Infinite Horizon Stochastic Differential Utility of Epstein-Zin Type

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Declaration

I hereby declare that, except where specific references have been made to the work of others, the content of this thesis is original and has not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work in collaboration with my supervisor, Professor Hao Xing, and contains no outcome of collaboration with others, except where noted.

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Abstract

In this thesis, we study stochastic differential utility of Epstein–Zin type in a general semimartingale setting. We show that the traditional characterization using the transversality condition identifies an incorrect Epstein–Zin utility in the empirical relevant parameter case. Instead, we present an original characterisation of Epstein–Zin utility in an infinite time horizon and provide sufficient conditions for its existence and uniqueness. In the second half of the thesis, we study an infinite horizon optimal consumption-investment problem in an incomplete, Brownian-driven market for an investor whose preferences are governed by Epstein–Zin utility.

Contents

CHAPTER **1**

INTRODUCTION

In classic portfolio optimisation frameworks, the representative agent is assumed to have a time-additive von Neumann-Morgenstein utility. In such a model, the relative risk aversion parameter, denoted γ, is forced to be equal to reciprocal of Elasticity of Intertemporal Substitution (EIS), denoted ψ. The former parameter measures the agent's attitude towards risk, and the latter his willingness to substitute consumption over time. This reciprocal relation turns out to be an advantage *and* disadvantage. On the one hand, this relationship has the attractive feature of mathematical tractability. On the other hand, this inflexibility causes time-additive models to perform poorly empirically. For example, a time-additive model suggests that the investor has to be implausibly risk-averse to justify the high average equity premium observed [59], a phenomenon known as the equity premium puzzle (see also [63]). Other asset pricing puzzles observed empirically include the risk free rate puzzle [63], excess volatility puzzle [58][44] and credit spread puzzle [14][18].

An alternative to time-additive utility is the so-called recursive utility, which dates back to the seminal papers of Kreps & Porteus [42],[43], Epstein & Zin [23], and Weil [64], who studied recursive utility in a discrete time setting. Its continuous time counterpart was developed by Duffie & Epstein [20] and shown to be the limit of the discrete model as the time grid converges to 0 by Kraft & Seifried [38]. Given a consumption plan c on the finite horizon $[0,T]$, the stochastic differential utility associated with it is defined as the solution of the Backwards Stochastic Differential Equation (BSDE)[1]:

$$V_t = \mathbb{E}_t\Big(U(c_T) + \int_t^T f(c_s, V_s)ds\Big), \quad t \in [0, T] \tag{1.1}$$

where f is an *aggregator* function and U a *bequest utility function*. Recursive utility generalises its time-additive counterpart in several ways. Firstly, it allows for the separation between risk aversion and EIS, which now can be parametrised individually. Secondly, it introduces the notion of resolution of uncertainty. An agent's temporal preference for information can be thought of as the result of two competing forces: first, his perhaps irrational aversion of future consequences, and thus delaying bad news, and second, the ability to plan ahead by using early information (cf. [60]). In the time additive case, the agent is indifferent.

In recent years, interest in recursive utility has been sparked by progresses towards explaining asset pricing puzzles, begun with the long-run risk model (LRRM) proposed by Bansal and Yaron [3][2], which explores the equity premium and risk-free rate puzzles and utilises a discrete time Epstein–Zin utility model in a critical way. In their settings, both risk aversion and EIS are calibrated to be greater than 1, indicating a preference for early resolution of uncertainty. Long run consumption growth risk becomes quantitatively important as a result, and stipulates an additional risk premium. Such a quantity is not present in the classical paradigm, where the agent is indifferent towards the moment in time where uncertainty is resolved. Other applications of recursive utility in explaining the aforementioned puzzles include [5] for the excess volatility puzzle and [6] for the credit spread puzzle. Motivated by these developments, we investigate the question of existence and uniqueness of stochastic differential utility of Epstein–Zin type, as well as the related portfolio optimisation problem, both of which are in infinite horizon.

The formal notion of Epstein–Zin utility in infinite horizon has not been satisfactorily established in the literature. Early work in stochastic differential utility (e.g. [20]) assumes the Lipschitz property for the aggregator, which is violated in the Epstein–Zin parametrisation. Similarly, Duffie et. al's work on infinite horizon (Appendix C [20]) stipulates the so-called uniform sector condition, which does not apply to the Epstein–Zin aggregator. Duffie & Lion [21] also studied stochastic differential utility for consumptions following a Markovian diffusion, using PDE methods. However, in the empirically relevant setting where $\gamma, \psi > 1$,[2] uniqueness becomes a delicate issue and requires c to be bounded

[1] All mathematical notions in the introduction will be defined formally later in the thesis.
[2] Corresponding to the case $\mu < -1$ in their paper.

from above and away from 0. A more recent treatment that addresses specifically Epstein–Zin utility is offered by Melnyk et. al [47]. However, the question of existence is not treated and, when $\gamma > \psi$, uniqueness is tied to the condition that long-term consumption cannot exceed current consumption. On the finite horizon, Schroder & Schiadas [55] studied Epstein–Zin utility's existence and uniqueness question with no terminal consumptions. Seiferling & Seifried [57] studied the same problem, but with terminal consumptions and extra parameter restrictions, under which the aggregator is either convex or concave. Xing [66] studied the case where $\gamma, \psi > 1$ and the filtration is generated by a Brownian Motion.

The existing literature on infinite horizon BSDE also does not readily answer the questions of existence and uniqueness for Epstein–Zin utility. An infinite horizon Epstein–Zin utility process V is can be defined by the equation:

$$V_t = \mathbb{E}\Big(V_T + \int_t^T f(c_s, V_s)ds \Big| \mathcal{F}_t\Big) \quad \text{for all } 0 \le t \le T < \infty. \tag{1.2}$$

Such a BSDE can be thought of as one with stopping terminal time τ such that $\mathbb{P}(\tau = \infty) = 1$. Darling & Pardoux [17], Briand & Hu [8] studied BSDEs with random but almost surely finite terminal time. Royer [54] allowed for infinite terminal time, but the question of uniqueness was only treated when $\mathbb{P}(\tau < \infty) = 1$. Existing results whose scope includes the case $\mathbb{P}(\tau = \infty) = 1$ have experienced setbacks when applied to the Epstein–Zin aggregator. For instance, results attained by Bahlali et. al [1], Hu & Tessitore [30] and Confortola & Briand [15] assume either linear growth or uniform Lipschitz condition in v. More recently, Papapantoleon et. al [49] studied random terminal time BSDEs with jumps under a so-called stochastic Lipschitz condition (condition (F3) therein). All these conditions are violated by the Epstein–Zin aggregator.

One contribution of this thesis is in our original method of characterising the Epstein–Zin utility process in infinite horizon. In the finite horizon case, Seifried & Seiferling [57] established the relationship between Epstein–Zin and additive utilities (cf. Section 2.3). In the referenced paper, they characterised the utility process V by equation (1.1). Then, if U^{γ} and $U^{\psi^{-1}}$ are additive utility processes of an agent with risk aversion γ and ψ^{-1}, respectively, then:

$$U^{\gamma \vee \psi^{-1}} \le V \le U^{\gamma \wedge \psi^{-1}}. \tag{1.3}$$

In extending to infinite horizon, our method is based on two observations. Firstly, as (1.3) displays a relationship between Epstein–Zin utility and different levels of risk aversion, it is desirable to preserve this ordering in an infinite horizon extension. Secondly, while extending V to infinite horizon might be tricky, it is relatively straightforward and intuitive to extend the definitions U^{γ} and $U^{\psi^{-1}}$. Therefore, we incorporate an infinite horizon version of (1.3) into the definition of Epstein–Zin utility. We essentially reverse the procedure: we take the inequalities (1.3) as given and use them to deduce the existence of Epstein–Zin utility. This a priori bound is utilised in answering both existence and uniqueness questions in infinite horizon. In this method, an existence result is obtained when the consumption process satisfies $\mathbb{E}(\int_0^\infty \delta e^{-\delta t} c_t^p \, dt) < \infty$ for $p = 2(1-\gamma), 2(1-\psi^{-1})$ and $2(\psi^{-1}-\gamma)$, where $\delta > 0$ is the agent's discounting rate.

In infinite horizon, the uniqueness question poses another challenge. In the literature on BSDEs with monotone driver and random terminal time, when $\partial_v f(c, V)$ is bounded above by k, one might consider a transversality condition of the type:

$$\lim_{T\to\infty} \mathbb{E}\big(e^{kT}|V_T|\big) = 0, \tag{1.4}$$

which helps to circumvent the absence of a terminal condition (e.g. Appendix C of [20], see also Section 2.6). However, in certain parametrisations where $\frac{1-\gamma}{1-\psi^{-1}} < 0$, (1.4) forces the unique transversal solution to equation (1.2) to be identically zero. Not only is it trivial and not open to any economic interpretation, it also excludes the solution implied by our existence theorem. We overcome this problem by proposing a limit condition in c, named the *uniqueness criterion* (cf. Theorem 2.4.2), under which the solution according to our characterisation is unique. An advantage of our formulation is that existence *and* uniqueness issues can be resolved by studying the consumption process. This condition can be considered a generalisation of (1.4) and overlaps in parts with the result of [47], which is discussed in detail in Section 2.6.

The portfolio and consumption for additive utility in finite time has been studied extensively, starting from Merton's 1971 paper [48], which utilised optimal control theory to study the investment decisions of a rational, utility-maximising investor. A martingale (duality) approach was introduced by by Pliska [52], Cox & Huang [16], Karatzas et al. [31], Karatzas et. al [32], and He & Pearson [29] (see also [40],[41] and [34]). As recursive utility allows one to break the relation $\gamma = \psi^{-1}$ and model the investor's rationale more closely, a natural evolution for the portfolio optimisation problem is to consider investors whose preferences are described by Epstein–Zin utility. In this direction, this problem has been studied by Schroder & Skiadas [55], [56], Chacko & Viceira [13], Kraft et al. [39], Kraft et al. [37] and Xing [66], utilizing stochastic control techniques. This approach often involves a Hamilton-Jacobi-Bellman (HJB) partial differential equation in the Markovian case or a BSDE in the non-Markovian case. Recently, Matoussi & Xing [46] also introduced a duality approach, which bypasses certain technical difficulties from the non-standard HJB equation. In the infinite horizon, the time-additive utility case has been considered by Hata & Sheu [27][28] and Guasoni & Wang [26]. To the best of our knowledge, however, the portfolio optimisation for Epstein–Zin utility in an infinite time horizon remains a gap in the literature, and this thesis will contribute to filling in that gap.

We tackled this problem through the method of stochastic control. In infinite horizon, the HJB equation is an elliptic quasilinear partial differential equation *without* boundary conditions. This absence of boundary data is the principal challenge in approaching this equation. However, in the presence of sub- and super-solution to the HJB equation, a solution can be established which is sandwiched between the sub- and super-solution. This is the main content of our existence theorem. We also provide technical conditions which furnish the required sub- and super-solutions to the HJB equation. In this approach, our method is an extension to recursive utility from the additive utility results in [27] and [26].

Regarding the question of verification, the classic technique in stochastic control is to establish the solution of the HJB equation as an upper bound, then verify the candidate control implied by said solution as a maximiser. This often involves extra regularity conditions. On the finite horizon, examples in the literature include the use of utility gradient by Kraft et. al [36] and BSDE comparison results by Kraft et. al [39] and Xing [66]. On the infinite horizon, verification for additive utilities were achieved by direct calculations in [28] and [26].

In our framework, finding the optimal Epstein–Zin utility process requires two ingredients: firstly, finding a candidate process that satisfies the dynamics (1.1), and secondly, verifying that it satisfies the power utility bounds (1.3). The second ingredient provides above turns out to also verify optimality of the candidate solution amongst a set of *permissible* strategies (cf. Theorem 5.2.2). The convenience of our result is that, *the characterisation of the candidate solution also characterises its optimality*. Moreover, confirming this second ingredient can be achieved by a limit condition on the candidate solution (cf. Lemma 5.2.1).

The thesis is structured as follows. The first half of the thesis concerns with a careful construction and treatment of Epstein–Zin stochastic differential utility. In Chapter 2, we present the characterisation of infinite horizon Epstein–Zin utilities, as well as the main existence and uniqueness result. The newly established results will be demonstrated in two consumption models in Chapter 3. Having established the concept of Epstein–Zin utility in infinite horizon, we address the question of portfolio optimisation in the second half. We formulate the problem and derive the HJB equation in Chapter 4, and solve it in Chapter 5. Some examples and numerical implementation are provided in Chapter 6.

CHAPTER 2

EPSTEIN–ZIN STOCHASTIC DIFFERENTIAL UTILITY IN INFINITE HORIZON: FORMULATION, EXISTENCE AND UNIQUENESS

2.1 Preliminaries & Notations

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in\mathbb{T}}, \mathbb{P})$. Here, $\mathbb{T}$ is a time index set, which is equal to $[0,T]$ for some $T > 0$ in the finite horizon setting, and $[0,\infty)$ in the infinite horizon setting. Moreover, when the time horizon is finite, we assume that the filtration $\{\mathcal{F}\}_{t\in\mathbb{T}}$ satisfies the usual conditions of completeness and right continuity. In the infinite horizon, we assume that for any $T > 0$, the restriction $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions. The usual conditions are standard in finite horizon. In particular, they were assumed in [57] & [66], whose work we build upon. However, $\{\mathcal{F}\}_{t\in\mathbb{T}}$ is not necessarily the augmentation of a filtration generated by a Brownian Motion.

Regarding probabilistic conventions, all adaptedness properties henceforth will be stated in relation to this filtration. For the rest of the thesis, we shall drop the 'almost sure' clarification whenever it is clear from the context. Moreover, when we compare two stochastic processes $\{X_t\}_{t\in\mathbb{T}}$ and $\{Y_t\}_{t\in\mathbb{T}}$, unless otherwise specified, it will always be in the 'pointwise' sense. That is, $X \geq Y$ if for all $t \in \mathbb{T}$, $X_t \geq Y_t$.

A *consumption process* or *consumption plan* in a finite time horizon $[0, T]$ consists of a continuously indexed stream of progressively measurable and positive instantaneous consumption $\{c_t\}_{t \in [0,T)}$ and a positive, $\mathcal{F}_T$-measurable terminal lump sum c_T. On the time horizon $[0, T]$, an agent's preferences over the space of consumption plans can be described by his *stochastic differential utility* (SDU). Let us describe the components necessary for its characterisation. Let $\delta > 0$ be the deterministic discounting rate, $0 < \gamma \neq 1$ the relative risk aversion, and $0 < \psi \neq 1$ the Elasticity of Intertemporal Substitution (EIS). By $(1-\gamma)\mathbb{R}^+$, we mean the positive half line $[0, \infty)$ when $\gamma < 1$ and the negative half line $(-\infty, 0]$ for $\gamma > 1$. It is also customary to denote by $\phi = \psi^{-1}$ the reciprocal of EIS and $\theta = (1-\gamma)/(1-\phi)$. The Epstein–Zin aggregator $f : (0, \infty) \times (1-\gamma)\mathbb{R}^+ \to \mathbb{R}$ is defined by:

$$f(c, v) = \frac{\delta}{1-\phi} c^{1-\phi} [(1-\gamma)v]^{1-\frac{1}{\theta}} - \delta\theta v. \tag{2.1}$$

This is a standard parametrisation, which was used by in [19], [24], [11], amongst others. Henceforth we will reserve the letter f for the Epstein–Zin aggregator, unless specified otherwise. Next, for p satisfying $p > 0$ and $p \neq 1$, let $u_p : \mathrm{dom}(p) \to (1-p)\mathbb{R}^+$ be a CRRA (constant relative risk aversion) utility function, i.e. $u_p(\cdot) = \frac{(\cdot)^{1-p}}{1-p}$, where $\mathrm{dom}(p) = (0, \infty)$ when $p > 1$ and $[0, \infty)$ when $p < 1$. This subsumes the additive utility case, as when $\gamma = \psi^{-1}$, this reduces to the additive utility aggregator of a agent with CRRA utility function with risk aversion γ. The bequest utility function is given by $U(\cdot) = u_\gamma(\cdot)$, i.e. the CRRA utility function with parameter γ.[1]

Definition 2.1.1. Let c be a consumption plan in the finite horizon $[0, T]$. The associated *Epstein–Zin utility process* is defined as the unique solution of the equation below:

$$V_t^c = \mathbb{E}_t\Big(\int_t^T f(c_s, V_s^c)ds + U(c_T)\Big) \quad t \in [0, T]. \tag{2.2}$$

In an infinite horizon, a consumption process is a stream of progressively measurable and positive instantaneous consumption over the positive half-line: $\{c_t\}_{t\geq 0}$. We wish to extend Definition 2.1.1 into the infinite horizon setting. A generalisation thereof can be:

[1]Some authors, e.g. [57] define the bequest utility as $U(x) = u_\gamma(\epsilon x)$, where $\epsilon > 0$ is the weight on terminal consumption. We set $\epsilon = 1$ firstly for the sake of notational simplicity, and secondly in the treatment of infinite horizon case, the value of ϵ is irrelevant. In the finite horizon case, the strategy is the same for $0 < \epsilon \neq 1$.

Pre-Definition 2.1.1. Let c be a consumption plan in the infinite horizon $[0, \infty)$. The associated Epstein–Zin utility process is defined as the unique solution of the equation:

$$V_t^c = \mathbb{E}_t\Big(\int_t^T f(c_s, V_s^c)ds + V_T^c\Big) \quad \text{for all } 0 \le t \le T < \infty. \tag{2.3}$$

This extension will be inadequate for our purpose (hence the phrase 'Pre-Definition'). In the sections that will follow, we will explore the drawbacks and challenges faced by existing works towards solving equation (2.3). After that, another definition will be proposed for Epstein–Zin utility.

Moreover, for the rest of this thesis, without rementioning, we focus on parametrisations of γ and ψ that satisfy either:

$$\gamma\psi, \psi > 1 \quad \text{or} \quad \gamma\psi, \psi < 1, \tag{2.4}$$

which was used in [57]. It overlaps with the context of Theorem 1 in [55], which requires either $\gamma > 1, 0 < \psi < 1$ or $0 < \gamma < 1, \psi > 1$. It also overlaps with the settings of [47], where $\gamma > 1$ and $\gamma \neq \psi^{-1}$. It also covers the empirically relevant configurations $\gamma, \psi > 1$ studied by Bansal & Yaron [3] and Xing [66].

2.2 Existing Results: An Overview

In this section, we explore attempts to resolve the question of existence and uniqueness for Stochastic Differential Utility, and the extent to which they apply to the Epstein–Zin case. The earliest work addressing SDU, although not of Epstein–Zin type, in infinite horizon is by Duffie, Epstein & Skiadas in Appendix C of [20]. Therein, the issue of existence is solved by repeatedly solving the following finite horizon BSDE:

$$V_t^{(T)} = \mathbb{E}_t\Big(\int_t^T f(c_s, V_s^{(T)})ds\Big), \quad t \in [0, T]. \tag{2.5}$$

The solution in infinite horizon is achieved by taking the limit $V_t \triangleq \lim_{T\to\infty} V_t^{(T)}$. Although construction in the upcoming sections will bear some of this flavour, namely the method of localisation, the generalisation to Epstein–Zin case is not straightforward. Firstly, in configurations where $\gamma > 1$, the terminal value of $V^{(T)}$ being nil is equivalent to an infinitely large terminal consumption. Secondly, even if we settled for such an untenably large terminal consumption, this would imply that the truncated processes $V^{(T)}$ are identically zero when $\theta < 0$. A more serious issue is in the regularity assumptions of the aggregator.

Assumption 2.2.1. The following assumptions on the aggregator f were assumed in Appendix C of [20]:

1. f satisfies linear growth in consumption. That is, there exists constants k_1 and k_2 for which $|f(c,0)| \leq k_1 + k_2|c|$.

2. f satisfies the so-called *uniform sector condition* in utility. That is, there exist real constants $-k < -\nu$ such that, for $h \in \mathbb{R}$ and $(c, v) \in (0, \infty) \times \mathbb{R}$:

$$- k \leq \frac{f(c, v+h) - f(c, v)}{h} \leq -\nu. \tag{2.6}$$

The second assumption is violated by the Epstein–Zin aggregator, which involves an a power term with exponent $1 - 1/\theta$ in v. Under the parameter restriction (2.4), this power is either negative or strictly greater than 1, and thus f has unbounded partial derivative in v.

The way in which uniqueness issue was addressed in [20] also faces another challenge. In order to circumvent the absence of terminal data, the following transversality condition was used:

$$\lim_{T\to\infty} \mathbb{E}(e^{-\nu T}|V_T|) = 0, \tag{2.7}$$

where $-\nu$ is an upper bound in equation (2.6). In settings where $\theta < 0$, the value of $-\nu$ is positive, which turns (2.7) into a decay condition, which forces the unique transversal solution to be identically zero. This is discussed at length in Section 2.6.

Another paper, which also dates back to the early 90s, that addresses infinite horizon SDU is that of Duffie & Lions [21]. Our formulation relates to theirs as follows. Suppose that $\{(V_t, Z_t)\}_{t\geq 0}$ satisfies the following dynamics:

$$dV_t = -f(c_t, V_t)dt + Z_t dB_t, \quad t \geq 0. \tag{2.8}$$

If we define $(\mathbb{V}_t, \mathbb{Z}_t) = (V_t^{\frac{1}{\theta}}, (1-\phi)[(1-\gamma)V_t]^{\frac{1}{\theta}-1}Z_t)$ for $t \geq 0$, then $(\mathbb{V}, \mathbb{Z})$ satisfies:

$$d\mathbb{V}_t = -\Big[\delta c_t^{1-\phi} - \delta\mathbb{V}_t + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2}{\mathbb{V}_t}\Big]dt + \mathbb{Z}_t dB_t, \quad t \geq 0. \tag{2.9}$$

In the notation of Duffie & Lions, $u(c) = \delta c^{1-\phi}$ and $\mu = \theta - 1 < 0$. The scope of their results, in several ways, do not encompass our settings. Firstly, they worked with a Brownian filtration and the consumption process is assumed to be a Markovian diffusion. By employing BSDE instead of PDE methods, we can establish Epstein–Zin utilities in infinite horizon without both of these assumptions. Secondly, their uniqueness results require u to be bounded from above (Theorem 8 therein) or below (Theorem 9 therein). In terms of c, this would translate into uniform boundedness conditions, which is unsatisfactory.

We also briefly review existing results in finite horizon concerning existence and uniqueness of Epstein–Zin utility. Schroder & Skiadas [55] and Seiferling & Seifreid [57] studied the existence of Epstein–Zin utility under integrability conditions of the type $\mathbb{E}(\int_0^T c_t^\beta dt + c_T^\beta) < \infty$ for all $\beta \in \mathbb{R}$. Xing [66] also addressed the issue of existence under a more satisfactory integrability condition: $\mathbb{E}(\int_0^T c_t^{1-1/\psi} dt + c_T^{1-\gamma}) < \infty$, which was achieved by focusing on configurations where $\gamma, \psi > 1$. As part of the roadmap towards solving the infinite horizon case, we will also review and refine these finite horizon results in a way suitable for our purpose.

The collection of existing results regarding the infinite horizon case in the literature is sparse, and does not characterise Epstein–Zin utility for an adequately wide class of consumption processes. This presents a gap in the literature, which the content of this chapter aims to fill. The limitations observed in existing results suggest two main difficulties: firstly, the question of characterising a BSDE without terminal data, and secondly, determining an alternative to the transversality condition. In the case of negative θ, the problem is exacerbated by the presence of trivial solutions, which naturally satisfy transversality conditions. These issues can be solved by imposing additional growth structures on solutions of equation (2.3). In the next section, we shall discuss how said structures arise naturally in the finite horizon, which both motivates and justifies our method.

2.3 Motivation From Finite Horizon Settings.

Let us motivate our theory with an informal consideration of discrete time recursive preferences with finitely many periods. Let $\{c_t, t = 0, 1, .., N\}$ be an adapted discrete time consumption process. Then, the recursive preference of Epstein–Zin type (cf. [23]) of a representative agent is given by:

$$\begin{aligned} U_t &= \left((1-\delta)c_t^{1-\phi} + \delta[\mathbb{E}_t(U_{t+1}^{1-\gamma})]^{\frac{1-\phi}{1-\gamma}}\right)^{\frac{1}{1-\phi}}, \quad t = 0, .., N-1, \\ U_T &= c_T . \end{aligned} \tag{2.10}$$

For concreteness of the following example, let us consider the empirically relevant setting, where $\phi < 1 < \gamma$. Assume additionally that $\delta < 1$. In this case, we can express U_t as sub- and super-solution of linear difference equations, as follows:

$$\begin{aligned} U_t^{1-\phi} &= (1-\delta)c_t^{1-\phi} + \delta\mathbb{E}_t(U_{t+1}^{1-\gamma})^{\frac{1-\phi}{1-\gamma}} \\ &\le (1-\delta)c_t^{1-\phi} + \delta\mathbb{E}_t(U_{t+1}^{1-\phi}). \end{aligned} \tag{2.11}$$

$$\begin{aligned} U_t^{1-\phi} &= (1-\delta)(c_t^{1-\gamma})^{\frac{1-\phi}{1-\gamma}} + \delta\mathbb{E}_t[U_{t+1}^{1-\gamma}]^{\frac{1-\phi}{1-\gamma}} \\ &\ge \left[(1-\delta)c_t^{1-\gamma} + \delta\mathbb{E}_t[U_{t+1}^{1-\gamma}]\right]^{\frac{1-\phi}{1-\gamma}}. \end{aligned} \tag{2.12}$$

The inequality of (2.11) follows from conditional Jensen's inequality, and (2.12) follows from Jensen's inequality applied to a two-point discrete distribution with probability masses $1-\delta$ and δ. We observe that the recursion in (2.11) and (2.12) are the difference dynamics of time-additive utilities for an agent with relative risk aversion ϕ and γ, respectively. (2.11) and (2.12) can be solved backwards in time with terminal condition $U_T = c_T$, resulting in the following discrete *Power Utility Bounds*:

$$\mathbb{E}_t\Big[\sum_{i=t}^{N}\delta^{i-t}u_\gamma(c_i)\Big] \le u_\gamma(U_t) \le u_\gamma \circ u_\phi^{-1}\Big(\mathbb{E}_t\Big[\sum_{i=t}^{N}\delta^{i-t}u_\phi(c_i)\Big]\Big) \tag{2.13}$$

This provides a natural bounds for recursive utility, which are expressible in terms of the more tractable additive utilities. This relationship has been extended to continuous time settings in [57]. We summarise their results below.

Theorem 2.3.1. *Suppose that c is a consumption process in a finite horizon* $[0,T]$ *such that* $\mathbb{E}(\int_0^T c_t^\beta dt + c_T^\beta) < \infty$ *for all* $\beta \in \mathbb{R}$*. Then, there exists a unique semimartingale* V^c *that satisfies* $\mathbb{E}(\sup_{t\in[0,T]}|V_t^c|^\beta) < \infty$ *for all* $\beta \in \mathbb{R}$ *that solves equation* (2.2)*. Moreover,* V^c *satisfies:*

$$U^{\gamma\vee\phi}(c) \le V^c \le U^{\gamma\wedge\phi}(c), \tag{2.14}$$

where:

$$
\begin{aligned}
U^{\gamma,T}(c)_t &= \mathbb{E}_t\Big(\int_t^T \delta e^{-\delta s} u_\gamma(c_s) + e^{-\delta(T-t)} u_\gamma(c_T)\Big),\\
U^{\phi,T}(c)_t &= u_\gamma \circ u_\phi^{-1}\left\{\mathbb{E}_t\Big(\int_t^T \delta e^{-\delta s} u_\phi(c_s) + e^{-\delta(T-t)} u_\phi(c_T)\Big)\right\}, \quad t \in [0,T].
\end{aligned} \tag{2.15}
$$

From the discussions above, we make two observations. Firstly, as the Power Utility Bounds hold for both discrete and continuous time in finite horizon, this suggests on a fundamental level the relationship between Epstein–Zin SDU and the classical time-additive utility at different levels of risk aversions. Thus, we expect any reasonable extension of this theory into infinite horizon to obey an analogous relationship. Therefore, we propose to incorporate an infinite horizon version of (2.14) into the definition of Epstein–Zin utility in infinite horizon. This will assist with its characterisation and rule out oddities such as identically zero solutions of (2.3). Secondly, an inspection of the proof of Theorem 2.3.1 [57] suggest that the stipulated integrability conditions are too stringent, and can be improved. As our approach will involve solving finite time approximations, a refinement of Theorem 2.3.1 will allow us to apply our construction to a wide class of consumption plans. These observations will be developed presently in the next section.

2.4 Epstein–Zin Utility in Infinite Horizon: The Main Results

As discussed in previous sections, the crux of our method is to explicitly demand an infinite horizon version of (2.14) as part of the Definition of Epstein–Zin utility. In order to do that, we first need a rigorous definition of our specified space of consumption plans, in both finite and infinite horizons, as well as power utility processes.

Definition 2.4.1. i. Given $T > 0$, we define the space C^T as the space of consumption processes c on the time interval $[0,T]$ that satisfy $\mathbb{E}(\int_0^T c_s^p ds + c_T^p) < \infty$ for *all* of the following values of p: $p = 2(1-\phi), 2(1-\gamma)$ and $p = 2(\phi - \gamma)$.

ii. On the infinite horizon, we define the space C^∞ as the space of infinite horizon consumption processes c that satisfy $\mathbb{E}(\int_0^\infty \delta e^{-\delta t} c_t^p dt) < \infty$ for *all* of the following values of p: $p = 2(1-\phi), 2(1-\gamma)$ and $p = 2(\phi-\gamma)$.

Definition 2.4.2. Let $u_p : (1-p)\mathbb{R}^+ \to \mathbb{R}$ be the CRRA utility function with relative risk aversion parameter p. For every $c \in C^T$, where $T > 0$ is finite, we define the additive utility process as follows:

$$Y^{p,T}(c)_t = \mathbb{E}_t\Big[\int_t^T \delta e^{-\delta(s-t)} u_p(c_t) + e^{-\delta(T-t)} u_\gamma(c_T)\Big], \quad t \in [0,T]. \tag{2.16}$$

For each $c \in C^\infty$, its infinite horizon version is defined by:

$$Y^{p,\infty}(c)_t = \mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta s} u_p(c_s) ds\Big], \quad t \geq 0. \tag{2.17}$$

Moreover, in both cases where $T < \infty$ and $T = \infty$, define:

$$U^{\gamma,T}(c) = Y^{\gamma,T}(c) \quad \text{and} \quad U^{\phi,T}(c) = u_\gamma \circ u_\phi^{-1}[Y^{\phi,T}(c)]. \tag{2.18}$$

The processes $U^{p,T}$ and $Y^{p,T}$ for $p = \gamma$ and ϕ, including the case $T = \infty$, will henceforth be referred to collectively as *Power Utility Processes*.

Before we are ready to state our definition of Epstein–Zin utility in infinite horizon, we need one more element: the space of potential solutions, which is defined below.

Definition 2.4.3. Given finite $T > 0$, we define $\mathcal{V}^T$ as the space of semimartingales $\{V_t\}_{t\in[0,T]}$ such that $\mathbb{E}(\int_0^T |V_t|^{2(1-1/\theta)} dt + \sup_{[0,T]} |V_t|^2) < \infty$. On the infinite horizon, define $\mathcal{V}^\infty$ as the space of semimartingales $\{V_t\}_{t\geq 0}$ such that the for all finite $T > 0$, the restriction $\{V_t\}_{t\in[0,T]}$ belongs to $\mathcal{V}^T$.

The integrability conditions in Definition 2.4.1 for $p = 2(1-\gamma)$ and $2(1-\phi)$ ensure that for any c within C^T, $U^{\gamma,T}(c)$ and $U^{\phi,T}(c)$, including the $T = \infty$ case, are well-defined. In fact, it establishes them as square-integrable semi-martingales. The last assumption for $p = 2(\phi - \gamma)$ is a technical assumption used to ensure that they are confined within the correct solution space (cf. Lemma 2.7.3). The solution space $\mathcal{V}^T$, loosely speaking, is designed to ensure that our power utility processes, and by extension, candidate Epstein–Zin utility processes, will possess certain desirable integrability conditions, which are sufficient for limit-based arguments to go through .

We will now present our proposed definition of Epstein–Zin utility in infinite horizon.

