all $i = 1, ..., N$*

$$\mathcal{J}^i(\alpha_i^*, \alpha_{-i}^*) \geq \mathcal{J}^i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \in \mathcal{A}_i \tag{1.3.3}$$

The notion of Nash equilibrium simply states that assuming that all agents are rational, in equilibrium none of the agents should regret their decision process given the fact that all the rest of the agents will use their best strategy. The Nash equilibrium is the fundamental element of game theory but it's generic form does not specify the information structure of the game. For example agents might have to commit to their initial decision strategy even in the case where they observe a deviation in the strategy of another agent. On the other hand agents in highly dynamic environments like algorithmic trading or in repeated auctions agents need to continuously adjust to the feedback they receive from their competitors and thus commitment to an initial strategy might not make sense. Therefore, we need to further specialize the definition of Nash equilibrium to case where the agents are either restricted (hence the model naturally restrict the available strategies) to play one type of strategy or the game itself contains different types of Nash equilibria. To this end we define two different notions of a Nash equilibrium [SY19, MNMS24, YP06, JFRZ18, MMKN23, BP09].

Definition 1.3.2 *(Open Loop Nash Equilibrium) An open loop Nash equilibrium is a set of strategies $a^* \in \mathcal{A}$ such that for every $i = 1, ..., N$*

$$\mathcal{J}^i(\alpha_i^*, \alpha_{-i}^*) \geq \mathcal{J}^i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \in \mathcal{A}_i \tag{1.3.4}$$

and all strategies α_i^*, α_i are of the form

$$\alpha_i = \phi_i(t, X_0, W_{[0,T]}), \quad i = 1, ..., N, \tag{1.3.5}$$

where $W_{[0,T]}$ denotes the full path of the Brownian motion from $t = 0$ to $t = T$ and $\{\phi_i\}_{i=1}^N$ deterministic functions

Hence, as the definition implies, the agent adapts her strategy only on the signal it receives from the exogenous process W. The fact, that the control takes into account the full path of the Brownian motion is just for the sake of generality and more approachable cases are when the control is deterministic $\alpha_i = \phi_i(t, X_0)$ or Markovian $\alpha_i = \phi_i(t, X_0, W_t)$.

Definition 1.3.3 *(Closed Loop Nash Equilibrium) A closed loop Nash equilibrium is a set of strategies $a^* \in \mathcal{A}$ such that for every $i = 1, ..., N$*

$$\mathcal{J}^i(\alpha_i^*, \alpha_{-i}^*) \geq \mathcal{J}^i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \in \mathcal{A}_i \tag{1.3.6}$$

and all strategies α_i^*, α_i *are of the form*

$$\alpha_i = \phi_i(t, X_0, X_{[0,T]}), \quad i = 1, ..., N, \tag{1.3.7}$$

where $X_{[0,T]}$ *denotes the full path of the state process from* $t = 0$ *to* $t = T$ *and* $\{\phi_i\}_{i=1}^N$ *deterministic functions*

Similarly to the previous case a more approachable decision process is the Markovian feedback form where for $i = 1, ..., N$ $\alpha_i = \phi_i(t, X_0, X_t)$.

Note that the existence and uniqueness of a Nash Equilibrium is based on the specific setting of the game as well as the type of Nash equilibrium we are considering. In any of the cases though the solution process is to reformulate the problem as a fixed point problem. In particular, assume for notational convenience that the best response of every agent to the other agent's fixed strategies α_{-i} is the function (as opposed to a set valued function in the case of multiple best responses) $\mathcal{B} : \prod_{j \neq i} \mathcal{A}_j \mapsto \mathcal{A}_i$ given by $\beta_i = \mathcal{B}_i(\alpha_i)$.

and let us define the mapping $\mathcal{B} : \mathcal{A} \mapsto \mathcal{A}$ given by

$$\mathcal{B}(\alpha_1, ..., \alpha_N) = (\beta_1, ..., \beta_N). \tag{1.3.8}$$

The Nash equilibrium of the game is the set of strategies $\alpha^* = (\alpha_1^*, ..., \alpha_N^*)$ which satisfy

$$\mathcal{B}(\alpha_1^*, ..., \alpha_N^*) = (\alpha_1^*, ..., \alpha_N^*). \tag{1.3.9}$$

The best response map is not always well defined and we must first impose appropriate assumptions to guarantee that the individual agent's optimization problems can provide an optimal strategy or at least a set of optimal strategies. In addition, as one can guess, it is not always the case that (1.3.9) has a fixed point. However, there are very useful theorems which give a positive answer under certain conditions. We mention some of the most famous fixed point theorems which are frequently used (and we will use them)

Theorem 1.3.4 *(Brouwer's Fixed-Point Theorem) Let $D \subset \mathbb{R}^n$ be a closed, bounded, and convex subset of $\mathbb{R}^n$. If $f : D \mapsto D$ is a continuous function, then there exists at least one point $x \in D$ such that*

$$f(x) = x.$$

Proof: for the proof see [Tes20] ■

Theorem 1.3.5 *(Kakutani's fixed point theorem) Let X be a non-empty, compact, convex subset of a finite-dimensional Euclidean space $\mathbb{R}^n$. Let $F : X \to 2^X$ be a set-valued function that satisfies the following conditions:*

- *For each $x \in X$, $F(x)$ is a non-empty, closed, and convex subset of X.*
- *F is upper semi-continuous, meaning that for any closed subset $C \subset X$, the set $\{x \in X \mid F(x) \subset C\}$ is closed in X.*

Then there exists a point $x^* \in X$ such that

$$x^* \in F(x^*).$$

Proof: For the proof see [Kak41] ■

Theorem 1.3.6 *(Knaster-Tarski's fixed-point theorem) Let $(L, \leq)$ be a complete lattice, and let $f : L \to L$ be an order-preserving (monotone) function. Then the set of fixed points of f,*

$$\{x \in L \mid f(x) = x\},$$

is non-empty and forms a complete lattice under the ordering $\leq$.

In particular, f has both a least fixed point and a greatest fixed point. The least fixed point can be expressed as

$$x^* = \inf\{x \in L \mid f(x) \leq x\},$$

and the greatest fixed point as

$$y^* = \sup\{x \in L \mid x \leq f(x)\}.$$

Proof: For the proof see [DP02] ■

Finally, we omit to discuss the ideas of mean-field game theory due to the fact that even a short exposition would need a lot of ideas and machinery to be introduced. However, we describe succinctly the central idea

Assume that a game takes place on a fixed finite time horizon $T > 0$ and agent i chooses a control process α_i from the set of controls $\mathcal{U}([0,T])$. the control process influences the evolution of the state process according to the following dynamics

$$\begin{aligned} dX_{it} =& b(X_{it}, \mu_t^N, \alpha_i)dt + \sigma(X_{it}, \mu_t^N, \alpha_i)dW_t \\ \mu_t^N =& \frac{1}{N}\sum_{k=1}^{N} \delta_{X_{it}}, \end{aligned} \tag{1.3.10}$$

where μ_t^N is called the empirical measure which represents the collective influence of all the other agents through their controls.

Each agent maximizes

$$\mathcal{J}_x^i(\alpha^i, \alpha^{-i}) = \mathbb{E}^x\left[\int_0^T f(s, X_s, \alpha_s^i, \alpha_s^{-i})ds + g(X_T)\right] \tag{1.3.11}$$

In the limit of $N \to \infty$, assume that $\mu_t^N \to \mu_t$ and the the influence of each agent i is negligible on the measure flow μ_t. Therefore, the problem reduces to a representative agent problem

$$\begin{aligned}\mathcal{J}_x^i(\alpha) =& \mathbb{E}^x\left[\int_0^T f(s, X_s, \alpha_s, \mu_s)ds + g(X_T, \mu_T)\right]\\ dX_t =& b(X_t, \mu_t, \alpha)dt + \sigma(X_t, \mu_t, \alpha)dW_t\end{aligned} \tag{1.3.12}$$

The concept of equilibrium for the mean field game (1.3.12) is to the the measure flow μ_t such that $\mu_t = \mathcal{L}(X_t^{\mu,\alpha^*}) = \mathbb{P} \circ X^{-1}$ for each $t \in \mathbb{R}^+$ and for some optimal control $\alpha^* \in \mathcal{U}([0,T])$. For more information we refer to [Car16]. Finally, a concept that is not directly relevant to this thesis but is important is that of ϵ-Nash equilibrium which is defined as follows

Definition 1.3.7 *an ε-Nash equilibrium is a strategy profile $\alpha^* \in \mathcal{A}$ such that for every player $i = 1, ..., N$,*

$$\mathcal{J}^i(\alpha_i^*, \alpha_{-i}^*) \geq \mathcal{J}^i(\alpha_i, \alpha_{-i}^*) - \varepsilon \quad \textit{for all } \alpha_i \in \mathcal{A}_i,$$

.

The ε-Nash equilibrium provides a practical relaxation, ensuring that no player can gain more than ε by deviating. In addition, its mathematical importance in regard to mean-field games is that under certain conditions one can approximate a finite N number of agents game with a mean field game and show that the obtained Nash equilibria are actually ε-Nash equilibria for the finite N agent game where ε depends on N (typically as $\mathcal{O}(1/N^{1/2})$ or $\mathcal{O}(1/N)$) [BFY+13]

Finally, throughout this thesis we will make use of Fatou's lemma, the monotone and dominated convergence theorems and thus we provide the statements

Lemma 1 (Fatou's Lemma) *Let* $(X, \mathcal{A}, \mu)$ *be a measure space and let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative measurable functions* $f_n : X \to [0, \infty]$. *Then*

$$\int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \;\le\; \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu.$$

Proof: For the proof see [SS09] ∎

Theorem 1.3.8 (Monotone Convergence Theorem) *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of nonnegative measurable functions on a measure space* $(X, \mathcal{A}, \mu)$ *such that*

$$f_1(x) \le f_2(x) \le \cdots \quad \text{for all } x \in X.$$

Define

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then f *is measurable, and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof: For the proof see [SS09] ∎

Theorem 1.3.9 (Dominated Convergence Theorem) *Let* $(X, \mathcal{A}, \mu)$ *be a measure space and let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of measurable functions* $f_n : X \to \mathbb{R}$ *such that* $f_n(x) \to f(x)$ *for* μ*-almost every* $x \in X$. *Suppose there exists* $g \in L^1(\mu)$ *with* $|f_n(x)| \le g(x)$ *for* μ*-almost every* x *and all* n. *Then* $f \in L^1(\mu)$ *and*

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \;=\; \int_X f \, \mathrm{d}\mu.$$

Moreover,

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu \;=\; 0.$$

Proof: For the proof see [SS09] ∎

1.4 Numerical schemes for SDEs

This section serves as a brief elementary introduction on numerical schemes for stochastic differential equations, intended to introduce concepts to the unfamiliar reader which will be used in chapter 4. In particular, we discuss the concepts of Euler-Maruyama discretization and implicit schemes. In addition, we introduce the concepts of strong and weak modes of convergence.

To begin with, let us consider a one-dimensional diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \; X_0 = x \tag{1.4.1}$$

where $b : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are deterministic functions. We wish to simulate values of X_T without the help of its distribution. This could be due to the fact that we cannot calculate the distribution or because simulating from the distribution is computationally unfeasible. In the case of ODEs the simplest discretization method is that of Euler. The intuitive extention, the Euler-Maruyama scheme is given by:

$$\hat{X}_{t_n} = \hat{X}_{t_{n-1}} + b(\hat{X}_{t_{n-1}}, t_n)\,\Delta t + \sigma(\hat{X}_{t_{n-1}}, t_n)\,\sqrt{\Delta t} Z_n; \; n = 1, ..., N \tag{1.4.2}$$

where the discretization times are $0 = t_0 < t_1 < ... < t_N = T$, $\Delta t = t_n - t_{n-1}$ and Z_n are $\mathcal{N}(0, 1)$ i.i.d.

Another way to discretize the SDE (1.4.1) is through an implicit scheme

$$\hat{X}_{t_n} = \hat{X}_{t_{n-1}} + b(\hat{X}_{t_n}, t_n)\,\Delta t + \sigma(\hat{X}_{t_n}, t_n)\,\sqrt{\Delta t} Z_n; \; n = 1, ..., N \tag{1.4.3}$$

Consequently, we define the two criteria of measuring the convergence of a discretized process $\hat{X} \equiv \{\hat{X}_0, \hat{X}_{t_1}, ..., \hat{X}_T\}$ to the original process $X \equiv (X_t; \; t \in [0, T])$

Definition 1.4.1 *(strong convergence) A general time discrete approximation $\hat{X}$ converges strongly to the solution X at time T if*

$$\lim_{\Delta t \to 0} \mathbb{E}\left[|X_T - \hat{X}_T|\right] = 0 \tag{1.4.4}$$

Definition 1.4.2 *(strong order of convergence) We say that a general time discrete approximation $\hat{X}$ has a strong order of convergence m if*

$$\mathbb{E}\left[|X_T - \hat{X}_T|\right] \leq \frac{K}{N^m}, \tag{1.4.5}$$

for some positive constant K and sufficiently large N.

Definition 1.4.3 *(weak convergence) A general time discrete approximation $\hat{X}$ converges weakly to the solution X at time T if*

$$\lim_{\Delta t \to 0} \left|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\hat{X}_T)]\right| = 0, \tag{1.4.6}$$

where $f : R \to R$ are appropriate smooth functions.

Let us mention that the weak convergence criterion compares the distributions and therefore we could have a small weak convergence error even if $\hat{X}$ and X live in a different probability space. This criterion is most relevant in financial application since we usually want to numerically evaluate the price of a derivative of the underlying process X.

Definition 1.4.4 *(weak order of convergence) We say that a general time discrete approximation $\hat{X}$ has a weak order of convergence d if*

$$\left|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\hat{X}_T)]\right| \leq \frac{K}{N^d}, \tag{1.4.7}$$

where $K > 0$, N *sufficiently large and* f *are functions whose derivatives of all orders up to* $2d+2$ *are polynomially bounded, i.e. there exist* $M > 0$ *and* $q \in \mathbb{N}$ *such that* $|f^{(k)}(x)| \leq M(1+|x|^q)$, $k = 1, ..., 2d+2$

Chapter 2

An equilibrium model for capacity expansion: The case of a monopolist.

In this chapter we consider an irreversible capacity expansion model for a single agent faced with an exogenous stochastic demand function. The agent decides the production schedule via the expansion of her capability (capacity) and the optimization problem is formulated as a singular stochastic control problem in which additional investment has an adverse effect on the price process of the underlying product. For this model, a closed form solution for the optimal schedule process as well as the value function is provided.

2.1 Introduction

The research area of capacity (capability) expansion under uncertainty is a decision-making process that firms regularly face particularly in industries such as manufacturing, energy, and technology[KK13, Jeo23, RSM98, SvJAdK21, MZ94, SS86]. Firms must decide when and how much capacity to add in response to future demand, technological shifts, and market changes, all of which are inherently uncertain. The standard approach to capacity expansion considers the trade-off between committing to large investments, which are thought as irreversible, and the potential for under-utilizing resources if demand does not meet expectations. In particular, in their seminal work Majd and Pindyck [MP87b] emphasized the importance of considering the option value of delaying investment until economic conditions are favorable. In particular a waiting period could be maximize cumulative profits since it allows firms to gather more information about future demand, costs, or more broadly speaking about market conditions. Capacity expansion, optimal trading and other related problems, can be formulated as stochastic control problems[GZ15, GKTY11, DDSV87, Man61, Kob93]. In particular, in capacity expansion the optimized quantity is the firm's profit and the control process is the capacity. In our model, a monopolist is faced with the problem of irreversibly increasing capacity when the production cost function is inversely proportional to the level of capacity and faced with a constant elasticity of substitution stochastic (CES) demand function. The irreversibility assumption is particularly plausible for industries where they have high upfront costs and re-selling of capital will result into significant financial losses

In particular, the problem is formulated as a singular stochastic control problem were the price process is adversely affected by the expansion of capacity. Singular control problems have been extensively studied due to both their mathematical complexity and interesting applications. A representative but definitely non-exhaustive list includes[BSW80, JJZ08, Ma92, HS95a, HS95b, Ben84, Kar83, DM04, Ban05, MZ07, DZ94, HHSZ15, FFS20]. Finally, irreversible capacity expansion models have been studied in many different settings and some related to our work are [AMZ17a, Øks00, Kob93, DAFF17, BC94, Alv10, BK96, CH05, LZ11, CF14].

The rest of the chapter is structured as follows. In section 1 we formulate the production problem of the agent, in section 2 we introduce some necessary assumptions and tools while in section 3 we solve the HJB equations and prove that indeed the solution that we found is optimal. Finally, in section 4 we make a small extension to the model to incorporate depreciation of capital.

2.2 Problem Formulation

Consider a monopolist with capability (capacity) C_t at time $t \in \mathbb{R}^+$ that has to decide the amount of product per unit time Q_t given a price level P_t and has a running profit function $\Pi : \mathbb{R}_0^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$\Pi(Q_t, C_t) = Q_t P_t(Q_t) - \frac{1}{\lambda^{\frac{1}{\beta-1}}} \frac{\beta - 1}{\beta^{1+\frac{1}{\beta-1}}} \frac{{Q_t}^{1+\frac{1}{\beta-1}}}{{C_t}^{\frac{\alpha}{\beta-1}}}, \tag{2.2.1}$$

where $\lambda > 0$ and $\beta > 1$.

The above profit function, models in a simple way the fact that increasing capability (capacity) decreases the cost of production of one unit of product. Moreover, note that the multiplicative parameter is arbitrary and is solely presented in such a way to simplify subsequent calculations, since everything could be absorbed to a new λ parameter.

The equation for the optimum quantity Q_t^* at time t can be found by differentiating $\Pi(Q, C)$ with respect to Q and is given by:

$$P_t(Q_t^*) + Q_t^* P_t'(Q_t^*) - \frac{1}{\lambda^{\frac{1}{\beta-1}}} \frac{\beta - 1}{\beta^{1+\frac{1}{\beta-1}}} \frac{{Q_t}^{\frac{1}{\beta-1}}}{{C_t}^{\frac{\alpha}{\beta-1}}} = 0. \tag{2.2.2}$$

In addition, we will make use of the market clearing condition

$$Q_t = D_t \tag{2.2.3}$$

where the demand process is given by

$$D_t = \frac{B_t}{P_t^\delta}, \tag{2.2.4}$$

with $B \equiv (B_t; t \in \mathbb{R}^+)$ being a base demand process and $\delta > 1$ is the price elasticity of demand. In particular we use that by differentiating (2.2.3) w.r.t. Q after substituting

$D_t = Q_t$ we obtain that

$$1 = -\delta \frac{B_t}{P_t^{\delta+1}} \tag{2.2.5}$$

Combining (2.2.2) and (2.2.5) we obtain that the optimum quantity process for the single agent is

$$Q_t^* = \beta\lambda(1-\frac{1}{\delta})^{\beta-1} {C_t}^{\alpha} P_t^{\beta-1} \tag{2.2.6}$$

Using the market clearing condition (2.2.3) and (2.2.4) we obtain that

$$P_t^{\beta+\delta-1} = \frac{1}{\beta^\gamma \lambda^\gamma (1-\frac{1}{\delta})^{\gamma(\beta-1)}} \frac{B_t}{C_t^\alpha}, \tag{2.2.7}$$

$$P_t = \frac{1}{(\beta\lambda(1-\frac{1}{\delta})^\gamma)^{\beta-1}} \frac{B_t^\gamma}{C_t^\theta}, \tag{2.2.8}$$

For notational we also temporarily absorb the factor $1/(\lambda(1-\frac{1}{\delta})^{\beta-1})^\gamma$ into the base demand by simply re-defining appropriately the initial level of the B process. Therefore,

$$Q_t^* = \beta(1-\frac{1}{\delta})^{\beta-1} {C_t}^{\alpha} P_t^{\beta-1}, \tag{2.2.9}$$

and

$$P_t = \frac{B_t^\gamma}{C_t^\theta}, \tag{2.2.10}$$

where $\gamma := \frac{1}{\beta+\delta-1}$ and $\theta := \alpha\gamma$

In order to model market uncertainty, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ satisfying the usual conditions of right continuity and augmentation by $\mathbb{P}$-negligible sets, and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion $W = (W_t;\ t \in \mathbb{R}^+)$. Consequently, we assume that the base demand $B = (B_t;\ t \in \mathbb{R}^+)$ is subject to market fluctuations given by means of the geometric Brownian motion (GBM), i.e.

$$dB_t = \mu_b B_t dt + \sigma_b B_t dW_t,\ \ B_0 = b_0 \tag{2.2.11}$$

The capability (capacity) process of the producer is a positive process which is given by

$$C_t = c + \xi_t,\ ;\ C_{0-} = c > 0 \tag{2.2.12}$$

where $\xi = (\xi_t; t \in \mathbb{R}^+) \in \mathcal{Z} := \{\xi \in \mathbb{R}^+ : \xi \ \text{ is cáglág } \mathcal{F}_t - \text{adapted non-decreasing with } \xi_0 = 0\}$ which is controlled by the agent.

The state space of the control problem of study is defined as

$$\mathcal{S} = \Big\{(p,c) \in \mathbb{R}^+ \times \mathbb{R}^+_{>0}\Big\} \tag{2.2.13}$$

Definition 1 *The set $\mathcal{A}$ of all admissible capacity expansion strategies is the family of all processes $\xi \in \mathcal{Z}$ such that*

$$\mathbb{E}\Big[\int_{[0,\infty)} k e^{-rt} d\xi_t\Big] < \infty \tag{2.2.14}$$

The agent's objective is to maximize the cumulative profit by deciding when to expand her capacity, taking into account that a unit of capacity has cost $\tilde{k}$. Therefore, she needs to maximize the following objective function

$$\mathcal{J}_{p,c}(C) = \mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt} C_t^\alpha P_t^\beta dt - \tilde{k}\int_0^\infty e^{-rt} d\xi_t\right], \tag{2.2.15}$$

where r is a discounting factor, reflecting the agent's preference of future payoffs relative to present payoffs. In particular, as r decreases the agent becomes more patient and thus the valuation of future payoffs increases

Moreover, for convenience the coefficient $\beta\lambda(1-\frac{1}{\delta})^{\beta-1}$ is absorbed into the cost per unit constant, i.e. $\tilde{k} \to \tilde{k}/(\beta\lambda\big(1-\frac{1}{\delta}\big)^{\beta-1})$.

Therefore, the agent's value function

$$v(p,c) = \beta\lambda\left(1-\frac{1}{\delta}\right)^{\beta-1}\max_{C\in\mathcal{A}}\mathcal{J}_{p,c}(C) \tag{2.2.16}$$

The agent's maximization problem is subject to the price dynamics which are determined by the Ito-Tanaka-Meyer formula[PP05]. Specifically, for any $f\in\mathcal{C}^{2,1}$ function we have

$$\begin{aligned} df(B_t,C_t) &= \frac{\partial f}{\partial b}dB_t + \frac{\partial f}{\partial c}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial b^2}(B_t,C_t)d\langle B_t.B_t\rangle \\ &+ \sum_{0\le s\le t}\left[f(B_s,C_s) - f(B_s,C_{s-}) - \frac{\partial f}{\partial c}(B_s,C_{s-})\Delta C_s\right]. \end{aligned} \tag{2.2.17}$$

Equivalently this can be written as

$$df(B_t,C_t) = \frac{\partial f}{\partial b}dB_t + \frac{\partial f}{\partial c}dC_t^c + \frac{1}{2}\frac{\partial^2 f}{\partial b^2}(B_t,C_t)d\langle B_t.B_t\rangle + \sum_{0\le s\le t}\left[f(B_s,C_s) - f(B_s,C_{s-})\right], \tag{2.2.18}$$

where the superscript c indicates that we tak into account only the continuous part of C.

Hence, the dynamics of the corresponding price process $P\equiv(P_t;t\in\mathbb{R}^+)$ are given by

$$dP_t = \mu P_t dt - \theta\frac{P_t}{C_t}dC_t^c - \theta\sum_{0\le s\le t}P_{s-}\int_0^{\xi_{s-}}\frac{C_{s-}^{\theta}}{(C_{s-}+z)^{\theta+1}}dz + \sigma P_t dW_t, \quad P_{0-}=p, \tag{2.2.19}$$

where, $\mu := \gamma\mu_b + \frac{1}{2}\gamma(\gamma-1)\sigma_b^2$ and $\sigma := \sigma_b\gamma$

Note that the jump term in (2.2.19) is derived as follows. Let us make, at $t-$ an arbitrary capacity jump ξ_{t-}, then we have that

$$f(B_s,C_s)-f(B_s,C_{s-}) = f(B_s,C_{s-}+\xi_{s-})-f(B_s,C_{s-}) = \int_0^{\xi_{s-}}\frac{\partial f(B_s,C_{s-}+u)}{\partial u}du, \tag{2.2.20}$$

where we just used the fact that $\int_a^b \frac{\partial f(u)}{\partial u} du = f(b) - f(a)$ for a suitable integrable f function

For future reference we define as P_t^0 the net price process, i.e. if the agent does not increase capacity above the initial level c, with dynamics given by

$$dP_t^0 = \mu P_t^0 dt + \sigma P_t^0 dW_t, \; P_{0-}^0 = p \tag{2.2.21}$$

For clarity, note that before any expansion by the producer the price process is given by $P_t = P_t^0 = B_t^\gamma / c^\theta$ while after the first expansion we have that $P_t = B_t^\gamma / C_{\tau_\mathcal{E}}^\theta$, where $\tau_\mathcal{E} := \{t \in \mathbb{R}^+ | C_t - C_{t-} > 0\}$ and is chosen by the producer.