Definition 2.4.4. Let c be an infinite horizon consumption plan in C^∞, a semi-martingale $V = V^c \in \mathcal{V}^\infty$ is said to be a *value process*, or *Epstein–Zin utility process* associated with c if it satisfies the BSDE:

$$V_t^c = \mathbb{E}_t\Big[V_T^c + \int_t^T f(c_s, V_s^c)ds\Big] \quad \text{a.s. for all } 0 \le t \le T < \infty, \tag{2.19}$$

and the power utility bounds:

$$U^{\phi\vee\gamma,\infty}(c)_t \le V_t^c \le U^{\phi\wedge\gamma,\infty}(c)_t \quad \text{a.s. for all } t \ge 0. \tag{2.20}$$

We will devote the rest of this chapter to the development of an infinite horizon Epstein–Zin utility process, the culmination of which is the following theorems concerning its existence and uniqueness. The development of Theorem 2.4.1 will be presented in Section 2.5, and a discussion and comparison between Theorem 2.4.2 and the transversality condition (2.7) will be included in Section 2.6.

Theorem 2.4.1. *Every consumption plan c in C^∞ has an Epstein–Zin utility process associated with it.*

Theorem 2.4.2. *Let c be a C^∞ consumption plan. Then:*

i. If $\theta \in (0,1)$, then its Epstein–Zin utility process is unique.

ii. If $\theta < 0$, then uniqueness holds under the additional uniqueness criterion:

$$\lim_{T\to\infty} \mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U^{\phi,\infty}(c)_s)ds\Big)|U^{\gamma,\infty}(c)_T|\Big) = 0. \tag{2.21}$$

Remark 2.4.5. The limit condition (2.21) can be equivalently expressed as:

$$\lim_{T\to\infty} \mathbb{E}\Big(e^{\delta(1-\theta)\int_0^T(1-\Phi_s)ds}\int_T^\infty \delta e^{-\delta s} c_s^{1-\gamma} ds\Big) = 0, \tag{2.22}$$

where $\Phi_t = \frac{c_t^{1-\phi}}{\mathbb{E}_t(\int_t^\infty \delta e^{-\delta(s-t)} c_s^{1-\phi} ds)}$. As this is somewhat more explicit, we shall henceforth refer to (2.22) as the *uniqueness criterion*.

Having settled the existence and uniqueness issues, we state below the few basic properties of Epstein–Zin utility. These properties have been established in, for instance, [57]. We extend them to our relaxed setting in finite horizon, as well as the infinite horizon setting.

Proposition 2.4.3. *Let c and $\tilde{c}$ be consumption plans in C^T. If $T = \infty$ and $\theta < 0$, then assume additionally that they satisfy the criterion (2.22). Then, the following holds:*

i. (Homotheticity) For every $\lambda > 0$, λc belongs to the class C^T and, in the case $T = \infty$ and $\theta < 0$, satisfies the criterion (2.21). Moreover, $V^{\lambda c} = \lambda^{1-\gamma} V^c$.

ii. (Monotonicity) If $\tilde{c} \geq c$, then $V^{\tilde{c}} \geq V^c$.

iii. (Concavity) If $\alpha \in (0,1)$ and $\alpha c + (1-\alpha)\tilde{c}$ belongs to C^T and satisfies (2.22) when $T = \infty$ and $\theta < 0$, then $V^{\alpha c + (1-\alpha)\tilde{c}} \geq \alpha V^c + (1-\alpha) V^{\tilde{c}}$.

2.5 Existence of Epstein–Zin utility in Infinite Horizon.

In this section, we present our development of Theorem 2.4.1. This is achieved by a localised construction. Given a consumption plan $c \in C^\infty$, we will define its local truncations in C^T for finite T. The associated truncated value processes then can be shown to converge pointwise monotonically to a process in $\mathcal{V}^\infty$. We then verify that this limiting process is the desired solution in the sense of Definition 2.4.4. We will make extensive use of the following result, which is a generalisation of Theorem 2.3.1:

Proposition 2.5.1. *For any consumption process $c \in C^T$, there exists a unique value process $V^c \in \mathcal{V}^T$ that satisfies:*

$$V_t^c = \mathbb{E}_t\Big(u_\gamma(c_T) + \int_t^T f(c_s, V_s^c) ds\Big) \quad a.s \text{ for } 0 \leq t \leq T, \tag{2.23}$$

Moreover, this value process satisfies the power utility bounds (2.14).

We explain the heuristics of localising consumption plans here. Given $c \in C^\infty$, equation (2.19) can be expressed as: $V_t = \mathbb{E}[u_\gamma \circ u_\gamma^{-1}(V_T) + \int_t^T f(c_s, V_s) ds]$. From a finite horizon point of view, the representative agent consumes at rate c_t up until time T, where he decides to terminate continuous consumption and consume the certainty equivalent of the 'look-ahead' value of the remaining utility. Since this remaining Epstein–Zin utility is, of course, unavailable, we shall approximate it with remaining power utilities. The formal definition is given below:

Definition 2.5.1. Let c be a consumption plan in C^∞. Its time T upper-truncation $c^{(T)}$ is defined by:

$$c_t^{(T)} \triangleq \begin{cases} c_t, & t \in [0, T) \\ u_\gamma^{-1}(U_T^{\phi \wedge \gamma, \infty}), & t = T. \end{cases} \tag{2.24}$$

We also define the lower-truncation $\tilde{c}_t^{(T)}$ with the same instantaneous stream on $[0,T)$, but terminal consumption $\tilde{c}_T^{(T)} = u_\gamma^{-1}(U_T^{\phi\vee\gamma,\infty})$.

Due to the power utility bounds (2.20), we expect $c_T^{(T)}$ and $\tilde{c}_T^{(T)}$ to represent the maximal and minimal look-ahead values, respectively. Therefore, the time T truncations will provide upper and lower bounds for the solution we will construct. Because these upper and lower truncations hold for all T, we will take limit in T later. We collect some of their desirable properties in Proposition 2.5.2 below. The first point states that the truncations $c^{(T)}$ and $\tilde{c}^{(T)}$ are consistent with our setting in finite horizon, which allows us to freely define the localised Epstein–Zin utility processes. The second and third points establish the integrability properties and ordering of the power utility processes.

Proposition 2.5.2. *Let c be a consumption plan in C^∞. For each $T > 0$, let $c^{(T)}$ and $\tilde{c}^{(T)}$ be the time T upper and lower truncation of c, respectively. Let $U^{\gamma,\infty} = U^{\gamma,\infty}(c)$ and $U^{\phi,\infty} = U^{\phi,\infty}(c)$ be power utility processes associated with c (cf. Definition 2.4.2). Then:*

i. *$c^{(T)}$ and $\tilde{c}^{(T)}$ belong to the class C^T.*

ii. *For any $T > 0$, the restrictions $\{U_t^{\gamma\wedge\phi,\infty}\}_{t\in[0,T]}$ and $\{U_t^{\gamma\vee\phi,\infty}\}_{t\in[0,T]}$ are equal to $U^{\gamma\wedge\phi,T}(c^{(T)})$ and $U^{\gamma\vee\phi,T}(\tilde{c}^{(T)})$, respectively. Therefore, the $\{U_t^{\gamma,\infty}\}_{t\in[0,T]}$ $\{U_t^{\phi,\infty}\}_{t\in[0,T]}$ belong to the semimartingale class $\mathcal{V}^T$ and $U^{\gamma,\infty}$ and $U^{\phi,\infty}$ belong to the semimartingale class $\mathcal{V}^\infty$.*

iii. *For any $t \geq 0$, $U_t^{\phi\vee\gamma,\infty} \leq U_t^{\phi\wedge\gamma,\infty}$ almost surely.*

Having established $c^{(T)}$ and $\tilde{c}^{(T)}$ as C^T consumption plans, we can apply Proposition 2.5.1 and define their associated Epstein–Zin utilities via the following BSDEs:

$$
\begin{aligned}
V_t^{(T)} &= \mathbb{E}_t\Big(U_T^{\phi\wedge\gamma,\infty} + \int_t^T f(c_s, V_s^{(T)})ds\Big) \quad t \in [0,T], \\
\tilde{V}_t^{(T)} &= \mathbb{E}_t\Big(U_T^{\phi\vee\gamma,\infty} + \int_t^T f(c_s, \tilde{V}_s^{(T)})ds\Big) \quad t \in [0,T].
\end{aligned}
\tag{2.25}
$$

We wish to take the limit of $V^{(T)}$ and $\tilde{V}^{(T)}$ as T diverges. The next proposition shows that these truncations are actually monotone in T, which allows for a simple and convenient way to achieve the desired convergence.

Proposition 2.5.3. *Let $V^{(T)}$ and $\tilde{V}^{(T)}$ be defined by equation (2.25). Then, they satisfy the following infinite horizon version of the power utility bounds:*

$$U_t^{\phi\vee\gamma,\infty} \leq \tilde{V}_t^{(T)} \leq V_t^{(T)} \leq U_t^{\phi\wedge\gamma,\infty}, \quad t \in [0,T]. \tag{2.26}$$

In particular, the upper and lower bound processes are independent of T. Moreover, for $t \in [0,T]$ and $S > T$, $V_t^{(S)} \leq V_t^{(T)}$ and $\tilde{V}_t^{(S)} \geq \tilde{V}_t^{(T)}$ almost surely.

We conclude this section with a result concerning taking limit of the localised value processes. The following result ensures that the limit processes belong to the correct solution space and solve the targeted BSDE. Moreover, they also form a natural bound for any potential Epstein–Zin utility process. With this result, we attain Theorem 2.4.1.

Proposition 2.5.4. *The limits $V_t = \lim_{T\to\infty} V_t^{(T)}$ and $\tilde{V}_t = \lim_{T\to\infty} \tilde{V}_t^{(T)}$ are well-defined and belong to the class $\mathcal{V}^\infty$. Moreover, they are Epstein–Zin utility processes associated with c, in the sense of Definition (2.4.4). Moreover, if v is another Epstein–Zin utility process, then:*

$$\tilde{V} \leq v \leq V. \tag{2.27}$$

2.5.1 The Case of Brownian Filtration.

In the problem of portfolio optimisation of an agent with Epstein–Zin preference, most popular market models are driven by a Brownian Motion (see, for instance, [39], [66],[37] and the applications therein). In a Brownian filtration, the standard formulation for a BSDE is of the form (El Karoui et al. has a survey paper [22]):

$$V_t = V_T + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s dB_s, \quad t \leq T. \tag{2.28}$$

The existence of the Z component in (2.28) is an application of the Martingale Representation Theorem in a Brownian setting (Theorem 3.4.15 [33]). This result is made straightforward by the fact that the aggregator f is independent of Z, and thus no continuity property of f is required the Martingale Representation Theorem. A construction of Z, however, is not generally available.

Theorem 2.5.5. *Suppose that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of a filtration generated by an $\mathbb{R}^d$-valued Brownian Motion. Let c be a consumption plan in C^∞ and $V \in \mathcal{V}^\infty$ be an Epstein–Zin utility process associated with it. Then, there exists an $\mathbb{R}^d$-valued progressively measurable process*

$\{Z_t\}_{t\geq 0}$ *such that for all* $T > 0$, $\int_0^T Z_s^2 ds < \infty$ *almost surely and:*

$$V_t = V_T + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s dB_s, \quad t \leq T. \tag{2.29}$$

2.6 Uniqueness of Epstein–Zin Utility in Infinite Horizon.

This proof of Theorem 2.4.2 is relatively straightforward, and will be deferred to Section 2.8.6. In this section, we shall heuristically describe its motivation behind our uniqueness result, and the way in which it refines the transversality condition in [20]. Since the partial derivative $\partial_v f(c, v)$ is uniformly bounded from above by[2] $-\delta\theta$, one might consider imposing the condition:

$$\lim_{T\to\infty} \mathbb{E}\Big(e^{-\delta\theta T}|V_T|\Big) = 0. \tag{2.30}$$

This is manifestly unnecessary when $\theta \in (0, 1)$, where uniqueness of the solution holds without extra assumptions other than those of C^∞ (see Theorem 2.4.2.i.). Moreover, this transversality condition can be naturally thought of as a growth condition on the value process. However, when $\theta < 0$, this is a decay condition, which requires the value process to vanish exponentially fast, which excludes even trivial cases such as constant consumption[3]. In fact, it excludes all cases of interest, as the result below shows:

Proposition 2.6.1. *Suppose that* $\theta < 0$. *Let* c *be a* C^∞ *consumption plan and suppose that* $V \in \mathcal{V}^\infty$ *satisfies the following BSDE for all positive constants* $t < T$:

$$V_t = \mathbb{E}_t\Big(V_T + \int_t^T f(c_s, V_s)ds\Big). \tag{2.31}$$

If V *satisfies the transversality condition* (2.30)*, then it is identically zero.*

Therefore, solutions satisfying (2.30) are not open to economic interpretation and exclude all processes sandwiched between $U^{\gamma,\infty}$ and $U^{\phi,\infty}$. Upon close inspection of the proof, uniqueness can

[2]See discussion following Definition 2.7.2b

[3]For example, if $c_t \equiv c > 0$, then one can verify directly that $V_t^c \equiv u_\gamma(c)$ is the associated Epstein–Zin utility process. In this case, $U^{\phi,\infty}(c)$ and $U^{\gamma,\infty}(c)$ are both constant and equal to $u_\gamma(c)$, implying that V^c is the unique solution according to Definition 2.4.4

be achieved by constructing a process α that simultaneously satisfies:

$$\alpha_t(V_t - \tilde{V}_t) \geq f(c_t, V_t) - f(c_t, \tilde{V}_t), \tag{2.32a}$$

$$\exp\Big(\int_0^T \alpha_s ds\Big)|V_T| \xrightarrow{L^1} 0. \tag{2.32b}$$

(2.32a) requires α to be sufficiently large so as to achieve an upper bound on $\partial_v f$ (cf. equation (2.75)). However, if α is too large, $\exp(\int_0^T \alpha_s ds)|V_T|$ will not converge, as required in (2.32b). The choice $\alpha \equiv -\delta\theta$ is an example that satisfies the first, but not the second requirement. The uniform, deterministic upper bound is not sharp enough. Therefore, we can think of the choice $\alpha = \partial_v f(c, U^{\phi,\infty}(c))$ as a dynamic generalisation of the transversality condition (2.30).

We will now discuss, in the case $\theta < 0$, a class of consumption processes for which uniqueness is achieved. Let us recall the uniqueness criterion

$$\lim_{T\to\infty} \mathbb{E}\Big(e^{\delta(1-\theta)\int_0^T (1-\Phi_s)ds} \int_T^\infty \delta e^{-\delta s} c_s^{1-\gamma} ds\Big) = 0, \tag{2.33}$$

where $\Phi_t = \frac{c_t^{1-\phi}}{\mathbb{E}_t(\int_t^\infty \delta e^{-\delta(s-t)} c_s^{1-\phi} ds)}$. Intuitively, the decay of (2.33) is the result of two competing forces: the fluctuation of $e^{\delta(1-\theta)\int_0^T (1-\Phi_s ds) ds}$ and the decay rate of $\int_T^\infty \delta e^{-\delta s} c_s^{1-\gamma} ds$. Therefore, uniqueness is attained if the first factor is bounded from above. One such situation is when $c_t^{1-\phi}$ is a super-martingale, as in this case:

$$\Phi_t = \frac{c_t^{1-\phi}}{\int_t^\infty \delta e^{-\delta(s-t)} \mathbb{E}_t(c_s^{1-\phi}) ds} \geq \frac{c_t^{1-\phi}}{\int_t^\infty \delta e^{-\delta(s-t)} c_t^{1-\phi} ds} = 1. \tag{2.34}$$

We draw comparison with the work in infinite horizon of Melnyk et. al [47], who, in the case where $\phi < \gamma$, restricts their studies to consumption processes that satisfy (Definition 3.1 therein):

$$V_t^c \leq u_\gamma(c_t) \quad \text{for all } t \geq 0. \tag{2.35}$$

In the empirically relevant case $\phi < 1 < \gamma$, the intuition behind conditions (2.34) and (2.35) are similar: uniqueness holds when future consumption is not 'too good' in comparison to current consumption. In fact, our method of acquiring uniqueness encompasses theirs in the following way. If we restrict our studies to consumption plans and Epstein–Zin processes satisfying (2.35), then, by using $u_\gamma(c_t)$ as an upper bound instead of $U^{\phi,\infty}(c)$ as an upper bound, we could have selected $\alpha_t = \partial_v f(c_s, u_\gamma(c_s))$ in

(2.32a). In this case, the (2.32b) reduces to:

$$e^{-\delta T}|V_T| \xrightarrow{L^1} 0, \tag{2.36}$$

which is their transversality condition.

2.7 Proofs of Finite Horizon Results.

2.7.1 Preliminary: Monotonicity of BSDEs and a Comparison Principle.

Below, we will recall a few mathematical devices that will come in handy in our proofs. The fundamental concept that underlies our theory is a sub- and super-solution for a backward stochastic differential equation with monotone generator.

Definition 2.7.1. Let $g : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ be $\mathcal{G}\otimes\mathcal{B}(\mathbb{R})$-measurable, where $\mathcal{G}$ denotes the progressive σ-field. Moreover let $\xi \in L^1(\mathbb{P})$ and suppose X is a semimartingale with $\mathbb{E}[\int_0^T |g(t,X_t)|dt + \sup_{t\in[0,T]}|X_t|] < \infty$. X is called a *subsolution* (resp. *supersolution*) of the $\mathtt{BSDE}(g,\xi)$ if:

$$dX_t = -g(t,X_t)dt + dM_t - dA_t, \quad X_T \leq \xi \text{ (resp. } X_T \geq \xi).$$

where M is a martingale and A a decreasing (resp. increasing) right-continuous process such that $A_0 = 0$. Moreover, X is a solution of $\mathtt{BSDE}(g,\xi)$ if it is both a subsolution and a supersolution.

Definition 2.7.2. *(Monotonicity).* Let $g : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ be $\mathcal{G}\otimes\mathcal{B}(\mathbb{R})$-measurable. g is said to satisfy the *monotonicity condition* if there exists a constant k such that for $d\mathbb{P}\otimes dt$-almost all $(\omega,t)\in\Omega\times[0,T]$:

$$g(\omega,t,x) - g(\omega,t,y) \leq k(x-y) \quad \text{for all } \omega\in\Omega \text{ and } x,y\in\mathbb{R} \text{ with } x\geq y \tag{2.37}$$

The constant k is referred to as the *constant of monotonicity* of g.

We consider the partial derivative in v of the aggregator: $\partial_v f(c,v) = \delta c^{1-\phi}(\theta-1)[(1-\gamma)v]^{-\frac{1}{\theta}} - \delta\theta$. Moreover, the restrictions (2.4) on γ and ψ can be categorised further into:

$$\phi < \gamma < 1, \Rightarrow 0 < \theta < 1; \qquad 1 < \gamma < \phi, \Rightarrow 0 < \theta < 1$$

$$\phi < 1 < \gamma, \Rightarrow \theta < 0; \qquad \gamma < 1 < \phi, \Rightarrow \theta < 0.$$

In all cases, $\theta < 1$ and $\partial_v f(c,v) \leq -\delta\theta$. Therefore, the Epstein–Zin aggregator is monotone with constant of monotonicity $-\delta\theta$.

The monotonicity condition can be thought of as a weaker version of the Lipschitz condition: the difference quotient is bounded above but not below. This condition has been utilised extensively in the BSDE literature, for instance, in [51], [8], [50] and [54]. It should be noted that, in the context of random terminal time, these authors formulated the monotonicity condition with negative values of k. Such a condition is violated by the Epstein–Zin aggregator when $\theta < 0$. Its effects on the uniqueness issues in infinite horizon is discussed Section 2.6.

We now state a comparison theorem, courtesy of Seiferling & Seifried [57].

Theorem 2.7.1. *Suppose that X is a sub-solution of $BSDE(g,\xi)$ and Y a super-solution of $BSDE(h,\eta)$, in the sense of definition 2.7.1. Suppose also that $\xi \leq \eta$. Then, for each $t \geq 0$, $X_t \leq Y_t$ almost surely if either of the following holds:*

i. *$g(\omega,t,Y_t) \leq f(\omega,t,Y_t)$ for $d\mathbb{P}\otimes dt$-almost all $(\omega,t) \in \Omega\otimes[0,T]$ and g satisfies the monotonicity condition; or*

ii. *$g(\omega,t,X_t) \leq f(\omega,t,X_t)$ for $d\mathbb{P}\otimes dt$-almost all $(\omega,t) \in \Omega\otimes[0,T]$ and f satisfies the monotonicity condition.*

An immediate corollary of Theorem 2.7.1 is the following monotonicity result of the Epstein–Zin value process. Intuitively, it states what one would expect: a dominating level of consumption corresponds to an accordingly dominating level of utility.

Corollary 2.7.2. *Let $c^{(i)}, i = 1,2$ be consumption processes in C^T. Suppose that $V^{(i)}, i = 1,2$ belong to $\mathcal{V}^T$ and satisfy:*

$$V_t^{(i)} = \mathbb{E}_t\Big(u_\gamma(c_t^{(i)}) + \int_t^T f(c_s^{(i)}, V_s^{(i)})ds\Big), \quad t \in [0,T]. \tag{2.38}$$

If $c_t^{(1)} \leq c_t^{(2)}$ $d\mathbb{P} \otimes dt$- almost everywhere, then $V_t^{(1)} \leq V_t^{(2)}$ almost surely for all t. As a consequence, if $c \in \mathcal{C}^T$, then its associated value process is unique in $\mathcal{V}^T$.

Proof. Denote $f^{(i)}(t,\cdot) = f(c_t^{(i)},\cdot)$. We assert that $V^{(i)}$ is a solution of $\mathtt{BSDE}(f^{(i)}, u_\gamma(c_T^{(i)}))$, in the sense of definition 2.7.1. Indeed, $\sup_{t\in[0,T]} |V_t^{(i)}|$ is square integrable, thanks to its membership in $\mathcal{V}^T$. Moreover, by Hölder's inequality:

$$\mathbb{E}\Big(\int_0^T |f^{(i)}(s,V_s)|ds\Big) \leq K\mathbb{E}\Big(\int_0^T \big[(c_s^{(i)})^{2(1-\phi)} + |V_s^{(i)}|^{2(1-1/\theta)} + |V_s^{(i)}|\big]ds\Big), \tag{2.39}$$

where K is a constant depending on δ, T, γ and ϕ.

By considering the first partial derivatives of the aggregator, $\frac{\partial}{\partial c} f(c,v) = \delta c^{-\phi}[(1-\gamma)v]^{1-1/\theta} \geq 0$ and $\frac{\partial}{\partial v} f(c,v) = \delta(\theta-1)[(1-\gamma)v]^{-\frac{1}{\theta}} - \delta\theta \leq -\delta\theta$ (Note that in all considered configurations, $\theta < 1$). Therefore f is increasing in c and satisfies the monotone condition with constant $-\delta\theta$. We can therefore apply Theorem 2.7.1 to obtain the conclusion of this corollary.

□

2.7.2 Proofs of Proposition 2.5.1.

Let us prove our result concerning existence and uniqueness of Epstein–Zin utility in finite time. We state below a lemma concerning the integrability of power utility processes that will be useful in the main result. Its proof will be deferred until after that of the main Proposition.

Lemma 2.7.3. *Given finite $T > 0$ and a consumption process $c \in \mathcal{C}^T$, the power utility processes $U^{\gamma,T}(c)$ and $U^{\phi,T}(c)$ belong to $\mathcal{V}^T$.*

Proof of Theorem 2.5.1. The uniqueness of Epstein–Zin utility is resolved in Corollary 2.7.2. Regarding its existence, we divide the proof into three parts with increasing levels of generality.

Part I. c is bounded above and away from 0. This case falls within the scope of Theorem 2.3.1, and the conclusion is immediate.

Part II. c is bounded away from 0. Now, suppose $c \in \mathcal{C}^T$ is a consumption process that is bounded away from 0 but not above, then $c^{(n)} = c \wedge n$ is a pointwise increasing sequence of bounded consumption processes that converges upwards to c. Moreover, thanks to Part I, there exists a unique $V^{(n)} \in \mathcal{V}^T$ such that:

$$V^{(n)} = \mathbb{E}_t\Big(u_\gamma(c_T^{(n)}) + \int_t^T f(c_s^{(n)}, V_s^{(n)})ds\Big) \quad t \in [0,T]. \tag{2.40}$$

Moreover, by Theorem 2.5.1 for consumption processes bounded above and away from zero, the following inequality holds almost surely for all $t \in [0,T]$:

$$U^{\phi\vee\gamma}(c^{(1)})_t \le U^{\phi\vee\gamma}(c^{(n)})_t \le V_t^{(n)} \le U^{\phi\wedge\gamma}(c^{(n)})_t \le U^{\phi\wedge\gamma}(c)_t. \tag{2.41}$$

By Corollary 2.7.2, $n \to V_t^{(n)}$ is monotone increasing, and the inequality (2.41) implies that the limit is finite. Therefore we can define $V_t^c = \lim_{n\to\infty} V_t^{(n)}$. A consequence of (2.41) is that $V \in \mathcal{V}^T$, By Lemma 2.7.3. We wish to take pass the limit through the expectation in (2.40). As $|u_\gamma(c_T^{(n)})| \le |u_\gamma(c_T^{(1)})| + |u_\gamma(c_T)|$, the first term follows readily from conditional dominated convergence theorem (DCT). The second term follows from conditional DCT, too, where the dominating random variable is provided by:

$$f(c_s^{(n)}, V_s^{(n)}) \le K\Big(c_s^{2(1-\phi)} + \big|U^{\phi\vee\gamma}(c^1)_t\big|^{2(1-1/\theta)} + \big|U^{\phi\wedge\gamma}(c)_t\big|^{2(1-1/\theta)} + \big|U^{\phi\vee\gamma}(c^1)_t\big| + \big|U^{\phi\wedge\gamma}(c)_t\big|\Big). \tag{2.42}$$

Denote by U_t the process on the right hand side above. By membership of c in C^T and Lemma 2.7.3, we have $\mathbb{E}(\int_t^T U_s ds) < \infty$, and thus $\int_t^T U_s ds < \infty$ almost surely. This justifies the following application of DCT on a set of probability 1:

$$\lim_{n\to\infty} \int_t^T f(c_s^{(n)}, V_s^{(n)})ds = \int_t^T \lim_{n\to\infty} f(c_s^{(n)}, V_s^{(n)})ds = \int_t^T f(c_s, V_s)ds. \tag{2.43}$$

By conditional DCT, with $\int_t^T U_s ds$ as the dominating random variable, we have:

$$\lim_{n\to\infty} \mathbb{E}_t\Big(\int_t^T f(c_s^{(n)}, V_s^{(n)})ds\Big) = \mathbb{E}_t\Big(\lim_{n\to\infty} \int_t^T f(c_s^{(n)}, V_s^{(n)})ds\Big). \tag{2.44}$$

Combinining equations (2.40), (2.43) and (2.44), we see that V^c satisfies the limit BSDE:

$$V_t^c = \mathbb{E}_t\Big(u_\gamma(c_T) + \int_t^T f(c_s, V_s^c)ds\Big). \tag{2.45}$$

Lastly, by conditional monotone convergence, we can take the limit as $n \to \infty$ in (2.41) to obtain the power utility bounds (2.14).

Part III. c is any process in C^T. If c is an arbitrary member of process in C^T, then $c^{(n)} = c \vee (1/n)$ is a pointwise decreasing sequence of consumption processes, each of which is bounded away from zero. From Part II, each $c^{(n)}$ has a unique associated utility process $V^{(n)}$ in $\mathcal{V}^T$, which, by corollary 2.7.2, is monotone decreasing in n. Moreover, for $t \in [0, T]$:

$$U^{\phi\vee\gamma}(c)_t \leq U^{\phi\vee\gamma}(c^{(n)})_t \leq V_t^{(n)} \leq U^{\phi\wedge\gamma}(c^n)_t \leq U^{\phi\wedge\gamma}(c^1)_t \tag{2.46}$$

Again, by monotonicity of the value process, we can define $V_t^c = \lim_{n\to\infty} V_t^{(n)}$, which thanks to the power utility bounds (2.46) belongs to $\mathcal{V}^T$. Similar to Part II, we want to pass to the limit in:

$$V_t^{(n)} = \mathbb{E}_t\Big(u_\gamma(c_T^n) + \int_t^T f(c_s^n, V_s^{(n)})ds\Big). \tag{2.47}$$

The argument for exchanging limit and conditional expectation is exactly the same as in Part II. Therefore, V^c is the utility process associated to c. To obtain the power utility bounds, let $n \to \infty$ in (2.46), which is justified by conditional dominated convergence.

□

Proof of Lemma 2.7.3. For simplicity of notation, we drop the T superscript and c argument from the power utility processes. In our calculations, we will use K to denote a generic constant that might change from line to line.

Part 1. $U^\gamma \in \mathcal{V}^T$. Under the restrictions (2.4), we have either $1 - 1/\theta < 0$ when $\theta \in (0, 1)$ or $1 - 1/\theta < 0$ when $\theta < 0$. In both cases, the mapping $x \to x^{2(1-1/\theta)}$ is convex for $x > 0$.

$$\begin{aligned}
|U_t^\gamma|^{2(1-1/\theta)} &= K\Big[\mathbb{E}_t\Big(\int_t^T \delta e^{-\delta(s-t)} c_s^{1-\gamma} ds + e^{-\delta(T-t)} c_T^{1-\gamma}\Big)\Big]^{2(1-1/\theta)} \\
&\leq K\mathbb{E}_t\Big[\Big(\int_t^T \delta e^{-\delta(s-t)} c_s^{1-\gamma} ds + e^{-\delta(T-t)} c_T^{1-\gamma}\Big)^{2(1-1/\theta)}\Big] \\
&\leq K\mathbb{E}_t\Big(\int_t^T \delta e^{-\delta(s-t)} c_s^{2(\phi-\gamma)} ds + e^{-\delta(T-t)} c_T^{2(\phi-\gamma)}\Big).
\end{aligned} \tag{2.48}$$

Above, the second inequality follows from the fact that $\delta e^{-\delta(s-t)}ds + e^{-\delta(T-t)}\delta_T$, where δ_T is the Dirac delta at point T, is a probability measure on $[t, T]$. Thus, we have:

$$\begin{aligned}\mathbb{E}\Big(\int_0^T |U_t^\gamma|^{2(1-1/\theta)}dt\Big) &\leq K\int_0^T \mathbb{E}\Big[\int_0^T c_s^{2(\phi-\gamma)}ds + c_T^{2(\phi-\gamma)}\Big]dt \\ &\leq KT\Big(\mathbb{E}\Big[\int_0^T c_s^{2(\phi-\gamma)}ds\Big] + \mathbb{E}(c_T^{2(\phi-\gamma)})\Big) < \infty.\end{aligned} \tag{2.49}$$

In order to prove that $\sup_{t\in[0,T]}|U_t^\gamma|$ is square integrable, we first observe that:

$$\begin{aligned}(1-\gamma)U_t^\gamma &= \mathbb{E}_t\Big[\int_t^T \delta e^{-\delta(s-t)}c_s^{1-\gamma}ds + e^{-\delta(T-t)}c_T^{1-\gamma}\Big] \\ &\leq \mathbb{E}_t\Big[\int_0^T \delta e^{-\delta(s-t)}c_s^{1-\gamma}ds + e^{-\delta(T-t)}c_T^{1-\gamma}\Big].\end{aligned} \tag{2.50}$$

As $\mathbb{E}([\int_0^T \delta e^{-\delta(s-t)}c_s^{1-\gamma}ds + e^{-\delta(T-t)}c_T^{1-\gamma}]^2) \leq \mathbb{E}(\int_0^T \delta e^{-\delta(s-t)}c_s^{2(1-\gamma)}ds + e^{-\delta(T-t)}c_T^{2(1-\gamma)}) < \infty$, the stochastic process $M_t \triangleq \mathbb{E}_t[\int_0^T \delta e^{-\delta(s-t)}c_s^{1-\gamma}ds + e^{-\delta(T-t)}c_T^{1-\gamma}]$ is a square-integrable martingale. By Doob's maximal inequality, $\mathbb{E}(\sup_{t\in[0,T]}|M_t|^2) < \infty$, and consequently, $\mathbb{E}[\sup_{t\in[0,T]}|U_t^\gamma|^2] < \infty$.

Part 2. $U^\phi \in \mathcal{V}^T$. This part's calculations are similar the Part I. Here, we utilise the convexity of the mapping $x \longrightarrow x^{2(\theta-1)}$.

$$\begin{aligned}|U_t^\phi|^{2(1-1/\theta)} &= K\mathbb{E}_t\Big[\int_t^T \delta e^{-\delta(s-t)}c_s^{1-\phi}ds + e^{-\delta(T-t)}c_T^{1-\phi}\Big]^{2(\theta-1)} \\ &\leq K\mathbb{E}_t\Big[\int_t^T \delta e^{-\delta(s-t)}c_s^{2(\phi-\gamma)}ds + e^{-\delta(T-t)}c_T^{2(\phi-\gamma)}\Big].\end{aligned} \tag{2.51}$$

Thus, we obtain the following estimate for the expectation of the time integral of $|U^\phi|$:

$$\begin{aligned}\mathbb{E}\Big(\int_0^T |U_t^\phi|^{2(1-1/\theta)}dt\Big) &\leq K\int_0^T \mathbb{E}\Big[\int_0^T c_s^{2(\phi-\gamma)}ds + c_T^{2(\phi-\gamma)}\Big]dt \\ &\leq KT\Big(\mathbb{E}\Big[\int_0^T c_s^{2(\phi-\gamma)}ds\Big] + \mathbb{E}(c_T^{2(\phi-\gamma)})\Big) < \infty.\end{aligned} \tag{2.52}$$

We now turn our attention to the last estimate, $\sup_{t\in[0,T]}|U_t^{\phi}|$. Suppose that $\theta \in (0,1)$, then there exist constants A_θ, B_θ such that $|x|^\theta \le A_\theta + B_\theta|x|$. Thus:

$$\sup_{t\in[0,T]}|U_t^{\phi}| = K\sup_{t\in[0,T]}|Y_t^{\phi}|^\theta \le A_\theta + B_\theta\sup_{t\in[0,T]}|Y_t^{\phi}|. \tag{2.53}$$

Analogous to U^γ, $\sup_{t\in[0,T]}|Y_t^{\phi}|$ belongs to $L^2(\mathbb{P})$, and thus so does $\sup_{t\in[0,T]}|U_t^{\phi}|$. When $\theta < 0$, we apply Jensen's inequality to obtain:

$$\begin{aligned} |U_t^{\phi}| &= K\mathbb{E}_t\Big(\int_t^T \delta e^{-\delta(s-t)}c_s^{1-\phi}ds + e^{-\delta(T-t)}c_T^{1-\phi}\Big)^\theta \\ &\le K\mathbb{E}_t\Big(\int_t^T c_s^{1-\gamma}ds + c_T^{1-\gamma}\Big) \\ &\le K\mathbb{E}_t\Big(\int_0^T c_s^{1-\gamma}ds + c_T^{1-\gamma}\Big). \end{aligned} \tag{2.54}$$

By the same Doob's maximal inequality argument as Part I, we conclude that $\sup_{t\in[0,T]}|U_t^{\phi}| \in L^2(\mathbb{P})$.

□

2.8 Proofs of Infinite Horizon Results.

2.8.1 Proof of Proposition 2.5.2

For the sake of concreteness, let us assume in Part i. and Part ii. below that $\gamma > \phi$. The case when $\phi < \gamma$ is proved similarly.

Part i. Due to the membership of c in C^∞, $\mathbb{E}(\int_0^T [c_s^{(T)}]^p ds) = \mathbb{E}(\int_0^T c_s^p ds) < \infty$ for the values of p required by C^T. Therefore, it remains only to show that the terminal condition $c_T^{(T)}$ satisfies $\mathbb{E}([c_T^{(T)}]^p) < \infty$ for $p = 2(1-\gamma), 2(1-\phi)$ and $2(\phi-\gamma)$. With K denoting a generic constant:

$$[c_T^{(T)}]^{2(1-\gamma)} = K\Big(\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)}c_s^{1-\gamma}ds\Big]\Big)^2 \le \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)}c_s^{2(1-\gamma)}ds\Big) \in L(\mathbb{P}). \tag{2.55}$$

Above, we used Jensen's inequality twice, firstly through the conditional expectation and secondly through the Lebesgue integral.

Next, we estimate $[c_T^{(T)}]^{2(1-\phi)}$ and $[c_T^{(T)}]^{2(\phi-\gamma)}$, using also Jensen's inequality in a manner similar to (2.55). We use below the fact that for any non-zero $\theta < 1$, the mapping $x \to x^{2/\theta}$ and $x \to x^{2(1-1/\theta)}$ are convex on $(0, \infty)$.

$$\begin{aligned} [c_T^{(T)}]^{2(1-\phi)} &= \Big(\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big]\Big)^{\frac{2}{\theta}} \le \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)} c_s^{2(1-\gamma)/\theta} ds\Big) \\ &= \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)} c_s^{2(1-\phi)} ds\Big) \in L(\mathbb{P}). \end{aligned} \tag{2.56}$$

$$\begin{aligned} [c_T^{(T)}]^{2(\phi-\gamma)} &= \Big(\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big]\Big)^{\frac{2(\phi-\gamma)}{1-\gamma}} = \Big(\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big]\Big)^{2(1-1/\theta)} \\ &\le \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)} c_s^{2(\phi-\gamma)} ds\Big) \in L(\mathbb{P}). \end{aligned} \tag{2.57}$$

We have shown that $c^{(T)}$ belong to the class C^T. Similarly, we need only to show that the terminal consumption $\tilde{c}_T^{(T)}$ satisfies $\mathbb{E}([\tilde{c}_T^{(T)}]^p) < \infty$ for $p = 2(1-\gamma), 2(1-\phi)$ and $2(\phi-\gamma)$. We observe that $[\tilde{c}_T^{(T)}]^{2(1-\phi)} = K\big(\mathbb{E}_T\big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\phi} ds\big]\big)^2$ and $[\tilde{c}_T^{(T)}]^{2(\phi-\gamma)} = K(\mathbb{E}_T\big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\phi} ds\big])^{2(\theta-1)}$. These quantities can be shown to be integrable using estimates similar to equations (2.55) and (2.56). Lastly, $[\tilde{c}_T^{(T)}]^{2(1-\gamma)} = K\big(\mathbb{E}_T\big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\phi} ds\big]\big)^{2\theta}$. If $\theta < 0$, the same convexity argument as in equation (2.56) shows that $[\tilde{c}_T^{(T)}]^{2(1-\gamma)}$ is integrable. If $\theta \in (0,1)$, then there exists constants A_θ, B_θ such that:

$$\begin{aligned} [\tilde{c}_T^{(T)}]^{2(1-\gamma)} &\le A_\theta + B_\theta\Big(\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\phi} ds\Big]\Big)^2 \\ &\le A_\theta + B_\theta \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)} c_s^{2(1-\phi)} ds\Big) \in L^1(\mathbb{P}). \end{aligned} \tag{2.58}$$

Part ii. For positive constants $t < T$, we can decompose the process $U^{\gamma,\infty}$ the following way, using the tower property of conditional expectation:

$$\begin{aligned} U_t^{\gamma,\infty} &= \mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta(s-t)} u_\gamma(c_s) ds\Big] \\ &= \mathbb{E}_t\Big[\int_t^T \delta e^{-\delta(s-t)} u_\gamma(c_s) ds + e^{-\delta(T-t)} \mathbb{E}_T\Big(\int_T^\infty \delta e^{-\delta(s-T)} u_\gamma(c_s) ds\Big)\Big]. \end{aligned} \tag{2.59}$$

This can be interpreted as the finite horizon power utility process associated with the upper truncation $c^{(T)}$. Specifically, $\{U^{\gamma,\infty}(c)_t\}_{t\in[0,T]} = U^{\gamma,T}(c^{(T)})$. In part i, we have proved that $c^{(T)} \in \mathcal{C}^T$. As a result, the restriction $\{U^{\gamma,\infty}(c)_t\}_{t\in[0,T]}$ belongs to $\mathcal{V}^T$ thanks to Lemma 2.7.3. The process $U^{\phi,\infty}$ has a similar decomposition:

$$
\begin{aligned}
U_t^{\phi,\infty} &= u_\gamma \circ u_\phi^{-1}\Big[\mathbb{E}_t\Big(\int_t^\infty \delta e^{-\delta(s-t)} u_\phi(c_s)ds\Big)\Big] \\
&= u_\gamma \circ u_\phi^{-1}\Big\{\mathbb{E}_t\Big(\int_t^T \delta e^{-\delta(s-t)} u_\phi(c_s)ds + e^{-\delta(T-t)}\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)} u_\phi(c_s)ds\Big]\Big)\Big\}.
\end{aligned} \tag{2.60}
$$

Similar to $U^{\gamma,\infty}$, the restriction on $[0,T]$ of $U^{\phi,\infty}$ can also be interpreted as a finite horizon power utility process. In particular, $\{U_t^{\phi,\infty}\}_{t\in[0,T]} = U^{\phi,\infty}(\tilde{c}^{(T)})$. As $\tilde{c}^{(T)} \in \mathcal{C}^T$, by Lemma 2.7.3, $\{U_t^{\phi,\infty}\}_{t\in[0,T]}$ belongs to $\mathcal{V}^T$.

Part iii. Lastly, we show that the ordering between power utility processes is preserved when we extend to infinite horizon, i.e. $U_t^{\phi\vee\gamma,\infty} \leq U_t^{\phi,\wedge,\gamma,\infty}$ for all $t \geq 0$. Recall that $U_t^{\phi,\infty} = u_\gamma \circ u_\phi^{-1}(\mathbb{E}_t \int_t^\infty \delta e^{-\delta(s-t)} u_\phi(c_s)ds)$. Consider the second derivative:

$$
\frac{d^2}{dx^2}(u_\gamma \circ u_\phi^{-1}(x)) = (\phi-\gamma)[(1-\phi)x]^{\theta-2}, \tag{2.61}
$$

which is concave on $(1-\phi)\mathbb{R}^+$ if $\phi < \gamma$ and convex on $(1-\phi)\mathbb{R}^+$ if $\phi > \gamma$. Therefore, we can apply Jensen's inequality to obtain:

$$
\begin{aligned}
U_t^{\phi,\infty} &= u_\gamma \circ u_\phi^{-1}\Big(\mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta(s-t)} u_\phi(c_s)ds\Big]\Big) \\
&\geq (\text{resp.} \leq)\ \mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta(s-t)} u_\gamma \circ u_\phi^{-1} \circ u_\phi(c_s)ds\Big] \\
&= U_t^{\gamma,\infty},
\end{aligned} \tag{2.62}
$$

when $\phi < \gamma$ (resp. $\phi > \gamma$). This concludes the lemma.

□

2.8.2 Proof of Proposition 2.5.3

Proof. For brevity, we will prove the result when $\phi < \gamma$. The other cases where $\gamma > \phi$ can be proved similarly. By Proposition 2.5.2, $U^{\gamma,\infty}(c)$ and $U^{\phi,\infty}(c)$ restricted on $[0, T]$ is equal to $U^{\gamma,T}(c^{(T)})$ and $U^{\phi,T}(\tilde{c}^{(T)})$, respectively. Therefore, by Power Utility Bounds in finite horizon (cf. Proposition 2.5.1), for $t < T$:

$$\begin{aligned} V_t^{(T)} &\leq U^{\phi,T}(\tilde{c}^{(T)})_t = U^{\phi,\infty}(c)_t, \quad \text{and} \\ \tilde{V}_t^{(T)} &\geq U^{\gamma,T}(\tilde{c}^{(T)})_t = U^{\gamma,\infty}(c)_t. \end{aligned} \tag{2.63}$$

As $V_T^{(T)} \geq \tilde{V}_T^{(T)}$, the ordering $V_t \geq \tilde{V}_t$ for $t < T$ is a straightforward consequence of a comparison principle (see Theorem 2.7.1). For the last statement, let $V^{(S)}$ and $\tilde{V}^{(S)}$ be defined by equation (2.25), with S in place of T. Then, similar to inequalities (2.26), we have:

$$U_T^{\phi\vee\gamma,\infty} \leq \tilde{V}_T^{(S)} \leq V_T^{(S)} \leq U_T^{\phi\wedge\gamma,\infty}. \tag{2.64}$$

On the the restricted horizon $[0, T]$, $V^{(S)}$ and $\tilde{V}^{(S)}$ has the same BSDE driver as (2.25), but with terminal conditions $V_T^{(S)}$ and $\tilde{V}_T^{(S)}$, respectively. Thus, we can combine inequalities (2.64) with Theorem 2.7.1 to attain the last statement of this proposition.

□

2.8.3 Proof of Proposition 2.5.4

Proof. It follows from the Proposition 2.5.3 that for $t < T$, $V_t^{(T)} \geq V_t^{(T+1)} \geq U_t^{\phi\wedge\gamma,\infty}$. As the lower bounding process is independent of T, we can define the downwards limit $V_t \triangleq\downarrow \lim_{T\to\infty} V_t^{(T)}$. Similarly, we can define $\tilde{V}_t \triangleq\uparrow \lim_{T\to\infty} \tilde{V}_t^{(T)}$, the upwards limit of its localisations. By construction, both V and $\tilde{V}$ are sandwiched between $U^{\phi\vee\gamma,\infty}$ and $U^{\phi\wedge\gamma,\infty}$. From Proposition 2.5.2, both $U^{\phi,\infty}$ and $U^{\gamma,\infty}$ belong to the semimartingale class $\mathcal{V}^\infty$, whence it follows that V and $\tilde{V}$ belong to the same class.

Having constructed two candidate solutions in the appropriate semimartingale classes, we now verify that they satisfy the target BSDE. Let $S > T$ be two positive constants, then:

$$V_t^{(S)} = \mathbb{E}_t\Big(V_T^{(S)} + \int_t^T f(c_s, V_s^{(S)})ds\Big). \tag{2.65}$$

Using the fact that $U^{\phi\vee\gamma,\infty} \leq V^{(S)} \leq U^{\phi\wedge\gamma,\infty}$ for all S, Proposition 2.5.2.ii. and conditional dominated convergence, we can pass to the limit[4] and obtain:

$$V_t = \mathbb{E}_t\Big(V_T + \int_t^T f(c_s, V_s)ds\Big). \tag{2.66}$$

Equation (2.66) with $\tilde{V}$ in place of V can be obtained using the same argument.

We will now prove the last statement. Since v satisfies the power utility bound, it belongs to $\mathcal{V}^\infty$. For any $T > 0$, $v_T \leq U_T^{\phi\wedge\gamma,\infty}$, and thus by comparison principle (Theorem 2.7.1), $v_t \leq V_t^{(T)}$ (cf. equation (2.25)) for all $t < T$. Letting T diverge, we obtain the upper bound in (2.27). The other inequality is proved similarly.

□

2.8.4 Proof of Theorem 2.5.5

Proof. By martingale representation theorem (Theorem 3.4.15 [33]), there exists an $\mathbb{R}^d$-valued progressively measurable process $\{Z^{(T)}\}_{t\leq T}$ such that $\int_0^T \|Z_s^{(T)}\|^2 ds < \infty$ and:

$$V_t = V_T + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s^{(T)} dB_s. \tag{2.67}$$

This representation is consistent across different horizon lengths, in the sense that the Z component in a longer horizon can also serve as the representing process for a shorter horizon. More specifically,

[4]See also Lemma 2.7.3, Part II. for a similar argument

for positive constants $t < T < S$:

$$
\begin{aligned}
V_t &= V_S + \int_t^S f(c_s, V_s)ds + \int_t^S Z_s^{(S)} dB_s \\
&= V_S + \int_T^S f(c_s, V_s)ds + \int_T^S Z_s^{(S)} dB_s + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s^{(S)} dB_s \\
&= V_T + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s^{(S)} dB_s.
\end{aligned}
\tag{2.68}
$$

Therefore, $\int_0^t Z_s^{(T)} dB_s = \int_0^t Z_s^{(S)} dB_s$ almost surely for $t \leq T$. By Lemma 2.8.1 below, $Z^{(T)}$ and $Z^{(S)}$ coincide $d\mathbb{P} \otimes dt$-almost everywhere on $[0, T] \times \Omega$. Thus, we can identify a process $\{Z_t\}_{t \geq 0}$ by defining $Z_t = Z_t^{(T)}$ for $t \leq T$, which satisfies the following BSDE:

$$
V_t = V_T + \int_t^T f(c_s, V_s)ds + \int_t^T Z_s dB_s, \quad t \leq T. \tag{2.69}
$$

□

Lemma 2.8.1. *Suppose the $\mathbb{R}^d$-valued progressively measurable processes Z and $\tilde{Z}$ are such that $\int_0^T \|Z\|_s^2 ds + \int_0^T \|\tilde{Z}\|_s^2 ds < \infty$ almost surely, and that $\{\int_0^t Z_s dB_s\}_{t \in [0,T]}$ and $\{\int_0^t \tilde{Z}_s dB_s\}_{t \in [0,T]}$ are indistinguishable, then $Z = \tilde{Z}$ $d\mathbb{P} \otimes dt$-almost everywhere on $[0, T] \otimes \Omega$.*

Define the stopping times $\tau_k \triangleq \inf\{t \geq 0, \int_0^t \|Z\|_s^2 ds \vee \int_0^t \|\tilde{Z}\|_s^2 ds \geq k\} \wedge T$. Then we have:

$$
\int_0^t Z_s \mathbb{I}_{s \leq \tau_k} dW_s = \int_0^{t \wedge \tau_k} Z_s dW_s = \int_0^{t \wedge \tau_k} \tilde{Z}_s dW_s = \int_0^t \tilde{Z}_s \mathbb{I}_{s \leq \tau_k} dW_s, \quad t \leq T. \tag{2.70}
$$

By Itô's isometry for square integrable martingales:

$$
\mathbb{E}\Big(\int_0^T \|(Z_t - \tilde{Z}_t)\mathbb{I}_{t \leq \tau_k}\|^2 dt \Big) = \mathbb{E}\Big(\Big[\int_0^T (Z_t - \tilde{Z}_t)\mathbb{I}_{t \leq \tau_k} dW_t \Big]^2 \Big) = 0. \tag{2.71}
$$

It follows that, almost surely, $Z_t(\omega) = \tilde{Z}_t(\omega)$ on $[0, \tau_k(\omega)]$. By letting $k \to \infty$ we have that $Z_t = \tilde{Z}_t$ for $t \leq T$ $dt \otimes d\mathbb{P}$-almost everywhere.

□

2.8.5 Proof of Proposition 2.6.1.

Suppose that V and $\tilde{V}$ are solutions of equation (2.31) that belong to $\mathcal{V}^\infty$ *and* satisfy the transversality condition (2.30). Then, V has the BSDE representation:

$$V_t = V_T + \int_t^T f(c_s, V_s)ds + (M_T - M_t), \tag{2.72}$$

where $M_t = \mathbb{E}_t(V_T + \int_0^T f(c_s, V_s)ds)$, a martingale. By Martingale Regularisation Theorem (Theorem 67.7, [65]), M has a càdlàg modification. Therefore, we can consider the càdlàg modification of V. $\tilde{M}$ is defined analogously and likewise, we also consider càdlàg versions of $\tilde{V}$ and $\tilde{M}$. Let us denote $\Delta V = V - \tilde{V}, \Delta M = M - \tilde{M}$ and $\Delta f = f(c, V) - f(c, \tilde{V})$. By Tanaka's formula for general semimartingales (Theorem 66, [53]), we have:

$$\begin{aligned} e^{-\delta\theta t}|\Delta V_t| = e^{-\delta\theta T}|\Delta V_T| &+ \int_t^T e^{-\delta\theta s}\left[\delta\theta|\Delta V_s| + g'(\Delta V_s)\Delta f_s\right]ds \\ &- \int_t^T e^{-\delta\theta s} g'(\Delta V_{s^-})d\Delta M_{s^-} - \int_t^T e^{-\delta\theta} dA^s, \end{aligned} \tag{2.73}$$

where $g(x) \triangleq |x|$ and $g'(x) = \mathbb{I}_{x>0} - \mathbb{I}_{x\leq 0}$ is its left-derivative, and $dA_t = (\Delta V_t - \Delta V_{t^-}) - g'(\Delta V_{t^-})(\Delta V_t - \Delta V_{t^-})$ is càdlàg non-decreasing. As the aggregator is monotone with $-\delta\theta$ being its constant of monotonicity, the Lebesgue integral above has a non-negative integrand. Moreover, the stochastic integral is a true martingale, as its integrand is uniformly bounded. Therefore, by taking expectation on both sides of (2.73):

$$\mathbb{E}\left[e^{-\delta\theta t}|\Delta V_t|\right] \leq \mathbb{E}\left[e^{-\delta\theta T}|\Delta V_T|\right]. \tag{2.74}$$

The right hand side of (2.74) vanishes as T diverges by transversality condition (2.30), which implies that $|\Delta V_t| = 0$ almost surely. We have demonstrated uniqueness of solutions satisfying the transversality condition. We observe also that when $\theta < 0$, $V \equiv 0$ is a $\mathcal{V}^\infty$ solution that satisfies (2.30). Therefore, all transversal solutions must be identically 0.

□

2.8.6 Proof of Theorem 2.4.2

Proof of Theorem 2.4.2. Suppose that c is a consumption plan in C^∞. Let V and $\tilde{V}$ denote the solutions constructed in Section 2.5. By Proposition 2.5.4, these actually define upper and lower bounds for all potential Epstein–Zin utility processes. Therefore, in order to prove uniqueness, it is sufficient to show that $V = \tilde{V}$.

Let $\{\alpha_t\}_{t\geq 0}$ be a progressively measurable process that shall be determined later. Define $M_t = \mathbb{E}_t(V_T + \int_0^T f(c_s, V_s))$ for $t < T$. $\tilde{M}_t$ is defined analogously with $\tilde{V}$. Denote by ΔV_t the non-negative process $V_t - \tilde{V}_t$, and by ΔM_t the difference $M_t - \tilde{M}_t$. The dynamics of $\exp(\int_0^t \alpha_s ds)\Delta V_t$ can be obtained by Itô's Lemma for non-continuous semimartingales (i.e. Theorem I.4.57 of [53]):

$$de^{\int_0^t \alpha_s ds}\Delta V_t = e^{\int_0^t \alpha_s ds}\big(\alpha_t \Delta V_t - [f(c_t, V_t) - f(c_t, \tilde{V}_t)]\big)dt + e^{\int_0^t \alpha_s ds} d\Delta M_t. \tag{2.75}$$

By mean value theorem, $f(c_t, V_t) - f(c_t, \tilde{V}_t) = \partial_v f(c_t, K)(V_t - \tilde{V}_t)$ for some $K \in [\tilde{V}_t, V_t]$. Moreover, by considering the second derivative $\partial^2_{vv} f(c, v) = \delta c^{1-\phi}(\gamma - \phi)[(1-\gamma)v]^{1-1/\theta}$, we observe that $\partial_v f$ is increasing in v when $\gamma > \phi$ and decreasing when $\gamma < \phi$. In both cases, $\partial_v f(c_t, K) \leq \partial_v f(c_t, U_t^{\phi,\infty})$. Therefore, if we set $\alpha_t = f_v(c_t, U_t^{\phi,\infty})$, the drift term in equation (2.75) is non-negative and $\exp(\int_0^t \alpha_s ds)\Delta V_t$ is a local submartingale. Moreover, as $\partial_v f(c, v) \leq -\delta\theta$, the exponential factor is locally bounded. $\exp(\int_0^t \alpha_s ds)\Delta V_t$ is thus of class (DL) and a true submartingale. For any positive constants $t < T$:

$$\mathbb{E}\big[e^{\int_0^t \alpha_s ds}\Delta V_t\big] \leq \mathbb{E}\big[e^{\int_0^T \alpha_s ds}\Delta V_T\big]. \tag{2.76}$$

As ΔV is non-negative, in order to show that $\Delta V_t = 0$, it is sufficient to show that the right hand side of inequality (2.76) vanishes as T diverges. We consider two separate cases.

<u>Case 1: $0 < \theta < 1$.</u> This covers the two cases ,$\phi < \gamma < 1$ and $< 1 < \gamma < \phi$. In both cases, we have from the power utility bounds:

$$0 \leq (1-\gamma)V_T \leq \left(\mathbb{E}_T\Big[\int_T^\infty e^{-\delta(s-T)} c_s^{1-\phi} ds\Big]\right)^\theta. \tag{2.77}$$

Therefore:

$$
\begin{aligned}
&\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)|(1-\gamma)V_T|\Big)\\
&\quad\le \mathbb{E}\Big(\exp\Big(\int_0^T \frac{\partial_v f(c_s, U_s^{\phi,\infty})}{\theta}ds\Big)\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)}c_s^{1-\phi}ds\Big]\Big)^\theta \qquad (2.78)\\
&\quad= \mathbb{E}\Big\{\exp\Big(\int_0^T \delta\Big(1-\frac{1}{\theta}\Big)[(1-\gamma)U_t^{\phi,\infty}]^{-\frac{1}{\theta}}dt - \delta T\Big)\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta(s-T)}c_s^{1-\phi}ds\Big]\Big\}^\theta.
\end{aligned}
$$

As $\theta \in (0,1)$, the term $\delta(1-1/\theta)[(1-\gamma)U_t^{\phi,\infty}]^{-1/\theta}$ is non-positive. It then follows from the previous inequality that:

$$
\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)|(1-\gamma)V_T|\Big) \le \mathbb{E}\Big(\int_t^\infty \delta e^{-\delta s}c_s^{1-\phi}ds\Big)^\theta. \qquad (2.79)
$$

We note that c satisfies $\mathbb{E}(\int_0^\infty \delta e^{-\delta s}c_s^{2(1-\phi)}ds) < \infty$ by its membership in C^∞. $c^{1-\phi}$ can be interpreted as belonging to the square-integrable space $L^2(\Omega\times\mathbb{R}^+)$, with probability measure $d\mathbb{P}\otimes\delta e^{-\delta t}dt$. Since this is a finite measure space, $c^{1-\phi}$ also belongs to $L^1(\Omega\times\mathbb{R}^+)$ with the same measure, which implies $\mathbb{E}(\int_0^\infty \delta e^{-\delta s}c_s^{1-\phi}ds) < \infty$. Thus, as T diverges, $\int_T^\infty \delta e^{-\delta s}c_s^{1-\phi}$ converges to 0 almost surely and in L^1, by dominated convergence. It therefore implies that:

$$
\lim_{T\to\infty}\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)|V_T|\Big) = 0. \qquad (2.80)
$$

The same result holds with $\tilde{V}$ in place of V, whence we conclude that the right hand side of (2.76) vanishes at infinity, which concludes Case 1.

Case 2: $\theta < 0$. This covers the two remaining configurations, $\phi < 1 < \gamma$ and $\gamma < 1 < \phi$. In both configurations, $|V_t|$ and $|\tilde{V}_t|$ are bounded above by $|U_t^{\gamma,\infty}|$. Therefore:

$$
\begin{aligned}
&\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)|V_T|\Big)\\
&\quad\le \frac{1}{|1-\gamma|}\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)\mathbb{E}_T\Big[\int_T^\infty \delta e^{-\delta s}c_s^{1-\gamma}ds\Big]\Big) \qquad (2.81)\\
&\quad= \frac{1}{|1-\gamma|}\mathbb{E}\Big(\exp\Big(\int_0^T \partial_v f(c_s, U_s^{\phi,\infty})ds\Big)\mathbb{E}_T\int_T^\infty \delta e^{-\delta s}c_s^{1-\gamma}ds\Big),
\end{aligned}
$$

which vanishes at infinity by the hypothesis of the theorem. The same result holds for $\tilde{V}$ in place of V, which concludes Case 2.

□

2.8.7 Proof of Proposition 2.4.3.

Proof of Proposition 2.4.3. We will now prove the properties of homotheticity, monotonicity and concavity of Epstein–Zin utilities, in both finite and infinite horizon.

Part I.1. Homotheticity, $T < \infty$. We can verify straightforwardly that if V^c solves equation (2.23), then:

$$\lambda^{1-\gamma} V^c = \mathbb{E}_t\Big(u_\gamma(\lambda c_T) + \int_t^T f(\lambda c_s, \lambda^{1-\gamma} V_s^c) ds\Big), \quad 0 \le t \le T, \tag{2.82}$$

which confirms the homotheticity of Epstein–Zin utilities.

Part I.2. Homotheticity, $T = \infty$. We can verify directly that the mappings $C^\infty \to \mathcal{V}^\infty : c \to U^{\gamma,\infty}$ and $c \to U^{\phi,\infty}$ are homothetic, i.e. $U^{\gamma,\infty}(\lambda c) = \lambda^{1-\gamma} U^{\gamma,\infty}(c)$ and $U^{\phi,\infty}(\lambda c) = \lambda^{1-\gamma} U^{\phi,\infty}(c)$. Therefore:

$$V^{(T)}(\lambda c) = \lambda^{1-\gamma} V^{(T)}(c), \tag{2.83}$$

where $V^{(T)}(c)$ and $V^{(T)}(\lambda c)$ are constructed via equation (2.25). As we take limit $T \to \infty$, this homotheticity property is preserved by the limit process. We observe also that the quantity Φ in (2.22) is invariant through scaling. Therefore λc satisfies the uniqueness criterion.

Part II. Monotonicity. When $T < \infty$, monotonicity of the mapping $C^T \to \mathcal{V}^T : c \to V^c$ is the result of Corollary 2.7.2. When $T = \infty$, $U^{\gamma,\infty}(\tilde{c}) \ge U^{\gamma,\infty}(c)$ when $\tilde{c} \ge c$. By the comparison principle 2.7.1:

$$V^{(T)}(\tilde{c} \ge V^{(T)}(c). \tag{2.84}$$

Letting $T \to \infty$, we attain the desired result.

Part III.1 Concavity, $T < \infty$. From the proof of Proposition 2.5.1, V^c can be obtained as the following limit:

$$V_t^c = \lim_{n\to\infty} \lim_{m\to\infty} V_t^{(c \vee \frac{1}{n}) \wedge m}. \tag{2.85}$$

$(c \vee \frac{1}{n}) \wedge m$ and $(\tilde{c} \vee \frac{1}{n}) \wedge m$ are bounded above and away from zero, and thus so are their convex combinations. This falls within the scope of Theorem 3.3 [57], whence we have:

$$V^{\alpha(c\vee\frac{1}{n})\wedge m+(1-\alpha)(\tilde{c}\vee\frac{1}{n})\wedge m} \leq \alpha V^{(c\vee\frac{1}{n})\wedge m} + (1-\alpha)V^{(\tilde{c}\vee\frac{1}{n})\wedge m}. \tag{2.86}$$

Letting $m, n \to \infty$, we obtain the concavity of the mapping $c \to V^c$ in finite horizon.

Part III.2. Concavity, $T = \infty$. Lastly, we observe that the mapping $c \to U^{\gamma,\infty}$ is concave, a consequence of concavity of CRRA utility functions. Therefore, the terminal condition of $V^{(T)}$ in equation (2.25) is concave in c. Therefore, by concavity in finite horizon:

$$V^{(T)}(\alpha c + (1-\alpha)\tilde{c}) \geq \alpha V^{(T)}(c) + (1-\alpha)V^{(T)}(\tilde{c}). \tag{2.87}$$

Letting $T \to \infty$, we attain concavity of infinite horizon Epstein–Zin utilities.

□

CHAPTER 3

APPLICATIONS

3.1 Example I - Geometric Brownian Motion Consumption.

We consider a simple model where the consumption follows Geometric Brownian Motion (GBM) dynamics:

$$dc_t = bc_t dt + \sigma c_t dB_t, \quad c_0 \in (0, \infty), \tag{3.1}$$

where b and σ are constants, σ is positive, and B is an $\mathbb{R}$-valued, $\{\mathcal{F}_t\}$-Brownian Motion. One virtue of this simple model is that every quantity of interest can be calculated explicitly. In particular, the integrability conditions of C^∞ and the uniqueness criterion (2.21) can be reduced to a set of easily verifiable inequalities.

Theorem 3.1.1. *Let c be a consumption process with Geometric Brownian Motion dynamics as defined in* (3.1)*. It belongs to the class* C^∞ *if and only if the following system of inequalities hold:*

$$\begin{cases} 2(1-\phi)\Big(b - \dfrac{\sigma^2}{2}\Big) + 2(1-\phi)^2\sigma^2 < \delta, & \text{(3.2a)} \\ 2(1-\gamma)\Big(b - \dfrac{\sigma^2}{2}\Big) + 2(1-\gamma)^2\sigma^2 < \delta, & \text{(3.2b)} \\ 2(\phi-\gamma)\Big(b - \dfrac{\sigma^2}{2}\Big) + 2(\phi-\gamma)^2\sigma^2 < \delta. & \text{(3.2c)} \end{cases}$$

Moreover, when $\theta < 0$, the uniqueness criterion (2.21) is satisfied if and only if the following additional inequality holds:

$$(1-\phi)\Big(b-\frac{1}{2}\sigma^2\Big)+\frac{1}{2}(\gamma-\phi)(1-\phi)\sigma^2+\frac{1}{2}(1-\gamma)^2\sigma^2<\delta. \tag{3.3}$$

3.2 Example II - Long Run Risk Model

We will now consider our adaptation of the so-called Long Run Risk Model (LRRM) developed by Bansal & Yaron [3][2]. The model specification is given below:

$$\begin{aligned} d\log(c_t) &= (\mu+X_t)dt+\sqrt{v_t}dB_t, \quad c_0>0, \\ dX_t &= -aX_tdt+bdW_t, \quad X_0=0, \\ dv_t &= \kappa(\eta-v_t)dt+\lambda\sqrt{v_t}dB_t^{\perp}, \quad v_0\in\mathbb{R}^+, \end{aligned} \tag{3.4}$$

where $a, b, \kappa, \eta, \lambda$ are positive constants, μ is a real number and W, B and $B^{\perp}$ are mutually independent $\mathbb{R}$-valued Brownian Motions.