Therefore, after the first expansion time and before the second expansion, one can re-write the price process as

$$P_t = P_t^0 \left(\frac{c}{C_{\tau_\mathcal{E}}} \right)^\theta \tag{2.2.22}$$

where $\tau_\mathcal{E} := \{t \in \mathbb{R}^+ | C_t - C_{t-} > 0\}$

2.3 Model Assumptions

To begin with, our analysis involves the general solution to the second order Euler's ODE

$$\sigma^2 p^2 u''(p) + bpu'(p) - ru(p) = 0 \tag{2.3.1}$$

which is given by

$$u(p) = Ap^n + Bp^m$$

where ,

$$n, m = \frac{-(b - \sigma^2) \pm \sqrt{(b - \sigma^2) + 4\sigma^2 r}}{2\sigma^2}, \tag{2.3.2}$$

and are solutions of the algebraic equation

$$\frac{\sigma^2}{2}t^2 + (\mu - \frac{\sigma^2}{2})t - r = 0. \tag{2.3.3}$$

In addition, as proven by [KMZ98, AMZ17a] one has that

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(P_t^0)^\lambda dt\right] < \infty, \tag{2.3.4}$$

iff $\lambda \in (-n, n)$ and that for $\lambda \in (0, n)$ there exists $\varepsilon, C > 0$ such that:

$$e^{-rT}\mathbb{E}\left[\left(\sup_{0\leq t\leq T} P_t^0\right)^\lambda\right] < Cp^\lambda e^{-\varepsilon T} \text{ and } \mathbb{E}\left[\left(\sup_{T\geq 0} e^{-rT} P_T^0\right)^\lambda\right] < Cp^\lambda \tag{2.3.5}$$

Assumption 2.3.1 *We will consider control processes from the following class of admissible controls*

$$\mathcal{A} = \left\{\xi_t \in \mathcal{Z} : \mathbb{E}[\int_{[0,T]} ke^{-rt}d\xi_t] < \infty\right\}, \tag{2.3.6}$$

where $\mathcal{Z}$ is the class of all non-decreasing cádglád adapted to the $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ filtration processes.

Assumption 2.3.2 *We assume that $r, \sigma > 0$ and $\delta, \beta > 1$ and $\alpha \in \mathbb{R}^+$. Furthermore we must have that*

$$\alpha > \beta\theta \tag{2.3.7}$$

$$\beta \in (0, n) \tag{2.3.8}$$

$$\frac{\beta}{1-\alpha} \in (-n, n) \tag{2.3.9}$$

Moreover, define $\nu := \frac{\beta}{\beta\theta+1-\alpha}$ and notice that the last condition implies that $\nu < n$ or equivalently that $n(\beta\theta - \alpha) + n - \beta > 0$.

Lemma 2.3.3 *For any initial condition* $(p,c) \in \mathcal{S}$ *if* $\beta\theta > \alpha$ *then the problem is trivially solved by the no investment strategy* $\xi_t = 0$*. In addition, if the rest of relations in assumption 2.3.2 do not hold, then for any initial condition* $(p,c) \in \mathcal{S}$ *the value function diverges.*

Proof: Firstly, let us assume that $\beta > n$ and consider the strategy where the single agent chooses to not expand, i.e. $\xi_t = 0, \ \forall t \in \mathbb{R}^+$ Then,

$$v(p,c) \geq c^{\alpha}\mathbb{E}\left[\int_0^{\infty} e^{-rt}(P_t^0)^{\beta}dt\right]. \tag{2.3.10}$$

However, since $\beta > n$ as shown in appendix B the expectation on the right hand side diverges.

Next, in order to reject the case where $a < \beta\theta$ but $\beta < n$ we consider an arbitrary admissible strategy ξ_t. Let us assume that we choose expand capacity by Z at $t = 0$ and then keep capacity constant, i.e., $\xi_t = Z$ By (2.2.22) we have that the impact on the price is given by $P_0 = P_0^0 \frac{c^{\theta}}{(c+Z)^{\theta}}$. Therefore,

$$\mathcal{J}_{p,c}(\xi) = \frac{c^{\theta\beta}}{(c+Z)^{\beta\theta-\alpha}}\mathbb{E}^{p,c}\left[\int_0^{\infty} e^{-rt}(P_t^0)^{\beta}dt\right] - k\mathbb{E}^{p,c}\left[\int_0^{\infty} e^{-rt}d\xi_t\right] \tag{2.3.11}$$

Evidently, in this case it is always optimal for the agent to not invest and thus, the problem is trivially solved by the strategy $\xi_t = 0, \ \forall t \in \mathbb{R}^+$.

Finally, let $|\frac{\beta}{1-\alpha}| > n$ and consider an investment strategy[MZ07] such that

$$C_t > (\overline{P}_t^0)^{\frac{n-\beta}{\alpha-\theta\beta}} \tag{2.3.12}$$

Then, by virtue of (2.2.22) and (2.3.4) observe that

$$\mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}C_t^\alpha P_t^\beta dt\right] \geq \mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}(P_t^0)^n dt\right] = \infty \tag{2.3.13}$$

Hence,

$$\mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}C_t^\alpha P_t^\beta dt\right] = \mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}(P_t^0)^n dt\right] = \infty. \tag{2.3.14}$$

Thus, the problem reduces to proving that $\mathbb{E}\left[\int_0^\infty e^{-rt}d\xi_t\right] < \infty$.

In particular,

$$\begin{aligned}\mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}d\xi_t\right] =& \mathbb{E}^{(p,c)}\left[\int_0^T e^{-rt}d\xi_t\right] = \lim_{T\to\infty}\mathbb{E}^{(p,c)}\left[r\int_{(0,T)} e^{-rt}\xi_t dt + e^{-rT}\xi_T\right] \\ &= \lim_{T\to\infty}\int_0^T e^{-rt}\mathbb{E}\left[\xi_t\right]dt + \lim_{T\to\infty} e^{-rT}\mathbb{E}\left[\xi_T\right] \\ &= \lim_{T\to\infty}\int_0^T e^{-rt}\mathbb{E}\left[(\overline{P}_t^0)^{\frac{n-\alpha}{\beta}}\right]dt + \lim_{T\to\infty}\mathbb{E}\left[e^{-rT}(\overline{P}_T^0)^{\frac{n-\alpha}{\beta}}\right].\end{aligned} \tag{2.3.15}$$

since $\frac{\beta}{1-\alpha} > n$ this implies that $\frac{n-\beta}{\alpha} < n$ and thus the use of (2.3.4) and (2.3.5) imply that $\mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt}d\xi_t\right] < \infty$ which concludes the argument.

∎

2.4 Solution to the Control Problem

In this section, using heuristics, we derive the HJB equations. Consequently, we directly solve the HJB equation and provide a candidate for the value function the optimal control. Finally, we verify that our solution is indeed optimal.

2.4.1 Heuristic Derivation of the HJB equation

To begin with, a heuristic analysis shows that there are two actions that the producer can make. The first one is to choose to wait, i.e. not expand capacity, while the second one is to expand. Let us assume that she chooses to take the first action which is to wait for a short period of time Δt and then continue optimally. Bellman's principle imply that this action is not necessarily optimal and therefore,

$$v(p,c) \geq \mathbb{E}^{p,c}\left[\int_0^{\Delta t} e^{-rt}c^{\alpha}P_t^{\beta}dt + e^{-r\Delta t}v(P_{\Delta t},c)\right] \tag{2.4.1}$$

Applying Ito's formula [PP05] to the second term of the right hand side(RHS), dividing by Δt and letting $\Delta t \to 0$, we obtain that we must have

$$\frac{\sigma^2}{2}p^2 v_{pp}(p,c) + \mu p v_p(p,c) - rv(p,c) + p^{\beta}c^{a} \leq 0,\ \ (p,c) \in \mathcal{S} \tag{2.4.2}$$

On the other hand, one could choose to increase capacity by $\varepsilon > 0$ and then continue optimally. In this case with the use of (2.2.22) we have that

$$v(p,c) \geq v\left(p\frac{c^{\theta}}{(c+\varepsilon)^{\theta}}, c+\varepsilon\right) - k \tag{2.4.3}$$

Expanding the integral on the RHS up two powers of ε we get that

$$v_c(p,c) - \theta\frac{p}{c}v_p(p,c) - k \leq 0,\ \ (p,c) \in \mathcal{S} \tag{2.4.4}$$

Due to the Markovian character of the problem it is guaranteed that one of these options should be optimal and one of (2.4.3),(2.4.4) should hold with equality at any point in the state space $\mathcal{S}$. It follows that the problem's value function v should identify with an appropriate solution w to the following HJB equation

$$\max\left\{\frac{\sigma^2}{2}p^2 v_{pp}(p,c) + \mu p v_p(p,c) - rv(p,c) + c^{a}p^{\beta},\right.$$
$$\left. v_c(p,c) - \theta\frac{p}{c}v_p(p,c) - k\right\} = 0, \quad \forall (p,c) \in \mathcal{S}. \tag{2.4.5}$$

Intermedio: Change of variables

Even though the previous formulation was more immediate from an economic point of view, and this is why we choose to start the presentation of the problem in that way, it is mathematically convenient to reformulate the problem using the base demand instead of the actual price process. We define as $D := (B_t^{\gamma}; t \in \mathbb{R}^+)$. Also for future reference we define $\overline{D}_\cdot := \sup_{s\leq \cdot} D_s$. As a small remark note that $D = cP^0$

In particular, the objective function can be re-written as

$$\mathcal{J}_{c,c}(C) = \mathbb{E}^{b,c}\left[\int_0^\infty e^{-rt} C_t^{\alpha-\beta\theta} D_t^{\beta} dt - k \int_0^\infty e^{-rt} d\xi_t\right], \tag{2.4.6}$$

while the HJB equation will become

$$\max\left\{\frac{\sigma^2}{2} d^2 v_{dd}(d,c) + \mu d v_d(d,c) - r v(d,c) + c^{a-\beta\theta} d^{\beta}, v_c(d,c) - k\right\} = 0, \quad \forall (d,c) \in \mathcal{S}. \tag{2.4.7}$$

Note that we retain the same symbol for the state space $\mathcal{S}$ since it is the same space but in a different coordinate system. In addition, the same constraints hold since the problem has the same economic and mathematical properties albeit written in a different coordinate system.

We will look for a classical solution $w : \mathcal{S} \to \mathbb{R}$ which identifies with the value function v of the control problem. The construction of the solution relies on the introduction of a strictly increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ which partitions the state space $\mathcal{S}$ into a waiting region $\mathcal{W}$ and an investment region $\mathcal{E}$, defined by

$$\mathcal{W} : \left\{\ (d,c) \in \mathcal{S} \mid d \leq G(c)\ \right\}$$

$$\mathcal{E} : \left\{\ (d,c) \in \mathcal{S} \mid d > G(c)\ \right\}$$

To begin with, in the waiting region $\mathcal{W}$ the solution w must satisfy the following equation.

$$\frac{\sigma^2}{2}d^2 w_{dd}(d,c) + \mu d w_d(d,c) - rw(d,c) + c^{a-\beta\theta}d^\beta = 0, \tag{2.4.8}$$

which implies that we have a solution of the form

$$w(d,c) = A(c)d^n + \Gamma d^\beta c^{\alpha-\beta\theta}, \tag{2.4.9}$$

where n is given by (2.3.2), while the negative solution is eliminated due to the fact that w should be finite when d approaches zero (transversality condition). In addition, the constant Γ is given by

$$\Gamma = \frac{2}{\sigma^2(\beta + |m|)(n-\beta)} \tag{2.4.10}$$

Moreover, in the investment region $\mathcal{E}$, w should satisfy that:

$$w_c(d,c) - k = 0, \tag{2.4.11}$$

Consequently, as we are looking for a classical solution we must impose that $w(d,c)$ is $\mathcal{C}^{2,1}$ along the free boundary $p = G(c)$

Therefore, on $d = G(c)$

$$w_c\left(G(c),c\right) - k = 0, \tag{2.4.12}$$

$$w_{cd}\left(G(c),c\right) = 0, \tag{2.4.13}$$

Hence,

$$\dot{A}(c)G^n(c) = k + \Gamma(\theta\beta - \alpha)G^\beta(c)c^{a-\beta\theta-1}, \tag{2.4.14}$$

$$\dot{A}(c)G^n(c) = \frac{\Gamma\beta}{n}(\theta\beta - \alpha)G^\beta(c)c^{a-\beta\theta-1} \tag{2.4.15}$$

Thus,

$$G(c) = \tilde{k}c^{\frac{1-a+\beta\theta}{\beta}} \tag{2.4.16}$$

where,

$$\tilde{k} = \left(\frac{kn}{\Gamma(\alpha - \beta\theta)(n - \beta)}\right)^{\frac{1}{\beta}} \tag{2.4.17}$$

Note that we can re-write the free boundary $G(c)$ in terms of the price p by using the transformation $d = pc^\theta$. We have to find the points in the state space such that $p = \tilde{G}(c)$. Hence, this leads to

$$\tilde{G}(c) = \tilde{k}c^{\frac{1-a}{\beta}} \tag{2.4.18}$$

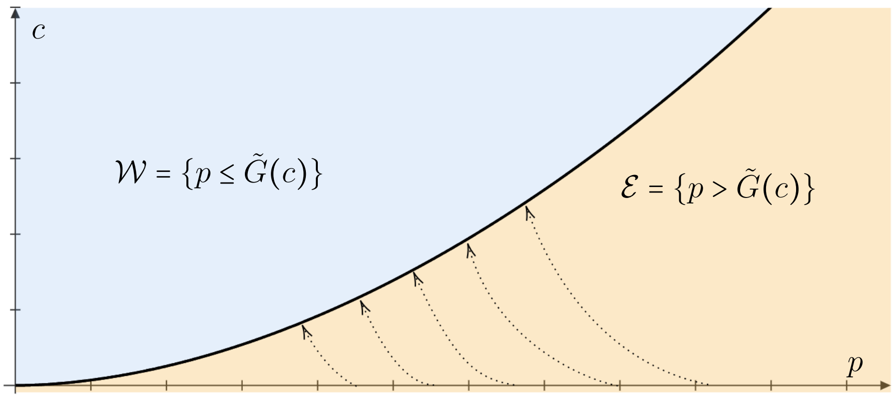

Figure 2.1 *Illustration of free boundary $\tilde{G}(c)$. The lines indicate the fact that increasing capacity decreases the price.* Note that on the plane (d, c) the free boundary has similar form but the capacity expansion lines should be vertical as they don't have any effect on the demand process

Note that in order for $G(c)$ to be well-defined we must have that

$$\alpha > \beta\theta, \quad n > \beta, \quad 0 < \alpha < 1 \tag{2.4.19}$$

At this point note that from (2.3.3) we have the identity $r = \frac{\sigma^2}{2}|m|n$ and thus re-arranging terms we get

$$\frac{n}{\Gamma(n-\beta)} = \frac{\sigma^2}{2}(n\beta + n|m|) > r \tag{2.4.20}$$

and consequently that

$$\tilde{k}^{\beta} = k\sigma^2 n\frac{\beta + |m|}{\alpha - \beta\theta} > rk \tag{2.4.21}$$

Finally, $A(c)$ will be given by

$$\dot{A}(c) = -k_a c^{n\frac{\alpha-\beta\theta-1}{\beta}} \tag{2.4.22}$$

where,

$$k_a = \frac{\Gamma\beta}{n}\tilde{k}^{\beta-n}(\alpha - \theta\beta) \tag{2.4.23}$$

Therefore, we need to evaluation the following for A

$$A(c) = k_a \int_c^{\infty} \frac{1}{u^{\theta n + \frac{(1-\alpha)n}{\beta}}} du, \tag{2.4.24}$$

which results in

$$A(c) = \frac{\beta}{\theta n\beta + n(1-\alpha) - \beta} k_a c^{1+\frac{n}{\beta}(\alpha-1)-\theta n}. \tag{2.4.25}$$

In addition to the previous condition, note that in order to obtain a well-defined result, i.e., that $w(d,c)$ is finite for finite c we also have to require the condition $\theta\beta + 1 - \alpha - \frac{\beta}{n} > 0$ otherwise $A(c)$ blows up. Moreover, note that this requirement makes $A(c) > 0$ which implies that w is a non-negative function.

At this point we introduce the optimal capacity expansion function $z(d) : \mathcal{E} \to \mathbb{R}_0^+$ which is given by

$$d = G(z(d)). \tag{2.4.26}$$

In the following lemma we prove that the function w is a classical solution of the HJB equation (2.4.5) and we provide some useful bounds.

Lemma 2.4.1 *The function $w : \mathcal{S} \to \mathbb{R}$ defined by*

$$w(d,c) = \begin{cases} A(c)d^n + \Gamma d^\beta c^{\alpha-\beta\theta} & \text{if } (d,c) \in \mathcal{W} \\ w\left(d, z(d)\right) - k(z(d) - c) & \text{if } (d,c) \in \mathcal{E} \end{cases} \tag{2.4.27}$$

where

$$A(c) = \frac{\Gamma\beta^2}{n}\frac{\alpha-\theta\beta}{n(\theta\beta-\alpha)+n-\beta}\tilde{k}^\beta \frac{c}{G^n(c)} \tag{2.4.28}$$

and $z(p) : \mathcal{E} \to \mathbb{R}^+$ is determined by

$$z(d) = \frac{1}{\tilde{k}^\nu}d^\nu \tag{2.4.29}$$

is a $\mathcal{C}^{2,1}$ solution to the HJB equation (2.4.5)

Finally, for all $(d,c) \in \mathcal{S}$

$$0 \le w(d,c) \le \Lambda(c + d^\nu) \tag{2.4.30}$$

where $\Lambda > 0$ depends only on the parameters of the problems

Proof: To begin with, $z(d) : \mathcal{E} \to \mathbb{R}^+$ is determined by

$$d = G(z(d)). \tag{2.4.31}$$

Solving for $z(p)$ we obtain that

$$z(d) = \frac{1}{\tilde{k}^{\nu}} d^{\nu} \tag{2.4.32}$$

where $\nu = \frac{\beta}{\beta\theta+1-\alpha}$.

Next, we are going to prove that $w(d,c)$ is $\mathcal{C}^{2,1}$ throughout the whole $\mathcal{S}$. To this end we must consider the continuity of w along the free-boundary G.

consequently, we calculate the derivative with respect to c

$$\begin{aligned} w_c(d,c) &= \frac{\partial}{\partial c}\left[w(d,z(d)) - k(z(d)-c)\right] \\ &= w_c(d,z(d)) \end{aligned} \tag{2.4.33}$$

Similarly, differentiating with respect to the d

$$w_d(d,c) = \frac{\partial}{\partial d}[w(d,z(d)) - k(z(d)-c)] \tag{2.4.34}$$

$$\begin{aligned} &+ w_d(d,z(d)) + \left[w_c(d,z(d)) - k\right] z_d(d) \\ &= w_d(d,z(d)) \end{aligned} \tag{2.4.35}$$

$$\begin{aligned} w_{dd}(d,c) &= \frac{\partial}{\partial d}\left[w_d(d,z(d))\right] \\ &= w_{dd}(d,z(d)) \end{aligned} \tag{2.4.36}$$

Next, we shall prove that $w(d,c)$ satisfies the HJB equation(3.4.18). To this direction, we must prove that the solution is sub-optimal in the complementary region.

To this direction, let $(d,c) \in \mathcal{W}$.

$$
\begin{aligned}
w_c(d,c) - k = & \\
&= \dot{A}(c)d^n + (\alpha - \theta\beta)\Gamma d^\beta c^{\alpha-1-\beta\theta} - k \\
&= (\alpha - \beta\theta)\tilde{k}^\beta \Gamma \left[\left(\frac{d}{G(c)}\right)^\beta - \frac{\beta}{n}\left(\frac{d}{G(c)}\right)^n \right] - k \le 0 \qquad (2.4.37)
\end{aligned}
$$

Where the last inequality follows from the fact that $d \le G(c)$ and assumptions (2.3.2). Note that strict equality holds for $d = G(c)$

Similarly, we must prove that that solution of the investment region is sup-optimal in the waiting region. Let $(d,c) \in \mathcal{E}$

$$
\begin{aligned}
&\frac{\sigma^2}{2} d^2 w_{dd}(p,c) + \mu d w_d(d,c) - r w(d,c) + c^{\alpha-\beta\theta} d^\beta \\
&= \frac{\sigma^2}{2} d^2 w_{dd}(d,z(d)) + \mu d w_d(d,z(d)) - r w(d,z(d)) + rk(z(d)-c) + c^{\alpha-\beta\theta} d^\beta \\
&= -z^{\alpha-\beta\theta}(d) z^\beta(d) + c^{\alpha-\beta\theta} d^\beta + rk(z(d)-c) \\
&= -z(d)^{\alpha-\beta\theta} G^\beta(z(d)) + c^{\alpha-\beta\theta} d^\beta + rk(z(d)-c) \\
&= -\int_c^{z(d)} \left[\frac{\partial}{\partial u}(u^{\alpha-\beta\theta}) d^\beta - rk \right] du \le 0 \qquad (2.4.38)
\end{aligned}
$$

where we have used the fact that for $(d,c) \in \mathcal{E}$ we have that $d > G(c)$ the relevant assumptions (2.3.2) and (2.4.20) to observe that the integrand is a non-negative quantity

Finally, we prove the relevant bounds for $w(d,c)$. Firstly, the lower bound is immediate since from (2.4.27) $w(d,c) > 0$. Regarding the upper bound let us take any point (d,c) in the waiting region $\mathcal{W}$.

$$
\begin{aligned}
w(d,c) &= \Gamma \tilde{k}^\beta \left[\left(\frac{d}{G(c)}\right)^\beta + \frac{\beta}{n} \frac{\alpha - \beta\theta}{n - \beta - n(\alpha - \beta\theta)} \left(\frac{d}{G(c)}\right)^n \right] c \\
&\le \Lambda c \qquad (2.4.39)
\end{aligned}
$$

where we have used that $d \leq G(c)$.

Next, for a point in the investment region $\mathcal{E}$ result follows again from (2.4.27)

$$w(d,c) \leq \tilde{\Lambda} z(d,c) = \tilde{\Lambda} d^{\nu} \tag{2.4.40}$$

■

2.4.2 Verification theorem

We know turn to the main theorem of this chapter. We prove that the function w identifies with the value function v and we give a closed form description of the optimal control.

Theorem 2.4.2 *Let us assume that all assumptions (2.3.2) hold. The value function of the control problem identifies with the classical solution (2.4.27) multiplied by* $\beta\lambda\left(1 - \frac{1}{\delta}\right)^{\beta-1}$. *In addition, the optimal capacity expansion is given by*

$$\xi_t^* = \left(\frac{\overline{D}_t}{\tilde{k}}\right)^{\nu} 1_{\{\bar{D}_t \geq G(c)\}}, \quad t > 0 \tag{2.4.41}$$

Moreover, the process $\xi^* \in \mathcal{A}$

Finally, the equilibrium price process P^* *can be found in terms of the base demand process* B *and is given by*

$$P_t^* = \frac{D_t}{c^{\theta} \vee \frac{1}{\tilde{k}^{\nu\theta}} \sup_{s \leq t} \overline{D}_s^{\nu\theta}}, \quad D_0 = d \tag{2.4.42}$$

Proof: Take any point $(d,c) \in \mathcal{S}$ with the assumption that the control C is admissible in the sense given by (2.3.1)

We begin by using the Ito's formula [PP05] on $e^{-rT} w(D_T, C_T)$.

$$\begin{aligned}
&e^{-rT}w(D_T, C_T)\\
&= w(d,c) + \int_0^T e^{-rt}\left[\frac{\sigma^2}{2}D_t^2 w_{dd}(D_t, C_t) + \mu D_t w_d(D_t, C_t) - rw(D_t, C_t)\right]dt\\
&+ \int_{[0,T)} e^{-rt} w_c(D_t, C_t) d\xi_t^c\\
&+ \sum_{0\leq s\leq T} e^{-rt}\left[w(D_t, C_t) - w(D_t, C_{t-})\right] + M_T,
\end{aligned} \tag{2.4.43}$$

where the process $M \equiv (M_t; t \in \mathbb{R}^+)$ is defined as

$$M_t := \sigma \int_0^t e^{-rs} D_s w_p(D_s, C_s) dW_s \tag{2.4.44}$$

Therefore,

$$\begin{aligned}
&\int_0^T e^{-rt} C_t^{\alpha-\beta\theta} D_t^\beta dt - \int_{[0,T)} k e^{-rt} d\xi_t + e^{-rT} w(D_T, C_T)\\
&= w(d,c) + \int_0^T e^{-rt}\left[\frac{\sigma^2}{2}D_t^2 w_{dd}(D_t, C_t) + \mu D_t w_d(D_t, C_t) - rw(D_t, C_t) + C_t^{\alpha-\beta\theta} D_t^\beta\right]dt\\
&+ \int_{[0,T)} e^{-rt}\left[w_c(D_t, C_t) - k\right] d\xi_t^c +\\
&+ \sum_{0\leq t\leq T} e^{-rt}\left[w(D_t, C_t) - w(D_t, C_{t-}) - k\Delta C_t\right] + M_T
\end{aligned} \tag{2.4.45}$$

Next, consider the term $w(D_t, C_t) - w(D_t, C_{t-})$ and an arbitrary positive change $\Delta C_t = z > 0$.

$$\begin{aligned}
w(D_t, C_t) - w(D_t, C_{t-}) &= \int_0^z \frac{d}{ds} w\left(D_t \frac{C_t^\theta}{(C_t + s)^\theta}, C_t + s\right) ds\\
&= \int_0^z \left[w_c\left(D_t \frac{C_t}{(C_t + s)^\theta}, C_t + s\right)\right] ds
\end{aligned} \tag{2.4.46}$$

Thus, using the fact that $w(d,c)$ satisfies the corresponding (HJB) inequalities we obtain

$$\int_0^T e^{-rt} C_t^{\alpha-\beta\theta} D_t^\beta dt - \int_{[0,T)} k e^{-rt} d\xi_t + e^{-rT} w(D_T, C_T) \leq w(d,c) + M_T \tag{2.4.47}$$

By Lemma 2.4.1 we have that

$$M_T \geq -w(d,c) - \int_{[0,T)} k e^{-rt} d\xi_t \tag{2.4.48}$$

Therefore, we obtain that $\mathbb{E}[\inf_{T\geq 0} M_T] > -\infty$ and thus the process M is a supermartingale with $\mathbb{E}[M_T] \leq 0,\ \forall T > 0$ which along with Fatou's lemma (1) gives that

$$\mathcal{J}_{(d,c)}(\xi) \leq \liminf_{T\to\infty} \mathbb{E}[\int_0^T e^{-rt} C_t^{\alpha-\beta\theta} D_t^\beta dt - \int_{[0,T)} k e^{-rt} d\xi_t] \leq w(d,c) + \liminf_{T\to\infty} e^{-rT} \mathbb{E}[-w(D_T, C_T)] \tag{2.4.49}$$

Thus, we conclude that

$$v(d,c) \leq w(d,c) \tag{2.4.50}$$

Consequently, we want to prove the reverse inequality and thus prove optimality. Apart from an possible initial jump at $t = 0-$, the optimal control should be such that the process $(D, C^\star)$ is reflecting in the free-boundary G in the positive direction. In particular,

$$D_t \leq G(C_t^\star) \quad \text{and} \quad \xi_t^\star - \xi_0^\star = \int_{(0,t)} \mathbf{1}_{\{D_s = G(C_s^\star)\}} d\xi_s^\star \quad \text{for all } t > 0.$$

while at $t = 0-$ the jump can occur iff

$$\xi_0^\star = \left(G^{-1}(d) - c\right)^+ > 0 \tag{2.4.51}$$

Thus,

$$C_t^\star = c\mathbf{1}_{\{\bar{D}_t < G(c)\}} + z(D_t)\mathbf{1}_{\{\bar{D}_t \geq G(c)\}} \tag{2.4.52}$$

The fact that $z(d) = G^{-1}(d) - c$ directly implies that $\xi_0^\star = z(d)$.