The dynamics of log consumption contains a long-run component X, which models random fluctuations in the state of the economy that alter expected growth of log consumption. This long run component is modelled by an Ornstein-Uhlenbeck (OU) process with stationary mean 0, representing the 'neutral' state of the economy. There is no loss of generality in letting X have zero long term mean, since it can always be adjusted via an affine shift. In Bansal & Yaron's model, random shocks have a long-lasting impact on the expected growth of log consumption. This persistence is modelled via the mean-reverting speed a.

There are a few aspects in which we deviate from their model in order to simplify the technical details incurred by continuous time. In their specification, up to scaling constants, the same stochastic volatility process is used for X, $\log(c)$ and an additional dividend process. First, we focus on establishing the associated Epstein–Zin utility rather than asset pricing, and thus we do not model dividends explicitly. Secondly, stochastic volatility only appears in the log consumption dynamics, which helps to avoid some technical challenges in estimating the moment generating function of X. Lastly, we model v with a Cox-Ingersoll-Ross (CIR) square-root process instead of an OU process to ensure positivity of volatility and at the same time keep the mean-reverting behaviour.

Similar to the case of Geometric Brownian Motion consumption, admissibility and uniqueness of Epstein–Zin utiltiy in an LRRM can be sufficed by a set of inequalities.

Theorem 3.2.1. *In the context of the Long Run Risk Model 3.4, the consumption process therein belongs to the class C^∞ if the following inequalities hold:*

$$\begin{cases} (1-\gamma)^2, (1-\phi)^2, (\phi-\gamma)^2 < \dfrac{\kappa^2}{4\lambda^2}, & (3.5a) \\ 2(1-\phi)\mu + 2(1-\phi)^2\dfrac{b^2}{a^2} + \dfrac{\kappa\eta}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 4\lambda^2(1-\phi)^2}) < \delta, & (3.5b) \\ 2(1-\gamma)\mu + 2(1-\gamma)^2\dfrac{b^2}{a^2} + \dfrac{\kappa\eta}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 4\lambda^2(1-\gamma)^2}) < \delta, & (3.5c) \\ 2(\phi-\gamma)\mu + 2(\phi-\gamma)^2\dfrac{b^2}{a^2} + \dfrac{\kappa\eta}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 4\lambda^2(\phi-\gamma)^2}) < \delta. & (3.5d) \end{cases}$$

The LRRM subsumes the Geometric Brownian Motion model (3.1) as a degenerate case. Heuristically, if we increase the mean-reverting velocity and decrease the volatility of a mean-reverting process, we suppress its variation and force it to behave more closely to a constant process. Therefore, if we vary the parameters so that $\frac{b}{a}$ and $\frac{\lambda}{\kappa}$ converge to 0, it stands to reason that X and v converge to their long term mean and the consumption process behave more closely to a Geometric Brownian Motion. Let us define the limiting consumption process by the equation:

$$dc_t = c_t\left(\mu + \frac{1}{2}\eta\right) + \sqrt{\eta}dB_t, \quad c_0 > 0. \tag{3.6}$$

The inequality system (3.5a)-(3.5d), in an appropriate sense, also converges to the system (3.2a)-(3.2b). Indeed, inequality (3.5a) is trivially satisfied when $\frac{\lambda}{\kappa} \to 0$. Moreover, as $\frac{\lambda}{\kappa} \to 0$, $\frac{\kappa\eta}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 4\lambda^2(1-\phi)^2}) \to 2(1-\phi)^2\eta$ (See Lemma 3.4.3). If $\frac{b}{a}$ converges to 0 additionally, then inequality (3.5b) becomes $2(1-\phi)\mu + 2(1-\phi)^2\eta < \delta$. This is simply inequality (3.2a) of Theorem 3.1.1 applied for the limiting consumption process (3.6). In the same way, inequalities (3.5c) & (3.5d) correspond to (3.2b) & (3.2c). We have demonstrated the consistency in the integrability conditions of the two models. Let us now state the uniqueness criterion for the LRRM.

Theorem 3.2.2. *Consider a LRRM such that the model constraints of Theorem 3.2.1 are satisfied and that $\theta < 0$. Let $\Gamma(z) \triangleq \sqrt{\kappa^2 - 2\lambda^2 z}$. Define:*

$$\zeta \triangleq (1-\phi)\mu + \frac{(1-\phi)^2 b^2}{2a^2} + \frac{\kappa\eta}{\lambda^2}(\kappa - \Gamma((1-\phi)^2/2)). \tag{3.7}$$

Then, the uniqueness criterion (2.21) holds if there exists a constant $m > 0$ and Hölder's conjugates p, q with $p \geq 2$ such that the following parameter restriction holds:

$$\begin{aligned}
&-\delta\theta - \frac{m}{p}\Big[1 + \log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big) - \frac{|1-\phi|}{4a}\Big] + \frac{1}{2p}\Big(a - \sqrt{a^2 - 2b^2\frac{m|1-\phi|}{a}}\Big) \\
&+ \frac{\kappa\eta}{p\lambda^2}\Big(\kappa - \sqrt{\kappa^2 - 2\lambda^2\frac{m(1-\phi)^2}{\kappa + \Gamma(\frac{1}{2}[1-\phi]^2)}}\Big) + (1-\gamma)\mu + \frac{q^2(1-\gamma)^2 b^2}{2qa^2} + \frac{\kappa\eta}{q\lambda^2}\Big[\kappa - \Gamma\Big(\frac{q^2[1-\gamma]^2}{2}\Big)\Big] < 0.
\end{aligned} \tag{3.8}$$

Theorem 3.2.2 appears quite obtuse with intertwining relationships between model parameters. The constant m was introduced by a parametrised family of lower bounds used in an estimate (cf. equation (3.39)). In principles, so as to assist with verifying the relation (3.8) and sharpen this bound, we could attempt to minimise in its left hand side in m and p. The first order condition, however, is unlikely to yield a closed-form solution for the turning point. Therefore, we shall choose m that simplifies inequality (3.8) and allows us to interpret it qualitatively.

Similar to the exposition following Theorem 3.2.1, we shall examine how inequality (3.8) subsumes (3.3) as a special case. Letting $a, \kappa \to \infty$ and $b, \lambda \to 0$, the left hand side of (3.8) becomes:

$$-\delta\theta - \frac{m}{p}\Big[1 + \log\Big(\frac{p(\delta-\tilde{\zeta})(1-\theta)}{m}\Big)\Big] + (1-\gamma)\mu + \frac{q^2}{2q}(1-\gamma)^2\eta, \tag{3.9}$$

where $\tilde{\zeta} = (1-\phi)\mu + \frac{1}{2}\eta(1-\phi)^2$. In particular, the third and fourth term of (3.8) vanish as we take limit (Lemma 3.4.3, Part ii.). Choosing $m = p(\delta - \tilde{\zeta})(1-\theta)$ in (3.9), when p is sufficiently large, the uniqueness criterion is reduced to:

$$(1-\phi)\mu + \frac{1}{2}(\gamma-\phi)(1-\phi)\eta + \frac{1}{2}(1-\gamma)^2\eta < \delta, \tag{3.10}$$

which is simply Theorem 3.1.1 applied to the limit consumption process (3.6).

We shall study also how the inequality (3.8) can be simplified under special configurations of the preference parameters γ and ϕ. Firstly, we shall consider the case where ϕ approaches 1, i.e. the unit

EIS case. Choosing $m = p(\delta - \zeta)(1-\theta)$ and let $\phi \to 1$, (3.8) simplifies to (Lemma 3.4.3, part iii.):

$$-\delta\Big(1 - \frac{|1-\gamma|}{4a}\Big) + \frac{1}{2p}\Big(a - \sqrt{a^2 - \frac{2b^2 p\delta|\gamma - 1|}{a}}\Big) + \frac{q(1-\gamma)^2 b^2}{2a^2} + \frac{\kappa\eta}{q\lambda^2}\Big[\kappa - \Gamma\Big(\frac{q^2[1-\gamma]^2}{2}\Big)\Big] < 0. \tag{3.11}$$

The first term in (3.11) is the only potential negative term. Therefore, for (3.11) to hold, we require $a > |1-\gamma|/4$. If such is the case, then, we observe that the *reduced uniqueness criterion holds when δ is large and either a is sufficiently large or b is sufficiently small.* If we further let $\gamma \to 1$, then it reduces to the trivial inequality $\delta > 0$. This is unsurprising, as the limiting case $\gamma, \phi \to 1$ correspond to the log utility case, where the investor is myopic.

3.3 Proofs for Section 3.1

Proof of Theorem 3.1.1. To verify the assumptions of C^∞, let us calculate $\mathbb{E}(\int_0^\infty e^{-\delta t} c_t^p \, dt)$ directly as follows:

$$\begin{aligned} \mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t} c_t^p \, dt\Big) &= \int_0^\infty \delta e^{-\delta t} \mathbb{E}[c_t^p] dt \\ &= \int_0^\infty \delta e^{-\delta t} e^{p(b - \frac{\sigma^2}{2})t + \frac{p^2\sigma^2}{2}t} dt. \end{aligned} \tag{3.12}$$

This integral is finite if and only if $p(b - \frac{\sigma^2}{2}) + \frac{1}{2}p^2\sigma^2 < \delta$. Substituting $2(1-\phi), 2(1-\gamma)$ and $2(\phi - \gamma)$ for p, we attain inequalities (3.2a)-(3.2c). For the uniqueness criterion, we rewrite $c^{1-\phi}$ in a more convenient form:

$$\begin{aligned} c_t^{1-\phi} &= c_0^{1-\phi} e^{(1-\phi)(b-\sigma^2/2)t} e^{(1-\phi)\sigma B_t} \\ &= c_0^{1-\phi} e^{((1-\phi)(b-\sigma^2/2) + \frac{1}{2}(1-\phi)^2\sigma^2)t} e^{(1-\phi)\sigma B_t - \frac{1}{2}(1-\phi)^2\sigma^2 t} \\ &\triangleq c_0^{1-\phi} e^{\lambda t} M_t, \end{aligned} \tag{3.13}$$

where M_t denotes the exponential martingale $e^{(1-\phi)\sigma B_t - \frac{1}{2}(1-\phi)^2\sigma^2 t}$ and λ denotes the constant $(1-\phi)(b-\frac{\sigma^2}{2}) + \frac{1}{2}(1-\phi)^2\sigma^2$. By conditional Fubini's theorem:

$$\begin{aligned}\Phi_t &= \frac{c_0^{1-\phi}e^{\lambda t}M_t}{\int_t^\infty \mathbb{E}_t[c_0^{1-\phi}e^{\lambda s}M_s]\delta e^{-\delta(s-t)}ds} = \frac{M_t}{\int_t^\infty e^{\lambda(s-t)}M_t\delta e^{-\delta(s-t)}ds} \\ &= \frac{1}{\int_0^\infty \delta e^{(\lambda-\delta)s}ds} = \frac{\delta-\lambda}{\delta}.\end{aligned} \tag{3.14}$$

We have calculated the explicit value for Φ_s for the case of constant coefficients. The uniqueness criterion is estimated below:

$$\begin{aligned}&\mathbb{E}\Big(e^{\delta(1-\theta)\int_0^T(1-\Phi_s)ds}\int_T^\infty \delta e^{-\delta s}c_s^{1-\gamma}ds\Big) = e^{(1-\theta)\lambda T}\mathbb{E}\Big(\int_T^\infty \delta e^{-\delta s}c_s^{1-\gamma}ds\Big) \\ &= \delta e^{(1-\theta)\lambda T}\int_T^\infty e^{((1-\gamma)(b-\frac{1}{2}\sigma^2)+\frac{1}{2}(1-\gamma)^2\sigma^2-\delta)s}ds \\ &= \delta e^{(1-\theta)\lambda T}e^{((1-\gamma)(b-\frac{1}{2}\sigma^2)+\frac{1}{2}(1-\gamma)^2\sigma^2-\delta)T}\int_0^\infty e^{((1-\gamma)(b-\frac{1}{2}\sigma^2)+\frac{1}{2}(1-\gamma)^2\sigma^2-\delta)s}ds,\end{aligned} \tag{3.15}$$

which vanishes at infinity when $(1-\theta)\lambda + (1-\gamma)(b-\frac{1}{2}\sigma^2) + \frac{(1-\gamma)^2\sigma^2}{2} - \delta < 0$. Substituting in the value of λ, we obtain the conclusion of the theorem.

3.4 Proofs for Section 3.2

In our calculations, we will make extensive use of the following result regarding the moment generating function (MGF) of the time integral of a CIR process. A result on its characteristic function can be found in Section 3 of [12]. Alternatively, a result on the joint conditional MGF of $(\int_s^t v_r dr, v_t)$ can be found in Theorem 4.8 of [9]. We state here an simplified result that suits our purpose:

Lemma 3.4.1. *Let v be the CIR process defined in the LRRM model specification* (3.4)*. Then, its moment generating function is given by:*

$$\mathbb{E}\Big[\exp\Big(z\int_0^t v_s ds\Big)\Big] = A(t,z)\exp(B(t,z)v_0), \quad \textit{for } z < \frac{\kappa^2}{2\lambda^2}, \tag{3.16}$$

where

$$
\begin{aligned}
A(t,z) &= \frac{\exp(\frac{\kappa^2\eta t}{\lambda^2})}{\Big(\cosh\big(\frac{\Gamma(z)t}{2}\big) + \frac{\kappa}{\Gamma(z)}\sinh\big(\frac{\Gamma(z)t}{2}\big)\Big)^{2\kappa\eta/\lambda^2}}, \\
B(t,z) &= \frac{2z}{\kappa + \Gamma(z)\coth\big(\frac{\Gamma(z)t}{2}\big)}, \\
\Gamma(z) &= \sqrt{\kappa^2 - 2\lambda^2 z}.
\end{aligned}
\tag{3.17}
$$

3.4.1 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. The consumption process can be solved explicitly, which yields:

$$
c_t = c_0 \exp\Big(\int_0^t (\mu + X_s)ds + \int_0^t \sqrt{v_s}dB_s\Big). \tag{3.18}
$$

We observe that X and v are strong solutions of their respective SDEs, which are driven by W and $B^\perp$. Moreover, the Brownian Motions W, B and $B^\perp$ are mutually independent. Thus $\exp(\int_0^t (\mu + X_s)ds)$ and $\exp(\int_0^t \sqrt{v_r}dB_r)$ are also independent. This allows us to split the expectation of product in the following:

$$
\begin{aligned}
\mathbb{E}(c_t^p) &= c_0^p \mathbb{E}\Big(\exp\Big[p\int_0^t (\mu + X_s)ds + p\int_0^t \sqrt{v_s}dB_s\Big]\Big) \\
&= c_0^p \mathbb{E}\Big(\exp\Big[p\int_0^t (\mu + X_s)ds\Big]\Big)\mathbb{E}\Big(\exp\Big[p\int_0^t \sqrt{v_s}dB_s\Big]\Big).
\end{aligned}
\tag{3.19}
$$

These expectations can be estimated by Lemma 3.4.2, which is deferred until after this proof. Combining equation (3.19) and Lemma 3.4.2, we have:

$$
\mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t} c_t^p dt\Big) \le K\int_0^\infty \delta \exp\Big(-\delta t + p\mu t + \frac{p^2 b^2}{2a^2}t + \frac{\kappa\eta}{\lambda^2}\big(\kappa - \Gamma(p^2/2)\big)t\Big)dt, \tag{3.20}
$$

where Γ is defined in Lemma 3.4.1. The integral on the right hand side is finite if and only if:

$$
p\mu + \frac{p^2 b^2}{2a^2} + \frac{\kappa\eta}{\lambda^2}(\kappa - \Gamma(p^2/2)) < \delta. \tag{3.21}
$$

Substituting $2(1-\gamma)$, $2(1-\phi)$ and $2(\phi-\gamma)$ for p, we obtain inequalities (3.5b)-(3.5d) of Theorem 3.2.1. Moreover, for $\Gamma(p^2/2)$ and thus the MGF of the time integral of v to be well defined, we require also $p^2/2 < \kappa^2/2\lambda^2$, which is the inequality (3.5a).

□

Lemma 3.4.2. *i. Let v be the CIR process defined in the LRRM model specification* (3.4)*. Then, for $p \leq \frac{\kappa^2}{2\lambda^2}$ and some positive constant K:*

$$\mathbb{E}\Big[\exp\Big(p\int_0^t v_r\,dr\Big)\Big] \leq K\exp\Big[\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\Gamma(p)\Big)t\Big] \tag{3.22}$$

ii. Additionally, let B be an $\mathbb{R}$-valued Brownian Motion independent of v. Then, for $p^2 \leq \frac{\kappa^2}{\lambda^2}$, $t \geq 0$, the following holds for some positive constant K:

$$\mathbb{E}\Big[\exp\Big(p\int_0^t \sqrt{v_r}\,dB_r\Big)\Big] \leq K\exp\Big[\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\Gamma\Big[\frac{p^2}{2}\Big]\Big)t\Big]. \tag{3.23}$$

iii. Let X be the Ornstein-Uhlenbeck process defined in (3.4)*, then, for any exponent $p \in \mathbb{R}$, the following estimate holds:*

$$\mathbb{E}\Big[\exp\Big(p\int_0^t X_s\,ds\Big)\Big] \leq \exp\Big(\frac{p^2b^2}{2a^2}t\Big). \tag{3.24}$$

Proof. Part i. For brevity, let us denote $\Gamma = \Gamma(p)$. We shall now examine the behaviour of A and B (cf. Lemma 3.4.1) as t diverges. First, $\downarrow \lim_{t\to\infty}\coth(\Gamma t/2) = 1$, for all positive t, $B(t) \leq B(\infty) = \frac{2p}{\kappa+\Gamma}$. Secondly, we estimate the denominator of $A(t,p)$:

$$\begin{aligned}\cosh\Big(\frac{\Gamma t}{2}\Big)+\frac{\kappa}{\Gamma}\sinh\Big(\frac{\Gamma t}{2}\Big) &= \frac{1}{2}\Big(1+\frac{\kappa}{\Gamma}\Big)e^{\frac{\Gamma t}{2}}+\frac{1}{2}\Big(1-\frac{\kappa}{\Gamma}\Big)e^{-\frac{\Gamma t}{2}}\\ &\geq \frac{1}{2}\Big(1+\frac{\kappa}{\Gamma}\Big)e^{\frac{\Gamma t}{2}}+\frac{1}{2}\Big(1-\frac{\kappa}{\Gamma}\Big)e^{\frac{\Gamma t}{2}} = e^{\frac{\Gamma t}{2}},\end{aligned} \tag{3.25}$$

where the inequality follows from the fact that $1-\kappa/\Gamma < 0$. An estimate for $A(t)$ can be achieved:

$$A(t,p) \leq \exp\Big(\frac{\kappa^2\eta t}{\lambda^2}\Big)\exp\Big(\frac{-\kappa\eta\Gamma t}{\lambda^2}\Big) = \exp\Big(\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\Gamma(p)\Big)t\Big). \tag{3.26}$$

Therefore, we achieve the final estimate for the moment generating function of $\int_0^t v_s ds$:

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(p\int_0^t \sqrt{v_r}dB_r\Big)\Big] &\leq K\exp\Big[\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\Gamma(p)\Big)t\Big] = A(t,p)\exp\Big(B(t,p)v_0\Big) \\ &\leq \exp\Big(\frac{2p}{\kappa+\Gamma(p)}v_0\Big)\exp\Big(\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\Gamma(p)\Big)t\Big).\end{aligned} \tag{3.27}$$

Part ii. We exploit the independence between $\{v_t\}_{t\in[0,\infty)}$ and $\{B_t\}_{t\in[0,\infty)}$ by conditioning the integral $\int_0^t \sqrt{v_s}dB_s$ on the path $\{v_s\}_{s\leq t}$. Heuristically, the added information on v tells us nothing new about B, and B remains a Brownian Motion. When v is known, $\int_0^t \sqrt{v_s}dB_s$ can be thought of as a stochastic integral with deterministic integrand, the distribution of which is well understood. Mathematically:

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(p\int_0^t \sqrt{v_r}dB_r\Big)\Big) &= \mathbb{E}\Big(\mathbb{E}\big(\exp\big(p\int_0^t \sqrt{v_r}dB_r\big)\Big|\sigma\{v_s, s\leq t\}\big)\Big) \\ &= \mathbb{E}\Big(\exp\Big(\frac{p^2}{2}\int_0^t v_r dr\Big)\Big),\end{aligned} \tag{3.28}$$

which can be bounded using the result attained in Part i., directly yielding the estimate (3.23).

Part iii. The exact distribution of the time integral $\int_0^t X_s ds$ is known and given in equation (1.8.4), Chapter 7.1, Part II of [7]. In particular, it is a Gaussian variable with mean zero and variance:

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t X_r dr\Big)^2\Big] &= \frac{b^2}{a^2}\Big(t+\frac{2}{a}e^{-at}-\frac{1}{2a}e^{-2at}-\frac{3}{2a}\Big) \\ &= \frac{b^2}{a^2}\Big(t+\Big(\frac{1}{a}e^{-at}-\frac{1}{a}\Big)+\Big(\frac{1}{a}e^{-at}-\frac{1}{2a}e^{-2at}-\frac{1}{2a}\Big)\Big) \\ &= \frac{b^2}{a^2}\Big(t-\frac{1}{a}(1-e^{-at})-\frac{1}{2a}(e^{-at}-1)^2\Big) \leq \frac{b^2}{a^2}t.\end{aligned} \tag{3.29}$$

Therefore, for any real number p:

$$\mathbb{E}\big(e^{p\int_0^t X_r dr}\big) \leq e^{\frac{p^2b^2}{2a^2}t}. \tag{3.30}$$

□

3.4.2 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. We recall below the definition of the uniqueness criterion for the convenience of the reader:

$$\lim_{T\to\infty} \mathbb{E}\Big(e^{-\delta\theta T} e^{\delta(1-\theta)\int_0^T -\Phi_s ds} \int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big) = 0, \tag{3.31}$$

where $\Phi_t = c_t^{1-\phi}\big(\mathbb{E}\int_t^\infty \delta e^{-\delta(s-t)} c_s^{1-\phi} ds\big)^{-1}$. Let Θ_T denote the expectation above, p, q be Hölder's conjugate, $p \geq 2$, we have:

$$\begin{aligned}
\Theta_T &= e^{-\delta\theta T}\mathbb{E}\Big(e^{\delta(1-\theta)\int_0^T -\Phi_s ds} \int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big) \\
&\leq e^{-\delta\theta T}\mathbb{E}\big(e^{p\delta(1-\theta)\int_0^T -\Phi_s ds}\big)^{\frac{1}{p}} \mathbb{E}\Big(\Big[\int_T^\infty \delta e^{-\delta(s-T)} c_s^{1-\gamma} ds\Big]^q\Big)^{\frac{1}{q}}.
\end{aligned} \tag{3.32}$$

As $\delta e^{-\delta(s-T)} ds$ defines a probability measure on $[T, \infty)$, we can apply Jensen's inequality to bound the second expectation in (3.32):

$$\Theta_T \leq e^{-\delta\theta T}\mathbb{E}\big(e^{p\delta(1-\theta)\int_0^T -\Phi_s ds}\big)^{\frac{1}{p}} \mathbb{E}\Big(\int_T^\infty \delta e^{-\delta(s-T)} c_s^{q(1-\gamma)} ds\Big)^{\frac{1}{q}}. \tag{3.33}$$

Part a. We begin by rewriting the form of Φ_t in a more workable form:

$$\begin{aligned}
\Phi_t &= \Big(\mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta(s-t)} e^{(1-\phi)\mu(s-t)} e^{(1-\phi)\int_t^s X_r dr} e^{(1-\phi)\int_t^s \sqrt{v_r} dB_r} ds\Big]\Big)^{-1} \\
&= \Big(\int_t^\infty \delta e^{-\delta(s-t)} e^{(1-\phi)\mu(s-t)} \mathbb{E}_t\Big[e^{(1-\phi)\int_t^s X_r dr} e^{(1-\phi)\int_t^s \sqrt{v_r} dB_r}\Big] ds\Big)^{-1} \\
&= \Big(\int_t^\infty \delta e^{-\delta(s-t)} e^{(1-\phi)\mu(s-t)} \mathbb{E}_t\Big[e^{(1-\phi)\int_t^s X_r dr}\Big] \mathbb{E}_t\Big[e^{(1-\phi)\int_t^s \sqrt{v_r} dB_r}\Big] ds\Big)^{-1}.
\end{aligned} \tag{3.34}$$

Above, the second line follows from conditional Fubini's theorem and the third line from conditional independence.

Conditional on $\mathcal{F}_t$ (or, due the Markovian property of X, it is equivalent to condition on X_t), $\{X_r\}_{r\in[t,s]}$ is an Ornstein-Uhlenbeck process with initial data X_t. Therefore, given $\mathcal{F}_t$, the distribution of $\int_t^s X_r dr$ is a Gaussian distribution with conditional mean $X_t \int_t^s e^{-a(r-t)} dr$ and conditional variance that is bounded above above by $\frac{b^2}{a^2}(s-t)$ (cf. Lemma 3.4.2). The conditional MGF of $\int_t^s X_r dr$ is bounded as follows:

$$
\begin{aligned}
\mathbb{E}_t\big(e^{(1-\phi)\int_t^s X_r dr}\big) &= e^{(1-\phi)\mathbb{E}_t(\int_t^s X_r dr)+\frac{1}{2}(1-\phi)^2\operatorname{Var}_t(\int_t^s X_r dr)} \\
&\leq e^{(1-\phi)\int_t^s e^{-a(r-t)}dr X_t} e^{\frac{(1-\phi)^2 b^2}{2a^2}(s-t)} \\
&\leq e^{\frac{|(1-\phi)X_t|}{a}} e^{\frac{(1-\phi)^2 b^2}{2a^2}(s-t)}.
\end{aligned}
\tag{3.35}
$$

Similarly, by conditioning on $\mathcal{F}_t$, $\{v_r\}_{r\in[t,s]}$ is a CIR process starting from v_t. We can use the same trick in Lemma 3.4.2, where we condition additionally on the path $\{v_u\}_{u\in[t,s]}$. Let $\sigma(v_u, t\leq u\leq s)$ be the σ-algebra generated by v between time $[t,s]$, and $\mathcal{F}_t \vee \sigma(v_u, t\leq u\leq s)$ be the smallest σ-algebra containing $\mathcal{F}_t$ and $\sigma(v_u, t\leq u\leq s)$, we have:

$$
\begin{aligned}
\mathbb{E}_t\Big(e^{(1-\phi)\int_t^s \sqrt{v_r}dB_r}\Big) &= \mathbb{E}_t\Big(\mathbb{E}\Big[e^{(1-\phi)\int_t^s \sqrt{v_r}dB_r}\big|\mathcal{F}_t \vee \sigma(v_u, t\leq u\leq s)\Big]\Big) \\
&= \mathbb{E}_t\Big(e^{\frac{(1-\phi)^2}{2}\int_t^s v_r dr}\Big|\mathcal{F}_t\Big) \\
&= A\Big(s-t, \frac{(1-\phi)^2}{2}\Big)\exp\Big(B\Big[s-t, \frac{(1-\phi)^2}{2}\Big]v_t\Big).
\end{aligned}
\tag{3.36}
$$

An estimate for the growth rate of A is provided by equation (3.26). B is bounded above by $(1-\phi)^2/(\kappa+\Gamma(\frac{1}{2}(1-\phi)^2))$. For brevity, we shall denote $\tilde{\Gamma} = \Gamma(\frac{1}{2}(1-\phi)^2)$:

$$
\mathbb{E}_t\big(e^{(1-\phi)\int_t^s \sqrt{v_r}dB_r}\big) \leq \exp\Big(\frac{\kappa\eta}{\lambda^2}(\kappa-\tilde{\Gamma})(s-t) + \frac{(1-\phi)^2}{\kappa+\tilde{\Gamma}}v_t\Big).
\tag{3.37}
$$

Combining equation (3.34) and inequalities (3.35) & (3.37), we achieve the following estimate for Φ_t:

$$
\begin{aligned}
\Phi_t &\geq \Big\{e^{\frac{|(1-\phi)X_t|}{a}} e^{\frac{(1-\phi)^2}{\kappa+\tilde{\Gamma}}v_t}\int_t^\infty \delta e^{-\delta(s-t)}e^{(1-\phi)\mu(s-t)}e^{\frac{(1-\phi)^2 b^2}{2a^2}(s-t)}e^{\frac{\kappa\eta}{\lambda^2}(\kappa-\tilde{\Gamma})(s-t)}ds\Big\}^{-1} \\
&= \Big\{\frac{\delta}{\delta-\zeta}e^{\frac{|(1-\phi)X_t|}{a}}e^{\frac{(1-\phi)^2}{\kappa+\tilde{\Gamma}}v_t}\Big\}^{-1} \\
&= \frac{\delta-\zeta}{\delta}\exp\Big(-\frac{|(1-\phi)X_t|}{a} - \frac{(1-\phi)^2}{\kappa+\tilde{\Gamma}}v_t\Big).
\end{aligned}
\tag{3.38}
$$

where $\zeta \triangleq (1-\phi)\mu + \frac{(1-\phi)^2 b^2}{2a^2} + \frac{\kappa\eta}{\lambda^2}(\kappa - \tilde{\Gamma})$. As a consequence of inequality (3.5b), $\zeta < \delta$, which ensures finiteness of the integral in (3.38). Φ_t can be estimated further by the following inequality[1]:

$$e^{-x} \geq e^{-\epsilon}(1+\epsilon) - e^{-\epsilon}x, \quad \forall \epsilon, x \in \mathbb{R}, \tag{3.39}$$

which leads to the following estimate:

$$\begin{aligned} e^{p\delta(1-\theta)\int_0^T -\Phi_t dt} \leq \exp\Big(&-p(\delta-\zeta)(1-\theta)e^{-\epsilon}(1+\epsilon)T + p(\delta-\zeta)(1-\theta)e^{-\epsilon}\frac{|1-\phi|}{a}\int_0^T |X_t| dt \\ &+ p(\delta-\zeta)(1-\theta)\frac{(1-\phi)^2}{\kappa+\tilde{\Gamma}}\int_0^T v_t dt\Big). \end{aligned} \tag{3.40}$$

To ease the notation, let us define $m = p(\delta-\zeta)(1-\theta)e^{-\epsilon}$, which reduces this inequality to:

$$\begin{aligned} e^{p\delta(1-\theta)\int_0^T -\Phi_t dt} \leq \exp\Big(&-m\Big[1+\log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big)\Big]T + \frac{m|1-\phi|}{a}\int_0^T |X_t| dt \\ &+ \frac{m(1-\phi)^2}{\kappa+\tilde{\Gamma}}\int_0^T v_t dt\Big). \end{aligned} \tag{3.41}$$

It is difficult to obtain the distribution of $\int_0^T |X_t|$, or indeed even an upper bound for its moment generating function. However, the distribution of $\int_0^T X_t^2 dt$ is known[2]. Using the inequality $|x| \leq x^2 + \frac{1}{4}$ for all real x, we obtain:

$$\begin{aligned} e^{p\delta(1-\theta)\int_0^T -\Phi_t dt} \leq \exp\Big(&-m\Big[1+\log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big) - \frac{|1-\phi|}{4a}\Big]T \\ &+ \frac{m|1-\phi|}{a}\int_0^T X_t^2 dt + \frac{m(1-\phi)^2}{\kappa+\tilde{\Gamma}}\int_0^T v_t dt\Big). \end{aligned} \tag{3.42}$$

[1] The right hand side of (3.39) is the tangent line of e^{-x} at the point $(\epsilon, e^{-\epsilon})$. The inequality follows from convexity of the mapping $x \to e^{-x}$.

[2] By Itô's formula: $dX_t^2 = 2a(\frac{b^2}{2a} - X_t^2)dt + 2b\sqrt{X_t^2}dW_t$, $X_0^2 = 0$. Therefore, it is a CIR process.

The expectation of the right hand side above can be computed readily by bounding the MGF of the time integral of a CIR process (cf. Lemma 3.4.1 and Lemma 3.4.2).

$$
\begin{aligned}
&\mathbb{E}(e^{p\delta(1-\theta)\int_0^T -\Phi_t dt}) \leq \exp\Big(-m\Big[1+\log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big)-\frac{|1-\phi|}{4a}\Big]T\Big)\times \\
&\quad \mathbb{E}\Big[\exp\Big(\frac{m|1-\phi|}{a}\int_0^T X_t^2 dt\Big)\Big]\mathbb{E}\Big[\exp\Big(\frac{m(1-\phi)^2}{\kappa+\tilde{\Gamma}}\int_0^T v_t\Big)\Big] \\
&\leq K\exp\Big(-m\Big[1+\log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big)-\frac{|1-\phi|}{4a}\Big]T+\frac{1}{2}\Big(a-\sqrt{a^2-2b^2\frac{m|1-\phi|}{a}}\Big)T \\
&\quad +\frac{\kappa\eta}{\lambda^2}\Big(\kappa-\sqrt{\kappa^2-2\lambda^2\frac{m(1-\phi)^2}{\kappa+\tilde{\Gamma}}}\Big)T\Big).
\end{aligned}
\tag{3.43}
$$

<u>Part b.</u> The second expectation in equation (3.33) is bounded from above by combining equation (3.19) and Lemma 3.4.2:

$$
\begin{aligned}
&\mathbb{E}\Big(\int_T^\infty \delta e^{-\delta(s-T)}c_s^{q(1-\gamma)}ds\Big) \leq K\int_T^\infty e^{-\delta(s-T)}e^{q(1-\gamma)\mu s+\frac{q^2(1-\gamma)^2b^2}{2a^2}s+\frac{\kappa\eta}{\lambda^2}\left(\kappa+\Gamma\left[\frac{q^2(1-\gamma)^2}{2}\right]\right)s} \\
&= Ke^{q(1-\gamma)\mu T+\frac{q^2(1-\gamma)^2b^2}{2a^2}T+\frac{\kappa\eta}{\lambda^2}\left(\kappa+\Gamma\left[\frac{q^2(1-\gamma)^2}{2}\right]\right)T} \\
&\quad \times\int_T^\infty e^{-\delta(s-T)}e^{q(1-\gamma)\mu(s-T)+\frac{q^2(1-\gamma)^2b^2}{2a^2}(s-T)+\frac{\kappa\eta}{\lambda^2}\left(\kappa+\Gamma\left[\frac{q^2(1-\gamma)^2}{2}\right]\right)(s-T)}ds \\
&= Ke^{q(1-\gamma)\mu T+\frac{q^2(1-\gamma)^2b^2}{2a^2}T+\frac{\kappa\eta}{\lambda^2}\left(\kappa+\Gamma\left[\frac{q^2(1-\gamma)^2}{2}\right]\right)T}.
\end{aligned}
\tag{3.44}
$$

Above, the second equality follows from a change of variable, which implies that the integral in the second line is independent of T.

<u>Final Estimate of Θ_T:</u> We are now ready to obtain the final upper bound of Θ_T. Combining the estimates (3.33), (3.43) and (3.44), we see that Θ_T is of the form e^{RT}, where R equals:

$$
\begin{aligned}
&-\delta\theta-\frac{m}{p}\Big[1+\log\Big(\frac{p(\delta-\zeta)(1-\theta)}{m}\Big)-\frac{|1-\phi|}{4a}\Big]+\frac{1}{2p}\Big(a-\sqrt{a^2-2b^2\frac{m|1-\phi|}{a}}\Big) \\
&\quad +\frac{\kappa\eta}{p\lambda^2}\Big(\kappa-\sqrt{\kappa^2-2\lambda^2\frac{m(1-\phi)^2}{\kappa+\Gamma(\frac{1}{2}[1-\phi]^2)}}\Big)+(1-\gamma)\mu+\frac{q^2(1-\gamma)^2b^2}{2qa^2}+\frac{\kappa\eta}{q\lambda^2}\Big[\kappa-\Gamma\Big(\frac{q^2[1-\gamma]^2}{2}\Big)\Big].
\end{aligned}
\tag{3.45}
$$

□

3.4.3 Proof of Lemma 3.4.3

Lemma 3.4.3. *Let κ and λ be positive constants, then:*

i. $\lim_{\frac{\lambda}{\kappa}\to 0} \frac{\kappa}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 2\lambda^2 z}) = z$;

ii. if $h(\kappa,\lambda)$ *satisfies* $\lim_{\kappa\to\infty,\lambda\to 0} h(\kappa,\lambda) = 0$, *then:*

$$\lim_{\kappa\to\infty,\lambda\to 0} \frac{\kappa}{\lambda^2}\left(\kappa - \sqrt{\kappa^2 - 2\lambda^2 h(\kappa,\lambda)}\right) = 0; \tag{3.46}$$

iii. $\lim_{\phi\to 1} \zeta(1-\phi)^{-1}$, *where ζ is defined in Theorem 3.2.2.*

Part i. The proof follows from an observation that the desired limit can be re-expressed as the difference quotient of a certain function, which allows use of calculus results:

$$\begin{aligned}\lim_{\frac{\lambda}{\kappa}\to 0} \frac{\kappa}{\lambda^2}(\kappa - \sqrt{\kappa^2 - 2\lambda^2 z}) &= \lim_{\frac{\lambda}{\kappa}\to 0} \frac{1 - \sqrt{1 - 2z\frac{\lambda^2}{\kappa^2}}}{\frac{\lambda^2}{\kappa^2}} \\ &= \frac{d}{dx}\sqrt{1-2zx}|_{x=0} \\ &= z.\end{aligned} \tag{3.47}$$

Part ii. Let $\epsilon > 0$ be arbitrarily small. Then, for sufficiently large κ and small λ, $h(\lambda,\kappa) \le \epsilon$. Therefore:

$$0 \le \limsup_{\kappa\to\infty,\lambda\to 0} \frac{\kappa}{\lambda^2}\left(\kappa - \sqrt{\kappa^2 - 2\lambda^2 h(\kappa,\lambda)}\right) \le \lim_{\kappa\to\infty,\lambda\to 0} \frac{\kappa}{\lambda^2}\left(\kappa - \sqrt{\kappa^2 - 2\lambda^2\epsilon}\right) = \epsilon. \tag{3.48}$$

As ϵ is arbitrary, the limit superior above is just a limit and equal to 0.

Part iii. The convergence of the first two terms of ζ is obvious. As for the last term:

$$\begin{aligned}\lim_{\phi\to 1} \frac{\kappa}{\lambda^2}\left(\kappa - \sqrt{\kappa^2 - \lambda^2(1-\phi)^2}\right)(1-\phi)^{-1} &= -\frac{\kappa}{\lambda^2}\lim_{\epsilon\to 0}\frac{\sqrt{\kappa^2-\lambda^2\epsilon^2}-\kappa}{\epsilon} \\ &= -\frac{d}{dx}\sqrt{\kappa^2-\lambda^2x^2}|_{x=0} = 0.\end{aligned} \tag{3.49}$$

□

CHAPTER **4**

PORTFOLIO OPTIMISATION: AN OVERVIEW

4.1 Preliminaries, Problem Formulation & Notations

Consider a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the usual augmentation of the filtration generated by B, an $\mathbb{R}^{n+k}$-valued Brownian Motion. Let us denote by W and $W^{\perp}$ the first k and last n dimensions of B, respectively.

We work with a financial market model consisting of a riskless asset S^0 and an n-tuple of risky assets $S = (S^1, ..., S^n)$. Their dynamics are given by the stochastic differential equations:

$$\begin{aligned}
dS_t^0 &= S_t^0 r(Y_t)dt, \\
dS_t &= \mathrm{diag}(S_t)\left[(r(Y_t)1_n + \mu(Y_t))dt + \sigma(Y_t)dW_t^{\rho}\right], \\
dY_t &= b(Y_t)dt + a(Y_t)dW_t, \quad Y_0 = y \in E.
\end{aligned} \tag{4.1}$$

Above, $\mathrm{diag}(S_t)$ is an n-dimensional diagonal matrix with $(S_t^1, ..., S_t^n)$ along the diagonal, 1_n is an n-dimensional vector with value 1 in every entry. W^{ρ} is an $\mathbb{R}^n$-valued defined by $W^{\rho} \triangleq \int_0^{\cdot} \rho(Y_s)dW_s + \int_0^{\cdot} \rho^{\perp}(Y_s)dW_s^{\perp}$, where $\rho : \mathbb{R}^k \to \mathbb{R}^{n\times k}$ and $\rho^{\perp} : \mathbb{R}^k \to \mathbb{R}^{n\times n}$ are correlation functions satisfying $\rho\rho' + \rho^{\perp}(\rho^{\perp})' = 1_{n\times n}$, the $n \times n$-identity matrix. By construction, W^{ρ} admits ρ as its instantaneous correlation-matrix with W. Moreover, Y is the state process valued in an open domain $E \subseteq \mathbb{R}^k$. Lastly, the model coefficients are the following functions: $r : E \to \mathbb{R}, \mu : E \to \mathbb{R}, \sigma : E \to \mathbb{R}^{n\times n}, b : E \to \mathbb{R}^k$ and $a : E \to \mathbb{R}^{k\times k}$. We also denote: $A = aa', \Sigma = \sigma\sigma'$ and $\Upsilon = \sigma\rho a'$.

An agent with initial wealth x invests in this financial market on an infinite horizon by choosing a progressively measurable investment-consumption strategy $(\pi_t, l_t)_{t\geq 0}$ (formally defined later in Definition 4.1.4). Here, $\pi_t = (\pi_t^1, .., \pi_t^n)$ is the proportion of his wealth invested in the n risky assets at time t, and $c_t \triangleq l_t X_t$ is the instantaneous consumption rate. The quantity l_t will henceforth be referred to as the *consumption-wealth ratio*. The resulting wealth process is given by:

$$dX_t^{\pi,l} = X_t^{\pi,l}[(r_t + \pi_t'\mu_t - l_t)dt + \pi_t'\sigma_t dW_t^\rho], \quad X_0^{\pi,l} = x. \tag{4.2}$$

Given a strategy (π, l), the agent's derived utility from it is given by the Epstein–Zin utility of the consumption process $c = lX^{\pi,l}$. In particular, let $V^{\pi,l}$ denote this utility process, it is defined as the solution to the infinite-horizon BSDE:

$$V_t^{\pi,l} = \mathbb{E}_t\Big(V_T^{\pi,l} + \int_t^T f(c_s, V_s^{\pi,l})ds\Big), \quad 0 \leq t \leq T, \tag{4.3}$$

which also satisfies the Power Utility Bounds (cf. Definition 2.4.4). If we restrict our studies only to strategies that admit a unique Epstein–Zin value process (cf. Theorem 2.4.1 & 2.4.2), then Epstein–Zin utility provides the agent with a method to rank different investment strategies. The agent, therefore, aims to maximising his derived utility by finding a strategy (π^*, l^*) such that:

$$V_0^* = V_0^{\pi^*,l^*} = \sup_{(\pi,l)\in\mathcal{A}} V_0^{\pi,c}, \tag{4.4}$$

where $\mathcal{A}$ is a suitable admissible class of investment-consumption strategies, which is defined formally in Definition 4.1.4.

Before we start approaching problem (4.4), let us define below the relevant real analysis notations, as well as the assumptions that will apply throughout.

Definition 4.1.1. For $d \in \mathbb{Z}^+$ and $O \subseteq \mathbb{R}^d$, let $C^m(E, O)$ be the space of m-times continuously differentiable functions from E to O. Moreover, let $C^{m,\alpha}(E, O)$ be the subspace of $C^m(E, O)$ such that all member functions and their partial derivatives up to m-th order are locally α-Hölder continuous. When the the co-domain is clear from the context, we suppress O for notational simplicity.

Assumption 4.1.2. Throughout the rest of the thesis, we assume that these model coefficients and preference parameters satisfy the following assumptions:

i. For some $\alpha \in (0,1)$, $r \in C^{1,\alpha}(E,\mathbb{R}^k)$, $b \in C^{1,\alpha}(E,\mathbb{R}^k)$, $\mu \in C^{1,\alpha}(E,\mathbb{R}^n)$, $A \in C^{2,\alpha}(E;\mathbb{R}^{k\times k})$, $\Sigma \in C^{2,\alpha}(E;\mathbb{R}^{n\times n})$, and $\Upsilon \in C^{2,\alpha}(E;\mathbb{R}^{n\times k})$. Moreover, assume that A and Σ are strictly positive definite for all $y \in E$.

ii. The factor process Y exists globally and does not escape its domain E in finite time.

iii. Let $\underline{\rho}$ and $\bar{\rho}$ denote the minimum and maximum eigenvalues for $\rho\rho'$, respectively. Assume that they satisfy the following inequality:

$$\phi|\theta|\gamma(1-\bar{\rho}) + \gamma\phi(1-\underline{\rho}) + \gamma(\underline{\rho}-\bar{\rho}) \geq 0. \tag{4.5}$$

iv. Lastly, we focus on the empirically relevant case where $\phi < 1 < \gamma$.

Remark 4.1.3. Assumption iii. above regarding the instantaneous correlation matrix ρ is naturally satisfied in the case $k = 1$, as in this case $\underline{\rho} = \bar{\rho}$.

Let us now formally define investment-consumption strategies (or simply strategies for short) and their admissible class.

Definition 4.1.4. A pair of progressively measurable stochastic processes $\{(\pi_t, l_t), t \geq 0\}$ is said to be a *strategy* if the following holds:

i. For all $t > 0$, $\int_0^t \left|\pi_s'\mu_s\right| ds < \infty$, $\int_0^t |l_s| ds$ and $\int_0^t \pi_s'\Sigma_s\pi_s ds < \infty$ $\mathbb{P}$-almost surely.

ii. $l_t \geq 0$ for all $t \geq 0$ almost surely.

A strategy (π, l) is said to belong to the admissible class $\mathcal{A}$ if the following holds:

iii. For all $t \geq 0$, $X_t^{\pi,l} \geq 0$.

iv. The resulting consumption process belongs to C^∞ and satisfies the uniqueness criterion (2.22) of Epstein–Zin utility.

The first condition ensures that all integrals in the wealth process is well-defined. The second and third condition together ensure that consumption remains non-negative. This implies that the strategy is self-financing and consumption is well-defined within the Epstein–Zin framework. The third condition implies that the investor can not employ a doubling strategy or consume without investing and remain in debt indefinitel, which is a standard admissibility requirement employed throughout the portfolio optimisation literature. The last condition ensures that the resulting Epstein–Zin utility process is always defined and unique (cf. Theorem 2.4.1 & Theorem 2.4.2).

In the next section, we will outline the method of deriving a candidate optimal strategy (π^*, l^*) via solving the associated Hamilton-Jacobi-Bellman equation. After that, this candidate will be verified to ensure that it is indeed admissible and optimal within a subset of the admissible class.

4.2 Deriving the Hamilton-Jacobi-Bellman equation

We will solve the optimisation problem (4.4) through a Dynamic Programming & Verification approach. First, we make an ansatz for the solution and heuristically derive the HJB equation. Once we confirm the existence of a solution. the candidate optimal strategy is then derived based on this solution and its derivative. Then we will formally verify that this candidate is indeed the optimal strategy.

Let us now derive the HJB equation in the Epstein–Zin case via a heuristic argument. First, we define the *optimal value function* $v(x, y) : (0, \infty) \times E \to \mathbb{R}$ via the following relation:

$$v(x, y) = \sup_{(\pi,l)\in\mathcal{A}} \mathbb{E}\Big(V_0^{\pi,l}\Big|X_0 = x, Y_0 = y\Big). \tag{4.6}$$

We will need v to be sufficiently regular for the next argument. Heuristically, let us assume that it is twice continuously differentiable in all of its arguments. Let (π, l) be an admissible strategy and $X^{\pi,l}$ be the resulting wealth process. The Dynamic Programming Principle suggests that $v(X^{\pi,l}, Y) + \int_0^{\cdot} f(c_t, v(X_t^{\pi,l}, Y_t))dt$ is a supermartingale for arbitrary strategies, and a martingale for the optimal one. For brevity, we write v_t for $v(X_t^{\pi,l}, V_t)$ and we suppress the time subscript as well as the dependence on Y_t of the model coefficients. Then, the drift term of $v + \int_0^\cdot f(c_s, v_s)ds$ is:

$$D_x v X^{\pi,l}(r + \pi'\mu - l) + (D_y v)'b + \frac{1}{2}\operatorname{tr}(AD^2_{yy}v) + \frac{1}{2}(X^{\pi,l})^2\pi'\Sigma\pi D^2_{xx}v + X^{\pi,l}(D_{xy}v)'\Upsilon'\pi + f(c, v). \tag{4.7}$$

By the homotheticity property of Epstein–Zin utility, we speculate that v takes the form $v(x, y) = \frac{x^{1-\gamma}}{1-\gamma}g(y)^{\phi\theta}$ for some $g \in C^2(E, (0, \infty))$. This homothetic decomposition is widely applied, sometimes with a different functional form in place of our $g(y)^{\phi\theta}$, for the stochastic control approach towards the portfolio optimisation problem (see, for instance, [39] , [66] and [26]). Substituting the derivatives of this conjectured v into equation (4.7), we obtain the drift of $\frac{(X^{\pi,l})^{1-\gamma}}{1-\gamma}g(Y)^{\phi\theta} + \int_0^\cdot f(c_s, \frac{(X^{\pi,l})_s^{1-\gamma}}{1-\gamma}g(Y_s)^{\phi\theta})ds$, as follows (time subscript and Y_t argument are suppressed for brevity):

$$\begin{aligned}(X_t^{\pi,l})^{1-\gamma}g^{\phi\theta}\Big\{&r + \pi'\mu - l - \frac{\gamma}{2}\pi'\Sigma\pi + \frac{\phi\theta\nabla g'b}{(1-\gamma)g} + \frac{\phi\theta(\phi\theta-1)\nabla g'A\nabla g}{2(1-\gamma)g^2} \\ &+ \frac{\phi\theta\operatorname{tr}(AD^2g)}{2(1-\gamma)g} + \phi\theta\frac{\nabla g'}{g}\Upsilon'\pi + \frac{\delta}{1-\phi}l^{1-\phi}g^{-\phi} - \frac{\delta}{1-\phi}\Big\}.\end{aligned} \tag{4.8}$$

From the heuristics discussed above, we expect this drift to be non-positive for arbitrary (π, l) and zero for the optimal pair, leading to the following equation:

$$\begin{aligned}r + \frac{\delta}{\phi-1} + \frac{\phi\theta\nabla g'b}{(1-\gamma)g} + \frac{\phi\theta(\phi\theta-1)\nabla g'A\nabla g}{2(1-\gamma)g^2} + \frac{\phi\theta\operatorname{tr}(AD^2g)}{2(1-\gamma)g}& \\ + \sup_{(\pi,l)}\left\{\pi'\mu + \phi\theta\pi'\Upsilon\frac{\nabla g}{g} - \frac{\gamma}{2}\pi'\Sigma\pi + \frac{\delta}{1-\phi}l^{1-\phi}g^{-\phi} - l\right\} &= 0.\end{aligned} \tag{4.9}$$

Under the parameter configuration $\phi < 1 < \gamma$, the optimisation problem in (4.9) is strictly concave globally and admits a maximiser in the interior, which is given by the first order condition:

$$\pi^* = \frac{1}{\gamma}\Sigma^{-1}\mu + \frac{\phi\theta}{\gamma}\Sigma^{-1}\Upsilon\frac{\nabla g}{g}, \quad l^* = \delta^\psi g^{-1}. \tag{4.10}$$

Substituting the first order conditions into (4.9), we attain the PDE for g, which henceforth we shall refer to as the *Epstein–Zin HJB equation*. Due to the role of g in the value function, we will consider only *positive* solution(s). Given a classical $C^2(E, (0, \infty))$ solution g, we will henceforth refer to the derived pair (π^*, l^*) in (4.10) as the *candidate optimal strategy*. We also introduce the notation for the Epstein–Zin HJB equation below:

$$\begin{aligned}\mathcal{H}^{\mathrm{EZ}}(y,g,\nabla g,D^2g|\phi,\gamma) &= \delta^{\psi}g^{-1}+\frac{\nabla g'}{g}\Big(b+\frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\Big)+\frac{1}{2}\frac{\nabla g'}{g}\Big\{(\phi\theta-1)A\\ &+\phi\theta\Big(\frac{1-\gamma}{\gamma}\Big)\Upsilon'\Sigma^{-1}\Upsilon\Big\}\frac{\nabla g}{g}+\frac{\operatorname{tr}(AD^2g)}{2g}+\Big(r+\frac{\delta}{\phi-1}+\frac{\mu'\Sigma^{-1}\mu}{2\gamma}\Big)\Big(\frac{1-\phi}{\phi}\Big)=0,\quad y\in E.\end{aligned}\tag{4.11}$$

We also remark that in equation (4.7), if we conjectured the form $v(x,y)=\frac{x^{1-\phi}}{1-\phi}g^{\phi}(y)$ and replaced the Epstein–Zin aggregator f with the mapping $f^{(\phi)}:(0,\infty)\times(0,\infty)\to\mathbb{R}$, $f^{(\phi)}(c,v)=\delta\Big(\frac{c^{1-\phi}}{1-\phi}-v\Big)$, it will lead to the following equation:

$$\begin{aligned}\mathcal{H}^{\mathrm{CRRA}}(y,g,\nabla g,D^2g|\phi) &= \delta^{\frac{1}{\phi}}g^{-1}+\frac{\operatorname{tr}(AD^2g)}{2g}+\frac{\nabla g'}{g}\Big(b+\frac{1-\phi}{\phi}\Upsilon\Sigma^{-1}\mu\Big)\\ &-\frac{1-\phi}{2}\frac{\nabla g'}{g}(A-\Upsilon'\Sigma^{-1}\Upsilon)\frac{\nabla g}{g}+\Big(r-\frac{\delta}{1-\phi}+\frac{\mu\Sigma^{-1}\mu}{2\phi}\Big)\Big(\frac{1-\phi}{\phi}\Big)=0.\end{aligned}\tag{4.12}$$

This equation corresponds the portfolio optimisation problem of an agent following CRRA utility with parameter ϕ. Henceforth, we shall refer to this equation as the *CRRA HJB equation*. [1]

Remark 4.2.1. We make the following remarks regarding the notation used for the rest of the thesis.

- We retain the superscripts in $\mathcal{H}^{\mathrm{EZ}}$ and $\mathcal{H}^{\mathrm{CRRA}}$ at all time to emphasise the type of HJB equations being considered.
- For brevity, we shall abbreviate $\mathcal{H}^{\mathrm{EZ}}(y,g,\nabla g,D^2g|\phi,\gamma)$ to $\mathcal{H}^{\mathrm{EZ}}(y,g,\nabla g,D^2g)$ when it is clear from the context. In later sections, when we need to transform the agent's preference parameters, we will make it explicit in the arguments of $\mathcal{H}^{EZ}$. This and the previous point will be important, as in the proofs of Chapter 5, we will be comparing utilities of agents with different aggregator functions and parameters.
- When we consider constant strategies, we will identify the constant stochastic process (π,l) with a point in $\mathbb{R}^{n+1}$.

[1] See [26] for an example of the CRRA HJB equation in infinite horizon. Here, ϕ plays the role of the risk aversion parameter, which is typically denoted γ. For our purpose, though, we will need to re-parametrise this risk aversion to ϕ (cf. the development of the supersolution in Chapter 5).

4.2.1 Existing Results: An Overview

In this section, let us survey existing results in the literature towards portfolio optimisation with Epstein–Zin utility in finite horizon, and the difficulties encountered in generalising them to infinite horizon. In the finite horizon case, the HJB equation is a semi-linear parabolic PDE in time and space, with boundary data at termination. Kraft et. al [39] studies this problem under an additional parameter restriction $\psi = 2 - \gamma + \frac{(1-\gamma)^2\rho^2}{\gamma}$ (condition (H) therein), which helps to linearise the HJB equation and facilitate a Feynman–Kac representation of the solution. Beside the obvious disadvantage of additional model constraints, this also excludes the empirically relevant case i.e. $\gamma, \psi > 1$. A more satisfactory approach is offered by Kraft et. al [37], where semi-linearity is resolved by combining traditional Feynman–Kac method for linear parabolic PDEs with a system of FBSDE(s) (see also [51] for the general method). This was achieved at the cost of boundedness of model coefficients. However, equation (4.11) poses a different challenge in comparison to its finite-horizon counterparts. The extension from finite to infinite horizon fundamentally changes the class of equations it belongs to. In this case, terminal data is removed and we are facing a *boundary-free* elliptic quasilinear equation in an open domain, which renders the aforementioned methods inapplicable.

From a BSDE perspective, in finite horizon, Xing [66] derives an *Epstein–Zin HJB BSDE* which parallels the HJB equation seen in [39] and [37]. An improvement over the [37] is that [66] allowed for models with unbounded coefficients, which encompass the Heston model and Kim & Ongberg model. However, if we were to generalise the method of [66], it would involve a strictly infinite horizon BSDE, which also removes its terminal data. One might consider repeatedly solving the Epstein–Zin BSDE with increasing horizon length and take limit. However, theoretically, it is unclear whether the finite horizon solution would converge when we let the horizon length T diverge; numerically, Xing remarked that in the empirically relevant setting, the convergence from finite to infinite horizon can be extremely slow, suggesting that finite horizon optimal strategies can substantially differ from their infinite horizon analogue.

Another approach that we have explored is the method of duality, which was studied by Matoussi & Xing [46] for finite horizon Epstein–Zin utility. The main idea therein is to show that, for an admissible class of consumption, Epstein–Zin utilities are bounded above by the so-called *stochastic differential dual* (SDD), and verify that certain strategy attains this upper bound and closes the *duality gap*. There are at least several difficulties with this approach. First, defining infinite horizon SDD, similar to

Epstein–Zin SDU, involves solving an infinite horizon BSDE with non-Lipschitz and non-monotone driver. The approach in Chapter 2 is not applicable, since we no longer have a priori bounds (power utility bounds, cf. Definition 2.4.4). Secondly, verification in this method requires optimising both the Epstein–Zin SDU *and* SDD, which requires too much regularity in infinite horizon.

CHAPTER **5**

PORTFOLIO OPTIMISATION: MAIN RESULTS

5.1 Existence of Solutions to the HJB Equation

In the construction of Epstein–Zin utility in infinite horizon, we have overcome this absence of boundary by imposing additional structures, namely the power utility bounds. A similar strategy is adapted and employed for the Epstein–Zin HJB partial differential equation[1]. We will tackle the question of existence by constructing a sub-solution and super-solution, through an appropriately chosen strategy (π, l) and risk premium η for unhedgeable risk. These sub- and super-solution will provide a basis for a sandwich-type construction. Let us begin with the definition of sub- and super-solution of a PDE.

Definition 5.1.1. Consider a boundary-free PDE of the form:

$$Qg = 0, \quad y \in E, \tag{5.1}$$

where E is an open domain and Q an elliptic quasilinear differential operator of the form $Qg = \frac{1}{2}\operatorname{tr}(AD^2g) + B(y, g, Dg)$. A function $\bar{g}$ (resp. $\underline{g}$): $E \to \mathbb{R}$ is said to be a *supersolution* (resp.

[1]Our approach towards solving equation (4.11) is inspired by the work in time-additive setting of Hata & Sheu [27] and Guasoni & Wang [26] and can be considered a generalisation of the methods therein.

subsolution) of 5.1 if it is $C^2(E)$ and:

$$Q\bar{g} \le 0 \quad (\text{resp. } Q\underline{g} \ge 0), \quad y \in E. \tag{5.2}$$

Moreover, if $\bar{g}(y) \ge \underline{g}(y)$ for all $y \in E$, we say that they are an *ordered pair of sub- and super-solution*. Moreover, if a function g is both a subsolution and a supersolution, it is a solution.

Definition 5.1.2. Let Q be the differential operator considered in Definition 5.1.1 and suppose that E is an open bounded domain. Consider the boundary value problem:

$$\begin{aligned} Qg &= 0, \quad y \in E_n; \\ g &= \tilde{g}, \quad y \in \partial E. \end{aligned} \tag{5.3}$$

A function $\bar{g}$ (resp. $\underline{g}$): $E \longrightarrow \mathbb{R}$ is said to be a *supersolution* (resp. *subsolution*) of (5.3) if it is $C^2(E)$ and:

$$\begin{aligned} Q\bar{g} \le 0 \quad &(\text{resp. } Q\underline{g} \ge 0), \quad y \in E, \\ \bar{g} \ge \tilde{g} \quad &(\text{resp. } \underline{g} \le \tilde{g}), \quad y \in \partial E. \end{aligned} \tag{5.4}$$

Moreover, if $\bar{g}(y) \ge \underline{g}(y)$ for all $y \in E$, we say that they are an *ordered pair of sub- and supersolution*. Moreover, if a function g is both a subsolution and a supersolution, it is a solution.

Remark 5.1.3. In the context of the Epstein–Zin HJB equation (4.11), we will only consider positive sub- and super-solutions.

We will now develop the sub- and super-solutions required for our sandwich argument. We first heuristically formulate two candidate functions, then facilitate the technical conditions under which they can be verified as sub- and super-solutions. Let (π, l) be an admissible control, $X^{\pi,l}$ be the resulting wealth process and $c = lX^{\pi,l}$ the resulting consumption process. Denote by $U^{\gamma,\infty}(c)$ the utility process derived by an agent with additive power utility with risk aversion γ (cf. Definition 2.4.2). By Definition 2.4.4, $U^{\gamma,\infty}(c)$ is a lower bound for $V^{\pi,l}$ and consequently, $U^{\gamma,\infty}(c)_0 \le V_0^*$. Thus, if we defined g_1 as follows:

$$\frac{x^{1-\gamma}}{1-\gamma} g_1(y)^{\phi\theta} = \mathbb{E}\Big(\int_0^\infty \delta e^{-\delta s} u_\gamma(c_s) ds \Big| Y_0 = y \Big), \tag{5.5}$$

then: $\frac{x^{1-\gamma}}{1-\gamma} g_1(y)^{\phi\theta} \leq \frac{x^{1-\gamma}}{1-\gamma} g^{\phi\theta}(y)$, and thus $g_1(y) \leq g(y)$. And therefore, we expect g_1 to be a sub-solution of the Epstein–Zin HJB equation.

Before we proceed further, we briefly review the following concepts. Let $\mathcal{E}$ denote the stochastic exponential. That is, if Z is a local martingale, then $\mathcal{E}(X) = \exp\left(X - \frac{1}{2}[X]\right)$. For every progressively measurable, $\mathbb{R}^k$-valued process η, referred to as the risk-premium for unhedgeable risk, $M^\eta S$ is a local martingale, where:

$$M_t^\eta \triangleq e^{-\int_0^t r_s ds} \mathcal{E}\Big(\int_0^\cdot (\mu'\Sigma^{-1} + \eta_s'\Upsilon'\Sigma^{-1})\sigma dW_s^\rho + \int_0^\cdot \eta_s' a dW_s \Big)_t. \tag{5.6}$$

From Lemma A.1 of [26], we have, for any η and admissible control (π, l):

$$\mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t} \frac{c_t^{1-\phi}}{1-\phi} dt \Big) \leq \frac{x^{1-\phi}}{1-\phi} \mathbb{E}\Big(\int_0^\infty \delta^\psi e^{-\frac{\delta}{\phi}t} (M_t^\eta)^{\frac{\phi-1}{\phi}} dt \Big)^\phi. \tag{5.7}$$

The left hand side quantity above is but $Y^{\phi,\infty}(c)$, the power utility of an agent with risk aversion parameter ϕ (cf. Definition 2.4.2). Also, by Power Utility Bounds (cf. Definition 2.4.4), $V^{\pi,l} \leq u_\gamma \circ u_\phi^{-1}(Y^{\phi,\infty})$. This, combined with the fact that (π, l) was arbitrary, gives an upper bound to the optimisation problem:

$$\sup_{(\pi,l)\in\mathcal{A}} V_0^{\pi,l} \leq \frac{x^{1-\gamma}}{1-\gamma} \Big(\int_0^\infty \delta^\psi e^{-\frac{\delta}{\phi}t} (M_t^\eta)^{\frac{\phi-1}{\phi}} dt \Big)^{\phi\theta}. \tag{5.8}$$

Therefore, if we define g_2 as follows:

$$g_2(y) \triangleq \mathbb{E}\Big[\int_0^\infty e^{-\frac{\delta}{\phi}t} (M_t^\eta)^{\frac{\phi-1}{\phi}} dt \Big| Y_0 = y \Big], \tag{5.9}$$

then $g(y) \leq g_2(y)$ and we conjecture that g_2 is a super-solution to equation (4.11). These heuristic notions are made formal in the following result.