In particular from (2.4.29), (2.3.5) assumption (2.3.2) one can immediately see that

$$e^{-r_i t}\mathbb{E}[C^*_{it}] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T e^{-r_i t} C^*_{it} dt\right] < \infty. \tag{2.4.53}$$

Using integration by parts we obtain that

$$e^{-r_i T} C_{iT} = c + r_i \int_0^T e^{-r_i t} C_{it} dt + \int_{[0,T)} e^{-r_i t} dC_{it} \tag{2.4.54}$$

Using the monotone convergence (1.3.8) on the first part of the left hand side, the estimates (2.3.5) and dominated convergence theorem (1.3.9) allow us to gives that

$$\mathbb{E}\left[\int_0^\infty e^{-r_i t} dC^*_{it}\right] < \infty, \tag{2.4.55}$$

concluding that the admisibility of the control process $\{C^*_t\}_{t\in\mathbb{R}^+}$.

Hence, from (2.4.1) and in view of the HJB equations (2.4.5) we can see that

$$\int_0^T e^{-rt}(C^*_t)^{\alpha-\beta\theta} D^\beta_t dt - \int_{[0,T]} ke^{-rt} d\xi^*_t + e^{-rT} w(D_T, C^*_T) = w(d,c) + M^*_T \tag{2.4.56}$$

Consequently,

$$\int_0^T e^{-rt}\,(C^*_t)^{\alpha-\beta\theta}\,D^\beta_t dt - \int_{[0,T]} ke^{-rt} d\xi_t + e^{-rT} w(D_T, C^*_T) = w(d,c) + M^*_{T-} \tag{2.4.57}$$

Hence,

$$\sup_{T\in\mathbb{R}^+} M^*_T \le \int_0^\infty e^{-rt}(C^*_t)^{\alpha-\beta\theta} D^\beta_t dt + \sup_{T\in\mathbb{R}^+} e^{-rT} w(P^*_T, C^*_T) \tag{2.4.58}$$

Using the Hölder inequality on $\int_0^T e^{-rt}(C_t^*)^{\alpha-\beta\theta} D_t^\beta \mathrm{d}t$, the estimates (2.3.4) and (2.4.30) and the fact that the C^* is admissible assumption (2.3.1) we obtain that

$$\mathbb{E}\left[\int_0^\infty e^{-rt} C_t^{*\alpha-\beta\theta} D_t^{*\beta} dt\right] \le \left(\mathbb{E}\left[\int_0^\infty e^{-rt} C_t^* dt\right]\right)^{\alpha-\beta\theta} \left(\mathbb{E}\left[\int_0^\infty e^{-rt} D_t^{\frac{\beta}{1-\alpha+\beta\theta}} dt\right]\right)^{1-\alpha+\beta\theta} < \infty \tag{2.4.59}$$

In addition using (2.4.30),

$$\begin{aligned} w(D_T, C_T^*) &\le \Lambda\left[C_T^* + D_T^\nu\right] \\ &= K\bar{D}_T^\nu, \end{aligned} \tag{2.4.60}$$

where K a positive constant.

Therefore, by (2.3.4) we obtain that

$$\mathbb{E}\left[\sup_{T\in\mathbb{R}^+} e^{-rT} w(D_T, C_T^*)\right] < \infty \tag{2.4.61}$$

The above discussion concluded that M^* is a submartingale and therefore,

$$\mathbb{E}\left[\int_0^T e^{-rt}(C_t^*)^{\alpha-\beta\theta} D_t^\beta dt - \int_{[0,T]} ke^{-rt} d\xi_t\right] + e^{-rT}\mathbb{E}\left[w(D_T, C_T^*)\right] \ge w(d,c) \tag{2.4.62}$$

where monotone convergence on the first part of the left hand side and the estimates (2.4.30) and the use of dominated convergence on the second part of the right hand side give the required inequality,

$$v(d,c) \ge w(d,c) \tag{2.4.63}$$

Hence, we have proved that $v(d,c) = w(d,c), \quad \forall (d,c) \in \mathcal{S}$.

Finally, the formula for the equilibrium price is immediate.

■

2.5 An extension : Optimal control in the presence of depreciation

In this section, we shortly present a simple extention of the control problem to the case where the producer also has to incorporate continuous wear and tear (depreciation) into her optimal strategy. This part is not incorporated into the main sections of this chapter because in 3 where the case of the continuum of agents is presented, a deprecation rate cannot be integrated.

Specifically, in this case

$$dC_t = -\lambda C_t dt + d\xi_t \tag{2.5.1}$$

Therefore, we have the following objective function:

$$\mathcal{J}_{p,c}(C) = \mathbb{E}^{p,c}\left[\int_0^\infty e^{-rt} C_t^{\alpha-\theta\beta} D_t^\beta + k\lambda C_t dt - k\int_0^\infty e^{-rt} dC_t\right], \tag{2.5.2}$$

Hence, the HJB equation in this case become:

$$\max\left\{\frac{\sigma^2}{2} d^2 v_{dd}(d,c) + \mu d v_d(d,c) - r v(d,c) + c^{\alpha-\beta\theta} d^\beta + k\lambda c \right.$$
$$\left. v_c(d,c) - k\right\} = 0, \quad \forall (p,c) \in \mathcal{S} \tag{2.5.3}$$

in the waiting region we have a solution of the form $w(p,c) = A(c)p^n + \Gamma p^\beta c^{\alpha-\beta\theta} - \frac{\lambda k}{r} c$

using the same procedure of smooth pasting between investment $\mathcal{E}$ and $\mathcal{W}$, the free boundary can be found to be

$$G(c) = \left(\frac{k(1+\frac{\lambda}{r})}{\Gamma(a-\beta\theta)n} \right)^{1/\beta} c^{\frac{\alpha-\theta\beta-1}{\beta}} \tag{2.5.4}$$

We conjecture that the agent still wants to increase capacity at the free boundary and therefore $z(d) : \mathcal{E} \to \mathbb{R}^+$ is given by:

$$z(d) = \frac{1}{\tilde{k}^\nu} d^\nu \tag{2.5.5}$$

Therefore, integrating (2.5.1) the investment area we get that the candidate optimal control is

$$C_t^* = ce^{-\lambda t} \vee \sup_{s \le t} \frac{1}{\tilde{k}^\nu} D_s^\nu e^{\lambda(s-t)} \tag{2.5.6}$$

In addition, we can define a candidate solution for the optimization problem

Lemma 2.5.1 *The function $w : \mathcal{S} \to \mathbb{R}$ defined by*

$$w(d,c) = \begin{cases} A(c)d^n + \Gamma d^\beta c^{\alpha-\beta\theta} - \frac{\lambda k}{r} c & \text{if } (d,c) \in \mathcal{W} \\ w\left(d, z(d,c)\right) - k[z(d) - c] & \text{if } (d,c) \in \mathcal{E} \end{cases} \tag{2.5.7}$$

where

$$A(c) = \frac{\Gamma\beta^2}{n} \frac{\alpha - \theta\beta}{n(\theta\beta - \alpha) + n - \beta} \tilde{k}^\beta \frac{c}{G^n(c)} \tag{2.5.8}$$

and $z(d) : \mathcal{E} \to \mathbb{R}^+$ is determined by

$$z(d) = \frac{1}{\tilde{k}^\nu} d^\nu \tag{2.5.9}$$

is a $\mathcal{C}^{2,1}$ solution to the HJB equation (2.5.3)

Proof: The proof is similar to the non-depreciation case ■

Finally, a similar verification theorem can be proven for the depreciation case

Theorem 2.5.2 *The value function of the problem is given by* (2.5.7) *while the optimal control is*

$$C_t^* = ce^{-\lambda t} \vee \sup_{s \le t} \frac{1}{\tilde{k}^{\nu}} D_s^{\nu} e^{\lambda(s-t)} \tag{2.5.10}$$

where $C_t^* \in \mathcal{A}$

Proof: The proof follows the same lines as in the non-depreciation case ■

Chapter 3

A equilibrium model for capacity expansion: The competitive market case.

In this chapter we consider an irreversible capacity expansion model for a continuum of heterogenous agents faced with an exogenous stochastic demand function. The agents decide the production schedule via the expansion of their capability (capacity) and the optimization problem is formulated as a singular stochastic control problem in which additional investment has collectively an adverse effect on the price process of the underlying product as opposed to the monopolist case. We obtain a closed form solution for the optimal schedule process as well as the value function of each individual agent and a closed-loop mean field Nash equilibrium is established.

3.1 Introduction

In the previous chapter, we examined the rationale behind a monopolist's decision to invest in capacity expansion in an economy where demand for the underlying commodity follows a CES (Constant Elasticity of Substitution) function. A natural extension of this analysis is to explore how individual firms make investment decisions in a competitive market. In such a multi-agent environment, each producer strategically chooses to increase investment capacity while anticipating the decisions of their competitors. Several models of irreversible investment under varying market structures, ranging from monopoly to perfect competition, have been studied in the literature, including works by [BP09, BK96, AMZ17a, Bal98, Gre02, Lea93, NM07, Ste12].

Recently, significant attention has been directed toward the macroeconomic implications of heterogeneity [KS98, Aiy94, LJM14, AHL+22, NM18] and its impact on individual decision-making processes in multi-agent systems. Furthermore, due to the complexity of multi-agent stochastic differential games, the focus has increasingly shifted toward Mean Field Games (MFG), where the influence of any single agent on the system is negligible, but the collective behavior of all agents plays a crucial role in shaping decision-making [LL07]. This approach allows for greater tractability while maintaining the indirect interaction between agents through a collective variable.

In this work, we consider a continuum of heterogeneous producers facing an exogenous stochastic CES demand function. Producers must decide how much investment to make in order to improve their production efficiency and they have to commit to their decision, meaning that investment is irreversible. This assumption is particularly plausible for industries where they have high upfront costs and re-selling of capital will result into significant financial losses. The decision problem for each producer is framed as a singular stochastic control problem, with heterogeneity reflected in their initial production capacity, individual discount rates, unit costs of capacity expansion, and production costs. Each producer acts as a price-taker, meaning they have no direct influence on prices. However, they account for the collective impact of all producers on the price dynamics, resulting in a model with dimensionality $d = 3$.

Interestingly, we derive closed-form solutions for both the value function and the optimal control strategy, and we demonstrate the existence of a mean field Nash equilibrium on the supply side of the economy. Additionally, due to the competitive nature of the model, we obtain the equivalence of open- and closed-loop Nash equilibria, related to the literature on stochastic differential games[MMKN23, Rei82, Ols02, BP09, Car16].

3.2 Framework

Let us assume that we have a continuum of price taking heterogeneous agents indexed from a probability space $(I, m, \mathcal{I})$ and assume that m is an atomless measure, i.e., $m\big(\{i\}\big) = 0$ for all $i \in I$. The initial configuration of the system is C_{0-}, a positive random variable, and W be a Brownian motion independent of C_{0-}. We denote the filtration generated by W as $\mathcal{F}^W \equiv (\mathcal{F}_t^W)_{t\in\mathbb{R}^+}$ and we work on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{R}^+})$ where the filtration is, the smallest generated by C_{0-} and $\mathcal{F}^W$ and $\mathcal{F}_\infty \equiv \bigvee_{t\in\mathbb{R}^+} \mathcal{F}_t$. The Brownian motion W will be the driver of the stochastic demand. Again, we assume that the demand process $D \equiv (D_t; t \in \mathbb{R}_+)$ is given by a constant elasticity of substitution (CES)

$$D_t = \frac{B_t}{P_t^\delta}, \;\; D_{0-} = d \tag{3.2.1}$$

where the base demand process $B \equiv (B_t; t \in \mathbb{R}_+)$ is given by a GBM

$$dB_t = \mu_b B_t dt + \sigma_b B_t dW_t, \;\; B_0 = b \tag{3.2.2}$$

3.3 Individual Problem Setup

We take the point of view of a single producer of product quantity per unit time Q_t, faced with external prices $P \equiv (P_t; t \in \mathbb{R}^+)$ with available capability (capacity) $C_{it} > 0$, the production function is

$$\Pi(Q_{it}, C_{it}) = Q_{it}P_t - \frac{1}{\lambda_i^{\frac{1}{\beta-1}}} \frac{\beta - 1}{\beta^{1+\frac{1}{\beta-1}}} \frac{Q_{it}^{1+\frac{1}{\beta-1}}}{C_{it}^{\frac{\alpha}{\beta-1}}} \tag{3.3.1}$$

The producer is a price taker and thus, the equation for the optimum quantity Q^*_{it} at time t is given by:

$$P_t - \frac{(Q^*_{it})^{\frac{1}{\beta-1}}}{C_{it}^{\frac{\alpha_i}{\beta-1}}} = 0, \tag{3.3.2}$$

where we immediately obtain that

$$Q^*_{it} = \lambda_i \beta C_{it}^{\alpha_i} P_t^{\beta-1}. \tag{3.3.3}$$

Since agent $i \in I$ is part of a price-taking continuum, the total production is given by $\mathcal{Q}^*_t = \beta P_t^{\beta-1} \int_{i \in I} \lambda_i C_{it}^{\alpha_i} m(di)$

In this case the market clearing condition is $D_t = \mathcal{Q}^*_t$ and using (3.2.1) we obtain that

$$P_t = \frac{B_t^\gamma}{H_t^\gamma}, \tag{3.3.4}$$

where $\gamma = 1/(\delta + \beta - 1)$ and $H_t \equiv \mathbb{E}_m\left[\beta \lambda C_t^\alpha | \mathcal{F}^W\right] = \beta \int_{i \in I} \lambda_i C_{it}^{\alpha_i} m(di)$ represents the mean field term incorporating the action from all the agents.

The producer needs to decide long term investment strategy thus in terms of capacity maximization is not myopic and thus needs to decide how the investment dC_{it} in expanding capacity. Assuming that the cost of unit expansion is $\lambda_i k_i$, the objective function of every individual is

$$\mathcal{J}^i(C_{it}|P) = \mathbb{E}_{p,C_{0-}}\left[\int_0^\infty e^{-r_i t} C_{it}^{\alpha_i} P_t^\beta dt - \int_{[0,\infty)} k_i e^{-r_i t} dC_{it}\right] \tag{3.3.5}$$

where $C_{0-} > 0$.

Remark 3.3.1 *Note that the objective function is given by*

$$\widetilde{\mathcal{J}}^i(C_i|P) = \lambda_i \mathcal{J}^i(C_i|P)$$

The problem takes the form of a competitive game where competition occurs through the price process, however the effect of every individual is negligible, and therefore each producer has to find an expansion schedule $C_i \equiv (C_{it}; t \in \mathbb{R}_+)$ taking as given the price process P, i.e. the producer needs to find

$$C^*_{it} = \arg \sup_{(C_{it};\, t \in \mathbb{R}^+)} \mathbb{E}^{p, C_{0-}} \Big[\int_0^\infty e^{-r_i t} C_{it}^{\alpha_i} P_t^\beta dt - \int_{[0,\infty)} k_i e^{-r_i t} dC_{it} \Big], \tag{3.3.6}$$

with P an exogenous process.

Before we start the analysis of the control problem let us introduce a set of necessary assumptions.

Definition 3.3.2 *For every producer $i \in I$ the class of admissible controls is given by*

$$\mathcal{A}_i = \Big\{ \xi_{it} \in \mathcal{Z} : \mathbb{E}\big[\int_{[0,\infty)} k_i e^{-r_i t} d\xi_{it} \big] < \infty \Big\}, \tag{3.3.7}$$

r_i, k_i are positive constants and $\mathcal{Z}$ is the class of all non-decreasing cádglad $\mathcal{F}_t$-adapted processes.

Assumption 3.3.3 *We assume that $\mu := \gamma\mu_b + \frac{1}{2}\gamma(\gamma - 1)\sigma_b^2$ and $\sigma := \sigma_b\gamma$ and are positive quantities. Furthermore, for each agent i we must have that*

$$0 \leq \alpha_i < 1 \tag{3.3.8}$$

$$\beta \in (0, n_i) \tag{3.3.9}$$

$$\frac{\beta}{1 - \alpha_i} \in (-n_i, n_i) \tag{3.3.10}$$

where $n_i := \frac{-\left(\mu-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\sigma^2 r_i}}{\sigma^2}$.

Assumption 3.3.4 *We assume that the indexing* $i \to c_i$ *is a measurable function and that*

$$\int_{i\in I} \lambda_i c_i^{\alpha_i} m(di) < \infty \tag{3.3.11}$$

for every $c_i \geq 0$

3.4 Individual producer's HJB equations

In this section we will derive the HJB equations for a rational producer $i \in I$, who participates in a multi-agent mean field game. To be more specific, producer i chooses her optimal strategy taking into account the strategies of all other producers. Thus, it is expected that producers's i value function is a function of C_{0-} as well as the initial price P_{0-}. In addition, we expect that the control process C_i will take into account the processes of all the other agents $\{C_{-i}\}_{i\in I}$ and previous chapter's intuition tell us that all producers should increase capacity when a new maximum price $\overline{P_t}$ is reached at time t. The maximum price is defined as

$$\overline{P}_t = \sup_{0\leq s\leq t} P_t; \tag{3.4.1}$$

Therefore, we restrict ourselves to strategies which are of the form $C_{it} = C_i(\overline{P_t})$, where the function C_i is assumed to be non-decreasing function that can be written as a difference of two convex functions and $C_i(0) = c_i$. To this end, we consider the following assumption

Assumption 3.4.1 *Assume that the strategies of all the other producers belong to* $\mathcal{V}$*. We write* $H : \mathbb{R}^+ \to \mathbb{R}^+$ *as defined by*

$$H(\overline{p}) := \left(\beta \int_{i\in I} \lambda_i C_i^{\alpha_i}(\overline{p}) m(di)\right)^{\gamma}. \tag{3.4.2}$$

The function $H : \mathbb{R}^+ \to \mathbb{R}^+$ *is known to producer* i*. Also, for future reference we define* $h := \ln H$

This assumption is necessary since we are looking for this kind of class of equilibria, i.e. equilibria where every producer utilizes a strategy where capacity will change when the the function reaches a new maximum. Notice that, even though we will not prove uniqueness of such equilibria, the works of [BEK04, BR01] hint towards this direction. In particular, in equilibrium we will search for controls of the

$$C_{it} = c_i \vee \Psi_i(\overline{P_t}) \tag{3.4.3}$$

where Ψ_i satisfy the following definition.

Assumption 3.4.2 *We define* $\mathcal{B}$ *to be the set of all families of functions* $\{\Psi_i,\ i \in I\}$ *such that each* Ψ_i *is the difference of two convex functions satisfying*

$$\Psi_i'(\overline{p}) > 0 \quad \textit{for all } \overline{p} > 0, \quad \lim_{\overline{p}\downarrow 0} \Psi_i(\overline{p}) = 0 \quad \textit{and} \quad \lim_{\overline{p}\uparrow\infty} \Psi_i(\overline{p}) = \infty \tag{3.4.4}$$

and such that $\int_{i\in I} \lambda_i \Psi_i^{\alpha_i}(\overline{p}) m(di) < \infty$ *for all* $\overline{p} > 0$ *holds true.*

In addition, (3.4.3) and (3.4.2) we can see that

$$\lim_{\overline{p}\to 0} H(\overline{p}) = M, \quad \lim_{\overline{p}\to\infty} H(\overline{p}) = \infty \tag{3.4.5}$$

and

$$H'(\overline{p}) < 0 \quad \text{for all } \overline{p} > \sup\{\overline{p} \geq 0 \mid H(\overline{p}) = M\}, \tag{3.4.6}$$

where M a positive constant

Assumption 3.4.3 *We say that* $H \in \mathcal{V}$ *if* H *is a difference of two convex functions and satisfies* (3.4.5), (3.4.6)

Finally, before we resume our main discussion, we prove the following lemma regarding the differentiability of H

Lemma 3.4.4 *We assume that the controls, of all producers, $\{C_i\}_{i\in I}$ are of the form are* (3.4.3) *and they also satisfy assumption 3.4.2 functions of $\overline{p}$ then H is absolutely continuous and left differentiable.*

Proof: Notice that $H(\overline{p}_2)^{\frac{1}{\gamma}}$ can be written as

$$\frac{1}{\beta}H^{1/\gamma}(\overline{p}_2) = \frac{1}{\beta}H^{1/\gamma}(\overline{p}_1) + \int_{i\in I}\left(\int_{\overline{p}_1}^{\overline{p}_2}\lambda_i(c_i \vee \Psi_i)^{\alpha_i}_{\overline{p}-}(y)dy\right)m(di) \tag{3.4.7}$$

where the subscript $f_{\overline{p}-}(\overline{p})$ denotes the left derivative of $f(\overline{p})$ with respect to $\overline{p}$. By the non-negativity of the derivative of $(c_i \wedge \Psi_i(\overline{p}))^{\alpha_i}$ we can use Tonelli's theorem resulting into

$$\frac{1}{\beta}\left(H^{1/\gamma}(\overline{p}_2) - H^{1/\gamma}(\overline{p}_1)\right) = \int_{\overline{p}_1}^{\overline{p}_2}\left(\int_{i\in I}\lambda_i(c_i \vee \Psi_i)^{\alpha_i}_{\overline{p}-}(y)m(di)\right)dy \tag{3.4.8}$$

Therefore, we conclude that H is absolutely continuous and a difference of two convex functions which implies that it is also differentiable a.s. and left/right-differentiable everywhere while the derivative is given by

$$H_{\overline{p}-}(\overline{p}) = \gamma\beta H^{\frac{\gamma-1}{\gamma}}(\overline{p})\int_{i\in I}\lambda_i(c_i^{\alpha_i} \vee \Psi_i^{\alpha_i})_{\overline{p}}(\overline{p})m(di) \tag{3.4.9}$$

∎

In view of the assumption (3.4.1) and (3.3.4), the price process P that every producer expects can be re-written as

$$P_t = B_t^{\gamma}e^{-h(\overline{P_t})},$$

Note that the corresponding price process SDE is given by

$$dP_t = \mu P_t dt + \sigma P_t dW_t - P_t dh(\overline{P_t}) + \sum_{0 \le s \le t} (P_s - P_{s-}) \tag{3.4.10}$$

or equivalently,

$$dP_t = \mu P_t dt + \sigma P_t dW_t - P_t h_{\overline{p}-}(\overline{P_t}) d\overline{P_t} + \sum_{0 \le s \le t} (P_s - P_{s-}) \tag{3.4.11}$$

We denote the price process that every producer $i \in I$ expects as P however it is implicit from (3.4.11) that P is influenced by H, in the sense that every producer monitors the price process given that the strategies of the rest of the producers are fixed.

Consequently, to obtain the HJB equations the following process

$$\zeta_t^i \equiv e^{-r_i t} v^i(C_{it}, P_t, \overline{P_t}) + \int_u^t e^{-r_i s} \left(C_{is}^{\alpha_i} P_s^{\beta} ds - k_i dC_{is} \right); \quad t \ge 0,$$

is a supermartingale for all C_{it}, and a martingale for the optimal C_{it}^*. Hence, by Ito's lemma we obtain that

$$\begin{aligned} e^{r_i t} d\zeta_t^i = & \left(\mathscr{L}_i v^i(P_t, \overline{P_t}, C_{it}) + P_t^{\beta} C_{it}^{\alpha_i} \right) dt \\ & + \left(v_{\overline{p}}^i(P_t, \overline{P_t}, C_{it}) - P_t v_p^i(P_t, \overline{P_t}, C_{it}) h_{\overline{p}-}(\overline{P_t}) \right) d\overline{P_t}^c \\ & + \left(v_c^i(P_t, \overline{P_t}, C_{it}) - k_i \right) dC_{it}^c + e^{-r_i t} \sigma P_t v_p^i(P_t, \overline{P_t}, C_{it}) dW_t \\ & + \sum_{0 \le s \le t} [v^i(P_t, \overline{P_t}, C_{it}) - v^i(P_{t-}, \overline{P}_{t-}, C_{it-})] \end{aligned}$$

where $\mathscr{L}_i := \frac{\sigma^2}{2} p^2 \partial_{pp} + \mu p \partial_p - r_i$ is the generator of the discounted diffusion process

Therefore, we get that in the continuous region the producer needs to solve the following HJB equation:

$$\max \left\{ \mathscr{L}_i v^i(p, \overline{p}, c) + p^{\beta} c^{\alpha_i}, v_c^i(p, \overline{p}, c) - k_i \right\} = 0; \quad \forall (p, \overline{p}, c) \in \mathcal{S}, \tag{3.4.12}$$

Incorporating the impact of all other producers in the additional boundary condition

$$v^i_{\overline{p}}(p,\overline{p},c) - ph_{\overline{p}-}(\overline{p})v^i_p(p,\overline{p},c) = 0; \quad \forall (p,\overline{p},c) \in \mathcal{S} \tag{3.4.13}$$

Let us mention that by the form of H, we can only have a single discontinuous jump between $t = 0-$ and $t = 0$. Therefore, for every $u > 0$ $P, \overline{P}$ do not have jumps.