Lemma 5.1.1. *Assume that there exists $l \in C^\alpha(E, \mathbb{R}^+)$, $\pi \in C^\alpha(E, \mathbb{R}^n)$ and $\eta \in C^\alpha(E, \mathbb{R}^k)$ such that:*

i. $\frac{\delta}{\gamma-1} + \pi'\mu - \frac{\mu'\Sigma^{-1}\mu}{2} - l + r \geq 0$,

ii. $\frac{\delta}{1-\phi} - r - \frac{\mu'\Sigma^{-1}\mu + \eta'(A-\Upsilon'\Sigma^{-1}\Upsilon)\eta}{2\phi} \geq 0$, *and*

iii. The functions g_1 and g_2 defined in (5.5) and (5.9) are finite and continuous for all $y \in E$.

Then, the following hold for $y \in E$:

$$\begin{cases} \mathcal{H}^{\mathrm{EZ}}(y, g_2, \nabla g_2, D^2 g_2 | \phi, \gamma) \leq \mathcal{H}^{\mathrm{CRRA}}(y, g_2, \nabla g_2, D^2 g_2 | \phi) \leq 0 & \text{(5.10a)} \\ \mathcal{H}^{\mathrm{EZ}}(y, g_1, \nabla g_1, D^2 g_1 | \phi, \gamma) \geq 0, \textit{ and} & \text{(5.10b)} \\ g_2 \geq g_1. & \text{(5.10c)} \end{cases}$$

Condition i. is satisfied by choosing π so that $\pi'\mu$ is sufficiently large. Condition ii. is satisfied if the interest rate and market price of risks are both bounded from above and δ is sufficiently large. Having facilitated the existence of appropriate sub- and super-solutions, we are now ready to state the main existence result for the Epstein–Zin HJB equation. Although our approach here is inspired by [26], the technical differences between additive and recursive utilities necessitate various modifications in the proof. Thus, a detailed proof will be provided in the Appendix.

Theorem 5.1.2. *Suppose that $g_1 \in C^2(E, (0, \infty))$ is a subsolution to the Epstein–Zin HJB equation and $g_2 \in C^2(E, (0, \infty))$ is a supersolution to the CRRA HJB equation. Assume additionally that $g_1(y) \leq g_2(y)$ for all $y \in E$. Then, there exists a twice continuously differentiable function $g : E \to (0, \infty)$ that solves the Epstein–Zin HJB equation (4.11). Moreover, g satisfies $g_1 \leq g \leq g_2$.*

5.2 Verification

In the last section, we have asserted the existence of a classical solution g to the Esptein-Zin HJB equation (4.11) under the presence of sub- and super-solutions. Moreover, we also provided technical conditions which furnish the sub- and super-solutions necessary for this existence result. The candidate optimal strategy (π^*, l^*) can then be expressed in terms of g via the first order condition (4.10). In this section, we will derive the conditions under which we can verify that (π^*, l^*) solves (4.4).

Let us denote $X^* = X^{\pi^*, l^*}$ and $V^* = \frac{(X_t^*)^{1-\gamma}}{1-\gamma} g(Y_t)^{\phi\theta}$, the candidate optimal value process. From the Epstein–Zin HJB equation (4.11) and Dynamic Programming Principle, it is evident that the V^* satisfies the dynamics of the Epstein–Zin BSDE[2]. However, this is not sufficient to characterise V^* as the value process associated with $l^* X^*$.[3] In order to verify V^* as the correct utility process, we

[2]The candidate solution V_t^* is an Itô process whose drift component equals $-f(c_t^*, V_t^*)dt$, where f is the Epstein–Zin aggregator.

[3]Especially in the empirically relevant case, where $\theta < 0$. See Section 2.6 for an example of non-unique solutions and the difficulties related to negative θ.

need to: i. confirm that (π^*, l^*) belongs to the admissible class $\mathcal{A}$, and ii. confirm that V^* satisfies the appropriate power utility bounds. A by-product achieved by confirming these two points is the sufficient regularity conditions for verifying optimality of (π^*, l^*) amongst a subset of $\mathcal{A}$. Let us name this subset of the set of *permissible* strategy and define it below.

Definition 5.2.1. Given a strictly positive solution g to equation (4.11), a strategy $(\pi, l) \in \mathcal{A}$ is said to be *permissible* with respect to g, denoted $(\pi, l) \in \mathcal{P}(g)$, if the following holds:

$$U^{\gamma,\infty}(c)_t \leq \frac{(X_t^{\pi,l})^{1-\gamma}}{1-\gamma} g(Y_t)^{\phi\theta} \leq U^{\phi,\infty}(c)_t, \qquad \text{a.s. for all } t \geq 0. \tag{5.11}$$

Assuming that the strategy (π^*, l^*) is admissible, then V^* is the Epstein–Zin utility associated with $c^* \triangleq l^* X^*$ if and only if (π^*, l^*) belongs to the class $\mathcal{P}(g)$. The scope of our verification argument will depend on the ability to verify permissibility of (π^*, l^*) as well as arbitrary strategies. This question will be addressed to some capacity in the next lemma, which suggests that a strategy (π, l) is permissible if $\frac{(X^{\pi,l})^{1-\gamma}}{1-\gamma} g(y)\phi\theta$ does not grow or vanish too fast in regards to the agent's discount factor δ.

Lemma 5.2.1. *Let g be a solution of the Epstein–Zin HJB equation (4.11), $(\pi, l) \in \mathcal{A}$ be an admissible strategy and denote $U^{\pi,l} = u_\gamma(X^{\pi,l}) g^{\phi\theta}(Y)$. Moreover, let us define the following process (time subscript and Y_t arguments are suppressed for brevity):*

$$n^{\pi,l} = r + \pi'\mu - l - \frac{\gamma\pi'\Sigma\pi}{2} + \frac{\phi\theta\nabla g' b}{(1-\gamma)g} + \frac{\phi\theta(\phi\theta - 1)\nabla g' A \nabla g}{2(1-\gamma)g^2} + \frac{\phi\theta \operatorname{tr}(AD^2 g)}{2(1-\gamma)g} + \phi\theta\frac{\nabla g'}{g}\Upsilon'\pi + \frac{\delta l^{1-\phi} g^{-\phi}}{1-\phi} - \frac{\delta}{1-\phi}. \tag{5.12}$$

Then:

i. If $|U^{\pi,l}|$ is of class (DL) and satisfies:

$$\liminf_{T\to\infty} \mathbb{E}(e^{-\delta T} |U_T^{\pi,l}|) = 0, \tag{5.13}$$

then $U^{\pi,l} \geq U^{\gamma,\infty}(c)$.

ii. If $|U^{\pi,l}|^{\frac{1}{\theta}}$ is of class (DL) and satisfies:

$$n_t^{\pi,l} + \frac{1}{2}(\gamma - \phi)\Big(\pi'\Sigma\pi + \frac{\phi^2}{(1-\phi)^2}\frac{\nabla g' A \nabla g}{g^2} + \frac{2\phi}{1-\phi}\frac{\nabla g'}{g}\Upsilon'\pi\Big) \geq 0 \quad \textit{a.s. for all } t \geq 0, \tag{5.14a}$$

$$\liminf_{T\to\infty} \mathbb{E}(e^{-\delta T} |U_T^{\pi,l}|^{\frac{1}{\theta}}) = 0, \tag{5.14b}$$

then, $U^{\pi,l} \leq U^{\phi,\infty}(c)$.

iii. *If g is bounded above and away from zero, then the limit conditions* (5.13) & (5.14b) *hold for any* $(\pi, l) \in \mathcal{A}$.

Condition (5.14a) can be equivalently written as:

$$n_t^{\pi,l} + \frac{(\gamma - \phi)}{2[(1-\gamma)U_t^{\pi,l}]^2} d\langle U^{\pi,l}\rangle_t \geq 0 \quad \text{a.s. for all } t \geq 0. \tag{5.15}$$

We recall from equation (4.8) and the discussion that follows that, $n_t^{\pi,l}$ is negative for arbitrary admissible strategies, and vanishes for the optimal pair. In our setting where $\gamma > 1 > \phi$, therefore, (5.14a) is naturally satisfied for (π^*, l^*). Thus, under mild integrability conditions, we can verify permissibility of (π^*, l^*) and non-emptiness of $\mathcal{P}(g)$. Part iii. of Lemma 5.2.1 provides a further simplification to the question of permissibility when g is sufficiently well-behaved.

Having addressed the question of permissibility for (π^*, l^*), we can now state our main verification result, which verifies its optimality amongst $\mathcal{P}(g)$ strategies:

Theorem 5.2.2. *Let g be a solution of the Epstein–Zin HJB equation* (4.11) *and (π^*, l^*) be defined by the first order condition* (4.10)*. Then, for every strategy $(\pi, l) \in \mathcal{P}(g)$:*

$$\frac{x^{1-\gamma}}{1-\gamma} g(y)^{\phi\theta} \geq V_0^{\pi,l}. \tag{5.16}$$

Moreover, if (π^*, l^*) also belongs to the class $\mathcal{P}(g)$, then it is optimal amongst $\mathcal{P}(g)$ strategies, i.e.:

$$\frac{x^{1-\gamma}}{1-\gamma} g(y)^{\phi\theta} = V_0^{\pi^*,l^*} = \sup_{(\pi,l)\in\mathcal{P}(g)} V_0^{\pi,l}. \tag{5.17}$$

5.3 Proofs from Section 5.1

The following ordering property between CRRA and Epstein–Zin HJB equations will be useful throughout the proofs of this chapter.

Lemma 5.3.1. *Under assumption 4.1.2.iii, for all $(y, g, z, p) \in E \times (0, \infty) \times \mathbb{R}^k \times \mathbb{R}^{k\times k}$, we have:*

$$\mathcal{H}^{\mathrm{CRRA}}(y, g, z, p|\phi) \geq \mathcal{H}^{\mathrm{EZ}}(y, g, z, p|\phi, \gamma). \tag{5.18}$$

Proof. For the convenience of the reader, let us recall below the relevant HJB equations:

$$\begin{aligned}
\mathcal{H}^{\mathrm{CRRA}}(y,g,z,p|\phi) &= \delta^{\psi}g^{-1} + \frac{z'}{g}\Big(b + \frac{1-\phi}{\phi}\Upsilon'\Sigma^{-1}\mu\Big) + \frac{(\phi-1)z'(A-\Upsilon'\Sigma^{-1}\Upsilon)z}{2g^2} + \frac{\mathrm{tr}(Ap)}{2g} \\
&\quad + \Big(\frac{1-\phi}{\phi}\Big)\Big(r + \frac{\delta}{\phi-1} + \frac{\mu'\Sigma^{-1}\mu}{2\phi}\Big), \\
\mathcal{H}^{\mathrm{EZ}}(y,g,z,p|\phi,\gamma) &= \delta^{\psi}g^{-1} + \frac{z'}{g}\Big(b + \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\Big) + \frac{1}{2}\frac{z'}{g}\Big\{(\phi\theta-1)A + \phi\theta\Big(\frac{1-\gamma}{\gamma}\Big)\Upsilon'\Sigma^{-1}\Upsilon\Big\}\frac{z}{g} \\
&\quad + \frac{\mathrm{tr}(Ap)}{2g} + \Big(r + \frac{\delta}{\phi-1} + \frac{\mu'\Sigma^{-1}\mu}{2\gamma}\Big)\Big(\frac{1-\phi}{\phi}\Big).
\end{aligned} \tag{5.19}$$

In order to prove the relation (5.18), we will show that $\mathcal{H}^{\mathrm{EZ}}(y,g,z,p|\phi,\gamma)-\mathcal{H}^{\mathrm{CRRA}}(y,g,z,p|\phi) \leq 0$. By substituting $\frac{z'}{p}$ with η in this difference, we see that the desired result is achieved by proving that the following mapping is non-positive for all $\eta \in \mathbb{R}^k$:

$$\eta \to \frac{(\phi-\gamma)\eta'\Upsilon'\Sigma^{-1}\mu}{\phi\gamma} + \frac{1}{2}\eta'\Big((\phi\theta-1)A + \Big[\frac{\phi\theta(1-\gamma)}{\gamma} - (1-\phi)\Big]\Upsilon'\Sigma^{-1}\Upsilon\Big)\eta + \frac{(1-\phi)(\phi-\gamma)\mu'\Sigma^{-1}\mu}{2\phi^2\gamma}. \tag{5.20}$$

We first begin by simplifying the quadratic coefficient, which readily shows that it is negative definite:

$$\begin{aligned}
&\frac{1}{2}\Big[(\phi\theta-\phi)A + \Big[\frac{\phi\theta(1-\gamma)}{\gamma} - (1-\phi)\Big]\Upsilon'\Sigma^{-1}\Upsilon\Big] \\
&\quad = \frac{1}{2}a\Big[(\phi\theta-\phi)I_{k\times k} + \Big[\frac{\phi\theta(1-\gamma)}{\gamma} - (1-\phi)\Big]\rho\rho'\Big]a' \\
&\quad = \frac{1}{2}a\Big[\phi\theta\Big(I_{k\times k} + \frac{1-\gamma}{\gamma}\rho\rho'\Big) - \phi I_{k\times k} - (1-\phi)\rho\rho'\Big]a' \\
&\quad = \frac{1}{2}a\Big[\phi\theta\Big(\rho^{\perp}(\rho^{\perp})' + \frac{1}{\gamma}\rho\rho'\Big) - \phi I_{k\times k} - (1-\phi)\rho\rho'\Big]a'.
\end{aligned} \tag{5.21}$$

A negative quadratic function of the form $\eta \to \frac{1}{2}\eta' A\eta + B\eta + C$ admits a global maximum value of $-\frac{1}{2}B'A^{-1}B + C$. The lemma is therefore sufficed by showing that:

$$\mu'(\sigma')^{-1}\Big\{\frac{(\phi-\gamma)^2}{2\phi^2\gamma^2}\rho\Big[\phi\theta\Big(I + \frac{1-\gamma}{\gamma}\rho\rho'\Big) - \phi I - (1-\phi)\rho\rho'\Big]^{-1}\rho' - \frac{(1-\phi)(\phi-\gamma)}{2\phi^2\gamma}I_{n\times n}\Big\}\sigma^{-1}\mu \geq 0, \tag{5.22}$$

or, equivalently:

$$\mu'(\sigma')^{-1}\Big\{\frac{\phi-\gamma}{\gamma}\rho\Big[\phi\theta\Big(I+\frac{1-\gamma}{\gamma}\rho\rho'\Big)-\phi I-(1-\phi)\rho\rho'\Big]^{-1}\rho'-(1-\phi)I_{n\times n}\Big\}\sigma^{-1}\mu\leq 0. \tag{5.23}$$

Denote by $\bar{\rho}$ and $\underline{\rho}$ the maximum and minimum eigenvalues, respectively, of the positive semidefinite matrix $\rho\rho'$. Then, they are bounded in the interval $[0,1]$, a direct consequence of the relation $\rho\rho+\rho^{\perp}(\rho^{\perp})'=1_{n\times n}$. Moreover, the eigenvalues of the matrix $\frac{\phi-\gamma}{\gamma}\rho\Big[\phi\theta\Big(I+\frac{1-\gamma}{\gamma}\rho\rho'\Big)-\phi I-(1-\phi)\rho\rho'\Big]^{-1}\rho'$ are bounded above by:

$$\frac{\phi-\gamma}{\gamma}\frac{\bar{\rho}}{\phi\theta\Big(1+\frac{1-\gamma}{\gamma}\bar{\rho}\Big)-\phi-(1-\phi)\underline{\rho}}. \tag{5.24}$$

Therefore, inequality (5.23) holds if the quantity in (5.24) is bounded above by $(1-\phi)$. We will simplify this relation. Below, every inequality is equivalent to each other.

$$\begin{aligned}\frac{(\phi-\gamma)\bar{\rho}}{\phi\theta(\gamma+(1-\gamma)\bar{\rho})-\gamma\phi-\gamma(1-\phi)\underline{\rho}}&\leq 1-\phi,\\ \frac{(1-\theta)\bar{\rho}}{\phi|\theta|(\gamma+(1-\gamma)\bar{\rho})+\gamma\phi+\gamma(1-\phi)\underline{\rho}}&\leq 1,\\ \phi|\theta|\gamma(1-\bar{\rho})+\gamma\phi(1-\underline{\rho})+\gamma(\underline{\rho}-\bar{\rho})&\geq 0,\end{aligned} \tag{5.25}$$

which has been assumed in assumption 4.1.2.iii.

□

5.3.1 Proof of Lemma 5.1.1.

Proof. From the proof of Lemma 3.1 in [26], conditions i. and iii. imply that $u(y)\triangleq\mathbb{E}(\int_0^\infty e^{-\delta t}u_\gamma(c_t)dt)$ satisfies the partial differential equation:

$$\Big(-\frac{\delta}{1-\gamma}+\pi\mu-\frac{\gamma\pi'\Sigma\pi}{2}-l+r\Big)u+\frac{\nabla u'b}{1-\gamma}+\pi'\Upsilon\nabla u+\frac{\mathrm{tr}(AD^2u)}{2(1-\gamma)}+\frac{l^{1-\gamma}}{1-\gamma}=0,\quad y\in E. \tag{5.26}$$

Since $g_1 = (\delta u)^{\frac{1}{\phi\theta}}$, we can verify that g_1 satisfies:

$$\begin{aligned} r &+ \frac{\delta}{\gamma - 1} + \frac{\phi\theta \nabla g_1' b}{(1-\gamma) g_1} + \frac{\phi\theta(\phi\theta - 1)\nabla g_1' A \nabla g_1}{2(1-\gamma) g_1^2} + \frac{\phi\theta \operatorname{tr}(A D^2 g_1)}{2(1-\gamma) g_1} + \Big\{ \pi'\mu - \frac{\gamma \pi' \Sigma \pi}{2} \\ &+ \phi\theta \frac{\nabla g_1'}{g_1} \Upsilon' \pi + \frac{\delta l^{1-\gamma} g^{-\phi\theta}}{1-\gamma} - l \Big\} = 0. \end{aligned} \tag{5.27}$$

By Corollary A.2 of [57], $f(c, v) \geq \delta(\frac{c^{1-\gamma}}{1-\gamma} - v)$ when $\phi < 1 < \gamma$ for $c > 0$ and $v < 0$. Upon setting $c = xl$ and $v = \frac{x^{1-\gamma}}{1-\gamma} g^{\phi\theta}(y)$, we have:

$$\frac{\delta l^{1-\gamma} g^{-\phi\theta}}{1-\gamma} - \frac{\delta}{1-\gamma} \leq \frac{\delta l^{1-\phi} g^{-\phi}}{1-\phi} - \frac{\delta}{1-\phi}. \tag{5.28}$$

Note that, since we only consider $g > 0$, this ensures that $v = \frac{x^{1-\gamma}}{1-\gamma} g(y)^{\phi\theta}$ has the correct sign needed to derive equation (5.28).

Combining (5.27) and (5.28), we have:

$$\begin{aligned} r &+ \frac{\delta}{\phi - 1} + \frac{\phi\theta \nabla g_1' b}{(1-\gamma) g_1} + \frac{\phi\theta(\phi\theta - 1)\nabla g_1' A \nabla g_1}{2(1-\gamma) g_1^2} + \frac{\phi\theta \operatorname{tr}(A D^2 g_1)}{2(1-\gamma) g_1} + \Big\{ \pi'\mu - \frac{\gamma \pi' \Sigma \pi}{2} \\ &+ \phi\theta \frac{\nabla g_1'}{g_1} \Upsilon' \pi + \frac{\delta l^{1-\phi} g^{-\phi}}{1-\phi} - l \Big\} \geq 0. \end{aligned} \tag{5.29}$$

Taking supremum in (π, l), we obtain $\mathcal{H}^{\mathrm{EZ}}(y, g_1, \nabla g_1, D^2 g_1) \geq 0$, thus confirming g_1 as a subsolution of equation (4.11).

Also, in Lemma 3.1 [26], under conditions ii. and iii., g_2 is shown to satisfy the following partial differential equation:

$$\begin{aligned} \delta^{\psi} g_2 &+ \frac{\nabla g_2}{g_2}\Big(b + \frac{1-\phi}{\phi} \Upsilon' \Sigma^{-1} \mu\Big) + \frac{1-\phi}{\phi} \Big\{ \frac{\eta'(A - \Upsilon'\Sigma^{-1}\Upsilon)\eta}{2\phi} + \frac{\nabla g_2'}{g_2} (\Upsilon' \Sigma^{-1} \Upsilon - A)\eta \Big\} \\ &+ \frac{\operatorname{tr}(A D^2 g_2)}{2 g_2} + \frac{1-\phi}{\phi}\Big(r + \frac{\delta}{\phi - 1} + \frac{\mu' \Sigma^{-1} \mu}{2\phi}\Big) = 0, \quad y \in E. \end{aligned} \tag{5.30}$$

We can minimise the quadratic term in η as follows: $\inf_{\eta\in\mathbb{R}^k}\{\frac{\eta'(A-\Upsilon'\Sigma^{-1}\Upsilon)\eta}{2\phi} + \frac{\nabla g_2'}{g_2}(\Upsilon'\Sigma^{-1}\Upsilon - A)\eta\} = -\frac{\phi\nabla g_2'(A-\Upsilon'\Sigma^{-1}\Upsilon)\nabla g_2}{2g_2^2}$. This yields:

$$
\begin{aligned}
&\delta^{\psi} g_2^{-1} + \frac{\nabla g_2'}{g_2}\Big(b + \frac{1-\phi}{\phi}\Upsilon'\Sigma^{-1}\mu\Big) + \frac{(\phi-1)\nabla g_2'(A-\Upsilon'\Sigma^{-1}\Upsilon)\nabla g_2}{2g_2^2} + \frac{\operatorname{tr}(AD^2 g_2)}{2g_2} \\
&\quad + \Big(\frac{1-\phi}{\phi}\Big)\Big(r + \frac{\delta}{\phi-1} + \frac{\mu'\Sigma^{-1}\mu}{2\phi}\Big) \le 0,
\end{aligned}
\tag{5.31}
$$

which confirms that g_2 is a supersolution of the CRRA HJB equation, i.e. $\mathcal{H}^{\mathrm{CRRA}}(y, g_2, \nabla g_2, D^2 g_2|\phi) \le 0$. By Lemma 5.3.1, it also satisfies $\mathcal{H}^{\mathrm{EZ}}(y, g_2, \nabla g_2, D^2 g_2|\phi, \gamma) \le 0$.

□

5.3.2 Proof of Theorem 5.1.2

For the convenience of the reader, let us recite the existence theorem for the HJB equation here:

Theorem 5.3.2. *Suppose that $g_1 \in C^2(E, (0,\infty))$ is a subsolution to the Epstein–Zin HJB equation and $g_2 \in C^2(E, (0,\infty))$ is a supersolution to the CRRA HJB equation. Assume additionally that $g_1(y) \le g_2(y)$ for all $y \in E$. Then, there exists a twice continuously differentiable function $g : E \to (0,\infty)$ that solves the Epstein–Zin HJB equation (4.11). Moreover, g satisfies $g_1 \le g \le g_2$.*

Proof. We first observe that, if g is a solution to (4.11) then the transform $u = \phi\theta\ln(g)$ satisfies the equation below[4]:

$$
\begin{aligned}
\mathcal{G}^{\mathrm{EZ}}(y, u, \nabla u, D^2 u|\phi,\gamma) &\triangleq \phi\theta\delta^{\psi} e^{-\frac{u}{\phi\theta}} + \nabla u'\Big(b + \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\Big) + \frac{1}{2}\nabla u'\Big(A + \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\Big)\nabla u \\
&\quad + \frac{\operatorname{tr}(AD^2 u)}{2} + \Big(r + \frac{\delta}{\phi-1} + \frac{\mu'\Sigma^{-1}\mu}{2\gamma}\Big)(1-\gamma) = 0.
\end{aligned}
\tag{5.32}
$$

We observe also that, if we define $u_i = \phi\theta\ln(g_i)$, $i = 1, 2$, then they satisfy the inequality:

$$
\mathcal{G}^{\mathrm{EZ}}(y, u_1, \nabla u_1, D^2 u_1|\phi,\gamma) \le 0 \le \mathcal{G}^{\mathrm{EZ}}(y, u_2, \nabla u_2, D^2 u_2|\phi,\gamma). \tag{5.33}
$$

[4]This is the resulting equation from parametrising the optimal value function as $v(x,y) = \frac{x^{1-\gamma}}{1-\gamma}e^{u(y)}$. This approach has been studied by Hata & Sheu (cf. [27] & [28]) for the additive utility case, and Xing for the Epstein–Zin utility case ([66]).

Moreover, if we found a solution u such that $\mathcal{G}^{\mathrm{EZ}}(y, u, \nabla u, D^2 u|\phi, \gamma) = 0$ in E and $u_2 \le u \le u_1$, then $g = \exp(\frac{u}{\phi\theta})$ is the solution required by the theorem. We will approach this equation by solving a local version of it, and apply Arzelà–Ascoli theorem to find a uniformly convergent subsequence which then solves (5.32).

For each $n \in \mathbb{N}$, since A is positive definite and continuous, its eigenvalues are bounded and bounded away from 0 on compact sets. Thus there exists $\underline{\lambda}_n < \bar{\lambda}_n$ such that for all $x \in \mathbb{R}^k$ and $y \in \bar{E}_n$, $\underline{\lambda}_n \|x\|^2 \le \sum_{i,j} A_{ij}(y) x_i x_j \le \bar{\lambda}_n \|x\|^2$. By Lemma 5.3.3 below, there exists a solution $u^{(n)}$ in $\bar{E}_n$ to the boundary value problem:

$$
\begin{aligned}
\mathcal{G}^{\mathrm{EZ}}(y, u^{(n)}, \nabla u^{(n)}, D^2 u^{(n)}|\phi, \gamma) = 0, \quad y \in E_n \\
u^{(n)} = u_2, \quad y \in \partial E_n.
\end{aligned}
\tag{5.34}
$$

Since $u_2 \le u_1$, by Comparison Theorem (Theorem 10.1 [25]), we have $u_2 \le u^{(n)} \le u_1$ in $\bar{E}_n$. The same holds for $m \ge n$, and thus $\{u^{(m)}\}_{m \ge n}$ are bounded uniformly in E_n.

We now derive a Hölder estimate for the gradient of $\{u^{(m)}\}_{m \ge n}$. By Theorem 13.6 [25], there exists $\alpha' \in (0, 1]$ and C such that:

$$
\left[\nabla u^{(m)}\right]_{\alpha', E_n} \le C, \tag{5.35}
$$

where C and α' depend only on $\sup_{y \in E_{n+1}} \left|u^{(m)}\right|$, $\underline{\lambda}_{n+1}$ and $\bar{\lambda}_{n+1}$, and $[g]_{\alpha, E_n} \triangleq \sup_{y, y' \in E_n; y \neq y'} \frac{g(y) - g(y')}{|y - y'|^{\alpha}}$. Without loss of generality, assume $\alpha = \alpha \wedge \alpha'$, for if it is not, we can reset the value of α to $\alpha \wedge \alpha'$.

Consider $u^{(m)}$ as the solution to the following linear problem:

$$
\mathcal{L}(y, u^{(m)}, \nabla u^{(m)}, D^2 u^{(m)}) = f_m(y), \tag{5.36}
$$

where:

$$
\begin{aligned}
\mathcal{L}(y, u^{(m)}, \nabla u^{(m)}, D^2 u^{(m)}) &= (\nabla u^{(m)})' \Big(b + \frac{1 - \gamma}{\gamma} \Upsilon' \Sigma^{-1} \mu\Big) + \frac{1}{2} \operatorname{tr}(A D^2 u^{(m)}), \\
f_m(y) &= -\phi\theta \exp\Big(-\frac{u^{(m)}}{\phi\theta}\Big) - \frac{1}{2} (\nabla u^{(m)})' \Big(A + \frac{1 - \gamma}{\gamma} \Upsilon' \Sigma^{-1} \Upsilon\Big) \nabla u^{(m)} \\
&\quad - \Big(r + \frac{\delta}{\phi - 1} + \frac{\mu' \Sigma^{-1} \mu}{2\gamma}\Big)(1 - \gamma).
\end{aligned}
\tag{5.37}
$$

The Schauder interior estimates (Corollary 6.3 [25]) imply that for $m > n$, with $d = \operatorname{dist}(E_n, \partial E_{n+1})$ and a constant D, where D is independent of m and the source term f_m:

$$d \max_{E_n}\left|\nabla u^{(m)}\right| + d^2 \max_{E_n}\left|D^2 u^{(m)}\right| + d^{2+\alpha}\left[D^2 u^{(m)}\right]_{\alpha,E_n} \leq D\left(\sup_{E_{n+1}}\left|u^{(m)}\right| + \sup_{E_{n+1}}|f_m| + [f_m]_{\alpha,E_{n+1}}\right). \tag{5.38}$$

The next step is to remove the dependence on m on the right hand side of (5.38). To this end, it is sufficient to find a bound for the gradient $\left|\nabla u^{(m)}\right|_{m>n}$. By Theorem 15.5 [25], the following holds:

$$\left|\nabla u^{(m)}(y)\right| \leq K(1 + \operatorname{dist}(y, \partial E_{n+1})^{-\frac{1}{\vartheta}}), \tag{5.39}$$

where ϑ is a constant in the *structural conditions* (cf. condition 15.3 [25]), and K is independent of m. Due to the assumptions on the nested domains, $\min_{y\in\bar{E}_n} \operatorname{dist}(y, \partial E_{n+1}) > 0$, and therefore:

$$\sup_{y\in\bar{E}_n}\left|\nabla u^{(m)}\right| \leq K(1 + \max_{y\in\bar{E}_n} \operatorname{dist}(y, \partial E_{n+1})^{-\frac{1}{\vartheta}}) \quad \text{for } m \geq n+1. \tag{5.40}$$

Combine this with the estimate (5.38), we have:

$$\begin{aligned} &d \max_{E_n}\left|\nabla u^{(m)}\right| + d^2 \max_{E_n}\left|D^2 u^{(m)}\right| + d^{2+\alpha}\left[D^2 u^{(m)}\right]_{\alpha,E_n} \leq \\ &D\left(\sup_{y\in E_{n+1}} |u_1(y)| \vee |u_2(y)| + \sup_{\substack{m>n \\ y\in E_{n+1}}} |f_m(y)| + \sup_{m>n}[f_m]_{\alpha,E_{n+1}}\right) < \infty. \end{aligned} \tag{5.41}$$

The estimates (5.35) and (5.41) imply that the the sequences $\{u^{(m)}\}_{m>n}$, $\{\nabla u^{(m)}\}_{m>n}$ and $\{D^2 u^{(m)}\}_{m>n}$ are bounded and equi-continuous in $\bar{E}_n$. By Arzelà–Ascoli Theorem, by passing to a subsequence, they converge uniformly to u, $\lim \nabla u^{(m)}$ and $\lim D^2 u^{(m)}$, respectively. Due to the uniformity of convergence, u is twice continuously differentiable and $\nabla u = \lim \nabla u^{(m)}$ and $D^2 u = \lim D^2 u^{(m)}$.

□

Lemma 5.3.3. *There exists a solution to the following boundary value problem:*

$$\begin{aligned} &\mathcal{G}^{\mathrm{EZ}}(y, u, \nabla u, D^2 u|\phi, \gamma) = 0, \quad y \in E_n, \\ &u(y) = u_2, \quad y \in \partial E_n. \end{aligned} \tag{5.42}$$

Proof. We adapt the strategy of Hata & Sheu ([27]) to our setting of Epstein–Zin utility. By Theorem 3.4 therein, it is sufficient to prove boundedness of solutions of a parameterised class of PDEs.

Define $\gamma(\tau) \triangleq (1-\tau)+\tau\gamma$, and $\phi(\tau) \triangleq (1-\tau)+\tau\phi$. We observe that, firstly, γ^τ and ϕ^τ still follow the general configuration, i.e. for $\tau > 0$, $\gamma(\tau) > 1 > \phi(\tau)$, and secondly, $\theta(\tau) = \frac{\tau(1-\gamma)}{\tau(1-\phi)} = \theta$ is independent of τ. By Theorem 3.4 [27], it is sufficient to prove that solutions to the following BVPs are bounded uniformly in $\tau \in [0,1]$.

$$
\begin{aligned}
&\mathcal{G}^{\mathrm{EZ}}(y, u^\tau, \nabla u^\tau, D^2 u^\tau | \phi(\tau), \gamma(\tau)) = 0 \\
&u^\tau(y) = \tau\phi\theta\ln(g_2), \quad y \in \partial E_n,
\end{aligned}
\tag{5.43}
$$

and:

$$
\begin{aligned}
&\frac{\operatorname{tr}(AD^2u)}{2} + \tau\Big(\theta\delta^\psi e^{-\frac{u}{\theta}} + \nabla u' b + \frac{1}{2}\nabla u' A \nabla u - \delta\theta\Big) = 0, \quad y \in E_n \\
&u = 0, \quad y \in \partial E_n.
\end{aligned}
\tag{5.44}
$$

We start with the BVP (5.44). In the case where $\delta \le 1$, we observe that $\bar{u} = 0$ is a super-solution and $\underline{u} = -\theta(1-\psi)\ln(\delta)$ is a sub-solution, and thus are the required lower and upper bound, respectively, for any solution of (5.44). Bounding the solutions of (5.43) requires more involved calculations, and is stated separately in Lemma 5.3.4.