3.4.1 Equivalence of P vs X formalism

Intermedio: Change of variables

As in chapter 2 the problem becomes mathematically easier if we choose to work with the base demand instead of the price. Hence, we reformulate the problem using the base demand instead of the actual price process. We define as $X := (B^\gamma_t; t \in \mathbb{R}^+)$. Also for future reference we define $\overline{X}_\cdot := \overline{x} \vee \sup_{s\leq \cdot} X_s$. Here, we allow for any initial condition $\overline{x} \geq x$ simply to accommodate the technical fact that we will solve the problem using dynamic programming.

Before we proceed we give the following definition

Definition 2 *We define* $\widetilde{H} : \mathbb{R}^+ \to \mathbb{R}^+$ *as*

$$\widetilde{H}(\overline{x}) := \left(\beta \int_{i\in I} \lambda_i \left(c_i \vee \Phi_i(\overline{x})\right)^{\alpha_i} m(di)\right)^\gamma. \tag{3.4.14}$$

where $\Phi_i \in \mathcal{B}$

Note that the above definition does not necessarily gets as an input the optimal functions but rather a broad class of functions that belong to $\mathcal{B}$.

Consequently, the price process can be re-written as

$$P_t = \frac{X_t}{H(\overline{P_t})}, \tag{3.4.15}$$

or equivalently using the above definition

$$P_t = \frac{X_t}{\widetilde{H}(\overline{X_t})} \Longleftrightarrow P_t = X_t e^{-\widetilde{h}(\overline{X_t})} \tag{3.4.16}$$

In particular, the objective function can be re-written as

$$\mathcal{J}_{x,c,\overline{x}}(C) = \mathbb{E}^{x,c,\overline{x}}\left[\int_0^\infty e^{-r_i t} C_t^{\alpha_i} e^{-\beta\widetilde{h}(\overline{X}_t)} X_t^\beta dt - k_i \int_0^\infty e^{-r_i t} d\xi_t\right], \tag{3.4.17}$$

while the HJB equation will become

$$\begin{aligned}\max\Big\{&\frac{\sigma^2}{2}x^2 v_{xx}(x,c,\overline{x}) + \mu x v_x(x,c,\overline{x}) - r v(x,c,\overline{x}) + c^{\alpha_i} e^{-\beta\widetilde{h}(\overline{x})} x^\beta \,, \\ &v_c(x,c,\overline{x}) - k_i\Big\} = 0, \quad \forall (x,c,\overline{x}) \in \mathcal{S}.\end{aligned} \tag{3.4.18}$$

while the boundary condition becomes

$$v_{\overline{x}}^i(x,\overline{x},c) = 0; \quad \forall (x,\overline{x},c) \in \mathcal{S} \tag{3.4.19}$$

The following result identifies pairs of expansion strategies of the form (3.1) and expansion strategies of the form (3.8) that are associated with the same price process P.

Lemma 3.4.5 *In the presence of assumption 3.3.4, the following statements hold true.*

(I) Consider a collection $\{\Phi_i,\ i \in I\} \in \mathcal{B}$ *and suppose that the function* $\varphi \in \mathcal{V}$ *given by* (3.4.14) *is such that*

$$(\overline{x}/\widetilde{H}(\overline{x}))' > 0 \text{ for all } \overline{x} > 0 \quad \text{and} \quad \lim_{\overline{x}\to\infty} \overline{x}/\widetilde{H}(\overline{x}) = \infty. \tag{3.4.20}$$

If we define

$$\chi_1(\overline{x}) = (\overline{x}/\widetilde{H}(\overline{x})), \quad \overline{x} > 0, \quad \text{and} \quad \Psi_i(\overline{p}) = \Phi_i(\chi_1^{\text{inv}}(\overline{p})), \quad \overline{p} > 0, \tag{3.4.21}$$

where χ_1^{inv} *is the inverse function of* χ_1*, then* $\{\Psi_i,\ i \in I\} \in \mathcal{B}$ *if and only if the integrability condition assumption 3.3.4 holds true. In this case, if* $H \in \mathcal{V}$ *is defined by for* Ψ_i*,* $i \in I$*, given by* (3.4.20)*, then*

$$H(\overline{p}) = (\widetilde{H} \circ \chi_1^{\mathrm{inv}})(\overline{p}), \quad \textit{for all } \overline{p} > 0. \tag{3.4.22}$$

(II) Consider a family $\{\Psi_i,\ i \in I\} \in \mathcal{B}$ *and let* $H \in \mathcal{V}$*. The function*

$$\chi_2(\overline{p}) = \overline{p}H(\overline{p}), \quad \overline{p} > 0, \tag{3.4.23}$$

is such that

$$(\chi_2)'(\overline{p}) = \overline{p}H(\overline{p}) > 1, \quad \textit{and} \quad \lim_{\overline{p}\to\infty} \chi_2(\overline{p}) = \lim_{\overline{p}\to\infty} \overline{p}H(\overline{p}) = \infty. \tag{3.4.24}$$

If we define

$$\Phi_i(\overline{x}) = \Psi_i\left(\frac{\overline{x}}{H \circ \chi_2^{\mathrm{inv}}(\overline{x})}\right), \quad \overline{x} > 0, \tag{3.4.25}$$

where χ_2^{inv} *is the inverse of the function* χ_2*, then* $\{\Phi_i,\ i \in I\} \in \mathcal{B}$*. Furthermore, if* $\widetilde{H} \in \mathcal{V}$ *for* Φ_i*,* $i \in I$*, given by* (3.4.25)*, then* $\widetilde{H}$ *satisfies* (3.4.20) *and*

$$\widetilde{H}(\overline{x}) = (H \circ \chi_2^{\mathrm{inv}})(\overline{x}), \quad \textit{for all } \overline{x} > 0. \tag{3.4.26}$$

(III) In the context of either (I) or (II),

$$\widetilde{H}(\overline{X}) = H(\overline{P}). \tag{3.4.27}$$

Proof: The claim that the family $\{\Psi_i,\ i \in I\}$ defined by (3.11) belongs to $\mathcal{B}$ if and only if assumption 3.3.4 holds true follows from the assumption that $\{\Phi_i,\ i \in I\} \in \mathcal{B}$. On the other hand, the function $H \in \mathcal{V}$ for Ψ_i, $i \in I$, given by (3.4.21), satisfies (3.4.22) because

$$H(\overline{p}) = \Big(\beta \int_I \lambda_i\big(c_i^{\alpha_i} \vee \Phi_i^{\alpha_i}(\chi_1^{\text{inv}}(\overline{p}))\big) m(di)\Big)^{\gamma} = \widetilde{H}(\chi_1^{\text{inv}}(\overline{p})).$$

The claims in (3.4.23) follow immediately from the fact that H satisfies (3.4.5) and (3.4.6). In turn, (3.4.23) and the observation that

$$\chi_2(\overline{p})/(H \circ \chi_2^{\text{inv}})(\chi_2(\overline{p})) = \overline{p} \quad \Longleftrightarrow \quad \overline{x}/(H \circ \chi_2^{\text{inv}})(\overline{x}) = \chi_2^{\text{inv}}(\overline{x}), \tag{3.4.28}$$

imply that the collection $\{\Phi_i,\ i \in I\}$ defined by (3.4.25) belongs to $\mathcal{B}$ thanks to the assumption that $\{\Psi_i,\ i \in I\} \in \mathcal{B}$ and the observation that

$$\int_I \lambda_i \Phi^{\alpha_i}(\overline{x})\, m(di) = \int_I \lambda_i \Psi^{\alpha_i}(\chi_2^{inv}(\overline{x}))\, m(di) \leq \int_I \lambda_i \Psi^{\alpha_i}(\overline{x})\, m(di) < \infty \quad \text{for all } \overline{x} > 0. \tag{3.4.29}$$

The function $\widetilde{H}$ for Φ_i, $i \in I$, given by (3.4.25) satisfies (3.4.25) because

$$\begin{aligned} \widetilde{H}(\overline{x}) =& \Big(\beta \int_I \lambda_i\big(c_i^{\alpha_i} \vee \Psi_i^{\alpha_i}\big(\frac{\overline{x}}{H \circ \chi_2^{\text{inv}}(\overline{x})}\big) m(di)\Big)^{\gamma} = \\ & H\left(\frac{\overline{x}}{H \circ \chi_2^{\text{inv}}(\overline{x})}\right) = H\left(\chi_2^{\text{inv}}(\overline{x})\right), \end{aligned} \tag{3.4.30}$$

Finally, (3.16) is an immediate consequence of the equivalences

$$\overline{P} = \overline{X}/H(\overline{P}) \iff \overline{P} = \chi_2^{\text{inv}}(\overline{X}),$$

and

$$\overline{P} = \frac{\overline{X}}{\widetilde{H}(\overline{X})} \iff \overline{X} = \chi_1^{\text{inv}}(\overline{P}),$$

which follow from (3.4) and (3.9). ∎

The usefulness of the above lemma is that it shows the equivalence between the two formalisms and will be used in a subsequent section.

3.5 Individual producer's control solution

The aim of this section is to determine the optimal strategy for a producer $i \in I$. We proceed directly solving the HJB equation using the assumption of continuity and differentiation. A free boundary participates the space into two waiting regions $\mathcal{W}_1^i, \mathcal{W}_2^i$ and an expansion region $\mathcal{E}^i$ as well as the optimal control. The proof of a verification theorem concludes that the conjectured optimal control is indeed optimal. Finally, in Appendix A conjecturing a candidate optimal control and using probabilistic arguments, we directly evaluate the value function. This method is particularly useful, as it is an immediate solution to the control problem. In addition, it provides an interesting alternative to the evaluation of the value function assuming that an educated guess for the optimal control has been made.

3.5.1 Solving the Hamilton Jacobi Bellman Equation

We will solve the individual agent's optimization problem by deriving the solution to the quasi-variational inequality stated in Problem 3.7.1 below. This involves a C^1 function $G_i : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\frac{\partial G_i(c,\overline{x})}{\partial c} > 0, \quad \lim_{c \downarrow 0} G_i(c,\overline{x}) = 0 \quad \text{and} \quad \lim_{c \uparrow \infty} G_i(c,\overline{x}) = \infty. \tag{3.5.1}$$

It also involves the unique solution $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ to the equation

$$G_i\big(\Phi_i(\overline{x}),\overline{x}\big) = \overline{x}, \ \ \overline{x} > 0 \tag{3.5.2}$$

where the function Φ_i is required to be C^1 and such that

$$\Phi_i'(\overline{x}) > 0 \quad \text{for all } \overline{x} > 0, \quad \lim_{\overline{x}\downarrow 0} \Phi_i(\overline{x}) = 0 \quad \text{and} \quad \lim_{\overline{x}\uparrow\infty} \Phi_i(\overline{x}) = \infty. \tag{3.5.3}$$

We define the following free-boundary surfaces

$$\mathcal{S}_i^1 = \big\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \leq \overline{x} \text{ and } x = G_i(c, \overline{x})\big\} \tag{3.5.4}$$

$$\text{and} \quad \mathcal{S}_i^2 = \big\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \leq \overline{x} \text{ and } c = \Phi_i(\overline{x})\big\} \tag{3.5.5}$$

To begin with we define the following surfaces which partition the control's problem state space

$$\mathcal{S} = \big\{(c, x, \overline{x}) \in \mathbb{R}_+^3 \mid\ 0 < x \leq \overline{x}\big\} \tag{3.5.6}$$

into the sets

$$\mathcal{E}_i = \big\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \leq \overline{x} \text{ and } c \leq \Gamma_i(x, \overline{x})\big\}, \tag{3.5.7}$$

$$\text{and} \quad \mathcal{W}_i^1 = \big\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \leq \overline{x} \text{ and } c \geq \Phi_i(\overline{x})\big\}, \tag{3.5.8}$$

$$\mathcal{W}_i^2 = \big\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \leq \overline{x} \text{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x})\big\} \tag{3.5.9}$$

where

$$\Gamma_i(\cdot, \overline{x}) \coloneqq G_i^{\text{inv}}(\cdot, \overline{x})$$

The idea of conjecturing such a partition of the state-space comes from the fact that contrary to the monopolist case where only two state variables were needed now we need also to include a third one and therefore without an extra partition of the state space the HJB equations will not be consistent. Alternatively, one can directly evaluate the value function using directly a conjecture for the optimal control as it was done in Appendix A . In this case one can see that the waiting region is naturally divided into two parts.

We postulate that the individual producer's optimal capacity schedule can be informally described as follows. At time 0, if the initial state $(c, x, \overline{x})$ is inside the region $\mathcal{E}_i$, then it is optimal for the individual producer to exert action so that they reposition the state process on the boundary surface $\mathcal{S}_i^2$. In view of standard singular stochastic control theory, such an action is associated with the requirement that the value function v^i should satisfy (3.7.26) as well as the inequality in (3.7.27). Beyond such a possible jump at time 0, it is optimal for the individual producer to exert minimal action so as to prevent the state process entering the interior of $\mathcal{E}_i$. On the other hand, it is optimal for the individual producer to exert no control effort while the state process takes values in the interior of the set $\mathcal{W}_i^1 \cup \mathcal{W}_i^2$, which is associated with the inequalities (3.7.24) and (3.7.25) as well as the equalities in (3.7.27). The significance of the surface $\mathcal{S}_i^2$ arises from the fact that, eventually, it is optimal for the individual producer to exert minimal effort so as to prevent the state process falling below the curve defined by $x = \overline{x}$ and $c = \Phi_i(\overline{x})$, which is the intersection of $\mathcal{S}_i^2$ with the boundary of the state space defined by $x = \overline{x}$. In particular, it is optimal that minimal control effort should be exercised so that the state process takes values on the surface $\mathcal{S}_i^2$ at all times after this surface has been reached.

Therefore, in terms of the HJB equations the problem can be succinctly states as:

Problem 3.5.1 *Determine a function $G_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ and a function $v^i : \mathcal{S} \to \mathbb{R}$, satisfying the following conditions.*

(I) *the functions $G_i(\cdot, \overline{x})$ is C^1, while $G_i(c, \cdot)$ and $\Phi(\cdot)$ are differences of two convex functions and satisfy. Furthermore they satisfy (3.5.1)–(3.5.3).*

(II) *The function $v^i(\cdot, \cdot, \overline{x})$ is $C^{1,2}$ in the interior of $\mathcal{S}$.*

(III) *The function* $v^i(c, x, \cdot)$ *is a difference of two convex functions in the interior of* $\mathcal{S}$*. Furthermore,*

$$w_{\overline{x}}(c, \overline{x}, \overline{x}) := \lim_{x \uparrow \overline{x}} w_{\overline{x}-}(c, x, \overline{x}) = 0 \quad \textit{for all} \quad 0 < \Phi_i(\overline{x}) \leq c \tag{3.5.10}$$

(IV) *The function* v^i *is such that*

$$v_c^i(c, x, \overline{x}) < k_i \quad \textit{for all } 0 < x < \overline{x} \textit{ and } \Phi_i(\overline{x}) \leq c, \tag{3.5.11}$$

$$v_c^i(c, x, \overline{x}) < k_i \quad \textit{for all } 0 < x \leq \overline{x} \textit{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x}), \tag{3.5.12}$$

$$v_c^i(c, x, \overline{x}) = k_i \quad \textit{for all } 0 < x \leq \overline{x} \textit{ and } c \leq \Gamma_i(x, \overline{x}), \tag{3.5.13}$$

and

$$\mathscr{L}_i v^i(c, x, \overline{x}) + x^\beta c^{\alpha_i} e^{-\beta h(\overline{x})} := \frac{1}{2}\sigma^2 x^2 v_{xx}^i(c, x, \overline{x}) + \mu x v_x^i(c, x, \overline{x}) - r_i v^i(c, x, \overline{x}) + x^\beta c^{\alpha_i} e^{-\beta h(\overline{x})}$$

$$\begin{cases} = 0 & \textit{for all } 0 < x < \overline{x} \textit{ and } 0 < \Phi_i(\overline{x}) \leq c, \\ = 0 & \textit{for all } 0 < x \leq \overline{x} \textit{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x}), \\ < 0 & \textit{for all } 0 < x < \overline{x} \textit{ and } c \leq \Gamma_i(x, \overline{x}). \end{cases} \tag{3.5.14}$$

Every solution to the ODE, $\mathscr{L}_i v(c, x, \overline{x}) + x^\beta c^{\alpha_i} e^{-\beta \widetilde{h}(\overline{x})} = 0$, associated with (3.7.27) is given by

$$v^i(c, x, \overline{x}) = \Delta_1(c, \overline{x}) x^{m_i} + \Delta_2(c, \overline{x}) x^{n_i} + \frac{1}{\rho_i} x^\beta c^{\alpha_i} e^{-\beta \widetilde{h}(\overline{x})}$$

for some functions Δ_1 and Δ_2, where $m_i < 0 < n_i$ are the solutions to the quadratic equation

$$\frac{1}{2}\sigma^2 k^2 + \left(\mu - \frac{1}{2}\sigma^2\right) k - r_i = 0,$$

given by

$$n_i, m_i = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r_i}}{\sigma^2}.$$

For future reference, we note that

$$r_i > \mu \iff n_i > \beta, \quad n_i + m_i - 1 = -\frac{2\mu}{\sigma^2}, \quad n_i m_i = -\frac{2r_i}{\sigma^2} \tag{3.5.15}$$

$$\text{and} \quad \rho_i = \frac{1}{2}\sigma^2 (n_i - \beta)(-m_i + \beta). \tag{3.5.16}$$

We consider a solution to Problem 3.7.1 of the form

$$v^i(c,x,\overline{x}) = \begin{cases} A_i(c,\overline{x}) x^{n_i} + \frac{1}{\rho_i} x^\beta c^{\alpha_i} e^{-\beta \widetilde{h}(\overline{x})}, & \text{if } (c,x,\overline{x}) \in \mathcal{W}_i^1, \\ B_i(c,\overline{x}) x^{n_i} + \frac{1}{\rho_i} x^\beta c^{\alpha_i} e^{-\beta \widetilde{h}(\overline{x})}, & \text{if } (c,x,\overline{x}) \in \mathcal{W}_i^2 \cup \mathcal{S}_i^1, \\ v^i\big(\Gamma_i(x,\overline{x}), x, \overline{x}\big) - k_i\big(\Gamma_i(x,\overline{x}) - c\big), & \text{if } (c,x,\overline{x}) \in \mathcal{E}_i \setminus \mathcal{S}_i^1, \end{cases} \tag{3.5.17}$$

To determine the function B_i and the free-boundary function G_i, we appeal to the so called "smooth-pasting condition" of singular stochastic control[1]. In particular, we require that $v^i(\cdot, x, \overline{x})$ should be C^2 along the free-boundary point $G_i(c,\overline{x})$, which suggests the

[1]Note that this is more of a conjecture. We are searching for a solution with this kind of regularity and consequently we have use a verification theorem to prove that indeed this is a solution to the problem. It is not ex-ante certain that such a solution should exist

equations

$$\lim_{x\uparrow G_i(c,\overline{x})} v_c^i(c,x,\overline{x}) = (B_i)_c(c,\overline{x})G_i^{n_i}(c,\overline{x}) + \frac{\alpha_i}{\rho_i}G_i^{\beta}(c,\overline{x})c^{\alpha_i-1}e^{-\beta\widetilde{h}(\overline{x})} = k_i \tag{3.5.18}$$

and $$\lim_{x\uparrow G_i(c,\overline{x})} v_{cx}^i(c,x,\overline{x}) = n_i(B_i)_c(c,\overline{x}) + \frac{\alpha_i\beta}{\rho_i}G_i^{\beta-n_i}(c,\overline{x})c^{\alpha_i-1}e^{-\beta\widetilde{h}(\overline{x})} = 0. \tag{3.5.19}$$

The solution to this system of equations is given by

$$G_i(c,\overline{x}) = \left(\frac{\rho_i n_i k_i}{\alpha_i(n_i-\beta)}\right)^{\frac{1}{\beta}} c^{\frac{1-\alpha_i}{\beta}}\, e^{\widetilde{h}(\overline{x})} \quad \text{and} \quad (B_i)_c(c,\overline{x}) = -\frac{k_i}{n_i-\beta}G_i^{-n_i}(c,\overline{x}). \tag{3.5.20}$$

For G_i given by the first of these expressions, we can see that the unique solution to equation (3.5.2) is given by

$$\Gamma_i(x,\overline{x}) = \left(\frac{\alpha_i(n_i-\beta)}{\rho_i n_i k_i}\right)^{\frac{1}{1-\alpha_i}} \left(x\, e^{-\widetilde{h}(\overline{x})}\right)^{\frac{\beta}{1-\alpha_i}}, \tag{3.5.21}$$

$$\Phi_i(\overline{x}) = \left(\frac{\alpha_i(n_i-\beta)}{\rho_i n_i k_i}\right)^{\frac{1}{1-\alpha_i}} \left(\overline{x}\, e^{-\widetilde{h}(\overline{x})}\right)^{\frac{\beta}{1-\alpha_i}}, \tag{3.5.22}$$

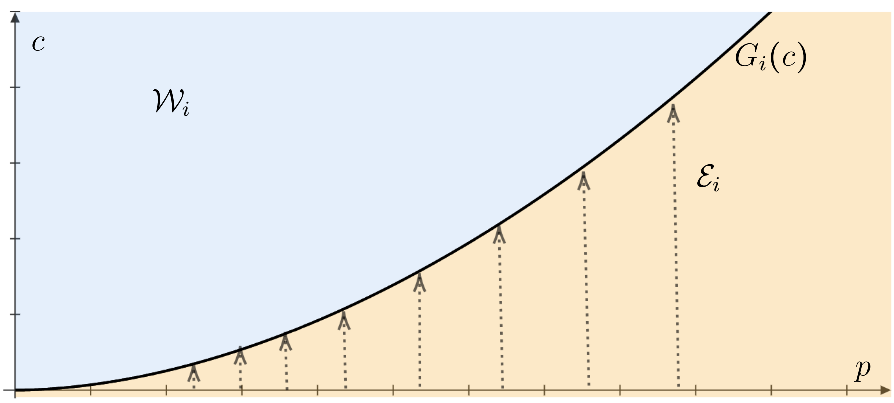

Figure 3.1 *Illustration of agent's i free boundary $G_i(c)$. The vertical lines indicate the fact that in the mean field setting the decision of an agent to increase capacity does not affect the price level of the product*

On the other hand, the solution to the ODE in (3.5.20) is given by

$$B_i(c,\overline{x}) = \widetilde{f}_i(\overline{x}) + \frac{k_i\beta^2}{(n_i-\beta)\big(n_i(1-\alpha_i)-\beta\big)}\frac{c}{G_i^n(c,\overline{x})}, \tag{3.5.23}$$

where $\widetilde{f}_i$ is a function to be determined.

In order to determine $\widetilde{f}_i$ we will use the fact that in the region $\{x=\overline{x}\}\cap\mathcal{W}_i^2$ we should have that

$$v_{\overline{x}}^i\big(\Phi_i(\overline{x}),x,\overline{x}\big)=0 \tag{3.5.24}$$

Therefore, substituting (3.5.23) into (3.5.24) we obtain that

$$\widetilde{f}(\overline{x}) = -\frac{k_i\beta^2}{(n_i-\beta)\big((1-\alpha_i)n_i-\beta\big)}\int_{\overline{x}}^{\infty} y^{-n_i}\widetilde{h}_{\overline{x}}(\overline{x})(y)\Phi_i(y)\,\mathrm{d}y + \frac{\beta}{\rho}\int_{\overline{x}}^{\infty} y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)\Phi_i^{\alpha_i}(y)e^{-\beta\widetilde{h}(y)}\,\mathrm{d}y. \tag{3.5.25}$$

or equivalently we obtain that

$$\widetilde{f}(\overline{x}) = -\frac{k_i\beta^2}{(n_i-\beta)\big((1-\alpha_i)n_i-\beta\big)}\int_{\overline{x}}^{\infty} y^{-n_i}\widetilde{h}_{\overline{x}}(\overline{x})(y)\Phi_i(y)\,\mathrm{d}y + \frac{n_ik_i\beta}{\alpha_i(n_i-\beta)}\int_{\overline{x}}^{\infty} y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)\Phi_i(y)\,\mathrm{d}y. \tag{3.5.26}$$

we should have $\lim_{\overline{x}\to\infty}\widetilde{f}_i(\overline{p})=0$

Consequently, to determine $A_i(c,\overline{x})$ we use again the fact that for $\mathcal{W}_i^1$ and $x\to\overline{x}$ we have that

$$\lim_{x\uparrow\overline{x}} v_{\overline{x}}^i\big(c,x,\overline{x}\big)=0 \tag{3.5.27}$$

Hence, we obtain that

$$A_i(c,\overline{x}) = -\int_{\overline{x}}^{G_i(c,\overline{x})}\frac{\beta}{\rho_i}y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)c^{\alpha_i}e^{-\beta\widetilde{h}(\overline{x})}\mathrm{d}y + f_i(c) \tag{3.5.28}$$

Next, the requirement that v^i should be continuous is reflected by the identity

$$\lim_{c\uparrow\Phi_i(\overline{x})} v^i(c,x,\overline{x}) = \lim_{c\downarrow\Phi_i(\overline{x})} v^i(c,x,\overline{x}),$$

which gives rise to the expression

$$\begin{aligned} &A_i(\Phi_i(\overline{x}),\overline{x}) = B_i\big(\Phi_i(\overline{x}),\overline{x}\big) \Rightarrow \\ &f_i(\Phi_i(\overline{x})) = B_i\big(\Phi_i(\overline{x}),\overline{x}\big) \end{aligned} \tag{3.5.29}$$

Therefore, we conclude that

$$\begin{aligned} f_i(c) = &-\frac{(1-\alpha_i)n_i k_i \beta}{(n_i-\beta)\big((1-\alpha_i)n_i-\beta\big)}\int_{\overline{x}}^{\infty} y^{-n_i}(\Phi_i)_{\overline{x}}(y)\,\mathrm{d}y + \frac{\beta}{\rho}\int_{\overline{x}}^{\infty} y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)\Phi_i^{\alpha_i}(y)e^{-\beta\widetilde{h}(y)}\,\mathrm{d}y \\ &+\frac{k_i\beta^2}{(n_i-\beta)\big(n_i(1-\alpha_i)-\beta\big)}\frac{c}{G_i^n(c,\overline{x})} \end{aligned} \tag{3.5.30}$$

Lemma 3.5.2 *The function $w^i : \mathcal{S} \to \mathbb{R}$ defined by*

$$w^i(c,x,\overline{x}) = \begin{cases} A_i(c,\overline{x})x^{n_i} + \frac{1}{\rho_i}x^{\beta}c^{\alpha_i}e^{-\beta\widetilde{h}(\overline{x})} & \textit{if } (c,x,\overline{x}) \in \mathcal{W}_i^1 \\ B_i(c,\overline{x})x^{n_i} + \frac{1}{\rho_i}x^{\beta}c^{\alpha_i}e^{-\beta\widetilde{h}(\overline{x})} & \textit{if } (c,x,\overline{x}) \in \mathcal{W}_i^2 \cup \mathcal{S}^1 \\ w^i\left(\Gamma_i(x,\overline{x}),x,\overline{x}\right) - k_i[\Gamma_i(x,\overline{x}) - c] & \textit{if } (c,x,\overline{x}) \in \mathcal{E}_i \end{cases} \tag{3.5.31}$$

$$\begin{aligned} A_i(c,\overline{x}) = &-\frac{k_i\beta^2}{(n_i-\beta)\big((1-\alpha_i)n_i-\beta\big)}\int_{\overline{x}}^{\infty} y^{-n_i}\widetilde{h}_{\overline{x}}(\overline{x})(y)\Phi_i(y)\,\mathrm{d}y - \frac{\beta}{\rho}\int_{\overline{x}}^{\infty} y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)\Phi_i^{\alpha_i}(y)e^{-\beta\widetilde{h}(y)}\,\mathrm{d}y \\ &+\frac{k_i\beta^2}{(n_i-\beta)\big(n_i(1-\alpha_i)-\beta\big)}\frac{c}{G_i^n(c,\overline{x})} + \int_{\overline{x}}^{G_i(c,\overline{x})}\frac{\beta}{\rho_i}y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)c^{\alpha_i}e^{-\beta\widetilde{h}(\overline{x})}\mathrm{d}y, \end{aligned} \tag{3.5.32}$$

$$B_i(c,\overline{x}) = \frac{k_i\beta^2}{(n_i-\beta)\big(n_i(1-\alpha_i)-\beta\big)}\frac{c}{G_i^n(c,\overline{x})} - \frac{k_i\beta^2}{(n_i-\beta)\big((1-\alpha_i)n_i-\beta\big)}\int_{\overline{x}}^{\infty} y^{-n_i}\widetilde{h}_{\overline{x}}(\overline{x})(y)\Phi_i(y)\,\mathrm{d}y$$
$$-\frac{\beta}{\rho}\int_{\overline{x}}^{\infty} y^{\beta-n_i}\widetilde{h}_{\overline{x}}(y)\Phi_i^{\alpha_i}(y)e^{-\beta\widetilde{h}(y)}\,\mathrm{d}y \tag{3.5.33}$$

and $\Gamma_i(x,\overline{x}):\mathcal{E}_i\to\mathbb{R}^+$, $\Gamma_i(x,\overline{x}):\mathcal{E}_i\to\mathbb{R}^+$ *are determined by*

$$\Gamma_i(x,\overline{x}) = \left(\frac{(n_i-\beta)\alpha_i}{\rho_i k_i n_i}\right)^{\frac{1}{1-\alpha_i}}\left(xe^{-\widetilde{h}(\overline{x})}\right)^{\frac{\beta}{1-\alpha_i}} \tag{3.5.34}$$

$$\Phi_i(\overline{x}) = \left(\frac{(n_i-\beta)\alpha_i}{\rho_i k_i n_i}\right)^{\frac{1}{1-\alpha_i}}\left(\overline{x}e^{-\widetilde{h}(\overline{x})}\right)^{\frac{\beta}{1-\alpha_i}} \tag{3.5.35}$$

is a $\mathcal{C}^{2,1}$ *solution to agent's* $i\in I$ *HJB equation* (3.4.12)*. Finally, for all* $(c,x,\overline{x})\in\mathcal{S}$

$$0\le w^i(c,x,\overline{x})\le\Lambda_i\left(c+c^{\alpha_i}\Phi_i^{1-\alpha_i}(\overline{x})+\Phi_i(\overline{x})\right) \tag{3.5.36}$$

where $\Lambda_i>0$ *depends only on the parameters of the problems*

Remark 3.5.3 *Note that by performing the coordinate transformation* $x\to pe^{h(\overline{p})}$ *and* $\overline{x}\to\overline{p}e^{h(\overline{p})}$ *and after a series of integrations by parts recover* (3.7.20) *(Appendix B) with*

$$\Psi_i(\overline{p}) = \left(\frac{(n_i-\beta)\alpha_i}{\rho_i k_i n_i}\right)^{\frac{1}{1-\alpha_i}}\overline{p}^{\frac{\beta}{1-\alpha_i}} \tag{3.5.37}$$

This also shows consistency between the two formulations of the problem.

3.5.2 Verification theorem

Now we turn our attention on proving that the previous discussion leads to an optimal solution for a producer $i\in I$.

Theorem 3.5.4 *Let us assume that assumptions Assumptions 3.3.2, Assumptions 3.3.3 hold for the producer $i \in I$. rise to the mean field process H satisfying (3.4.4). The value function of the control problem identifies with the solution (3.5.2). In addition, the optimal capacity expansion strategy is given by*

$$C^*_{it} = c_i \vee \Phi_i(\overline{X}_t) \tag{3.5.38}$$

Moreover, $C^*_i \in \mathcal{A}, \;\; \forall i \in I$

Proof:

$$\begin{aligned} &e^{-r_iT} w^i(C_{iT}, X_T, \overline{X}_T) \\ &= w^i(c, x, \overline{x}) + \int_0^T e^{-r_it} \mathscr{L}_i e^{-r_it} w^i(C_{it}, X_t, \overline{X_t}) dt + \int_{[0,T)} e^{-r_it} w^i_c(C_{it}, X_t, \overline{X_t}) dC^c_{it} \\ &+ \int_0^T e^{-r_it} w^i_{\overline{x}}(C_{it}, X_t, \overline{X_t}) d\overline{X_t} \\ &+ \sum_{0 \le s \le T} e^{-r_it} \left[w^i(C_{it}, X_t, \overline{X_t}) - w^i(C_{it-}, X_t, \overline{X_t}) \right] + M^i_T \end{aligned} \tag{3.5.39}$$

where,

$$M^i_T = \sigma \int_0^T e^{-r_it} X_t w^i_p(C_{it}, X_t, \overline{X_t}) dW_t. \tag{3.5.40}$$

Therefore,

$$\begin{aligned} &\int_0^T e^{-r_it} X^\beta_t C^{\alpha_i}_{it} e^{-\beta \widetilde{h}(\overline{X_t})} dt - \int_{[0,T]} k_i e^{-r_it} dC_{it} + e^{-r_iT} w^i(C_{iT}, X_T, \overline{X}_T) \\ &= w^i(c, x, \overline{x}) + \int_0^T e^{-r_it} \left[\mathscr{L}_i w^i(C_{it,}, X_t, \overline{X_t}) + X^\beta_t C^{\alpha_i}_{it} e^{-\beta \widetilde{h}(\overline{X_t})} \right] dt \\ &+ \int_{[0,T]} e^{-r_it} \left[w^i_c(C_{it}, X_t, \overline{X_t}) - k_i \right] dC^c_{it} + \int_0^T e^{-r_it} w^i_{\overline{x}}(C_{it}, X_t, \overline{X_t}) d\overline{X_t} \\ &+ \sum_{0 \le t \le T} e^{-r_it} \left[w^i(C_{it}, X_t, \overline{X_t}) - w^i(C_{it-}, X_t, \overline{X_t}) - k_i \Delta C_{it} \right] + M_T \end{aligned} \tag{3.5.41}$$

To evaluate the jump term $w^i(C_{it}, X_t, \overline{X_t}) - w^i(C_{it-}, X_t, \overline{X_t})$, consider that producer i makes and arbitrary expansion $\Delta C_{it} = z > 0$

$$
\begin{aligned}
w^i(C_{it}, X_t, \overline{X_t}) - w^i(C_{it-}, X_t, \overline{X_t}) &= \int_0^z \frac{d}{ds} w^i\left(C_{it} + s, X_t, \overline{X_t}\right) ds \\
&= \int_0^z w^i_c\left(C_{it} + s, X_t, \overline{X_t}\right) ds
\end{aligned}
\tag{3.5.42}
$$

Thus, using the fact that $w^i(c, x, \overline{x})$ satisfies the corresponding HJB (3.4.12) equation we obtain

$$
\int_0^T e^{-r_i t} X_t^{\beta} C_{it}^{\alpha_i} e^{-\beta \widetilde{h}(\overline{X_t})} dt - \int_{[0,T]} k_i e^{-r_i t} dC_{it} + e^{-r_i T} w^i(C_{iT}, X_T, \overline{X}_T) \leq w^i(c, x, \overline{x}) + M_T
\tag{3.5.43}
$$

Consequently, consider a localizing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tau_n = \infty$ and the process $\{M_{T\wedge\tau_n}\}_{n\in\mathbb{N}}$ is a local martingale. Taking expectations and using the fact that (3.5.36) from Lemma 3.5.2 we have that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-r_i t} X_t^{\beta} C_{it}^{\alpha_i} e^{-\beta \widetilde{h}(\overline{X_t})} dt - \int_{[0,T\wedge\tau_n]} k_i e^{-r_i t} dC_{it}\right] &\leq w^i(c, x, \overline{x}) \\
&\quad - e^{-r_i T\wedge\tau_n} \mathbb{E}[w^i\left(C_{i(T\wedge\tau_n)-}, X_{T\wedge\tau_n}, \overline{X}_{T\wedge\tau_n}\right)]
\end{aligned}
\tag{3.5.44}
$$

Hence, we get that

$$
v^i(c, x, \overline{x}) \leq w^i(c, x, \overline{x})
\tag{3.5.45}
$$

Next, the reverse inequality will be proven by making use of the control process given by (3.5.38). We start by showing that it is indeed an admissible strategy in the sense of

(3.3.2). In particular from (3.5.34) (3.5.35) and assumption (3.3.4) one can immediately see that

$$C^*_{it} \leq c_i + \Phi_i(\overline{x}) + \Phi_i(\overline{X_t}), \tag{3.5.46}$$

Therefore, using (3.5.35) combined with the assumptions (3.3.2) we have that

$$e^{-r_i t}\mathbb{E}[C^*_{it}] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T e^{-r_i t} C^*_{it} dt\right] < \infty. \tag{3.5.47}$$

Using integration by parts we obtain that

$$e^{-r_i T} C_{iT} = c + r_i \int_0^T e^{-r_i t} C_{it} dt + \int_{[0,T)} e^{-r_i t} dC_{it} \tag{3.5.48}$$

which along with the monotone convergence theorem allow us to gives that

$$\mathbb{E}\left[\int_0^\infty e^{-r_i t} dC^*_{it}\right] < \infty, \tag{3.5.49}$$

concluding that the admisibility of the control process C^*.

Consequently, following similar arguments as in (2.4.2) regarding the optimality of the control and in view of the transformed HJB (3.4.12) we get that

$$\int_0^T e^{-r_i t} X_t^\beta (C^*_{it})^{\alpha_i} e^{-\widetilde{h}(\overline{X_t})} dt - \int_{[0,T]} k_i e^{-r_i t} dC^*_{it} + e^{-r_i T} w(C^*_{iT}, X_T, \overline{X_t}) = w(c, x, \overline{x}) + M^*_T \tag{3.5.50}$$

Again from a localization argument and taking expectations we obtain that

$$\mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-r_i t} X_t^\beta (C^*_{it})^{\alpha_i} e^{-\beta\widetilde{h}(\overline{X_t})} dt - \int_{[0,T\wedge\tau_n)} k_i e^{-r_i t} dC^*_{it}\right] + \mathbb{E}\left[e^{-r_i T\wedge\tau_n} w(C^*_{iT}, X_T, \overline{X_t})\right] = w(c, x, \overline{x}) \tag{3.5.51}$$

Therefore, by the monotone convergence theorem we can take take the limits $n \to \infty$, $T \to \infty$ we get that

$$\mathcal{J}^i_{(c,x,\overline{x})}(C_i^*) + \lim_{T \to \infty} \mathbb{E}\left[e^{-r_i T \wedge \tau_n} w(C_{iT}^*, X_T, \overline{X}_T)\right] = w^i(c,x,\overline{x}) \tag{3.5.52}$$

Next, using (3.5.36) and assumption (3.3.4) and (2.3.5) we obtain that

$$\mathbb{E}\left[w(C_{iT}^*, X_T, \overline{X}_T)\right] \leq K_i \mathbb{E}\left[c_i + \Phi_i(\overline{x}) + \Phi_i(\overline{X_t}) + \Phi_i^{1-\alpha_i}(\overline{X_t})\right] < \infty. \tag{3.5.53}$$

Thus, using the dominated convergence theorem we can pass the limit and conclude that $\mathcal{J}^i_{(c,x,\overline{x})}(C_i^*) = w^i(c,x,\overline{x})$ which implies that $v^i(c,x,\overline{x}) = w^i(c,x,\overline{x})$

■

Remark 3.5.5 *Finally, the true value function for producer $i \in I$ is given by*

$$\widetilde{v}^i(c,x,\overline{x}) = \lambda_i v^i(c,x,\overline{x})$$

3.6 Nash Equilibrium for the multi-agent game

In the previous section we proved that given the strategies C_{-i} of the other producers, producers $i \in I$ optimal strategy is given by theorem 3.5.4 and the value function by lemma 3.5.2. Using lemma (3.4.5) we show that the optimal control in terms of $\overline{X}$ process is equivalent with a control in terms of the $\overline{P}$ process. Hence, in this section we form the market equilibrium by assuming that also all other producers uses their optimal strategy C_{it}^* which gives rise to the mean field H. As a side note, let us mention that since producers form a continuum, the notion of open/closed-loop strategies Appendix are not so interesting from a practical point of view. In our case, we have to follow the idea of a mean field game equilibrium. In particular, since $P = X/H$, we need to find the mean field H such that when all players use their optimal expansion schedules based on the H we get back H. i.e. we have the following fixed point equation

$$H(\overline{p}) = \left(\beta \int_{i\in I} \lambda_i C_i^{\alpha_i}(\overline{p}; H(\overline{p}))m(di)\right)^{\gamma}, \ \forall \overline{p} \in \mathbb{R}^+ \tag{3.6.1}$$

In conjunction with theorem (3.5.4) which shows the optimal expansion schedule for every producer is not directly dependent through H but only through the price process P we conclude with the following Nash equilibrium for the market which concludes this section

Theorem 3.6.1 *(Market Equilibrium)*

Define the increasing function $H^* : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$H^*(\overline{p}) := \left(\beta \int_{i\in I} \lambda_i \left(c_i^{\alpha_i} \vee \left(\frac{\alpha_i(n_i - \beta)}{\rho_i n_i k_i}\overline{p}^{\beta}\right)^{\frac{\alpha_i}{1-\alpha_i}}\right) m(\mathrm{d}i)\right)^{\gamma}, \ \overline{p} \in \mathbb{R}^+.$$

Then, the individual optimal schedule process given by

$$C_{it}^* = c_i \vee \left(\frac{(\alpha_i(n_i - \beta)}{\rho_i n_i k_i}\bar{P}_t^{\beta}\right)^{\frac{1}{1-\alpha_i}}, \tag{3.6.2}$$

and the price process P is given by $P_t = \widetilde{g}^{-1}(X_t)$ where $\widetilde{g}$ is defined through $\widetilde{g}(\overline{p})e^{-\widetilde{g}(\overline{p})} := \overline{p}$ form the multi-producer's equilibrium

Proof: Using (3.3.4) for $C_{it} = C_{it}^*$ $P_t = X_t/H(\overline{P_t})$ which implies that $\overline{P_t} = \overline{X_t}/H(\overline{P_t})$. Hence,

$$\overline{P_t} = f^{-1}(\overline{X_t}), \tag{3.6.3}$$

where $f(\overline{p}) = \overline{p}H(\overline{p})$.

However, we have shown that for every $H : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the assumption (3.4.1)

The optimal expansion is given by (3.6.2) Thus, the solution of the fixed point equation H should be equal to H^* which concludes the proof ■

To help the reader understand the above proof we briedly outline the idea of the proof.

Assume that with the possible exception of the producer labeled by $i \in I$, suppose that every other producer, say $j \in I \setminus \{i\}$, adopts the capacity expansion strategy

$$C_j(t) = c_j \vee \Phi_j(\bar{X}_t).$$

In the competitive setting that we have considered, this assumption and the clearing condition give rise to the price process

$$P = X / \widetilde{H}(\overline{X_t}). \tag{3.6.4}$$

However, producer i, who has been singled out, is faced with the task of maximising the performance index

$$\widetilde{J}^{(i)}_{C_i}(C_i \mid P) = J^{(i)}_{C_i,x,\overline{x}}(C_i),$$

over all $C_i \in \mathcal{A}$, where $\widetilde{J}^{(i)}_{C_i}(C_i \mid P)$ and $J^{(i)}_{C_i,x,\overline{x}}(C_i)$ are the performance criteria defined by (3.3.5) and (3.4.17) while by the Theorem (3.5.4) the expansion strategy

$$\begin{aligned} C^*_{it} &= \left(\frac{\alpha_i(n_i - \beta)}{(r_i - \mu) n_i k_i} \right)^{\frac{\beta}{1-\alpha_i}} \left(X_t e^{-\widetilde{h}(\overline{X_t})} \right)^{\frac{\beta}{(1-\alpha_i)}} \\ &= c_i \vee \Phi_i(\overline{X_t}) = c_i \vee \Psi^\star_i(\overline{P}_t), \end{aligned}$$

where

$$\Psi^\star_i(\overline{p}) = \left(\frac{\alpha_i(n_i - \beta)}{(r_i - \mu) n_i k_i} \right)^{\frac{\beta}{1-\alpha_i}} \overline{p}^{\frac{\beta}{1-\alpha_i}}, \tag{3.6.5}$$

is the optimal behavior for the ith agent given the optimal response of for every $j \in I - \{i\}$. Therefore,

$$H^*(\overline{p}) := \left(\beta \int_{i \in I} \lambda_i \left(c_i^{\alpha_i} \vee \left(\frac{\alpha_i (n_i - \beta)}{\rho_i n_i k_i} \overline{p}^{\beta} \right)^{\frac{\alpha_i}{1-\alpha_i}} \right) m(\mathrm{d}i) \right)^{\gamma}, \ \overline{p} \in \mathbb{R}^+.$$

is indeed a solution to the fixed point problem and thus $P = X/H^*$ along with the optimal expansion strategies $C_{it} = c_i \vee \Psi_i(\overline{P_t})$ must comprise a competitive production equilibrium.

3.7 Appendix

Appendix A: Proof of lemma 3.5.2

In this part we prove lemma 3.5.2

Proof: By construction $w : \mathcal{S} \to \mathbb{R}$ is a continuous function, $w(c, x, \cdot) : [x, \infty) : \mathbb{R}$ is a difference of two convex functions and therefore absolutely continuous. Moreover, w_c is continuous along the surface $\mathcal{S}_i^1$. Hence, we must only prove the continuity of the continuity of w_x and w_{xx} across the surface $\mathcal{S}_i^1$. Let $(c, x, \overline{x}) \in \mathcal{E}_i \setminus \mathcal{S}_i^1$,

$$\begin{aligned} w_x^i(c,x,\overline{x}) &= \lim_{c \downarrow \Gamma_i(x,\overline{x})} w_x^i(c,x,\overline{x}) + \Big(w_c\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big) - k_i \Big)(\Gamma_i)_x(x,\overline{x}) \\ &= \lim_{c \downarrow \Gamma_i(x,\overline{x})} w_x^i(c,x,\overline{x}) = w_x^i\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big) \qquad (3.7.1) \end{aligned}$$

$$\begin{aligned} \text{and} \quad w_{xx}^i(c,x,\overline{x}) &= \lim_{c \downarrow \Gamma_i(x,\overline{x})} w_{xx}^i(c,x,\overline{x}) + w_{cx}^i\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big)(\Gamma_i)_x(x,\overline{x}) \\ &= \lim_{c \downarrow \Gamma_i(x,\overline{x})} w_{xx}(c,x,\overline{x}) = w_{xx}\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big), \qquad (3.7.2) \end{aligned}$$

where we have used the definition (3.5.31) of w^i as well as (3.5.18) and (3.5.19). Consequently, due to (3.5.31) and the expression (3.5.33) we have that w_x and w_{xx} are also continuous across the surface $\mathcal{S}^2$. On the other hand, in view of (3.5.32) and (3.5.33) we obtain that

$$\lim_{c\downarrow\Phi_i(\overline{x})} w_c^i(c,x,\overline{x}) = A_c(\Phi_i(\overline{x}),\overline{x})x^n + \frac{\alpha_i}{\rho_i}x^\beta\Phi_i^{\alpha_i-1}(\overline{x})e^{-\beta\widetilde{h}(\overline{x})}$$
$$= B_c(\Phi_i(\overline{x}),\overline{x})x^n + \frac{\alpha_i}{\rho}x^\beta\Phi_i^{\alpha_i-1}(\overline{x}) = \lim_{c\uparrow\Phi_i(\overline{x})} w_c^i(c,x,\overline{x}).$$

Therefore, we have concluded that the function $w(\cdot,\cdot,\overline{x})$ is $C^{2,1}$ in the interior of $\mathcal{S}$ for all $\overline{x} > 0$.

Next we prove that w^i provides a solution to Problem 3.7.1, i.e we show that

$$w_c^i(c,x,\overline{x}) < k_i \quad \text{for all } 0 < x < \overline{x} \text{ and } \Phi_i(\overline{x}) < c, \tag{3.7.3}$$

$$w_c^i(c,x,\overline{x}) < k_i \quad \text{for all } 0 < x \le \overline{x} \text{ and } \Gamma_i(x,\overline{x}) < c \le \Phi_i(\overline{x}), \tag{3.7.4}$$

$$\text{and} \quad \mathscr{L}_i w^i(c,x,\overline{x}) + x^\beta c^{\alpha_i} \le 0 \quad \text{for all } 0 < x \le \overline{x} \text{ and } c \le \Gamma_i(x,\overline{x}). \tag{3.7.5}$$

To begin with, let $(c,x,\overline{x}) \in \mathcal{W}_i^2$, i.e. a point in $\mathcal{S}$ such that $0 < x \le \overline{x}$ and $\Gamma_i(x,\overline{x}) < c \le \Phi_i(\overline{x})$. In this case, we get that

$$w_c^i(c,x,\overline{x}) = \frac{\alpha_i x^\beta c^{\alpha_i-1}}{\rho_i}\left(1 - \frac{\beta}{n_i}G_i^{\beta-n}(c,\overline{x})x^{n_i-\beta}\right) \le k_i, \tag{3.7.6}$$

where we have used the fact that $p \le G_i(c,\overline{x})$ and that $G^\beta(c,\overline{x}) = \frac{n_i\rho_i k_i}{\alpha_i(n_i-\beta)}c^{1-\alpha_i}e^{\widetilde{h}(\overline{x})}$ and equality is satisfied for $c \downarrow \Gamma_i(x,\overline{x})$.

Similarly, for any point $(c,x,\overline{x}) \in \mathcal{W}_i^1$

$$w_c^i(c,x,\overline{x}) \le k_i, \tag{3.7.7}$$

where equality is satisfied in the limit of $x \uparrow \overline{x}$ and $c \downarrow \Phi_i(\overline{x})$

Next, to establish (3.7.5)., we use the definition (3.5.31) of w_i, as well as (3.7.1) and (3.7.2), to obtain. In particular let $(c,x,\overline{x}) \in \mathcal{E}_i$

$$\begin{aligned}\mathscr{L}_i w^i(c,x,\overline{x}) + c^{\alpha_i} x^\beta e^{-\beta \widetilde{h}(\overline{x})} &= \mathscr{L}_i w^i(\Gamma_i(x,\overline{x}),x,\overline{x}) + r_i k_i\Big(\Gamma_i(x,\overline{x}) - c\Big) + c^{\alpha_i} x^\beta e^{-\beta \widetilde{h}(\overline{x})} \\ &= -\Gamma_i^{\alpha_i}(x,\overline{x}) x^\beta e^{-\beta \widetilde{h}(\overline{x})} + r_i k_i\Big(\Gamma_i(x,\overline{x}) - c\Big) + c^{\alpha_i} x^\beta e^{-\beta \widetilde{h}(\overline{x})} \\ &= -\int_c^{\Gamma_i(x,\overline{x})} \Big(\alpha_i u^{\alpha_i - 1} x^\beta e^{-\beta \widetilde{h}(\overline{x})} - r_i k_i\Big)\, du \quad \text{for all } c < \Gamma_i(x,\overline{x}).\end{aligned}$$

In view of (3.5.15) and (3.5.16), we can see that the free-boundary point $G_i(c,\overline{x})$ given by (3.5.20) is the unique solution to the equation[2]

$$\int_0^{G_i(c,\overline{x})} u^{-m_i - 1}\Big(\alpha u^{\alpha_i - 1} x^\beta e^{-\beta \widetilde{h}(\overline{x})} - r_i k_i\Big)\, du = 0.$$

Therefore, $\alpha_i c^{\alpha_i - 1} x^\beta e^{-\beta \widetilde{h}(\overline{x})} - r_i k_i > 0$ for all $x \geq G_i(c,\overline{x})$, which implies that

$$\alpha_i c^{\alpha_i - 1} x^\beta e^{-\beta \widetilde{h}(\overline{x})} - r_i k_i > 0 \quad \text{for all } c \leq \Gamma_i(x,\overline{x}),$$

because $\Gamma_i(\cdot,\overline{x})$ is the inverse of the strictly increasing function $G_i(c,\overline{x})$. However, this conclusion imply that w_i satisfies (3.7.5).