□

Lemma 5.3.4. *If u^τ is a solution to the following boundary value problem:*

$$
\begin{aligned}
&\mathcal{G}^{\mathrm{EZ}}(y, u^\tau, \nabla u^\tau, D^2 u^\tau | \phi(\tau), \gamma(\tau)) = 0, \quad y \in E_n, \\
&u^\tau(y) = \tau u_2(y), \quad y \in \partial E_n,
\end{aligned}
\tag{5.45}
$$

then it admits the following bounds:

$$
\theta\ln\Big\{\tau g_2^\phi(y) + (1-\tau)\Big\} \le u^\tau \le \phi|\theta|\ln\Big(\frac{C}{\phi|\theta|\delta^\psi} \vee 1\Big) + \sup_{y\in\bar{E}_n}\Big|\ln(g_2^{\phi\theta}(y))\Big|, \tag{5.46}
$$

where $C_n = \sup_{\tau\in[0,1], y\in\bar{E}_n}\Big(r + \frac{\mu'\Sigma^{-1}\mu}{2\gamma(\tau)}\Big)(1-\gamma(\tau)) - \delta\theta.$

Proof. We start with the upper bound. Define the function, where the constant C_n is defined in the lemma:

$$\bar{u}_n = \phi|\theta| \ln\left(\frac{C_n}{\phi|\theta|\delta^{\psi}} \vee 1\right) + \sup_{y \in \bar{E}_n} \left| \ln(g_2^{\phi\theta}(y)) \right|, \tag{5.47}$$

For any $\tau \in [0, 1]$, $\bar{u}_n$ is a sub-solution, and therefore $u^\tau \leq \bar{u}_n$.

Next, we will show that it is bounded from below. The scheme of the proof is as follows: firstly, we will show that, for $(x, y) \in (0, \infty) \times \bar{E}_n$ and $u_2 = \phi \ln(g_2)$:

$$\frac{x^{\tau(1-\phi)} e^{\frac{u^\tau}{\theta}} - 1}{\tau(1-\phi)} \leq \frac{x^{1-\phi} e^{u_2} - 1}{1-\phi}. \tag{5.48}$$

Once this is established, we can simply set $x = 1$ to obtain:

$$\begin{aligned} \frac{1}{\tau} e^{\frac{u^\tau}{\theta}} - \frac{1}{\tau} &\leq e^{u_2(y)} - 1 \\ e^{\frac{u^\tau}{\theta}} &\leq \tau e^{u_2(y)} + (1-\tau)1, \quad \text{and thus} \\ u^\tau &\geq \theta \ln\left\{\tau g_2^{\phi} + (1-\tau)1\right\}. \end{aligned} \tag{5.49}$$

We will obtain the relation (5.48) by contradiction. We will derive a relationship that is known to hold, and show that such a relationship cannot be true without (5.48).

Step 0. Review of the time-additive case. Before we proceed, we recollect some facts from the additive utility case and establish some notations. Suppose that the agent's utility is defined by a CRRA utility function with relative risk aversion ϕ, i.e. $u_\phi : x \to \frac{x^{1-\phi}}{1-\phi}$ and the optimal value function is parametrised by $\frac{x^{1-\phi}}{1-\phi} g(y)^\phi$, then the associated HJB equation is (see also [26]):

$$\begin{aligned} \mathcal{H}^{\mathrm{CRRA}}(y, g, \nabla g, D^2 g|\phi) &= \delta^{\frac{1}{\phi}} g^{-1} + \frac{\mathrm{tr}(AD^2 g)}{2g} + \frac{\nabla g'}{g}\Big(b + \frac{1-\phi}{\phi}\Upsilon\Sigma^{-1}\mu\Big) \\ &- \frac{1-\phi}{2}\frac{\nabla g'}{g}(A - \Upsilon'\Sigma^{-1}\Upsilon)\frac{\nabla g}{g} + \Big(r - \frac{\delta}{1-\phi} + \frac{\mu\Sigma^{-1}\mu}{2\phi}\Big)\Big(\frac{1-\phi}{\phi}\Big) = 0. \end{aligned} \tag{5.50}$$

On the other hand, if we let the agent have utility function u_ϕ but parametrise the optimal value function as $\frac{x^{1-\phi}}{1-\phi}e^{u(y)}$, then the resulting HJB equation is:

$$\begin{aligned}\mathcal{G}^{\mathrm{CRRA}}(y,u,\nabla u,D^2u|\phi) &= \phi\delta^{\psi}e^{-\frac{u}{\phi}} + \frac{1}{2}\operatorname{tr}(AD^2u) + \frac{1}{2}\nabla u'\Big(A + \frac{1-\phi}{\phi}\Upsilon'\Sigma^{-1}\Upsilon\Big)\nabla u \\ &+ \nabla u'\Big(b + \frac{1-\phi}{\phi}\Upsilon'\Sigma^{-1}\mu\Big) + \Big(r - \frac{\delta}{1-\phi} + \frac{\mu\Sigma^{-1}\mu}{2\phi}\Big)(1-\phi) = 0.\end{aligned} \tag{5.51}$$

Moreover, these two parametrisations are equivalent in the sense that, if we set $u = \phi\ln(g)$, then $\mathcal{H}^{\mathrm{CRRA}}(y,g,\nabla g,D^2g|\phi) = \frac{1}{\phi}\mathcal{G}^{\mathrm{CRRA}}(y,u,\nabla u,D^2u|\phi)$. Moreover, we have shown in Lemma 5.3.1 that under assumption 4.1.2.iii, for all $(y,g,z,p) \in E\times(0,\infty)\times\mathbb{R}^k\times\mathbb{R}^{k\times k}$, $\mathcal{H}^{\mathrm{CRRA}}(y,g,z,p|\phi) \geq \mathcal{H}^{\mathrm{EZ}}(y,g,z,p|\phi,\gamma)$.

Step 1. Derive a relation for contradiction.

Now, let u_τ be defined by equation (5.45) and g^τ by the relation $e^{u^\tau} = (g^\tau)^{\phi(\tau)\theta}$, then g^τ satisfies the Epstein–Zin HJB equation parametrised by $\phi(\tau)$ and $\gamma(\tau)$ in the domain E_n. More specifically:

$$\mathcal{H}^{\mathrm{EZ}}(y,g^\tau,\nabla g^\tau,D^2g^\tau|\phi(\tau),\gamma(\tau)) = 0, \quad y\in E_n. \tag{5.52}$$

Using the relationship $\mathcal{H}^{\mathrm{CRRA}}(\cdot|\phi(\tau)) \geq \mathcal{H}^{\mathrm{EZ}}(\cdot|\phi(\tau),\gamma(\tau))$ (see Lemma 5.3.1), we achieve the relation $\mathcal{H}^{\mathrm{CRRA}}(y,g^\tau,\nabla g^\tau,D^2g^\tau|\phi(\tau)) \geq 0$. Thus, the mapping $\phi(\tau)\ln(g^\tau) = \frac{u^\tau}{\theta}$ satisfies the relation:

$$\mathcal{G}^{\mathrm{CRRA}}\Big(y,\frac{u^\tau}{\theta},\frac{\nabla u^\tau}{\theta},\frac{D^2u^\tau}{\theta}|\phi(\tau)\Big) \geq 0. \tag{5.53}$$

If we define $V^\tau = \frac{x^{\tau(1-\phi)}}{\tau(1-\phi)}e^{\frac{u^\tau(y)}{\theta}}$, then the following holds:

$$\begin{aligned}&\frac{\operatorname{tr}(AD^2_{yy}V^\tau)}{2} + D_y(V^\tau)'b + \sup_{(\pi,l)}\Big\{D_xV^\tau x(r+\pi'\mu-l) + \frac{x^2}{2}\pi'\Sigma\pi D^2_{xx}V^\tau + xD_{xy}(V^\tau)'\Upsilon'\pi \\ &+ \delta\Big(\frac{c^{\tau(1-\phi)}}{\tau(1-\phi)} - V^\tau\Big)\Big\} \geq 0.\end{aligned} \tag{5.54}$$

On the other hand, we know that g_2 satisfies $\mathcal{H}^{\text{CRRA}}(y, g_2, \nabla g_2, D^2 g_2|\phi) \leq 0$. If we define u_2 by: $u_2 = \phi \ln(g_2)$, then $\mathcal{G}^{\text{CRRA}}(y, u_2, \nabla u_2, D^2 u_2|\phi) \leq 0$. Now, let us define $V = \frac{x^{1-\phi}}{1-\phi} e^{u_2}$, we have:

$$\begin{aligned}&\frac{\operatorname{tr}(AD^2_{yy}V)}{2} + D_y V' b + \sup_{(\pi,l)} \Big\{ D_x V x(r + \pi'\mu - l) + \frac{x^2}{2}\pi'\Sigma\pi D^2_{xx}V + xD_{xy}V'\Upsilon'\pi \\ &\quad + \delta\Big(\frac{c^{1-\phi}}{1-\phi} - V\Big)\Big\} \leq 0.\end{aligned} \tag{5.55}$$

By choosing (π, l) that attains the supremum in equation (5.54), we attain the following relationship between V^τ and V:

$$\begin{aligned}&\frac{\operatorname{tr}(AD^2_{yy}(V^\tau - V))}{2} + D_y(V^\tau - V)'b + D_x(V^\tau - V)'x(r + \pi'\mu - l) + \frac{x^2}{2}\pi'\Sigma\pi D^2_{xx}(V^\tau - V) + \\ &xD^2_{xy}(V^\tau - V)'\Upsilon'\pi + \delta\Big(\frac{c^{\tau(1-\phi)}}{\tau(1-\phi)} - \frac{c^{1-\phi}}{1-\phi}\Big) - \delta(V^\tau - V) \geq 0.\end{aligned} \tag{5.56}$$

We observe that the mapping $\beta \to \beta^{-1}(c^\beta - 1)$ is increasing for $\beta > 0$. Therefore, $\frac{c^{\tau(1-\phi)}}{\tau(1-\phi)} - \frac{c^{1-\phi}}{1-\phi} \leq \frac{1}{\tau(1-\phi)} - \frac{1}{1-\phi}$. This, combined with inequality (5.56) yields the following:

$$\begin{aligned}&\frac{\operatorname{tr}(AD^2_{yy}(V^\tau - V))}{2} + D_y(V^\tau - V)'b + D_x(V^\tau - V)'x(r + \pi'\mu - l) + \frac{x^2}{2}\pi'\Sigma\pi D^2_{xx}(V^\tau - V) + \\ &xD^2_{xy}(V^\tau - V)'\Upsilon'\pi + \delta\Big(\frac{1}{\tau(1-\phi)} - \frac{1}{1-\phi}\Big) - \delta(V^\tau - V) \geq 0.\end{aligned} \tag{5.57}$$

We can simplify equation (5.57) by absorbing the constant term $\frac{1}{\tau(1-\phi)} - \frac{1}{1-\phi}$ in an affine transformation. To achieve that, we define $V_0^\tau = \frac{x^{\tau(1-\phi)}e^{\frac{u^\tau}{\theta}} - 1}{\tau(1-\phi)}$ and $V_0 = \frac{x^{1-\phi}e^{u_2-1}}{1-\phi}$. Combine this with inequality (5.57), one can verify straightforwardly that V_0^τ and V_τ satisfies the relation (5.58) below. We will use this inequality to set up our contradiction argument in Step 2.

$$\frac{\operatorname{tr}(AD^2_{yy}(V_0^\tau - V_0))}{2} + D_y(V_0^\tau - V_0)'b + D_x(V_0^\tau - V_0)'x(r + \pi'\mu - l) + \frac{x^2}{2}\pi'\Sigma\pi D^2_{xx}(V_0^\tau - V_0) +$$
$$xD^2_{xy}(V_0^\tau - V_0)'\Upsilon'\pi + -\delta(V_0^\tau - V_0) \geq 0. \tag{5.58}$$

Step 2. Contradiction step. We are now ready to prove relation (5.48), i.e. $V_0 \geq V$ for all $(x, y) \in (0, \infty) \times \bar{E}_n$. We will show below that if (5.48) does not hold, it would violate relation (5.58) derived in the previous step. For the sake of a contradiction, let us assume the opposite, i.e.:

$$\sup_{(x,y)\in(0,\infty)\times\bar{E}_n} (V_0^\tau - V_0) > 0. \tag{5.59}$$

We make the following observation on the boundary, using again the fact that $\beta \rightarrow \beta^{-1}(c^\beta - 1)$ is increasing in β for $\beta > 0$ and $c > 0$: for $y \in \partial E_n$:

$$V_0^\tau(x, y) = \frac{x^{\tau(1-\phi)}e^{\tau u_2(y)} - 1}{\tau(1 - \phi)} \leq \frac{x^{1-\phi}e^{u_2(y)} - 1}{1 - \phi} = V_0(x, y). \tag{5.60}$$

Moreover, for any $y \in E_n$, we have $V_0^\tau(x, y) \leq V_0(x, y)$ for either $x = 0$ or x large enough. These observations on the boundary behaviour imply that the supremum in (5.59) is attained at an interior point. Specifically, there exists some $(x_0, y_0) \in (0, \infty) \times E_n$ such that $V_0^\tau(x_0, y_0) > V_0(x_0, y_0)$, $D_x V_0^\tau(x_0, y_0) = D_x V_0(x_0, y_0)$, $D_y V_0^\tau(x_0, y_0) = D_y V_0(x_0, y_0)$ and $D^2(V_0^\tau - V_0)$ is a negative semi-definite matrix. These facts, however, would imply that the left hand side of (5.58) is strictly negative, a contradiction.

□

5.4 Proofs from Section 5.2

5.4.1 Proof of Lemma 5.2.1

For the proof of Lemma 5.2.1, we will need the following lemma concerning the growth rate of power utility processes.

Lemma 5.4.1. *Let c be a consumption plan in C^∞ and $U^{\gamma,\infty}(c)$ and $Y^{\phi,\infty}(c)$ be its associated power utility processes (cf. Definition 2.4.2). Then:*

$$\lim_{T\to\infty} \mathbb{E}(e^{-\delta T} U^{\gamma,\infty}(c)_T) = \lim_{T\to\infty} \mathbb{E}(e^{-\delta T} Y^{\phi,\infty}(c)_T) = 0. \tag{5.61}$$

Proof. By definition $\mathbb{E}[e^{-\delta T} U^{\gamma,\infty}(c)_T] = \mathbb{E}(\int_T^\infty \delta e^{-\delta t} c_t^{1-\gamma} dt)$. As $c \in C^\infty$, $c^{1-\gamma}$ is a square integrable random variable on the product space $(\Omega \times \mathbb{R}^+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+))$ with probability measure $d\mathbb{P} \otimes \delta e^{-\delta t} dt$. Since this product space is a finite measure space, $c^{1-\gamma}$ is also integrable in the product L^1 norm, i.e. $\mathbb{E}(\int_0^\infty \delta e^{-\delta t} c_t^{1-\gamma} dt) < \infty$. By Dominated Convergence Theorem with $\int_0^\infty \delta e^{-\delta t} c_t^{1-\gamma} dt$ as the dominating random variable:

$$\lim_{T\to\infty} \mathbb{E}\Big[\int_T^\infty \delta e^{-\delta t} c_t^{1-\gamma} dt\Big] = 0, \tag{5.62}$$

which concludes the first convergence of the lemma. The second convergence can be proved similarly, using square integrability of $c^{1-\phi}$ instead.

□

Proof of Lemma 5.2.1.

Part I. Show that $U^{\pi,l} \geq U^{\gamma,\infty}(c)$. Following from equation (4.8), $U^{\pi,l}$ satisfies the dynamics:

$$dU_t^{\pi,l} = -f(c_t, U_t^{\pi,l})dt + (1-\gamma)U_t^{\pi,l} n_t^{\pi,l} dt + (1-\gamma)U_t^{\pi,l}\Big[\pi'\sigma dW_t^\rho + \frac{\phi\nabla g' a}{(1-\phi)g} dW_t\Big]. \tag{5.63}$$

We shall denote the local martingale term above more succinctly by M^1. On the other hand, the dynamics of $U^{\gamma,\infty}(c)$ satisfies:

$$dU_t^{\gamma,\infty} = -\delta(u_\gamma(c_t) - U^{\gamma,\infty}(c)_t)dt + dM_t^2, \tag{5.64}$$

where M^2 is a local martingale. By product rule, we can compute the dynamics of $e^{-\delta t}(V_t^* - U_t^{\gamma,\infty})$ as follows:

$$\begin{aligned} d[e^{-\delta t}(U_t^{\pi,l} - U_t^{\gamma,\infty})] &= -e^{-\delta t}(f(c_t, U_t^{\pi,l}) - \delta[u_\gamma(c_t) - U_t^{\gamma.\infty}])dt + e^{-\delta t}(1-\gamma)U_t^{\pi,l}n_t^{\pi,l}dt \\ &\quad - \delta e^{-\delta t}(U_t^{\pi,l} - U_t^{\gamma,\infty})dt + e^{-\delta t}(dM_t^1 - dM_t^2) \\ &= -e^{-\delta t}[f(c_t, U_t^{\pi,l}) - \delta(u_\gamma(c_t) - U_t^{\pi,l})]dt \\ &\quad + e^{-\delta t}(1-\gamma)U_t^{\pi,l}n_t^{\pi,l}dt + e^{-\delta t}(dM_t^1 - dM_t^2). \end{aligned} \tag{5.65}$$

When $\gamma\psi > 1$, $f(c, v) \geq \delta(u_\gamma(c) - v)$ for all $c > 0$ and $(1-\gamma)v > 0$ (Corollary A.2 in [57]). Because g satisfies the Epstein–Zin HJB equation (see Dynamic Programming Principle after equation (4.8)), $n^{\pi,l}$ is non-positive for arbitrary strategies. Therefore, $e^{-\delta t}(U_t^{\pi,l} - U_t^{\gamma,\infty})$ has non-positive drift component and is a local supermartingale. Let $\{\tau_n\}_{n\geq 0}$ be a reducing sequence of stopping times and $t < T$ be positive constants, we have:

$$e^{-\delta(t\wedge\tau_n)}(U_{t\wedge\tau_n}^{\pi,l} - U_{t\wedge\tau_n}^{\gamma,\infty}) \geq \mathbb{E}_{t\wedge\tau_n}\big(e^{-\delta(T\wedge\tau_n)}(U_{T\wedge\tau_n}^{\pi,l} - U_{T\wedge\tau_n}^{\gamma,\infty})\big). \tag{5.66}$$

The hypothesis of this lemma states that $U^{\pi,l}$ is of class (DL). Moreover, since (π, l) is assumed to be admissible, c belongs to the class C^∞ and as a result $U^{\gamma,\infty}(c)$ belongs to the class $\mathcal{V}^\infty$ (Proposition 2.5.2). Thus, $U^{\gamma,\infty}$ belongs to class (DL) as well. Therefore, we can apply Proposition 5.4.2 below to take the limit $n \to \infty$ on the right hand side of (5.66). The left hand side limit follows simply from continuity. We have:

$$e^{-\delta t}(U_t^{\pi,l} - U^{\gamma,\infty}(c)_t) \geq \mathbb{E}_t[e^{-\delta T}(U_T^{\pi,l} - U^{\gamma,\infty}(c)_T)]. \tag{5.67}$$

By the limit condition (5.13) in the hypothesis of the lemma, there exists a divergent sequence T_n such that $\lim_{n\to\infty} e^{-T_n}\mathbb{E}(U_{T_n}^{\pi,l}) = 0$. Combine this observation with lemma 5.4.1, the right hand side of (5.67) also converges to 0 in L^1 along $\{T_n\}$. By passing to a fast convergent subsequence, also denoted $\{T_n\}$, it converges almost surely, too. Taking limit of (5.67) along this subsequence implies that $U_t^{\pi,l} \geq U^{\gamma,\infty}(c)_t$ almost surely.

Part II. Show that $U_t^{\pi,l} \leq U^{\phi,\infty}(c)_t$. Since the mapping $y \to u_\phi \circ u_\gamma^{-1}(y)$ is increasing for $y < 0$, it is sufficient to show that $Y^{\pi,l} \triangleq u_\phi \circ u_\gamma^{-1}(U^{\pi,l}) \leq Y^{\phi,\infty}$.

$$\begin{aligned} dY_t^{\pi,l} &= -\delta(u_\phi(c_t) - Y_t^{\pi,l})dt + dM_t^1 \\ &\quad + [(1-\gamma)U_t^{\pi,l}]^{\frac{1}{\theta}}\Big[n_t + \frac{1}{2}(\gamma-\phi)\Big(\pi'\Sigma\pi + \frac{\phi^2}{(1-\phi)^2}\frac{\nabla g' A \nabla g}{g^2} + \frac{2\phi}{1-\phi}\frac{\nabla g'}{g}\Upsilon'\pi\Big)\Big]dt, \end{aligned} \tag{5.68}$$

where the last term above is non-negative by assumption (5.14a). Let us denote this term by p_t for brevity. On the other hand, where M^2 is a local martingale:

$$dY^{\phi,\infty}(c)_t = -\delta(u_\phi(c_t) - Y^{\phi,\infty}(c)_t)dt + dM_t^2. \tag{5.69}$$

Following similar calculations to Part I, we have:

$$d(e^{-\delta t}(Y^{\phi,\infty}(c)_t - Y_t^{\pi,l})) = e^{-\delta t}d(M_t^1 - M_t^2) - e^{-\delta t}p_t dt. \tag{5.70}$$

Therefore, $e^{-\delta t}(Y^{\phi,\infty}(c) - Y_t^{\pi,l})$ is a local supermartingale. The same argument in Part I can be applied here to show that $Y_t^{\pi,l} \leq Y^{\phi,\infty}(c)_t$ almost surely for all $t \geq 0$.

Part III. The special case of bounded g. If the solution g is bounded above and away from zero, then there exists a constant K for which $|U_t^{\pi,l}| \leq K c_t^{1-\gamma}$. As $c^{1-\gamma}$ is belongs to the product space $L^2(\Omega \times \mathbb{R}^+)$ endowed with the probability measure $d\mathbb{P} \otimes \delta e^{-\delta t}dt$, it is L^1, too. Therefore:

$$\int_0^\infty \delta e^{-\delta t}\mathbb{E}[(c_t)^{1-\gamma}]dt = \mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t}(c_t)^{1-\gamma}dt\Big) < \infty. \tag{5.71}$$

Thus for Lebesgue-almost all $t > 0$, $\mathbb{E}(e^{-\delta(nt)}[c_{nt}]^{1-\gamma}) \to 0$ as $n \to \infty$ (Theorem 1 of [45]). Therefore, $e^{-\delta(nt)}\mathbb{E}(U_{nt}^{\pi,l})$ vanishes as $n \to \infty$ for almost all $t > 0$, too. The result for $|U^{\pi,l}|^{\frac{1}{\theta}}$ follows a similar argument, utilising instead the integrability of $(c)^{1-\phi}$.

□

Proposition 5.4.2. *If V is a continuous and adapted stochastic process and of class (DL) and $\{\tau_n\}_{n\geq 0}$ is an increasing sequence of stopping time such that $\tau_n \to \infty$ almost surely, then:*

$$\mathbb{E}\big(V_{T\wedge\tau_n}|\mathcal{F}_{t\wedge\tau_n}\big) \to \mathbb{E}\big(V_T|\mathcal{F}_t\big), \quad \text{in } L^1. \tag{5.72}$$

Proof. We have the following L^1 estimate:

$$\begin{aligned}\|\mathbb{E}(V_{T\wedge\tau_n}|\mathcal{F}_{t\wedge\tau_n}) - \mathbb{E}(V_T|\mathcal{F}_t)\|_{L^1} \leq \|\mathbb{E}(V_{T\wedge\tau_n}|\mathcal{F}_{t\wedge\tau_n}) - \mathbb{E}(V_T|\mathcal{F}_{t\wedge\tau_n})\|_{L^1} \\ + \|\mathbb{E}(V_T|\mathcal{F}_{t\wedge\tau_n}) - \mathbb{E}(V_T|\mathcal{F}_t)\|_{L^1}\end{aligned} \tag{5.73}$$

The first term above is bounded above by $\mathbb{E}(|V_{T\wedge\tau_n} - V_T|)$. As $n \to \infty$, $V_{T\wedge\tau_n} - V_t \to 0$ almost surely due to continuity. Thanks to the (DL) property, we can apply Dominated Convergence Theorem to conclude that it converges in L^1 too. The second term also vanishes, following from Martingale Convergence Theorem.

□

5.4.2 Proof of Theorem 5.2.2

Proof. Denote $U_t^{\pi,l} \triangleq \frac{(X_t^{\pi,l})^{1-\gamma}}{1-\gamma} g(Y_t)^{\phi\theta}$. Following from the Epstein–Zin HJB equation (see equation (4.8) and the discussion that follows), as (π, l) is an arbitrary strategy, $U^{\pi,l}$ is a super-solution of the Epstein–Zin BSDE, in the sense that:

$$dU_t^{\pi,l} = -f(c_t, U_t^{\pi,l})dt + Z_t^U dB_t - p_t dt, \quad t \geq 0; \tag{5.74}$$

for some positive, progressive process p_t and progressive Z_t^U. On the other hand, denote by $V_t^{\pi,l}$ the value process associated with $c = lX^{\pi,l}$, which satisfies the BSDE:

$$dV_t^{\pi,l} = -f(c_t, V_t^{\pi,l})dt + Z_t^{\pi,l} dB_t, \quad t \geq 0. \tag{5.75}$$

For a progressively measurable process $\{\alpha_t, t \geq 0\}$ which will be defined later, the dynamics of $e^{\int_0^t \alpha_s ds}(U_t - V_t^{\pi,l})$ is:

$$\begin{aligned}d\big(e^{\int_0^t \alpha_s ds}(U_t^{\pi,l} - V_t^{\pi,l})\big) = -e^{\int_0^t \alpha_s ds}\big[f(c_t, U_t^{\pi,l}) - f(c_t, V_t^{\pi,l}) - \alpha_t(U_t^{\pi,l} - V_t^{\pi,l})\big]dt \\ + e^{\int_0^t \alpha_s ds}(Z_t^U - Z_t^{\pi,l})dB_t - e^{\int_0^t \alpha_s ds} p_t dt.\end{aligned} \tag{5.76}$$

By setting $\alpha_t = \frac{f(c_t, U_t^{\pi,l}) - f(c_t, V_t^{\pi,l})}{U_t^{\pi,l} - V_t^{\pi,l}} \mathbb{I}_{\{U_t^{\pi,l} \neq V_t^{\pi,l}\}} + f'(c_t, U_t^{\pi,l})\mathbb{I}_{\{U_t^{\pi,l} = V_t^{\pi,l}\}}$, the first drift term vanishes and thus $e^{\int_0^t \alpha_s ds}(U_t^{\pi,l} - V_t^{\pi,l})$ is a local supermartingale. Let $\{\sigma_n\}_{n\geq 0}$ be a reducing sequence for the stochastic integral, then:

$$U_0^{\pi,l} - V_0^{\pi,l} \geq \mathbb{E}\big[e^{\int_0^{T\wedge\sigma_n} \alpha_s ds}(U_{T\wedge\sigma_n}^{\pi,l} - V_{T\wedge\sigma_n}^{\pi,l})\big]. \tag{5.77}$$

By Mean Value theorem, $\alpha_t = f_v(c_t, \xi)$ for some $\xi \in [U_t^{\pi,l} \wedge V_t^{\pi,l}, U_t^{\pi,l} \vee V_t^{\pi,l}]$. We recall that $\partial_v f(c,v) \leq -\delta\theta$ uniformly (see discussion after Definition 2.7.2). Therefore, the exponential factor in (5.77) is locally bounded. Moreover, $U^{\pi,l}$ and $V^{\pi,l}$ are of class (DL) since they are sandwiched between $U^{\gamma,\infty}$ and $U^{\phi,\infty}$, both of which are (DL) processes. Therefore, we can let $n \to \infty$ in (5.77) to obtain:

$$U_0^{\pi,l} - V_0^{\pi,l} \geq \mathbb{E}\Big(e^{\int_0^T \alpha_s ds}(U_T^{\pi,l} - V_T^{\pi,l})\Big). \tag{5.78}$$

We assert that the right hand side of (5.78) vanishes as $T \to \infty$. By considering the second derivative of f: $\partial^2_{vv} f(c,v) = \delta(\gamma - \phi)c^{1-\phi}[(1-\gamma)v]^{-1-1/\theta}$, ∂f_v is increasing in v when $\gamma > \phi$. Therefore, the application of Mean Value Theorem above implies $\alpha_t \leq \partial_v f(c_t, U_t^{\pi,l} \vee V_t^{\pi,l}) \leq \partial_v f(c_t, U^{\phi,\infty}(c)_t)$. Then, by uniqueness criterion (2.21), we achieve the estimate:

$$\mathbb{E}\Big(e^{\int_0^T \alpha_s ds}\Big|U_T^{\pi,l}\Big|\Big) \leq \mathbb{E}\Big(e^{\int_0^T \partial_v f(c_s, U^{\phi,\infty}(c)_s) ds}\Big|U_T^{\gamma,\infty}\Big|\Big) \to 0 \text{ as } T \to \infty. \tag{5.79}$$

The same convergence to 0 holds for $V^{\pi,l}$ in place of $U^{\pi,l}$. Letting T diverge in (5.78), we obtain an upper bound for permissible strategies:

$$\frac{x^{1-\gamma}}{1-\gamma} g(y)^{\phi\theta} = U_0^{\pi,l} \geq V_0^{\pi,l}. \tag{5.80}$$

Lastly, if (π^*, l^*) belongs to the permissible class, then $V^* = U^{\pi^*,l^*}$ is verified as the Epstein–Zin utility associated with $c^* = l^* X^{\pi^*,l^*}$, thanks to the Power Utility Bounds (5.11) in Definition 5.2.1. It is trivial to confirm that $V_0^* = u_\gamma(x) g(y)^{\phi\theta}$, which attains the upper bound for $\mathcal{P}(g)$ strategies.

□

CHAPTER **6**

EXAMPLES

6.1 Example I - Geometric Brownian Motion Price.

We will demonstrate the results developed in previous sections in certain market models. Similar to Chapter 4, we also start with the Merton's model as a baseline, where the factor process is constant. This is subsumed as a degenerate case of our factor model by setting $y = 1, b(y) \equiv 0$ and $a(y) \equiv 0$. We also denote g, r, μ, σ for $g(1), r(1), \mu(1)$ and $\sigma(1)$ respectively. Equation (4.11) reduces to:

$$\mathcal{H}^{\mathrm{EZ}}(y, g, \nabla g, D^2 g) = \delta^{\psi} g^{-1} + \left(r + \frac{\mu^2}{2\gamma\sigma^2}\right)\left(\frac{1-\phi}{\phi}\right) - \frac{\delta}{\phi} = 0. \tag{6.1}$$

This yields directly the solution for the candidate optimal strategy:

$$\pi^* = \frac{\mu}{\gamma\sigma^2}, \quad l^* = \delta^{\psi} g^{-1} = \frac{\delta - r(1-\phi)}{\phi} - \frac{(1-\phi)\mu^2}{2\gamma\phi\sigma^2}. \tag{6.2}$$

In this simple model, the admissibility and permissibility of the candidate optimal strategy can be reduced to a set of inequalities. Moreover, this is the only case where we can derive a necessary *and* sufficient condition for verification of the candidate solution.