Finally, it is straightforward to find the relevant bounds of $w^i(c,x,\overline{x})$. In particular, from the expressions (3.5.32) and (3.5.33) as well as the expression for (3.5.35) and the fact that for $(c,x,\overline{x}) \in \mathcal{W}_i$ we have that $c > \Phi_i(\overline{x})$ we obtain that

$$A_i(c,\overline{x}) \leq \Lambda_A \left(c\overline{x}^{-n_i} + (\overline{x} e^{-\widetilde{h}(\overline{x})})^{\frac{\beta}{1-\alpha_i}}\right), \tag{3.7.8}$$

where $\Lambda_A > 0$ a positive constant.

[2]This is essentially the same argument as in the proof of the corresponding lemma in chapter 2

Similarly we obtain that obtain that

$$B_i(c,\overline{x}) \leq \Lambda_B \left(c\overline{x}^{-n_i} + (\overline{x}e^{-\widetilde{h}(\overline{x})})^{\frac{\beta}{1-\alpha_i}} \right), \tag{3.7.9}$$

where $\Lambda_B > 0$ a positive constant.

In addition, note that from (3.5.35) one can see that $(\overline{x}e^{-\widetilde{h}(\overline{x})})^{\frac{\beta}{1-\alpha_i}} \leq K_i\Phi_i(\overline{x})\ \overline{x}^{\beta}e^{-\beta\widetilde{h}(\overline{x})} \leq K_i\Phi_i^{1-\alpha_i}(\overline{x})$ with K a suitable positive constant therefore for both regions $\mathcal{W}_i^1, \mathcal{W}_i^2$ we can write them as

$$w^i(c,x,\overline{x}) \leq \Lambda_i \left(c + c^{\alpha_i}\Phi_i^{1-\alpha_i}(\overline{x}) + \Phi_i(\overline{x}) \right),\ (c,x,\overline{x}) \in \mathcal{W}_i \tag{3.7.10}$$

Finally, for $(c,x,\overline{x},) \in \mathcal{E}_i$ we have that $c \leq \Phi_i(\overline{x})$ and therefore we have that

$$w^i(c,x,\overline{x}) \leq \Lambda_i\Phi_i(\overline{x}),\ (c,x,\overline{x}) \in \mathcal{E}_i \tag{3.7.11}$$

Hence, as required we get (3.5.36). Finally, the positivity of $w^i(c,x,\overline{x})$ becomes a simple observation after using the remark (3.5.3) since the integral form is immediately shown to be positive ■

Appendix B: Solution of the HJB equation using the P process

Heuristic solution to agent's control problem

A heuristic method is used to calculate the conjectured value function by conjecturing the optimal strategy and directly computing the objective function. To make the exposition easier we will use log-prices, i.e. $P_t = e^{Z_t}$. Therefore, the SDE for $Z \equiv (Z_t; t \in \mathbb{R}^+)$ is

$$dZ_t = (\mu - \frac{\sigma^2}{2})dt + \sigma \mathrm{d}W_t - \mathrm{d}h(\overline{Z_t}) \tag{3.7.12}$$

We conjecture that the agent's $i \in I$ optimal strategy is $C_{it} = c_i \vee \widetilde{\Psi}_i(\overline{Z_t})$ while the other player's strategies give rise to the mean field process $\{H(\overline{Z_t})\}_{t \in \mathbb{R}^+}$ which are assumed to be fixed. In the following lines we will directly evaluate the objective function with the aforementioned expansion schedule.

$$v^i(c,z,\overline{z}) = \mathbb{E}_z\left[\int_0^\infty e^{-r_i t+\beta Z_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})dt - k_i\int_0^\infty e^{-r_i t}\mathrm{d}\widetilde{\Psi}_i(\overline{Z_t})\right] \tag{3.7.13}$$

where $\widetilde{\Psi}_i$ belong to a class of non-decreasing differentiable functions

First of all,

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_i t}d\widetilde{\Psi}_i(\overline{Z_t})\right] = \mathbb{E}_z\left[\int_z^\infty e^{-r_i\tau_y}\mathrm{d}\widetilde{\Psi}_i(y)\right],$$

where $\tau_y \equiv \{t \in \mathbb{R}^+ | Z_t = y\}$ is the first hitting time at level $y \in \mathbb{R}^+$ of Z. Furthermore, by Girsanov,

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_i t+\beta Z_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})\mathrm{d}t\right] = e^{\beta(z+h(z))}\mathbb{E}_z^{\mathbb{Q}}\left[\int_0^\infty e^{-\rho_i t-\beta h(\overline{Z_t})}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})\mathrm{d}t\right]$$

where

$$\rho_i := r_i - \beta(\mu - \frac{\sigma^2}{2}) - \frac{(\beta\sigma)^2}{2} > 0.$$

and $Z_t = \mu' t + \sigma W_t'$ under $\mathbb{Q}$, where $\mu' := \mu - \frac{\sigma^2}{2} + \beta\sigma^2$. Integration-by-parts gives

$$\rho_i\mathbb{E}_z^{\mathbb{Q}}\left[\int_0^\infty e^{-\rho_i t-\beta h(\overline{Z_t})}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})\mathrm{d}t\right] = e^{-\gamma h(z)}\widetilde{\Psi}^{\alpha_i}(z) + \mathbb{E}_z^{\mathbb{Q}}\left[\int_z^\infty e^{-\rho_i\tau_y}\mathrm{d}\left(e^{-\beta h(y)}\widetilde{\Psi}_i^{\alpha_i}(y)\right)\right]$$

Therefore,

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_i t+\beta X_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})dt\right] = \frac{1}{\rho_i}e^{\beta z}\widetilde{\Psi}_i^{\alpha_i}(z) + \frac{1}{\rho_i}e^{\beta(z+h(z))}\mathbb{E}_z^{\mathbb{Q}}\left[\int_z^\infty e^{-\rho_i\tau_y}\mathrm{d}\left(e^{-\beta h(y)}\widetilde{\Psi}_i^{\alpha_i}(y)\right)\right]. \tag{3.7.14}$$

For a generic process with $\mathrm{d}Z_t = a\mathrm{d}t + \sigma\mathrm{d}W_t - \mathrm{d}h(\overline{Z_t})$ under $\mathbb{P}$, we have that $e^{b(Z_t+h(\overline{Z_t})-(z+h(z)))-bat-(b\sigma)^2/2t}$ being a martingale, and optional sampling gives

$$\mathbb{E}_z^{\mathbb{P}}\left[e^{-(ba+b^2\sigma^2/2)\tau_y}\right] = e^{-b(y+h(y)-z-h(z))}$$

Therefore, define

$$n_i := \sqrt{\frac{2r_i}{\sigma^2} + \left(\frac{2\mu-\sigma^2}{2\sigma^2}\right)^2} - \frac{2\mu-\sigma^2}{2\sigma^2};$$

we can show that

$$\mathbb{E}_z\left[e^{-r_i\tau_y}\right] = e^{-n_i(y+h(y)-x-h(z))};$$

using

$$\xi_i(y) := e^{-n_i(y+h(y))}$$

this gives

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_it}\mathrm{d}\widetilde{\Psi}_i(\overline{Z_t})\right] = \frac{1}{\xi_i(z)}\int_z^\infty \xi_i(y)\mathrm{d}\widetilde{\Psi}_i(y), \tag{3.7.15}$$

i.e.,

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_it}\mathrm{d}\widetilde{\Psi}_i(\overline{Z_t})\right] = -\widetilde{\Psi}_i(z) + \frac{1}{\xi_i(z)}\int_z^\infty \widetilde{\Psi}_i(y)\xi_i(y)n_i(1+H_{\overline{z}}(y))dy,$$

as well as

$$\mathbb{E}_z^{\mathbb{Q}}\left[e^{-\rho_i\tau_y}\right] = e^{(n_i-\beta)(y+h(y)-z-h(z))},$$

which gives that equals

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_it+\beta Z_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})\mathrm{d}t\right] = \frac{1}{\rho_i}e^{\gamma z}\widetilde{\Psi}_i^{\alpha_i}(z) + \frac{1}{\rho_i}\frac{1}{\xi_i(z)}\int_z^\infty \xi_i(y)e^{\beta(y+h(y))}\mathrm{d}\left(e^{-\beta h(y)}\widetilde{\Psi}_i^{\alpha_i}(y)\right), \tag{3.7.16}$$

where by performing an integration by parts is further equal to

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_it+\beta Z_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})\mathrm{d}t\right] = \frac{1}{\rho_i\xi_i(z)}\int_z^\infty \widetilde{\Psi}_i^{\alpha_i}(y)\xi_i(y)e^{\beta y}(n_i-\beta)(1+h_{\overline{z}}(y))dy, \tag{3.7.17}$$

In total, equals

$$\mathbb{E}_z\left[\int_0^\infty e^{-r_it+\beta Z_t}\widetilde{\Psi}_i^{\alpha_i}(\overline{Z_t})dt - \int_0^\infty e^{-r_it}k\mathrm{d}\widetilde{\Psi}_i(\overline{Z_t})\right] =$$
$$= \frac{1}{\xi_i(z)}\int_z^\infty \left(\frac{1}{\rho_i}\widetilde{\Psi}_i^{\alpha_i}(y)(n_i-\beta)e^{\beta y} - kn_i\widetilde{\Psi}_i(y)\right)\xi_i(y)(1+h_{\overline{z}}(y))dy + k\widetilde{\Psi}_i(z), \tag{3.7.18}$$

Therefore, a pointwise maximization inside the integral gives that the optimal $\widetilde{\Psi}$ is

$$\widetilde{\Psi}_i(y) = \left(\frac{(n_i-\beta)\alpha_i}{\rho_ik_in_i}\right)^{\frac{1}{1-\alpha_i}} e^{\frac{\beta}{1-\alpha_i}y}. \tag{3.7.19}$$

and thus, we conjecture that the agent's optimal strategy is $C^*_{it} := C_{0-} \vee \widetilde{\Psi}_i(\overline{X_t})$ which in conjunction the direct evaluation of (3.7.13) allows for a closed-form expression of the conjectured value function.

In particular, let

$$m_i(y,c) := \max_{\widetilde{\Psi}\geq c}\{\frac{n_i-\beta}{\rho_i}e^{\gamma y}\widetilde{\Psi}^\alpha - k_in_i\widetilde{\Psi}\}$$

Therefore, if $c \leq \widetilde{\Psi}(y)$, the maximum occurs at $\widetilde{\Psi}_i(y)$

$$m_i(y,c) = (1-\alpha_i)\left(\frac{\alpha_i}{k_in_i}\right)^{\frac{\alpha_i}{1-\alpha_i}}\left(\frac{n_i-\beta}{\rho_i}\right)^{\frac{1}{1-\alpha_i}} e^{\frac{\beta}{1-\alpha_i}y},$$

On the other hand, if $c > \widetilde{\Psi}_i(y)$, the optimum occurs at $\widetilde{\Psi} = c$. Therefore, it follows that

$$m_i(y,c) = \left(\frac{n_i-\beta}{\rho_i}e^{\beta y}c^{\alpha_i} - k_in_ic\right)1_{\{\widetilde{\Psi}_i(y)<c\}} + \left((1-\alpha_i)\left(\frac{\alpha_i}{k_in_i}\right)^{\frac{\alpha_i}{1-\alpha_i}}\left(\frac{n_i-\beta}{\rho}\right)^{\frac{1}{1-\alpha_i}} e^{\frac{\beta}{1-\alpha_i}y}\right)1_{\{c\leq\widetilde{\Psi}_i(y)\}}.$$

Therefore, the value function on the continuation region $\mathcal{W} = \{(c, z, \overline{z}) | \ c > \widetilde{\Psi}_i(z)\}$ is defined as

$$w^i(c, z, \overline{z}) := ck_i + \int_z^\infty m_i(y, c) e^{-n_i(y - z + (h(y) - h(\overline{z}))_+)} (1 + h_{\overline{z}}(y) 1_{\{\overline{z} < y\}}) \mathrm{d}y.$$

By integration-by-parts, we obtain

$$w^i(c, z, \overline{z}) := \frac{n_i - \beta}{\rho_i n_i} e^{\beta z} c^{\alpha_i} + \int_z^\infty j_i(y, c) e^{n_i(z + h(\overline{z}) - h(y \vee \overline{z}) - y)} \mathrm{d}y$$

where

$$j_i(y, c) = \frac{n_i - \beta}{\rho_i n_i} \beta e^{\beta y} c^{\alpha_i} 1_{\{\widetilde{\Psi}_i(y) < c\}} + \beta \left(\frac{\alpha_i}{k_i} \right)^{\frac{\alpha_i}{1 - \alpha_i}} \left(\frac{n_i - \beta}{\rho_i n_i} e^{\beta y} \right)^{\frac{1}{1 - \alpha_i}} 1_{\{c \leq \widetilde{\Psi}_i(y)\}}.$$

Moreover, in the investment region $\mathcal{E}_i = \{(c, z, \overline{z}) | \ c \leq \widetilde{\Psi}_i(z)\}$, we define

$$w^i(c, z, \overline{z}) = w^i(\widetilde{\Psi}_i(z), z, \overline{z}) - k_i(\widetilde{\Psi}_i(z) - c).$$

Finally, we can re-write the conjectured value function in terms of prices, using the fact that $p = e^z$ and noting that $\widetilde{\psi}_i(\ln z) = \Psi_i(p)$

$$w^i(c, p, \overline{p}) = \begin{cases} \frac{n_i - \beta}{\rho_i n_i} p^\beta c^{\alpha_i} + p^{n_i} \int_p^\infty j_i(y, c) e^{n_i(h(\overline{p}) - h(y \vee \overline{p}))} dy & \text{if } (c, p, \overline{p}) \in \mathcal{W}_i \\ w^i\left(\Psi_i(p), p, \overline{p}\right) - k_i[\Psi_i(p) - c] & \text{if } (c, p, \overline{p}) \in \mathcal{E}_i \end{cases} \tag{3.7.20}$$

where,

$$j_i(y, c) = \frac{n_i - \beta}{\rho_i n_i} \beta p^{\beta - n_i - 1} c^{\alpha_i} 1_{\{\Psi_i(y) < c\}} + \beta \frac{n_i - \beta}{\rho_i n_i} p^{\beta - n_i - 1} \Psi_i^{\alpha_i}(y) 1_{\{c \leq \Psi_i(y)\}} \tag{3.7.21}$$

and

$$\Psi_i(\overline{p}) = \left(\frac{(n_i - \beta)\alpha_i}{\rho_i k_i n_i}\right)^{\frac{1}{1-\alpha_i}} \overline{p}^{\frac{\beta}{1-\alpha_i}} \tag{3.7.22}$$

Formal solution to the HJB equations

In this section we provide a solution of the HJB equations using the P process. This simple corresponds to the transformation $x \to pe^{h(\overline{p})}$ and $\overline{p} \to \overline{p}e^{h(\overline{p})}$ of the problem and corresponds to

Problem 3.7.1 *Determine a function $G_i : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $v^i : \mathcal{S} \to \mathbb{R}$, satisfying the following conditions.*

(I) *the functions G_i and Ψ_i are C^1 and satisfy (3.5.1)–(3.5.3).*

(II) *The function $w^i(\cdot,\cdot,\overline{p})$ is $C^{2,1}$ in the interior of $\mathcal{S}$.*

(III) *The function $w^i(c,p,\cdot)$ is a difference of two convex functions in the interior of S.*

(IV)

The function w^i is such that

$$w^i_{\overline{p}-}(c,p,\overline{p}) = h_{\overline{p}-}(\overline{p})pw^i_p(c,p,\overline{p}) \quad \textit{for all } 0 < \Psi_i(\overline{p}) \le c, \tag{3.7.23}$$

$$w^i_c(c,p,\overline{p}) < k_i \quad \textit{for all } 0 < p < \overline{p} \textit{ and } \Psi_i(\overline{p}) \le c, \tag{3.7.24}$$

$$w^i_c(c,p,\overline{p}) < k_i \quad \textit{for all } 0 < p \le \overline{p} \textit{ and } \Psi(p) < c < \Psi_i(\overline{p}), \tag{3.7.25}$$

$$w^i_c(c,p,\overline{p}) = k_i \quad \textit{for all } 0 < p \le \overline{p} \textit{ and } c \le \Psi(p), \tag{3.7.26}$$

where $w^i_{\overline{p}-}\big(\overline{p},\overline{p},c\big) = \lim_{p\uparrow\overline{p}} v^i_{\overline{p}-}\big(p,\overline{p},c\big)$, *and*

$$\mathscr{L}_i v^i(c,p,\overline{p}) + p^\beta c^{\alpha_i} := \frac{1}{2}\sigma^2 p^2 w^i_{pp}(c,p,\overline{p}) + \mu p w^i_p(c,p,\overline{p}) - r_i v^i(c,p,\overline{p}) + p^\beta c^{\alpha_i}$$
$$\begin{cases} = 0 & \textit{for all } 0 < p < \overline{p} \textit{ and } 0 < \Psi_i(\overline{p}) \le c, \\ = 0 & \textit{for all } 0 < p \le \overline{p} \textit{ and } \Psi_i(p) < c < \Psi_i(\overline{p}), \\ < 0 & \textit{for all } 0 < p < \overline{p} \textit{ and } c \le \Psi_i(p). \end{cases} \tag{3.7.27}$$

We consider a solution to Problem 3.7.1 of the form

$$w^i(c,p,\overline{p}) = \begin{cases} A_i(c,\overline{p})p^{n_i} + \frac{1}{\rho_i}p^\beta c^{\alpha_i}, & \text{if } (c,p,\overline{p}) \in \mathcal{W}^1_i, \\ B_i(c,\overline{p})p^{n_i} + \frac{1}{\rho_i}p^\beta c^{\alpha_i}, & \text{if } (p,\overline{p},c) \in \mathcal{W}^2_i \cup \mathcal{S}^1_i, \\ v^i\big(\Psi_i(\overline{p}),p,\overline{p}\big) - k_i\big(\Psi_i(\overline{p}) - c\big), & \text{if } (p,\overline{p},c) \in \mathcal{E}_i \setminus \mathcal{S}^1_i, \end{cases} \tag{3.7.28}$$

To determine the function B_i and the free-boundary function G_i, we appeal to the so called "smooth-pasting condition" of singular stochastic control. In particular, we require that $w^i(\cdot,p,\overline{p})$ should be C^2 along the free-boundary point $G_i(c)$, which suggests the equations

$$\lim_{p\uparrow G_i(c)} v^i_c(c,p,\overline{p}) = (B_i)_c(c,\overline{p})G_i^{n_i}(c) + \frac{\alpha_i}{\rho_i}G_i^\beta(c)c^{\alpha_i-1} = k_i \tag{3.7.29}$$

$$\text{and} \quad \lim_{p\uparrow G_i(c)} v^i_{cp}(c,p,\overline{p}) = n_i(B_i)_c(c,\overline{p}) + \frac{\alpha_i\beta}{\rho_i}G_i^{\beta-n_i}(c)c^{\alpha_i-1} = 0. \tag{3.7.30}$$

The solution to this system of equations is given by

$$G_i(c) = \left(\frac{\rho_i n_i k_i}{\alpha_i(n_i-\beta)}\right)^{\frac{1}{\beta}} c^{\frac{1-\alpha_i}{\beta}} \quad \text{and} \quad (B_i)_c(c,\overline{p}) = -\frac{k_i}{n_i-\beta}G_i^{-n_i}(c). \tag{3.7.31}$$

For G_i given by the first of these expressions, we can see that the unique solution to equation (3.5.2) is given by

$$\Psi_i(p) = \left(\frac{\alpha_i(n_i-\beta)}{\rho_i n_i k_i}\right)^{\frac{1}{1-\alpha_i}} p^{\frac{\beta}{1-\alpha_i}}, \tag{3.7.32}$$

On the other hand, the solution to the ODE in (3.7.31) is given by

$$B_i(c,\overline{p}) = \widetilde{f}_i(\overline{p}) + \frac{k_i\beta^2}{(n_i-\beta)\big(n_i(1-\alpha_i)-\beta\big)}\frac{c}{G_i^n(c)}, \tag{3.7.33}$$

where $\widetilde{f}_i$ is a function to be determined.

In order to determine $\widetilde{f}_i$ we will use the fact that in the region $\{p=\overline{p}\}\cap\mathcal{W}_i^2$ we should have that

$$v_{\overline{p}}^i\big(\Psi(\overline{p}),\overline{p},\overline{p}\big) = h_{\overline{p}-}(\overline{p})v_p^i\big(\Psi(\overline{p}),\overline{p},\overline{p}\big) \tag{3.7.34}$$

Therefore,

$$\widetilde{f}_i'(\overline{p}) - nh_{\overline{p}-}(\overline{p})\widetilde{f}_i(\overline{p}) = \frac{1}{\rho_i}\frac{\beta}{n_i}\frac{(n_i-\beta)(1-\alpha_i)}{n_i(1-\alpha_i)-\beta}\overline{p}^{\beta-n_i}\Psi_i^{\alpha_i}(\overline{p})h_{\overline{p}-}(\overline{p}) \tag{3.7.35}$$

Thus, integrating we obtain that

$$\widetilde{f}_i(\overline{p}) = \int_{\overline{p}}^{\infty}\frac{1}{\rho_i}\frac{\beta}{n_i}\frac{(n_i-\beta)(1-\alpha_i)}{n_i(1-\alpha_i)-\beta}y^{\beta-n_i}\Psi_i^{\alpha_i}(y)n_ih_{\overline{p}-}(y)e^{n_i(h(\overline{p})-h(y))}dy, \tag{3.7.36}$$

where we used the fact that we should have $\lim_{\overline{p}\to\infty}\widetilde{f}_i(\overline{p})e^{nh(\overline{p})} = 0$

Performing an integration by parts we obtain that

$$\begin{aligned}\widetilde{f}_i(\overline{p}) = &\frac{1}{\rho_i}\frac{\beta(n_i-\beta)}{n_i}\frac{1-\alpha_i}{\beta-n_i(1-\alpha_i)}\overline{p}^{\beta-n_i}\Psi_i^{\alpha_i}(\overline{p})\\ &+\int_{\overline{p}}^{\infty}\frac{1}{\rho_i}\frac{\beta(n_i-\beta)}{n_i}y^{\beta-n_i-1}\Psi_i^{\alpha_i}(y)e^{n_i(h(\overline{p})-h(y))}dy\end{aligned} \tag{3.7.37}$$

Consequently, to determine $A_i(c,\overline{p})$ we use again the fact that for $\mathcal{W}_i^1$ and $p \to \overline{p}$ we have that

$$\lim_{p \uparrow \overline{p}} w^i_{\overline{p}}\big(c,p,\overline{p}\big) = h_{\overline{p}-}(\overline{p}) \lim_{p \uparrow \overline{p}} w^i_p\big(c,p,\overline{p}\big) \tag{3.7.38}$$

Hence, we obtain that

$$A_i(c,\overline{p}) - A_i(c,G(c))e^{-n_i(h(\overline{p})-h(G_i(c)))} = \frac{1}{\rho_i}\frac{\beta}{n_i}G_i(c)^{\beta-n_i}c^{\alpha_i}e^{n_i(h(\overline{p})-h(G_i(c)))} - \frac{1}{\rho_i}\frac{\beta}{n_i}\overline{p}^{\beta-n_i}c^{\alpha_i} + \int_{\overline{p}}^{G_i(c)} \frac{1}{\rho_i}\beta\frac{n_i-\beta}{n_i}y^{\beta-n_i-1}c^{\alpha}e^{n(h(\overline{p})-h(y))}dy + f_i(c)e^{nh(\overline{p})} \tag{3.7.39}$$

Next, the requirement that v^i should be continuous is reflected by the identity

$$\lim_{c \uparrow \Psi_i(\overline{p})} w^i(c,p,\overline{p}) = \lim_{c \downarrow \Psi_i(\overline{p})} w^i(c,p,\overline{p}),$$

which gives rise to the expression

$$\begin{aligned} &A_i(\Psi_i(\overline{p}),\overline{p}) = B_i\big(\Psi_i(\overline{p}),\overline{p}\big) \Rightarrow \\ &f_i(\Psi_i(\overline{p})) = \int_{\overline{p}}^{\infty} \frac{1}{\rho_i}\frac{\beta(n_i-\beta)}{n_i}y^{\beta-n_i-1}\Psi_i^{\alpha_i}(y)e^{-n_ih(y)}dy \end{aligned} \tag{3.7.40}$$

Therefore, we conclude that

$$f_i(c) = \int_{G_i(c)}^{\infty} \frac{1}{\rho_i}\frac{\beta(n_i-\beta)}{n_i}y^{\beta-n_i-1}\Psi_i^{\alpha_i}(y)e^{-n_ih(y)}dy \tag{3.7.41}$$

To conclude, as a consistency check, we recover the form of (3.7.20) let $(p, \overline{p}, c) \in \mathcal{W}_i^1$. Then

$$\begin{aligned} w^i(c,p,\overline{p}) = A_i(c,\overline{p})p^{n_i} + \frac{1}{\rho_i}p^\beta c^{\alpha_i} = \\ -\frac{1}{\rho_i}\frac{\beta}{n_i}\overline{p}^{\beta-n_i}c^{\alpha_i} + \int_{\overline{p}}^{G_i(c)} \frac{1}{\rho_i}\beta\frac{n_i-\beta}{n_i}y^{\beta-n_i-1}c^\alpha e^{n_i(h(\overline{p})-h(y))}dy \\ + \int_{G_i(c)}^\infty \frac{1}{\rho_i}\frac{\beta(n_i-\beta)}{n_i}y^{\beta-n_i-1}\Psi_i^{\alpha_i}(y)e^{n_i(h(\overline{p})-h(y))}dy + \frac{1}{\rho_i}p^\beta c^{\alpha_i} \end{aligned} \tag{3.7.42}$$

This can be clearly re-written as

$$w^i(c,p,\overline{p}) = \frac{1}{\rho_i}\frac{n_i-\beta}{n_i}p^\beta c^{\alpha_i} + p^n\int_p^\infty j(y,c)e^{n_i(h(\overline{p})-h(y))_+}dy, \quad (c,p,\overline{p}) \in \mathcal{W}_i^1 \tag{3.7.43}$$

Similarly, for $(c,p,\overline{p}) \in \mathcal{W}_i^2 \cup \mathcal{S}_i^1$ we recover the same compact form verifying (3.7.20)

Appendix C: Asymptotic Results

This section supplements the main part and, in particular, provides some elementary asymptotic results regarding the asymptotic behavior of the H (or correspondingly $\widetilde{H}$) function.

To begin with we define

$$\bar{\alpha} = m\text{-ess sup } \alpha = \inf\left\{a \in]0,1] \,\middle|\, m(\alpha > a) = 0\right\}.$$

where $\alpha = \{\alpha_i\}_{i\in I}$

Lemma 3.7.2 *For any $\mathcal{I}$-measurable functions $\eta : I \to [0,\infty)$, $\zeta : I \to (0,\infty)$ and $\xi : I \to \mathbb{R}$ such that*

$$\int_I \zeta_i\, \iota(di) < \infty \quad \textit{and} \quad O(y) := \int_I 1_{\{\eta_i<y\}}\zeta_i y^{\xi_i}\, m(di) < \infty \quad \textit{for all } y > 0,$$

it holds that

$$\lim_{y\to\infty} \frac{\ln O(y)}{\ln y} = m\text{-}ess\ sup\ \xi =: \bar{\xi}.$$

Proof: We first note that

$$\limsup_{y\to\infty} \frac{\ln O(y)}{\ln y} = \limsup_{y\to\infty} \frac{1}{\ln y} \ln \int_I \zeta_i e^{\xi_i \ln y}\, m(di) = \limsup_{y\to\infty} \frac{1}{\ln y} \ln \int_I e^{\xi_i \ln y}\, \widetilde{m}(di)$$

$$= \limsup_{y\to\infty} \left\| e^{\xi_\cdot} \right\|_{L^{\ln y}(\widetilde{m})} = \widetilde{m}\text{-ess sup } \xi = m\text{-ess sup } \xi,$$

where $\widetilde{m}$ is the *finite* measure on $(I, \mathcal{I})$ that is equivalent to m, with Radon-Nikodym derivative given by $d\widetilde{m}/dm = \zeta$.

Next, for $k \in \mathbb{N}$ define $I_k := \{\eta < k\} \in \mathcal{I}$, and note that

$$\liminf_{y\to\infty} \frac{\ln O(y)}{\ln y} \geq \liminf_{y\to\infty} \frac{1}{\ln y} \ln \int_I 1_{I_k} e^{\xi_i \ln y}\, \widetilde{m}(di) = \liminf_{y\to\infty} \left\| 1_{I_k} e^{\xi_\cdot} \right\|_{L^{\ln y}(\widetilde{m})} = m\text{-ess sup } (1_{I_k}\xi).$$

Combining this result with the fact that $\bigcup_{k=1}^{\infty} I_k = I$, we obtain

$$\liminf_{y\to\infty} \frac{\ln O(y)}{\ln y} \geq m\text{-ess sup } \xi.$$

This last inequality and (3.7) imply (3.7.2)

Using this lemma and the equivalence

$$\lim_{y\to\infty} \frac{\ln Q(y)}{\ln y} = -\ell \quad \Leftrightarrow \quad \lim_{y\to\infty} y^{\ell} Q(y) = \begin{cases} 0, & \text{if } \xi < \ell, \\ \infty, & \text{if } \xi > \ell, \end{cases}$$

where Q is a strictly positive function, we can establish the following asymptotic result.

■

Lemma 3.7.3 *Suppose that Assumption (3.3.4) holds true and let* $\Psi = \{\Psi_i,\ i \in I\} \in \mathcal{B}$ *be such that*

$$\Psi_i(\overline{p}) = K_i \overline{p}^{\frac{\beta}{1-\alpha_i}},$$

for some constant $\beta > 0$ *and some* $\mathcal{I}$*-measurable functions* $K > 0$ *and* α *such that* $\alpha_i \in (0,1)$ *for all* $i \in I$*. Also, let* $\chi_2, \widetilde{H}, \Phi = \{\Phi_i,\ i \in I\}$ *be as in lemma (3.4.5)(II)*

The functions H *and* $\widetilde{H}$ *are such that*

$$\lim_{\overline{p}\to\infty} \frac{\ln H(\overline{p})}{\ln \overline{p}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}}, \qquad \lim_{\overline{p}\to\infty} \frac{\ln H'(\overline{p})}{\ln \overline{p}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}} - 1,$$

$$\lim_{\overline{x}\to\infty} \frac{\ln \widetilde{H}(\overline{x})}{\ln \overline{x}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}+\overline{\alpha}\beta\gamma}, \qquad \lim_{\overline{x}\to\infty} \frac{\ln(\widetilde{H}'(\overline{x}))}{\ln \overline{x}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}+\overline{\alpha}\beta\gamma} - 1,$$

where $\overline{\alpha}$ *is defined by (3.7).*

Proof: In view of the definition of H, we can see that

$$\beta \int_I \lambda_i \Psi_i^{\alpha_i}(\overline{p})\, m(di) \le H^{1/\gamma}(\overline{p}) \le \beta\kappa_0 + \beta \int_I \lambda_i \Psi_i^{\alpha_i}(\overline{p})\, m(di),$$

where $\kappa_0 = \int_{i\in I} \lambda_i c_i^{\alpha_i} m(di) < \infty$. These inequalities, lemma (3.7.2) and the fact that the function $(0,1) \ni \alpha \mapsto \alpha/(1-\alpha)$ is increasing imply the first limit in (3.7.3).

On the other hand, combining the expression

$$\ln(H'(\overline{p})) = \ln(\gamma\beta) + \frac{\gamma-1}{\gamma}\ln H(\overline{p}) + \ln \int_I 1_{\{(c_i/K_i)^{(1-\alpha_i)/\beta} < \overline{p}\}} \frac{\alpha_i\beta\lambda_i K_i^{\alpha_i}}{1-\alpha_i} \overline{p}^{\alpha_i\beta/(1-\alpha_i)-1}\, m(di),$$

which follows from (3.4.9), with lemma (3.7.2) and the first limit in (3.7.3), we obtain the second limit in (3.7.3).

Using the definitions of χ_2 and $\widetilde{H}$ as well as (3.7.3), we can see that

$$\lim_{\overline{p}\to\infty} \frac{\ln \chi_2(\overline{p})}{\ln \overline{p}} = 1 + \lim_{\overline{p}\to\infty} \frac{\ln H(\overline{p})}{\ln \overline{p}} = 1 + \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}},$$

$$\lim_{\overline{x}\to\infty} \frac{\ln \chi_2^{\text{inv}}(\overline{x})}{\ln \overline{x}} = \lim_{\overline{x}\to\infty} \frac{\ln \chi_2^{\text{inv}}(\overline{x})}{\ln \chi_2(\chi_2^{\text{inv}}(\overline{x}))} = \left(1 + \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}}\right)^{-1},$$

and $$\lim_{\overline{x}\to\infty} \frac{\ln \widetilde{H}(\overline{x})}{\ln \overline{x}} = \lim_{\overline{x}\to\infty} \frac{\ln H(\chi_2^{\text{inv}}(\overline{x}))}{\ln \chi_2^{\text{inv}}(\overline{x})} \cdot \frac{\ln \chi_2^{\text{inv}}(\overline{x})}{\ln \overline{x}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}+\overline{\alpha}\beta\gamma}.$$

The last two of these limits and the equivalence in (3.7.2) imply that

$$\lim_{\overline{p}\to\infty} \overline{p}^{\xi}(\chi_2)'(\overline{p}) = \lim_{\overline{p}\to\infty} \overline{p}^{\xi}\left(H(\overline{p}) + \overline{p}H'(\overline{p})\right) = \begin{cases} 0, & \text{if } \xi < -\frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}}, \\ \infty, & \text{if } \xi > -\frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}} \end{cases}$$

Combining this observation with (3.7), we obtain

$$\lim_{\overline{p}\to\infty} \frac{\ln(\chi_2)'(\overline{p})}{\ln \overline{p}} = \frac{\overline{\alpha}\beta\gamma}{1-\overline{\alpha}}$$

In view of the limits that we have derived thus far and the calculation

$$\lim_{\overline{x}\to\infty} \frac{\ln(\widetilde{H}'(\overline{x}))}{\ln \overline{x}} = \lim_{\overline{x}\to\infty} \left(\frac{\ln(H')\big(\chi_2^{\text{inv}}(\overline{x})\big)}{\ln \chi_2^{\text{inv}}(\overline{x})} - \frac{\ln \chi_2\big(\chi_2^{\text{inv}}(\overline{x})\big)'}{\ln \chi_2^{\text{inv}}(\overline{x})} \right) \frac{\ln \chi_2^{\text{inv}}(\overline{x})}{\ln \overline{x}},$$

we can see that the second limit in (3.7.3) also holds true.

■

Chapter 4

Weak convergence rate for the Cox-Ingersoll-Ross process

In this chapter, we study the weak convergence approximation rate of the Cox-Ingersoll-Ross (CIR) process in the regime where the process is positive, using a drift implicit method. Using a simple argumentation we were are able to obtain a convergence rate of order one under mild conditions on the payoff function and despite the fact that the coefficients of the underlying stochastic differential equation are not Lipschitz.

4.1 Introduction

The CIR process has the following form:

$$dX_t = k(\mu - X_t)dt + \theta\sqrt{X_t}dW_t, X_0 = x;\ t \in \mathbb{R}_+, \tag{4.1.1}$$

where $W := (W_t; t \in \mathbb{R}_+)$ is a one-dimensional Brownian motion, $k \geq 0$ is the speed of adjustment, $\mu \geq 0$ is the long term mean and $\theta > 0$ the diffusion parameter, with all parameters being non-negative. At this point, let us mention that from now on we will use parameter $\alpha := k\mu$ instead of k, μ separately. The CIR model is a prominent model for the short-term prediction of interest rates since it satisfies three main properties[CIR85, KS91]: (i) It is well known that admits a unique strong solution which is non-negative, (ii) volatility decreases when itself the interest rate increases, (iii) An equilibrium state exists. Additionally, Feller's test of explosions[KS91] ensures that if $X_0 > 0$

$$\mathbb{P}(\hat{X}_t > 0, t \geq 0) = 1 \tag{4.1.2}$$

provided that $2\alpha \geq \theta^2$

At the core of financial applications is the pricing of derivatives which in a one-dimensional setting is simply translated as the ability to evaluate $\mathbb{E}[f(X_T)]$, where f represents the payoff function and T is the maturity time. In most cases, although the aforementioned solution is integrable, it cannot be evaluated in closed form and therefore a suitable numerical scheme should be suggested.

Even though the increments of the CIR process are non-central chi-squared random variables and thus the process can be simulated exactly, the exact simulation is computationally unfeasible and thus an approximation scheme is preferred. For example, there are many available methods such as a Monte Carlo method or the so-called Walk on Spheres algorithm[Mul56, Mil97] or finally a discretization method, such as the Euler-Maruyama.

In general, the Euler-Maruyama method is a straightforward and simple method in which time is turned into a discretized grid, say of N points, and all continuous quantities are substituted by their discrete counterparts. In fact, it is a widely used method for numerically solving ordinary and partial differential equations while extensions of this method provide us with the well known Runge-Kutta method.

At this point let us point out that in the case of killed diffusions, it was shown by Göbet [Göb00] that the weak convergence rate is $\mathcal{O}(N^{-1/2})$ with this rate being exact and intrinsic to the problems arising from the discretization of the killing stopping time.

Interestingly, Çetin & Hok 2022, Çetin [cH21, Ç18] using a recurrent transformation managed, under mild conditions on the diffusion process and the barrier payoff to bring back the convergence rate to $\mathcal{O}(1/N)$. In the case of the CIR process, recurrence is a simple parameter adjustment since by the Feller test it is ensured that $X_t > 0, \mathbb{P}$-a.s. if $2\alpha\mu \geq \theta^2$ provided that $X_0 > 0$. The Imposition of non-negativity is crucial and should be respectively obeyed by the discretized stochastic process $\hat{X}$. However, if we consider an explicit Euler-Maruyama scheme

$$\hat{X}_{t_{n+1}} = \hat{X}_{t_n} + (\alpha - k\hat{X}_{t_n})\frac{T}{N} + \theta\sqrt{\hat{X}_{t_n}}(W_{t_{n+1}} - W_{t_n}), \tag{4.1.3}$$

where $t_n = n\frac{T}{N}$ for $n = 1, \ldots, N$, it becomes evident that positivity is not preserved since the Gaussian process can take arbitrarily large negative values. Solutions to this problem have been given in [DD98, BD07], though the most natural solution is to realize an implicit scheme. In particular, an appropriate implicit scheme was proposed by Brigo & Alfonsi [BA05]

Alternatively, consider the implicit scheme originating from the SDE which drives the square-root process, [Alf05]

$$\sqrt{\hat{X}_t} = \sqrt{\hat{X}_{t_n}} + \frac{\alpha - \theta^2/4}{2\sqrt{\hat{X}_t}}(t - t_n) - \frac{k}{2}\sqrt{\hat{X}_t}(t - t_n) + \frac{\theta}{2}(W_t - W_{t_n}), \tag{4.1.4}$$

Hence, $\hat{X}_t$ is the solution of a second order algebraic equation

$$[2 + k(t - t_n)]\hat{X}_t - \left[\theta(W_t - W_{t_n}) + \sqrt{\hat{X}_{t_n}}\right]\sqrt{\hat{X}_t} - \frac{\alpha - \theta^2/4}{2}(t - t_n) = 0, \tag{4.1.5}$$

which for $4\alpha > \sigma^2$ has a unique positive root[Alf05].

The purpose of this article is to prove that the weak convergence rate of the CIR process scales as $\mathcal{O}(1/N)$. Let us mention that the strong convergence rate has been an extensively studied subject [DD98, BD07, Alf05, NS14, DNS12] and was found to optimally be at the order of 1. On the other hand, the weak convergence rate was studied by [Alf05] where using convergence hypotheses for the numerical scheme it was shown that the rate scales as $\mathcal{O}(1/N)$ while recently in [MN21] using an appropriate stochastic discretization inspired by the stochastic trapezoidal rule used in [Zhe17] they were able to obtain a weak convergence rate of 1 when $2\alpha \geq \theta^2$ and the payoff is an appropriate four times differentiable function.

The present work serves as a proof of concept since only elementary arguments and mild assumptions on the payoff function were used in order to find the optimal convergence rate. Hence, hoping to make the evaluation of the optimal convergence rate a straightforward task, it would be interesting to extend this method to a broader class of stochastic process. This will be the study of a subsequent paper.

4.2 Weak Convergence of the CIR model

Lemma 4.2.1 *Consider the implicit scheme defined by (4.1.4) with $2\alpha \geq \theta^2$. Then,*

$$d\hat{X}_t = \frac{\sigma(\hat{X}_t)}{F(\hat{X}_t, t; t_n)} dW_t + \frac{\sigma^2(\hat{X}_t)}{F^2(\hat{X}_t, t; t_n)} \left[g(\hat{X}_t, t; t_n) + \frac{b(\hat{X}_t)}{\sigma^2(\hat{X}_t)} \right] dt, \quad t \in (t_n, t_{n+1}] \qquad (4.2.1)$$

where $\sigma(\hat{X}_t) = \theta\sqrt{X}_t, \quad b(\hat{X}_t) = \alpha - k\hat{X}_t,$

$$F(x, t; t_n) = 1 + \frac{k}{2}(t - t_n) + \frac{(4\alpha - \theta^2)}{8x}$$
$$g(x, t; t_n) = -\frac{k(t - t_n)}{2\theta^2} + \frac{1}{2x} - \frac{2(2 + k(t - t_n))}{4x(2 + 2k(t - t_n)) + (4\alpha - \theta^2)(t - t_n)} - \frac{(4\alpha - \theta^2)^2(t - t_n)}{32\theta^2 x^2},$$

Additionally, $\frac{1}{F(x,t;t_n)} \leq 1$ *and* $|g(x, t; t_n)| \leq K\left(1 + \frac{1}{x} + \frac{t - t_n}{x^2}\right)$, *with* K *a positive constant depending only on* α, θ *and* T

Proof: Let us define $H(x,t;t_n) := (2+k(t-t_n))\sqrt{x} - (\alpha - \frac{\theta^2}{4})\frac{(t-t_n)}{\sqrt{x}}$. To begin with, note that we cannot directly apply Ito's lemma due to the fact that H is not $\mathcal{C}^2$. To this end, let us consider the stopping time $\tau_M := \inf\{t \geq 0 | \hat{X}_t \leq \frac{1}{M}\}$ and localize the corresponding stochastic process $\{H(\hat{X}_t, t; t_n)\}_{t\geq 0}$.

$$H(\hat{X}_{t\wedge\tau_M}, t\wedge\tau_M; t_n) = H(X_0, 0; t_n) + \int_0^{t\wedge\tau_M} \partial_s H(\hat{X}_s, s; t_n) ds + \int_0^{t\wedge\tau_M} \partial_x H(\hat{X}_s, s; t_n) d\hat{X}_s + \int_0^{t\wedge\tau_M} \frac{1}{2}\partial_{xx} H(\hat{X}_s, s; t_n)\ d\langle\hat{X}_s\rangle \tag{4.2.2}$$

Consequently, due to the fact that the root solution of the implicit scheme is always positive, $\lim_{M\to\infty} \tau_M \to \infty$ $\mathbb{P}$-a.s. implying that Ito's lemma can be applied safely throughout the interval $[0,T]$.

Consequently, using (4.1.5) and re-arranging the terms leads to the desired form

$$d\hat{X}_t = \frac{\sigma(\hat{X}_t)}{F(\hat{X}_t, t; t_n)} dW_t + \frac{\sigma^2(\hat{X}_t)}{F^2(\hat{X}_t, t; t_n)}\left[g(\hat{X}_t, t; t_n) + \frac{b(\hat{X}_t)}{\sigma^2(\hat{X}_t)}\right] dt, \tag{4.2.3}$$

with

$$g(x,t;t_n) := -\frac{1}{2}\frac{\partial_{xx} H(x,t;t_n)}{\partial_x H(x,t;t_n)} - \frac{1}{\theta^2}\partial_x H(x,t;t_n)\partial_t H(\hat{X}_t, t; t_n) - \frac{b(x)}{\sigma^2(x)},$$

where $F(x,t;t_n) := \sqrt{x}\partial_x H(x,t;t_n)$. Note that from Ito's lemma we have identified the quadratic variation as $d\langle\hat{X}_t\rangle = \sigma(\hat{X}_t)/F(\hat{X}_t, t; t_n)$

Simple computations lead to $g(x,t;t_n)$ and $F(x,t;t_n)$ given at (4.2.1). In addition, application of the triangle inequality and the fact that $4\alpha > \theta^2$ give the desired bounds on $g(x,t;t_n)$ and $F(x,t;t_n)$.

Finally, we calculate $\partial_x g(x,t;t_n)$

$$\partial_x g(x,t;t_n) = \frac{8\left((t-t_n)k+2\right)^2}{\left(4a(t-t_n) - (t-t_n)\theta^2 + 4(t-t_n)kx + 8x\right)^2} - \frac{(t-t_n)\left(\theta^2 - 4a\right)^2}{16\theta^2 x^3} - \frac{1}{2x^2} \tag{4.2.4}$$

using elementary inequalities we get that $\partial_x g(x,t;t_n) \leq 0$ concluding that $\inf_{x\geq 0} g(x,t;t_n) = -\frac{k}{2\theta^2}(t-t_n) \geq -\frac{-k}{2\theta^2}T$. ■

The next lemma allow us to show that certain (inverse) moments of the discretized process are finite. This is crucial in order to control the estimates that are responsible for the weak convergence rate of $\{\hat{X}_t\}_{t\geq 0}$ towards the original process $\{X_t\}_{t\geq 0}$

Lemma 2 *Let $\hat{X}$ be the process defined by (4.1.4) and $2\alpha \geq \theta^2$. If $0 \leq p < \frac{2\alpha}{\theta^2}$ then*

$$\sup_{t\leq T,N} \mathbb{E}^x \left[\int_{t_n}^t \frac{1}{\hat{X}_s^p} \frac{1}{F^2(\hat{X}_s, s; t_n)} ds \right] \leq K(t-t_n), \tag{4.2.5}$$

for some constant $K > 0$. Additionally, the process $\hat{X}$ does not have explosions. Finally, $\forall m \geq 0$

$$\sup_{t\leq T,N} \mathbb{E}^x[\hat{X}_t^m] < \infty \tag{4.2.6}$$

Proof: From (4.2.1)

$$d\hat{X}_t = \frac{\sigma(\hat{X}_t)}{F(\hat{X}_t,t;t_n)} dW_t + \frac{\sigma^2(\hat{X}_t)}{F^2(\hat{X}_t,t;t_n)} \left[g(\hat{X}_t,t;t_n) + \frac{\alpha/\theta^2}{\hat{X}_t} - \frac{k}{\theta^2} \right] dt, \tag{4.2.7}$$

Consider the process $\hat{Y}$ defined by $\hat{Y}_t = \hat{X}_{A_t^{-1}}$, where

$$dA_t = \frac{1}{F^2(\hat{X}_t,t;t_n)} dt, \tag{4.2.8}$$

and $A_t^{-1} := \{s \geq 0 \mid \int_0^s \frac{1}{F^2(\hat{X}_s,s;t_n)} ds = t\}$ denotes the stopping time for which A_t becomes t. Firstly, A_t is well defined since by Lemma 4.2.1 we have that $\frac{1}{F(\hat{X}_t,t;t_n)} \leq 1$ which in turns gives that $A_t \leq t$ $\mathbb{P}$-a.s

Consequently, Dambis, Dubins-Schwarz theorem [RY99] yields

$$d\hat{Y}_t = \theta\sqrt{\hat{Y}_t} dB_t + \theta^2 \left[\left(g(\hat{Y}_t,t;t_n) - \frac{k}{\theta^2} \right) \hat{Y}_t + \frac{\alpha}{\theta^2} \right] dt, \quad t \in (t_n, t_{n+1}] \tag{4.2.9}$$

where B is a standard Brownian motion adapted to the filtration $(\mathcal{F}_{A_t^{-1}})_{t\geq 0}$

Therefore, if we define Y as the CIR process starting for X_0, i.e.,

$$Y_t = X_0 + \int_0^t \theta\sqrt{Y}_s dB_s + \int_0^t (\alpha - |c|Y_s)ds, \quad c = -\frac{kT}{2\theta^2}, \tag{4.2.10}$$

we can apply the comparison theorem of stochastic differential equations[cD18, KS91], since $g(x,t;t_n) \geq \inf_x g(x,t;t_n) \geq -\frac{kT}{2\theta^2}$, as given by Lemma 4.2.1 to obtain that

$$\mathbb{P}(\hat{Y}_t \geq Y_t,\ t \leq T) = 1. \tag{4.2.11}$$

Immediate use of the above identity gives that

$$\sup_{t\leq T,N} \mathbb{E}^x \left[\int_{t_n}^t \frac{1}{\hat{X}_s^p} \frac{1}{F^2(\hat{X}_s, s; t_n)} ds \right] \leq \theta \sup_{t\leq T} \mathbb{E}^x \left[\int_{t_n}^{A_t} \frac{1}{Y_s^p} ds \right]. \tag{4.2.12}$$

Therefore, using the fact that $\sup_{t\leq T} \mathbb{E}[\frac{1}{Y_t^p}] < \infty$ iff $p < \frac{2\alpha}{\theta^2}$ [DNS12, HK08] combined with Fubini's theorem and the fact that $A_t \leq t$ $\mathbb{P}-$a.s. we conclude the first statement.

Next, for the second statement of the lemma. It is already proven that $\forall t \in [0,T]$ $\hat{X}_t > 0$ $\mathbb{P}-$a.s provided that $X_0 > 0$. Thus, we are left to show that explosions do not occur for the (4.2.1) scheme. From Lemma 4.2.1 we know that $g(x.t;t_n)$ can be be bounded by $\sup_{t\leq T,N} g(x.t;t_n) \leq K\left(\frac{1}{x} + \frac{1}{x^2}\right)$ where $K > 0$ and depends only on α, θ and T. In addition, note that the bound on the total drift term is $K(1 + \frac{1}{x}) + b(x)$ which we can see that it is locally Lipschitz.

Consequently, consider a general time inhomogeneous SDE and define its scale function $s(x) = \exp(-2\int_c^y d(\xi)\beta(\xi)d\xi)dy$, where $d : \mathbb{R} \to \mathbb{R}$ is the drift coefficient, $\beta : \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient and $c \in int\mathcal{D}$ where $\mathcal{D}$ is the domain where the diffusion lives. Then, if

$s(x) = \infty$ it implies that x is an inaccessible point[KS91]. Therefore, if we consider the process $\{Z_t\}_{t\geq 0}$ starting from X_0 we obtain by a comparison theorem argument[KS91, cD18] that $\mathbb{P}(\hat{Y}_t \leq Z_t, \forall t \in \mathbb{R}_+) = 1$. Hence, a straightforward computation of the scale function for the process Z shows that infinity is an inaccessible point and hence explosions do not occur throughout the whole $[0, T]$ which is immediately translated for $\hat{X}$.

For the next assertion, we define the stopping time $\zeta_M := \inf\{t \geq 0 | \hat{X}_t > M\}$.

Again from (4.2.1) we get that

$$\mathbb{E}^x\left[\hat{X}_{t\wedge\zeta_M}\right] = \mathbb{E}\left[\int_{t_n}^{t\wedge\zeta_M} \frac{\sigma(\hat{X}_t)}{F(\hat{X}_t, t; t_n)} dW_t + \int_{t_n}^{t\wedge\zeta_M} \frac{\sigma^2(\hat{X}_t)}{F^2(\hat{X}_t, t; t_n)} \left[g(\hat{X}_t, t; t_n) + \frac{b(\hat{X}_t)}{\sigma^2(\hat{X}_t)}\right] dt\right] \tag{4.2.13}$$

Evidently, this sequence is a localizing sequence which reduces the local martingale term to a true martingale one. In addition, using the first statement of the lemma we obtain that

$$\sup_{t\leq T,N} \mathbb{E}^x[\hat{X}_{t\wedge\zeta_m}] < K\left[1 + \int_{t_n}^{t\wedge\tau_M} \mathbb{E}^x[\hat{X}_s] ds\right], \tag{4.2.14}$$

where K is a positive constant. Finally, since explosions do not occur $\lim_{M\to\infty} \zeta_M \to \infty$ $\mathbb{P}$-a.s. and thus, application of Fatou's lemma on the left hand side and monotone convergence on the right hand side along with Grönwall's inequality results to

$$\sup_{t\leq T,N} \mathbb{E}^x[\hat{X}_t] < \infty. \tag{4.2.15}$$

Consequently, let us assume that

$$E(m) := \sup_{t\leq T,N} \mathbb{E}^x[\hat{X}_t^m] < \infty, \quad m \geq 2 \tag{4.2.16}$$

Once again, using Ito's Lemma we obtain

$$
\begin{aligned}
\hat{X}_t^{m+1} &= \hat{X}_{t_n}^{m+1} + (m+1)\int_{t_n}^{t} \hat{X}_s^m \frac{\sigma(\hat{X}_s)}{F(\hat{X}_s, s; t_n)} dW_s \\
&+ (m+1)\int_{t_n}^{t} \hat{X}_s^m \frac{\sigma(\hat{X}_s)^2}{F^2(\hat{X}_s, s; t_n)} \left(g(\hat{X}_s, s; t_n) - \frac{k}{\theta^2} + \frac{\frac{\alpha}{\theta^2}}{\hat{X}_t} \right) ds \\
&+ \frac{1}{2} m(m+1)\int_{t_n}^{t} \hat{X}_s^{m-1} \frac{\sigma(\hat{X}_s)^2}{F^2(\hat{X}_s, s; t_n)} ds, \quad t \in (t_n, t_n + 1] \qquad (4.2.17)
\end{aligned}
$$

Therefore, taking expectations for the corresponding stopped process and using the bounds of for $g(x, t; t_n), F(x, t; t_n)$ we obtain

$$
\mathbb{E}[\hat{X}_{t\wedge\zeta_M}^{m+1}] \leq \mathbb{E}[\hat{X}_{t_n\wedge\zeta_M}^{m+1}] + K\int_{t_n}^{t\wedge\zeta_M} \mathbb{E}[\hat{X}_s^m] ds + K\int_{t_n}^{t\wedge\zeta_M} \mathbb{E}[\hat{X}_s^{m-1}] ds \qquad (4.2.18)
$$

where $K > 0$ a constant depending on the parameters of the problem and T. Note that the local martingale term is eradicated by the localization. In view of Fatou's lemma and monotone convergence in the left and right hand side respectively, the inequality $x^{m-1} \leq 1 + x^m$ and using (4.2.18) recursively we are left with

$$
E(m+1) \leq \hat{X}_0^{m+1} + KE(m) < \infty. \qquad (4.2.19)
$$

Finally, $\sup_{t\leq T,N} \mathbb{E}^x[\hat{X}_t^m] < \infty, \quad \forall m \in \mathbb{N}$ trivially implies that the extension for all $m \geq 0$ holds. ∎

Next, a PDE expression for the expectation of the payoff-function is needed since we must compare the difference between the actual result and the numerical approximation which subsequently going to give rise to differential terms. Alfonsi [Alf05] using the analytical formula of the CIR transition density proved the following proposition

Proposition 4.2.1 *Let $f \in \mathcal{C}^{(q)}((0,\infty),\mathbb{R})$, $m \geq q$ such that there is $K > 0$ and $m \geq q$, $m \in \mathbb{N}$ such that*

$$
\forall x \geq 0, \quad |f^{(q)}(x)| \leq K(1 + |x|^m). \qquad (4.2.20)
$$

Then if $v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ *defined by* $v(x,t) \equiv \mathbb{E}^x[f(X_{T-t})]$ *it holds that* v *satisfies the following PDE:*

$$\partial_t v + b(x)\partial_x v + \frac{\sigma(x)^2}{2}\partial_{xx} v = 0, \tag{4.2.21}$$

$v(T,x) = f(x)$. *Additionally, derivatives of* $v(x,t)$ *are uniformly bounded and there exists* $K > 0$ *such that*

$$\forall (x,t) \in \mathbb{R}_+ \times [0,T], \qquad |\partial_x^k \partial_t^r v(x,t)| \leq K(1 + x^{m+q+r}), \quad k + 2r \leq q \tag{4.2.22}$$

Consequently, we are ready to proceed to the main result of this paper.

Theorem 4.2.2 *Let* $2\alpha\mu \geq \theta^2$ *and* $f \in \mathcal{C}_b^{(2)}((0,\infty),\mathbb{R})$ *such that* $|f^{(2)}(x)| \leq K(1 + x^m)$, *where* $m \geq 2$ *Then,*

$$|\mathbb{E}^x[f(X_t)] - \mathbb{E}^x[f(\hat{X}_t)]| \sim \mathcal{O}(\frac{1}{N}) \tag{4.2.23}$$

Proof: In order to evaluate $|\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(\hat{X}_T)]$ we partition it into small intervals as follows:

$$\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(\hat{X}_T)] = \mathbb{E}^x[v(T,\hat{X}_T)] - v(0,X_0)] = \sum_{n=0}^{N-1} \mathbb{E}^x[v(t_{n+1},\hat{X}_{t_{n+1}}) - v(t_n,\hat{X}_{t_n})] \tag{4.2.24}$$

In view of Ito's lemma and (4.2.21) we obtain that

$$v(t_{n+1},\hat{X}_{t_{n+1}}) - v(t_n,\hat{X}_{t_n}) = M_{t_{n+1}} - M_{t_n} + I_1^n + I_2^n, \tag{4.2.25}$$

where

$$I_1^n \equiv \int_{t_n}^{t_{n+1}} \partial_t v(\hat{X}_t, t) \left(1 - \frac{1}{F^2(\hat{X}_t, t; t_n)}\right) dt \tag{4.2.26}$$

$$I_2^n \equiv \int_{t_n}^{t_{n+1}} \partial_x v(\hat{X}_t, t) \ g(\hat{X}_t, t; t_n) \frac{\sigma(\hat{X}_t)^2}{F^2(\hat{X}_t, t; t_n)} dt \tag{4.2.27}$$

$$M_t \equiv \int_0^t \partial_x v(\hat{X}_t, t) \frac{\sigma(\hat{X}_t)}{F(\hat{X}_t, t; t_n)} dW_t \tag{4.2.28}$$

with M being a *local* martingale.

At this point, before we proceed to the main calculations, we define two polynomial functions which will be used further down

$$\begin{aligned} Q(x) &:= 1 + x^{m+2} \\ P(x) &:= 1 + x^{m+3}. \end{aligned} \tag{4.2.29}$$

To begin with, we show that M is a *true* martingale. Indeed,

$$\mathbb{E}^x[\langle M \rangle_t] = \mathbb{E}^x \left[\int_0^t \left(\partial_x v(\hat{X}_t, t)\right)^2 \frac{\sigma(\hat{X}_t)^2}{F^2(\hat{X}_t, t; t_n)} dt \right] \leq \theta \int_0^t \sup_{t \leq T} \mathbb{E}^x[\left(\partial_x v(\hat{X}_t, t)\right)^2 \hat{X}_t] dt \tag{4.2.30}$$

Since $|\partial_x v(x,t)| \leq Q(x)$, Lemma 2 immediately shows that M is a true martingale and thus $\mathbb{E}^x[M_{t_{n+1}} - M_{t_n}] = 0$.

Next,

$$|\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(\hat{X}_T)]| = \left| \sum_{n=0}^{N-1} \mathbb{E}^x[I_1^n + I_2^n] \right| \leq \sum_{n=0}^{N-1} (\mathbb{E}^x[|I_1^n|] + \mathbb{E}^x[|I_2^n|]) \tag{4.2.31}$$

Hence, we need to evaluate $\mathbb{E}^x[|I_1^n|]$ and $\mathbb{E}^x[|I_2^n|]$.

At this point we outline the idea for bounding I_j^n, $j = 1, 2$. Firstly, we shall make use of Ito's lemma in order to prove that $I_j^n \sim \mathcal{O}(\frac{1}{N^2})$ which consequently gives us integrals of the form

$$\mathbb{E}^x\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\mathcal{S}(X_t,t;t_n)\frac{1}{F^2(\hat{X}_s,s;t_n)}ds\ dt\right]. \tag{4.2.32}$$

Hence, if the $\mathcal{S}(x,t;t_n)$ does not diverge quicker that $1/x^{\frac{2\alpha}{\theta^2}}$ when x approaches zero then Fubini's theorem and the first part of Lemma 2 ensures the required scaling with respect to N. In addition, if $\mathcal{S}(x,t;t_n)$ has a polynomial behavior then the second part of the aforementioned proof along with Fubini's theorem shall be used to obtain the same bound. Additionally, Proposition 4.2.1 tell us that $\forall(x,t) \in \mathbb{R}_+ \times [0,T]$ $|\partial_x v(x,t)| \leq Q(x)$ and $|\partial_t v(x,t)| \leq P(x)$ $\forall t \in \mathbb{R}_+$, and thus we only need to appropriately bound the terms which come from the inherited properties of the process $\hat{X}$.

Let us start with evaluation of I_1^n estimate. Direct use of $\sup_{t\leq T}|\partial_s v(x,t)| \leq P(x)$ and Ito's lemma results to

$$\left|\int_{t_n}^{t_{n+1}}\partial_s v(\hat{X}_t,t)\left(1-\frac{1}{F^2(\hat{X}_t,t;t_n)}\right)dt\right| \leq \int_{t_n}^{t_{n+1}}P(\hat{X}_t)\left(1-\frac{1}{F^2(\hat{X}_t,t;t_n)}\right)dt$$

$$= \int_{t_n}^{t_{n+1}}\left[\int_{t_n}^{t}P(\hat{X}_s)\partial_s\Psi(\hat{X}_s,s;t_n)ds\right.$$
$$+\int_{t_n}^{t}P(\hat{X}_s)\partial_x\Psi(\hat{X}_s,s;t_n)d\hat{X}_s+\int_{t_n}^{t}\partial_x P(\hat{X}_s)\Psi(\hat{X}_s,s;t_n)d\hat{X}_s$$
$$+\int_{t_n}^{t}\partial_{xx}P(\hat{X}_s)\Psi(\hat{X}_s,s;t_n)\frac{\sigma^2(\hat{X}_s)}{2F^2(\hat{X}_s,s;t_n)}ds \tag{4.2.33}$$
$$+\int_{t_n}^{t}P(\hat{X}_s)\partial_{xx}\Psi(\hat{X}_s,s;t_n)\frac{\sigma^2(\hat{X}_s)}{2F^2(\hat{X}_s,s;t_n)}ds$$
$$\left.+\int_{t_n}^{t}\partial_x P(\hat{X}_s)\partial_x\Psi(\hat{X}_s,s;t_n)\frac{\sigma^2(\hat{X}_s)}{F^2(\hat{X}_s,s;t_n)}ds\right]dt, \tag{4.2.34}$$

where we used the fact that $F^2(x,t_n;t_n) = 1$ and Ψ is defined as $\Psi(x,t;t_n) := 1 - \frac{1}{F^2(x,t;t_n)}$

In order to demonstrate how Lemma 2 controls our particular estimates to an appropriately convergent term, let us calculate the fifth term of the right hand side. Since $\frac{1}{F^2(x,t;t_n)} \leq 1$ we get that $|1 - \frac{1}{F^2(x,t;t_n)}| \leq 1$ and thus,

$$\mathbb{E}^x\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\left|\partial_{xx}P(\hat{X}_s)\Psi(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)^2}{2F^2(\hat{X}_s,s;t_n)}\right| ds\ dt\right] \leq \frac{\theta}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\mathbb{E}^x\left[\partial_{xx}P(\hat{X}_s)\hat{X}_s\right] ds\ dt$$
$$\leq K\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds\ dt \sim \mathcal{O}(\frac{1}{N^2}),$$

The reasoning for the rest of the terms is similar and is exhibited in Appendix A where with the help of Lemma 2 we prove that all terms are $\mathcal{O}(\frac{1}{N^2})$ which immediately gives that $\mathbb{E}^x[|I_1^n|] \sim \mathcal{O}(\frac{1}{N^2})$.

For the second estimate I_2^n we follow a similar argument where Proposition 4.2.1 provides the bound $|\partial_x v(x,t)| \leq Q(x)$. Additionally, as shown in Lemma 4.2.1 the last term of $g(x,t;t_n)$ is $\mathcal{O}(t-t_n)$ and thus we separate this term from $g(x,t;t_n)$ and apply Ito's lemma to the rest:

$$\left|\int_{t_n}^{t_{n+1}} \partial_x v(\hat{X}_t,t) g(\hat{X}_t,t;t_n) \frac{\sigma(\hat{X}_s)^2}{F^2(\hat{X}_t,t;t_n)} dt\right| \leq \int_{t_n}^{t_{n+1}} Q(\hat{X}_t)|g(\hat{X}_t,s;t_n)| \frac{\sigma(\hat{X}_t)}{F^2(\hat{X}_t,t;t_n)} dt$$
$$\leq \int_{t_n}^{t_{n+1}} \Bigg[Q(\hat{X}_t)(t-t_n) \left(\frac{\frac{4\alpha-\theta^2}{16} + \hat{X}_t}{2\theta^2 \hat{X}_t} \right) \frac{\sigma(\hat{X}_t)^2}{F^2(\hat{X}_t,t;t_n)} + \int_{t_n}^{t} Q(\hat{X}_s) \partial_s Z(\hat{X}_s,s;t_n) ds$$
$$+ \int_{t_n}^{t} \partial_x Q(\hat{X}_s) Z(\hat{X}_s,s;t_n) d\hat{X}_s + \int_{t_n}^{t} Q(\hat{X}_s) \partial_x Z(\hat{X}_s,s;t_n) d\hat{X}_s$$
$$+ \int_{t_n}^{t} \partial_{xx} Q(\hat{X}_s) Z(\hat{X}_s,s;t_n) \frac{\sigma^2(\hat{X}_s)}{2F^2(\hat{X}_s,s;t_n)} ds$$
$$+ \int_{t_n}^{t} Q(\hat{X}_s) \partial_{xx} Z(\hat{X}_s,s;t_n) \frac{\sigma(\hat{X}_s)^2}{2F^2(\hat{X}_s,s;t_n)} ds$$
$$+ \int_{t_n}^{t} \partial_x Q(\hat{X}_s) \partial_x Z(\hat{X}_s,s;t_n) \frac{\sigma^2(\hat{X}_s)}{2F^2(\hat{X}_s,s;t_n)} ds \Bigg] dt, \tag{4.2.35}$$

where $Z(x,t;t_n) := \tilde{g}(x,t;t_n)\frac{\sigma^2(x)}{F^2(x,t;t_n)}$ with $\tilde{g}(x,t;t_n) := \frac{1}{2x} - \frac{2+k(t-t_n)}{2x(2+k(t-t_n))+(4\alpha-\theta^2)(t-t_n)}$. Additionally, note that we also used the triangle inequality along with the fact that $\tilde{g}(x,t;t_n) \geq 0$ with $g(x,t_n;t_n) = 0$

The $\mathcal{O}(\frac{1}{N^2})$ convergence rate of the first term of the right hand side is immediate from Lemma 2. while for the rest of the terms we show in appendix A that all integrated terms can be controlled by finite expectation quantities and thus also, $\mathbb{E}^x[|I_2^n|] \sim \mathcal{O}(\frac{1}{N^2})$

Finally, from (4.2.31) we get that

$$|\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(\hat{X}_T)]| \sim \mathcal{O}(\frac{1}{N}) \tag{4.2.36}$$

■

4.3 Appendix A

Intended to clarify the obtained bounds in Theorem 4.2.2 we provide all quantities which appear in the estimation of $I_1^n,\ I_2^n$.

4.3.1 Calculations for I_1^n

Let us start with the relevant quantities for I_1^n

$$F_t(x,t;t_n) = \frac{4\alpha-\theta^2}{8x} + \frac{k}{2} \Rightarrow \frac{F_t(x,t;t_n)}{F(x,t;t_n)} \le \frac{4\alpha-\theta^2}{8x} + \frac{k}{2}$$

Thus, application of Lemma 2 results to

$$\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\big|P(\hat{X}_s)\partial_s W(\hat{X}_t,t;t_n)\big|\,ds\ dt\Big]| \sim \mathcal{O}(\frac{1}{N^2})$$

Next,

$$\partial_x\Psi(x,t;t_n) = \partial_x\left(1-\frac{1}{F^2(x,t;t_n)}\right) = \frac{128(t-t_n)x\,(\theta^2-4a)}{\left[(4a-\theta^2)(t-t_n)+4x(2+k(t-t_n))\right]^3}$$

Hence, using the triangle inequality and the fact that $4\alpha > \theta^2$ we obtain the following two inequalities

$$|\partial_x\Psi(x,t;t_n)| \le \frac{K}{t-t_n}$$
$$|\partial_x\Psi(x,t;t_n)| \le \frac{K}{x}, \tag{4.3.1}$$
$$\tag{4.3.2}$$

where K depends only on α,θ,T.

The construction of two separate inequalities allow us to efficiently control the term

$\partial_x \Psi(x,t;t_n)\left(g(x,t;t_n) + \frac{b(x)}{\sigma^2(x)}\right)\sigma^2(x)$. At this point, let us justify the purpose of the above construction. As shown below, the first inequality of (4.3.1) will be used to control the $\frac{t-t_n}{x^2}$ term contained in $g(x,t;t_n)$ while the second one will be used to control the rest of the terms thus allowing us to get a finite result. For instance, if we were to use only the first inequality we would get a divergence from the integration of $\frac{1}{t-t_n}$ while if instead we chose to use only the second one we would get a $\frac{1}{x^2}$ total term in our integrated quantity which, unless we further further impose that $\alpha > \sigma^2$, has a divergent expected value.

Thus, using the aforementioned inequalities we obtain

$$\left|\partial_x \Psi(x,t;t_n)\frac{(4\alpha-\theta^2)^2(t-t_n)}{32\theta^2x^2}\right| \le \frac{K}{x^2}$$
$$\left|\partial_x \Psi(x,t;t_n)\left(\frac{b(x)}{\sigma^2(x)} + g(x,t;t_n) - \frac{(4\alpha-\theta^2)^2(t-t_n)}{32\theta^2x^2}\right)\right| \le K\left(\frac{1}{x}+\frac{1}{x^2}\right),$$

Getting things together we obtain an integrand of the form $K(1+\frac{1}{x})\frac{1}{F^2(x,t;t_n)}$ which ensures that

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\left|P(\hat{X}_s)\partial_x\Psi(\hat{X}_s,s;t_n)\left(g(\hat{X}_s,s;t_n)+\frac{b(\hat{X}_s)}{\sigma^2(\hat{X}_s)}\right)\frac{\sigma^2(\hat{X}_s)}{F^2(\hat{X}_s,s;t_n)}\right|ds\ dt\right] \sim \mathcal{O}\left(\frac{1}{N^2}\right)$$

The next term of concern is

$$\partial_{xx}\Psi(x,t;t_n) = 128(t-t_n)(4\alpha-\theta^2)\frac{8x(1+2k(t-t_n))-(t-t_n)(4\alpha-\theta^2)}{(8x(1+2k(t-t_n))+(t-t_n)(4\alpha-\theta^2))^4}$$

Again, using elementary algebra one can obtain that

$$|\partial_{xx}\Psi(x,t;t_n)| \le \frac{128(t-t_n)(4\alpha-\theta^2)}{(8x(1+2k(t-t_n))+(t-t_n)(4\alpha-\theta^2))^3} \le \frac{K}{x^2}$$

Therefore, immediate application of the same arguments results to

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\left|P(\hat{X}_s)\partial_{xx}\Psi(\hat{X}_s,s;t_n)\frac{\sigma^2(\hat{X}_s)}{2F^2(\hat{X}_s,s;t_n)}\right|ds\ dt\right]\sim\mathcal{O}(\frac{1}{N^2})$$

Concluding calculations for I_1^n we show that the local martingale terms of (4.2.33),

$$\int_{t_n}^{t}P(\hat{X}_t)\partial_x\Psi(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}dW_s$$
$$\int_{t_n}^{t}\partial_xP(\hat{X}_t)\Psi(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}d\hat{W}_s$$

are true martingales and thus have vanishing expectations. Indeed, their corresponding quadratic variations

$$\mathbb{E}^x\left[\int_{t_n}^{t}\left(P(\hat{X}_t)\partial_x\Psi(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}\right)^2ds\right]\leq K\mathbb{E}^x\left[\int_{t_n}^{t}\frac{P(\hat{X}_s)/\hat{X}_s}{F^2(\hat{X}_s,s;t_n)}ds\right]<\infty$$
$$\mathbb{E}^x\left[\int_{t_n}^{t}\left(\partial_xP(\hat{X}_t)\Psi(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}\right]^2ds\right)\leq K\int_{t_n}^{t}\mathbb{E}^x[\partial_xP(X_s)\hat{X}_s]ds<\infty,$$

where for the first quadratic variation we used the second (4.3.1).

Note that, for the rest of the terms of (4.2.33) it is immediate, using the aforementioned techniques, to prove that they converge as $\mathcal{O}(\frac{1}{N^2})$

4.3.2 Calculations for I_2^n

Next, let us move on to the estimates regarding I_2^n

$$\partial_t Z(x,t;t_n) = \partial_t\left(\tilde{g}(x,t;t_n)\frac{\sigma^2(x)}{F^2(x,t;t_n)}\right) = \frac{\frac{\theta^2}{x}\left(-\frac{2}{F^2(x,t;t_n)} + \frac{(2+(t-t_n)k)\left(4a-\theta^2+4kx\right)}{F^2(x,t;t_n)} - \frac{32k^2}{\theta^2}x^2\right)}{F^2(x,t;t_n)} + \frac{\theta^2\left(4a-\theta^2+4kx\right)\left(\frac{(2+(t-t_n)k)}{xF(x,t;t_n)} + \frac{(t-t_n)k^2}{\theta^2} - \frac{1}{x}\right)}{8F^3(x,t;t_n)}$$

The above quantity can be bounded as

$$|\partial_t Z(x,t;t_n)| \leq K(\frac{1}{x}+1+x)\frac{1}{F^2(x,t;t_n)} \tag{4.3.3}$$

Therefore,

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} Q(\hat{X}_s)\left|\partial_s Z(\hat{X}_s,s;t_n)\right| ds\ dt\right] \sim \mathcal{O}(\frac{1}{N^2})$$

Next, since $|Z(x,t;t_n)| = \tilde{g}(x,t;t_n)\frac{\sigma^2(x)}{F^2(x,t;t_n)} \leq K(1+x)$ and $|g(x,t;t_n) + \frac{b(x)}{\sigma^2(x)}| \leq K(1+\frac{1}{x}+\frac{t-t_n}{x^2})$ it is evident that their product will be of the form

$$\left|Z(x,t;t_n)\left(g(x,t;t_n)+\frac{b(x)}{\sigma^2(x)}\right)\frac{\sigma^2(x)}{F^2(x,t;t_n)}\right| \leq K\left(1+\frac{1}{x}+x+x^2\right),$$

Concluding that

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\left|\partial_x Q(\hat{X}_s)Z(\hat{X}_s,s;t_n)\frac{\sigma^2(\hat{X}_s)}{F^2(\hat{X}_s,s;t_n)}\left(g(\hat{X}_s,s;t_n)+\frac{b(\hat{X}_t)}{\sigma^2(\hat{X}_s)}\right)\right| ds\ dt\right] \sim \mathcal{O}(\frac{1}{N^2}).$$

Consequently, the next term is bounded by

$$\left|\partial_x Z(x,t;t_n)\left(g(x,t;t_n)+\frac{b(x)}{\sigma^2(x)}\right)\right| \leq K\left(1+\frac{1}{x}+\frac{1}{x^2}\right),$$

Thus, the term

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\left|Q(\hat{X}_s)\partial_x Z(\hat{X}_s,s;t_n)\left(g(\hat{X}_s,s;t_n)+\frac{b(\hat{X}_s)}{\sigma^2(\hat{X}_s)}\right)\frac{\sigma^2(\hat{X}_s)}{F^2(\hat{X}_s,s;t_n)}\right| ds\ dt\right]$$

is also $\mathcal{O}(\frac{1}{N^2})$

The remaining terms are in the same spirit and thus, it remains to show that the corresponding local martingales are true martingales. This is immediate by the previous discussion

$$\mathbb{E}^x\left[\int_{t_n}^t\left(Q(\hat{X}_s)\partial_x Z(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}\right)^2 ds\right]<\infty$$
$$\mathbb{E}^x\left[\int_{t_n}^t\left(\partial_x Q(\hat{X}_s)Z(\hat{X}_s,s;t_n)\frac{\sigma(\hat{X}_s)}{F(\hat{X}_s,s;t_n)}\right)^2 ds\right]<\infty,$$

Concluding that is also $I_2^n \sim \mathcal{O}(\frac{1}{N^2})$

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