Proposition 6.1.1. *We consider a market model with Geometric Brownian Motion price. Let (π, l) be a constant strategy. Then, for any exponent $p \in \mathbb{R}$, the resulting consumption process c satisfies $\mathbb{E}(\int_0^\infty \delta e^{-\delta t} c_s^p \, ds) < \infty$ if:*

$$\delta - p(r - l) - p\pi\mu - \frac{p^2 - p}{2}\pi^2\sigma^2 > 0. \tag{6.3}$$

Moreover, it satisfies the uniqueness criterion (2.21) if and only if:

$$\delta - (1 - \phi)(r - l) - (1 - \phi)\pi\mu - \frac{(\gamma - 1)\gamma - \phi(\gamma - \phi)}{2}\pi^2\sigma^2 > 0. \tag{6.4}$$

As a corollary, the candidate optimal strategy (π^*, l^*) defined by (6.2) belongs to $\mathcal{A}$ if and only if the following system of inequalities hold:

$$\begin{cases} \delta + \dfrac{2(1-\phi)(\delta - r)}{\phi} - \Big(\dfrac{1-\phi}{\gamma}\Big)\Big(\dfrac{1}{\phi} + \dfrac{1 - 2\phi + \gamma}{\gamma}\Big)\dfrac{\mu^2}{\sigma^2} > 0, & \text{(6.5a)} \\ \delta + \dfrac{2(\phi-\gamma)(\delta - r)}{\phi} - \Big(\dfrac{\phi-\gamma}{\gamma}\Big)\Big(\dfrac{1}{\phi} + \dfrac{2\phi - \gamma - 1}{\gamma}\Big)\dfrac{\mu^2}{\sigma^2} > 0, & \text{(6.5b)} \\ \dfrac{\delta - (1-\phi)r}{\phi} - \Big[\dfrac{(1-\phi)}{2\gamma\phi} + \dfrac{(\gamma-\phi)^2}{2\gamma^2}\Big]\dfrac{\mu^2}{\sigma^2} > 0. & \text{(6.5c)} \end{cases}$$

A by-product of Proposition 6.1.1 is that it also shows the permissible set $\mathcal{P}(g)$ is non-empty and non-trivial. Let us recall from Lemma 5.2.1, when g is bounded from above and below, a strategy (π, l) is permissible if it belongs to $\mathcal{A}$ and satisfies:

$$n_t^{\pi,l} + \frac{1}{2}(\gamma - \phi)\pi^2\sigma^2 \geq 0 \quad \text{for all } t \geq 0. \tag{6.6}$$

Provided that $\mu > 0$, the left hand side above is strictly positive at (π^*, l^*). Because of joint continuity of the mapping $(\pi, l) \to n^{\pi,l} + \frac{1}{2}(\gamma - \phi)\pi'\Sigma\pi$, there exists an open set $O \subseteq \mathbb{R}^2$ containing (π^*, l^*), such that for all $(\pi, l) \in O$, relation (6.6) is satisfied. Similarly, if inequalities (6.5a), (6.5b) & (6.5c) hold, then there exists an open neighbourhood around it, also denoted O, where such that all constant strategies are permissible.

The verification result in this market model can be summarised as follows:

Theorem 6.1.2. *We consider a market model with Geometric Brownian Motion price and constant factor process. Let (π^*, l^*) be defined by (6.2). Then, if the system of inequalities (6.5a), (6.5b) & (6.5c) hold, it belongs to the permissible class $\mathcal{P}(g)$ and is optimal within it.*

6.2 Example II - Models With Bounded Coefficients.

We build upon the general results obtained in previous sections by imposing boundedness additional conditions on the coefficients and their derivatives. Roughly speaking, these additional assumptions will enable us to derive uniform boundedness for the candidate optimal control (π^*, l^*), which in turn translates to more-or-less explicit conditions for its admissibility.

For convenience, let us briefly recall the market model here. The financial market model consists of a riskless asset S^0 and an n-tuple of risky assets $S = (S^1, ..., S^n)$. Their dynamics are given by the following equations:

$$
\begin{aligned}
dS_t^0 &= S_t^0 r(Y_t)dt, \\
dS_t &= \operatorname{diag}(S_t)\left[(r(Y_t)1_n + \mu(Y_t))dt + \sigma(Y_t)dW_t^\rho\right], \\
dY_t &= b(Y_t)dt + a(Y_t)dW_t, \quad Y_0 = y \in E.
\end{aligned}
\tag{6.7}
$$

We refer the reader to chapter 4.1 for a detailed discussion of the smoothness assumptions of the coefficients. In additional to the assumptions already made there, we also impose the following additional assumptions for the remainder of section 6.2.

Assumption 6.2.1.

i. The domain E is the real space $\mathbb{R}^k$;

ii. (uniform ellipticity) there exists $\underline{\lambda} > 0$ such that the matrices A and Σ satisfy, for all $y \in E$ and $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^n$:

$$
\sum_{ij} A_{ij}(x)\xi_i\xi_j \geq \underline{\lambda}|\xi|^2, \quad \sum_{ij} \Sigma_{ij}(x)\eta_i\eta_j \geq \underline{\lambda}|\eta|^2. \tag{6.8}
$$

iii. the coefficients are globally bounded, i.e.:

$$\begin{aligned}
&\sup_{y\in E}|r|, \quad \sup_{\substack{y\in E\\ i=1..n}}|\mu_i|, \quad \sup_{\substack{y\in E\\ i=1..n}}|b_i| < \infty;\\
&\sup_{\substack{y\in E\\ i,j=1..k}}\left|a_{ij}\right| < \infty, \quad \sup_{\substack{y\in E\\ i,j=1..n}}\left|\sigma_{ij}\right| < \infty.
\end{aligned} \tag{6.9}$$

iv. the coefficients have bounded derivatives, i.e.:

$$\sup_{\substack{y\in E\\ i,j,l}}\left|D_l a_{ij}(y)\right|, \sup_{\substack{y\in E\\ i,j,l}}\left|D_l \sigma_{ij}(y)\right|, \sup_{\substack{y\in E\\ i}}\left|D_i r\right|, \sup_{\substack{y\in E\\ i,j}}\left|D_i \mu_j\right|, \sup_{\substack{y\in E\\ i,j}}\left|D_i b_j\right| < \infty, \tag{6.10}$$

v. and finally, the interest rate is bounded away from 0: $\inf_{y\in E} r(y) > 0$.

In what follows, we will derive bounds for the solution g of the HJB equation and its derivative, which will translate into uniform bounds for the optimal strategy (π^*, l^*). Regarding bounding g, this is achieved by making an appropriate choice for g_1 and g_2, the sub- and super-solution constructed in Section 5.1. Theorem 5.1.2 asserts a solution g sandwiched between g_1 and g_2. Therefore, if g_2 is bounded above and g_1 away from zero, g will inherit these boundedness properties. We will choose g_1 and g_2 in such a way that helps to ease the calculations. For g_1, we choose the strategy where the investor ceases his trading activities and consumes the earnings from interest rate. Specifically, we set $\pi_t \equiv 0$ and $l_t = r_t$, in which case, $X_t \equiv x$. For g_2, we simply set $\eta_t \equiv 0$. Bounding g_1 and g_2 will help to bound l^*. It is more challenging to prove the uniform boundedness of π^*. From the first order condition (4.10) and the boundedness of model coefficients, it follows that uniform boundedness of π^* would follow from that of the ratio $\frac{\nabla g}{g}$. This can be derived from interior gradient bounds of an elliptic quasilinear PDE (cf. Chapter 15 of [25]). We state the result formally below.

Theorem 6.2.1. *1. Define $g_1(y)$ and $g_2(y)$ by:*

$$\frac{x^{1-\gamma}}{1-\gamma} g_1^{\phi\theta}(y) = \mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t} u_\gamma(r_t x)dt \Big| Y_0 = y\Big), \tag{6.11a}$$

$$g_2(y) = \mathbb{E}\Big(\int_0^\infty e^{-\frac{\delta}{\phi}t + \frac{1-\phi}{\phi}\int_0^t r_s ds} \mathcal{E}\Big(-\int_0^\cdot \mu(\sigma')^{-1} dW_s^\rho\Big)_t^{\frac{\phi-1}{\phi}} \Big| Y_0 = y\Big). \tag{6.11b}$$

Suppose that the following conditions hold:

i. $\frac{\delta}{\gamma-1} - \frac{\|\mu'\Sigma^{-1}\mu\|_\infty}{2} \geq 0$, *ii.* $\frac{\delta}{1-\phi} - \|r\|_\infty - \frac{\|\mu'\Sigma^{-1}\mu\|_\infty}{2\phi} > 0$,

iii. g_1 *and* g_2 *are continuous in* $\mathbb{R}^k$.

Then, g_1 *and* g_2 *are bounded in* $\mathbb{R}^k$*. Moreover,* g_1 *is bounded away from zero. Consequently, a solution* g *exists for the equation* (4.11) *that is bounded above and away from zero. Moreover, it satisfies* $\sup_{y\in\mathbb{R}^k} \left\| \frac{\nabla g(y)}{g(y)} \right\| < \infty$.

2. *Let* (π^*, l^*) *be defined by the first order condition* (4.10)*. Then,* l^* *is bounded above and away from zero and* $\|\pi^*\|$ *is bounded above.*

The next result concerns the admissibility of the candidate strategy (π^*, l^*). We have chosen to report this result for the special case where the factor process is one-dimensional. The chosen dimensionality helps ease the notations in our calculations. The methods remain virtually unchanged when Y has dimension greater than 1. From the point of view of applications, this does not present a serious handicap, as most popular models of the financial market employs a one-dimensional factor process.

Proposition 6.2.2. *Suppose that assumption 6.2.1 and the conditions of Theorem 6.2.1 hold. Assume additionally that* $\phi > \frac{1}{2}$*. Let* g_1*,* g_2 *be defined by equation* (6.11a) *and* (6.11b)*, respectively and* g *be the solution to the HJB equation* (4.11) *sandwiched between* g_1 *and* g_2*, which is facilitated by Theorem 5.1.2. Denote* $\underline{g} = \inf_{y\in\mathbb{R}^k} g_1(y)$ *and* $\bar{g} = \sup_{y\in\mathbb{R}^k} g_2(y)$*. Let* (π^*, l^*) *be defined by the first order condition* (4.10) *and* c^* *be the resulting consumption process. Then,* c^* *belongs to the class* C^∞ *if the following system of inequalities hold:*

$$\begin{cases} 2(\phi-\gamma)(\inf_{y\in E} r - \delta^\psi \underline{g}^{-1}) + \dfrac{(\phi-\gamma)(2\phi-1)}{\gamma^2} \inf_y \left[\dfrac{\mu^2}{\sigma^2}\right] + \left|\dfrac{2\phi\theta(\phi-\gamma)(2\phi-\gamma-1)\rho}{\gamma^2}\right| \left\|\dfrac{\nabla g}{g}\right\|_\infty \|\dfrac{\mu a}{\sigma}\|_\infty \\ \qquad + \left|\dfrac{(\phi-\gamma)(2\phi-2\gamma-1)\phi^2\theta^2}{\gamma^2}\right| \left\|\dfrac{\nabla g}{g}\right\|_\infty^2 \|a\|_\infty^2 < \delta, & (6.12a) \\ 2(1-\phi)(\sup_{y\in E} r - \delta^\psi \bar{g}^{-1}) + \dfrac{(1-\phi)(2\gamma-2\phi+1)}{\gamma^2} \sup_y \left[\dfrac{\mu^2}{\sigma^2}\right] \\ \qquad + \left|\dfrac{2\phi(1-\gamma)(\gamma+1-2\phi)\rho}{\gamma^2}\right| \left\|\dfrac{\nabla g}{g}\right\|_\infty \|\dfrac{\mu a}{\sigma}\|_\infty < \delta. & (6.12b) \end{cases}$$

Moreover, c^* satisfies the uniqueness criterion if the following additional inequality holds:

$$(1-\theta)\left[\delta - \left(\frac{\bar{g}}{\underline{g}}\right)^{\phi-1}(\delta - R)\right] + \delta\theta - \theta\delta^{\psi}\underline{g}^{-1} - \delta < 0, \text{ where}$$

$$R = (1-\phi)(\sup_{y\in E} r - \delta^{\psi}\bar{g}^{-1}) + \frac{(1-\phi)(2\gamma-\phi)}{2\gamma^2}\left\|\frac{\mu^2}{\sigma^2}\right\|_\infty + \left|\frac{\phi(1-\gamma)(\gamma-\phi)\rho}{\gamma^2}\right|\left\|\frac{\nabla g}{g}\right\|_\infty\left\|\frac{\mu a}{\sigma}\right\|_\infty. \tag{6.13}$$

Remark 6.2.2. The inequality system attained in Proposition 6.2.2 subsumes that obtained in Proposition 6.1.1. If we set g and all model coefficients as constant, $\nabla g = 0$ and $\underline{g} = \bar{g} = g$, then inequalities (6.12a),(6.12b) and (6.13) reduce to (6.5a), (6.5b) and (6.5c).

Remark 6.2.3. The bounds attained in Proposition 6.2.2 are very crude bounds. They key message that we want to demonstrate here is that there exist verifiable inequalities with which we can confirm our verification result, even if the inequalities are somewhat convoluted. This is demonstrated in the next section, where we numerically solve the Epstein–Zin HJB equation in a non-trivial market model. A refinement of these bounds can be a direction for potential future research.

Theorem 6.2.1 and Proposition 6.2.2 can be combined to yield the following verification result in a model with bounded coefficients.

Theorem 6.2.3. *Consider a one-dimensional market model satisfying assumption 6.2.1 and the conditions of Theorem 6.2.1, which implies a solution g for the Epstein–Zin HJB equation. Suppose $\phi > \frac{1}{2}$ and the system of inequalities (6.12a), (6.12b) & (6.13) hold, then the strategy (π^*, l^*) defined by the first order condition (4.10) belongs to the permissible class $\mathcal{P}(g)$ and is optimal within this class.*

6.3 Numerical Implementation of A Linear Diffusion Model

6.3.1 Introduction & Market Model

When it comes to implementing the theory we have developed to market models, there are several technical hurdles. The solution of the HJB equation (4.11) that we constructed in Section 5.1 is furnished by an abstract subsequence. That is, it is an existential rather than constructive proof. Therefore, estimates related to g and its derivatives can be difficult to obtain. The problem of upper and lower bounds of g can be mitigated by certain choices of g_1 and g_2 in equations (6.11a)-(6.11b). However, boundedness of the ratio $\frac{\nabla g}{g}$ is obtained via an abstract argument (cf. Theorem 15.5 [25]) and an explicit form is unlikely to be available.

In this section, we propose a truncation scheme for an adaptation of the linear diffusion factor model, and solve it numerically. Estimates concerning the solution g, in particular the ratio $\frac{\nabla g}{g}$ will also be obtained numerically, too, and the inequalities of Proposition 6.2.2 will be verified based on numerical estimates. In the model considered below, the risky asset's volatility is constant but its return depends on the factor process, which itself follows a mean-reverting Ornstein-Uhlenbeck dynamics. This model specification has been studied by Kim & Ongberg [35], Wachter [62] in the time separable utility settings, Campbell & Viceira [10] in a recursive utility, discrete time setting, and Xing [66] for continuous time Epstein–Zin utility in finite horizon. The precise model specification is:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r(Y_t) + \mu(Y_t))dt + \sigma dW_t^{\rho}, \\ dY_t &= b(Y_t)dt + a dW_t, \end{aligned} \tag{6.14}$$

where $y \in \mathbb{R}, r(y) = r_0 + r_1 y, \mu(y) = \sigma(\lambda_0 + \lambda_1 y)$ and $b(y) = -b_0 y$.

Remark 6.3.1. There is no loss of generality in defining $\lambda_0 = 0$, since the constant term of $\mu(y)$ can be incorporated by the interest rate term. Specifically, we could define $r(y) = (r_0 + \sigma\lambda_0) + r_1 y$. For simplicity, however, we shall define $r_1 = 0$. We use the numerical values from [4], [62] and [66]:

$$r_0 = 0.02, \quad r_1 = 0, \quad \sigma = 0.0436, \quad \lambda = 0, \quad \lambda_1 = 1, \quad b_0 = 0.0226, \quad a = 0.0189. \tag{6.15}$$

6.3.2 Numerical Output: Optimal Strategy

We present below the numerical output for the optimal strategy (π^*, l^*) for different value combinations of γ and ϕ. The algorithm and method of truncation is reported in section 6.3.3. The resulting strategies are reported in Figure 6.1 and Figure 6.2.

We first observe how this representative agent responds to the state of the economy. The general shape of π^* suggests that when $y > 0$ (resp. $y < 0$), the risky asset is performing better (resp. worse) than inflation, and the agent enters a long (resp. short) position in order to benefit from that discrepancy. Similarly, when y diverges from 0 in either direction, the consumption-wealth ratio decreases. Therefore, whenever the economy departs from its neutral state, it represents an investment opportunity and the agent trades his current consumption for the prospect of future earning.

Secondly, we observe how the candidate optimal strategies change in response to changes in the preference parameters. With everything else being equal, for larger values of γ, π^* has flatter slope and l^* increases. At a greater level of risk aversion, the agent reacts by reducing the proportion of wealth invested in the risky asset, either in long or short position. The consumption-wealth ratio also goes up, suggesting that he increases immediate consumption to counteract his dislike of late resolution of uncertainty.

On the other hand, higher values of ϕ imply lower EIS. In this case, the agent becomes less patient. He prefers immediate consumption instead of substituting it for investment opportunities. This is portrayed in Figure 6.2, where consumption-wealth ratio is increasing in ϕ. However, for differing values of ϕ and everything else equal, the numerical result suggests that it exerts little effect, if at all, on the investment-wealth ratio. This insensitivity of π^* with respect to the EIS parameter is explicit in the GBM model (cf. equation (6.2)). The same phenomenon has been observed by Xing [66], who observed that in a finite-horizon Heston model, the optimal investment-wealth ratio is insensitive to ϕ. Intuitively, this is consistent with the fact that investment decision is governed by risk aversion γ. Since this phenomenon is observed across multiple different market models, we conjecture that this might hold for a wider class of market models. However, as of the time of writing this thesis, no result in this direction has been established, even in finite horizon. If this conjecture were true, it would help to disentangle the effects of preference parameters on the investor's behaviour: while both risk aversion and EIS determine how much wealth is consumed immediately, only the risk aversion parameter affects the allocation of the invested fund.

Fig. 6.1 Optimal Strategies for different values of γ. In all cases, $\phi = 2/3$ and $\delta = 0.02$.

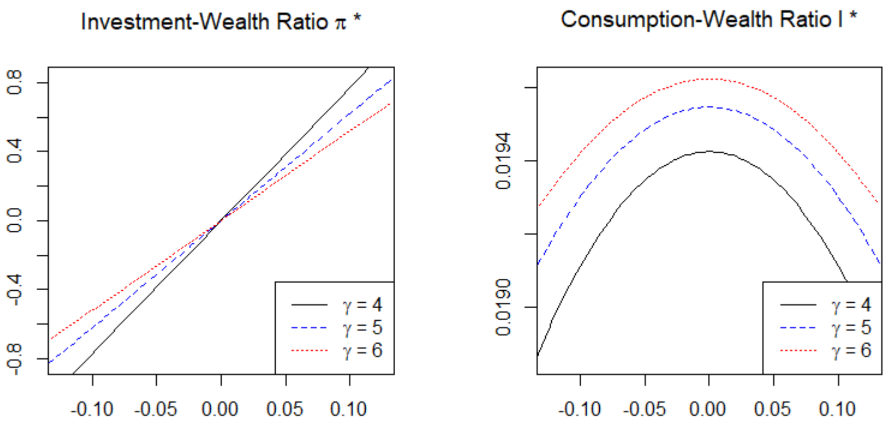
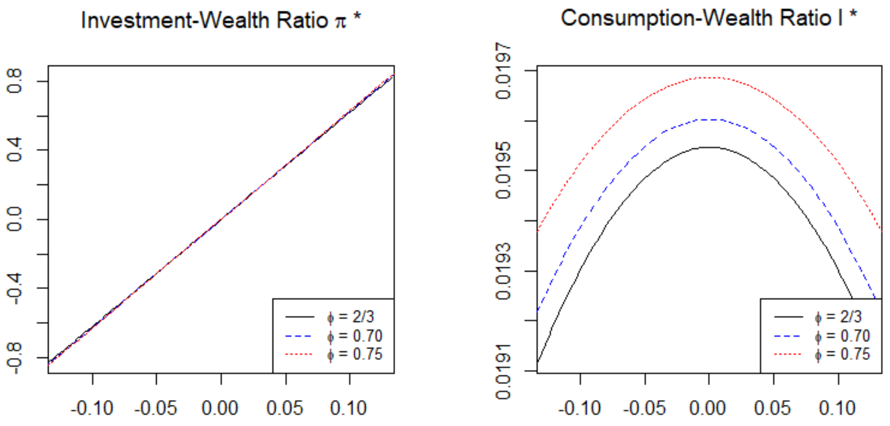


Fig. 6.2 Optimal Strategies for different values of ϕ. In all cases, $\gamma = 5$ and $\delta = 0.02$.



6.3.3 Truncation of the HJB equation

It is more convenient to work with the exponential parametrisation of the value function. Let us recall that, if g is the solution to equation (4.11), then the transform $u = \phi\theta \ln(g)$ satisfies the equation:

$$\mathcal{G}^{\mathrm{EZ}}(y, u, u', u'') = \phi\theta\delta^{\psi} \exp\left(-\frac{u}{\phi\theta}\right) + u'\left(b + \frac{1-\gamma}{\gamma}\frac{\mu a \rho}{\sigma}\right) + \frac{1}{2}(u')^2 a^2\left(1 + \frac{1-\gamma}{\gamma}\rho^2\right) + \frac{1}{2}a^2 u''$$
$$+ \left(r + \frac{\delta}{\phi - 1} + \frac{\mu^2}{2\gamma\sigma^2}\right)(1-\gamma) = 0, \quad y \in \mathbb{R}. \tag{6.16}$$

The idea behind the truncation we propose is simple. We let the factor process starts at its long term mean (0 in this case) and allow it to perturb in a small region around it. As it leaves its permitted region, we immediately terminate its dynamics. More specifically, for a cut-off value $K > 0$, let us define the truncated coefficients:

$$\begin{aligned} &\sigma^{(K)}(y) = \sigma[(y \wedge K) \vee (-K)], \quad \mu^{(K)}(y) = \mu[(y \wedge K) \vee (-K)], \\ &a^{(K)}(y) = a\mathbb{I}_{\{y \in (-K,K)\}}, \quad b^{(K)}(y) = b\mathbb{I}_{\{y \in (-K,K)\}}. \end{aligned} \tag{6.17}$$

The truncated HJB equation is obtained by replacing the model coefficients in (6.16) with their truncated versions. Outside the permitted region, all derivative terms in equation (6.16) vanish, which allows us to explicitly determine the following boundary data:

$$u(K) = u(-K) = \phi\theta \ln \left(\frac{\phi\delta^{\psi}}{\delta - (1-\phi)r - \frac{1-\phi}{2\gamma}\frac{\mu^2(K)}{\sigma^2}} \right). \tag{6.18}$$

By defining $u_1 = u$ and $u_2 = u'$, equation (6.16) can be reduced to the following first-order, non-linear ODE for $y \in [-K, K]$ with boundary data (6.18):

$$\begin{aligned} u_1'(y) &= u_2, \\ u_2'(y) &= -\frac{2\phi\theta}{a^2}\delta^{\psi} \exp\left(-\frac{u_1}{\phi\theta}\right) - \frac{2u_2}{a^2}\left(b(y) + \frac{1-\gamma}{\gamma}\frac{\mu a \rho}{\sigma}\right) - \frac{1}{2}u_2^2\left(1 + \frac{1-\gamma}{\gamma}\rho^2\right) \\ &\quad - \frac{2(1-\gamma)}{a^2}\left(r + \frac{\delta}{\phi - 1} + \frac{\mu^2}{2\gamma\sigma^2}\right). \end{aligned} \tag{6.19}$$

The solving of the above boundary value problem is implemented with the package `deSolve` in `R` [61], which is built upon `Fortran` routines. There are a few points to note:

- The numerical stipulates initial conditions of u_1 and u_2 at $y = -K$. However, the truncation only provides boundary data for u_1 at $y = -K$ and $y = K$. We circumvent this problem as follows. The numerics suggest that $u_1(K)$ is increasing with respect to the initial value $u_2(-K)$. We take advantage of this by using an simple interval bisection algorithm to determine numerically the appropriate value for $u_2(-K)$, so that $u_1(K)$ matches the terminal data in (6.18) We use a simple interval bisection algorithm to determine the appropriate value for $u_2(0)$ so that $u_1(K)$ matches the terminal data in (6.18).

Algorithm 6.3.1. *An interval bisection algorithm/pseudo-code for solving equation of the type $g(x) = 0$ for a continuous, increasing function g is presented below.*

Step 1. *Initilisation: Select values a_0 and b_0 such that $a_0 < b_0$ and $g(a_0) < 0 < g(b_0)$.*

Step 2. *Iterative step: At n-th iteration, we should have lower and upper bounds $a_n < b_n$ such that $g(a_n) < 0 < g(b_n)$. If $g(\frac{a_n+b_n}{2}) < 0$, then let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$. If $g(\frac{a_n+b_n}{2}) > 0$, let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$.*

Step 3. *If $|g(\frac{a_n+b_n}{2})| < \epsilon$ for a predetermined error level ϵ, stop the iteration and report $\frac{a_n+b_n}{2}$ as the numerical solution.*

- For the boundary data in (6.18) to be well-defined, the denominator within logarithm has to be strictly positive. This gives an upper bound to the cut off value:

$$K < \sqrt{\frac{2\gamma}{\lambda_1^2}\Big(\frac{\delta}{1-\phi} - r\Big)} \triangleq \text{Cut-off limit.} \tag{6.20}$$

We characterize K by what we shall name the cut-off factor, defined as follows:

$$K = \text{cut-off factor} \times \text{cut-off limit.} \tag{6.21}$$

As the cut-off factor approaches 1, K approaches its upper bound and the boundary data (6.18) diverges. In particular, the algorithm becomes unstable and highly sensitive with respect to $u_2(-K)$ as the cut-off factor grows.

6.3.4 Numerical Output: Truncation Level Selection

The numerical results reported below suggest that there exists a threshold for the cut-off factor, under (resp. over) which the truncated model satisfies (resp. fails) inequalities (6.12a),(6.12b) and (6.13) of Theorem 6.2.2. Let us collectively refer to these inequalities as the *Sufficiency* condition. Given a set of model parameters, we let the cut-off factor increase in steps of 0.005 until it no longer passes the sufficiency check. In section 6.3.3, for each configuration of γ and ϕ, we select the maximal cut-off factor using this procedure.

One way to assess the goodness of the truncation K is by comparing it to the long-term standard deviation of the factor process. In particular, the Ornstein-Uhlenbeck process Y has a Gaussian asymptotic distribution with zero mean and standard deviation $\frac{a^2}{2b_0}$. Therefore, we also report the ratio $K/\sqrt{a^2/2b_0}$ in our findings. In table 6.1, we report our findings under baseline parameters given in (6.15) and preference parameters suggested in [3].

Table 6.1 Model parameters: $r_0 = 0.02, r_1 = 0, \sigma = 0.0436, \lambda = 0, \lambda_1 = 1, b_0 = 0.0226, a = 0.0189$. Preference paramaters: $\gamma = 5, \phi = 2/3, \delta = 0.02$.

Cut Off Factor	K	$K/\sqrt{\frac{a^2}{2b_0}}$	$u_2(-K)$	Sufficiency
0.2000	0.1265	1.4229	2.3199	✓
0.2050	0.1297	1.4584	2.4778	✓
0.2100	0.1328	1.4940	2.6421	✓
0.2150	0.1360	1.5296	2.8129	X
0.2200	0.1391	1.5652	2.9903	X
0.2250	0.1423	1.6007	3.1744	X

Table 6.2 Model parameters are the same as Table 6.1. Preference parameters: $\gamma = 1.2, \phi = 0.9$ and $\delta = 0.02$.

Cut Off Factor	K	$K/\sqrt{\frac{a^2}{2b_0}}$	$u_2(-K)$	Sufficiency
0.4300	0.2826	3.1792	8.9243	✓
0.4350	0.2859	3.2162	9.3025	✓
0.4400	0.2892	3.2531	9.6939	✓
0.4450	0.2925	3.2901	10.0991	X
0.4500	0.2958	3.3271	10.5185	X

The uniqueness criterion is more forgiving for γ and ϕ both close to 1, in the sense that we can now achieve a truncation over three times the asymptotic standard deviation of the factor process (see Table 6.2). This makes intuitive sense, because as $\gamma \to 1^+$ and $\phi \to 1^-$, we are approximating the case of a myopic investor with logarithmic utility. We can also raise the ratio $K/\sqrt{\frac{a^2}{2b_0}}$ if the factor process is more 'tempered', i.e. when b is large and a is small. By halving the constant a, we achieve a maximum ratio of approximately 2.561. If we additionally double b, this ratio only raises slightly to 2.817.

6.4 Proofs for Section 6.1

Proof of Proposition 6.1.1. Given a constant strategy (π, l), let us estimate the growth rate of c For any generic exponent $p \in \mathbb{R}$:

$$
\begin{aligned}
c_t^p &= \delta^{p\psi} g^{-p} (X_t^{\pi,c})^p \\
&= \delta^{p\psi} g^{-p} \exp\Big(\int_0^t \Big[p(r-l) + p\pi\mu + \frac{p^2-p}{2}\pi^2\sigma^2\Big] ds\Big)\mathcal{E}\Big(p\int_0^\cdot \pi\sigma\Big)_t
\end{aligned}
\tag{6.22}
$$

Therefore, $\mathbb{E}(\int_0^\infty \delta e^{-\delta(s-t)}(c_t^*)^p dt)$ is finite if and only if:

$$
-\delta + p(r-l) + p\pi\mu + \frac{p^2-p}{2}\pi^2\sigma^2 < 0. \tag{6.23}
$$

Setting $p = 2(\phi - \gamma)$ and $p = 2(1-\phi)$ and substituting the formula of π^* and l^* (see equation (6.2)) into (6.23), we obtain inequalities (6.5a) and (6.5b). Moreover, in the considered configuration where $\phi < 1 < \gamma$, $L^{2(\gamma-\phi)}(\mathbb{R}^+ \times \Omega) \subseteq L^{2(\gamma-1)}(\mathbb{R}^+ \times \Omega)$, where the product space is endowed with the probability measure $d\mathbb{P} \otimes \delta e^{-\delta t} dt$. Therefore, the integrability of $(c^*)^{2(\phi-\gamma)}$ also implies that of $(c^*)^{2(1-\gamma)}$.

Next, we estimate the uniqueness criterion of c. For the convenience of the reader, let us recall the uniquness criterion of a given consumption process $c \in C^\infty$ (see equation (2.22)):

$$
\lim_{T\to\infty} \mathbb{E}\Big[e^{\delta(1-\theta)\int_T^\infty (1-\Phi_t)dt} \int_T^\infty e^{-\delta s} c_s^{1-\gamma} ds\Big] = 0, \text{ where } \Phi_t = \frac{c_t^{1-\phi}}{\mathbb{E}_t\big(\int_t^\infty \delta e^{-\delta(s-t)} c_t^{1-\phi} ds\big)}. \tag{6.24}
$$

In the special case of constant (π, l), the quantity Φ_t in the uniqueness criterion can be computed explicitly as follows:

$$
\begin{aligned}
\Phi_t^{-1} &= \mathbb{E}_t\Big[\int_t^\infty \delta e^{-\delta(s-t)}\Big[\frac{c_s}{c_t}\Big]^{1-\phi} ds\Big] \\
&= \mathbb{E}_t\Big\{\int_t^\infty \delta e^{-\delta(s-t)} \exp\Big(\int_t^s \Big[(1-\phi)(r-l) + (1-\phi)\pi\mu - \frac{\phi(1-\phi)}{2}\pi^2\sigma^2\Big] ds\Big)\mathcal{E}\Big(p\int_0^\cdot \pi\sigma dW^\rho\Big)_{t,s}\Big\} \\
&= \delta\Big(\delta - (1-\phi)(r-l) - (1-\phi)\pi\mu + \frac{\phi(1-\phi)}{2}\pi^2\sigma^2\Big)^{-1}.
\end{aligned}
\tag{6.25}
$$

Therefore, the first exponential factor in the uniqueness criterion is:

$$e^{\delta(1-\theta)\int_0^T(1-\Phi_t)dt} = \exp\Big(\Big[(\gamma-\phi)(r-l)+(\gamma-\phi)\pi\mu-\frac{\phi(\gamma-\phi)}{2}\pi^2\sigma^2\Big]T\Big). \tag{6.26}$$

Moreover:

$$\begin{aligned}\mathbb{E}\Big(\int_T^\infty e^{-\delta s}c_s^{1-\gamma}ds\Big) &= \int_T^\infty \delta^{\psi(1-\gamma)}g^{\gamma-1}e^{-\delta s}\exp\Big(\Big[(1-\gamma)(r-l)+(1-\gamma)\pi\mu+\frac{(\gamma-1)\gamma}{2}\pi^2\sigma^2\Big]s\Big)ds\\ &= K\exp\Big(\Big[-\delta+(1-\gamma)(r-l)+(1-\gamma)\pi\mu+\frac{(\gamma-1)\gamma}{2}\pi^2\sigma^2\Big]T\Big)\end{aligned} \tag{6.27}$$

where $K = \delta^{\psi(1-\gamma)}g^{\gamma-1}\int_0^\infty \exp\big([-\delta+(1-\gamma)(r-l)+(1-\gamma)\pi\mu+\frac{(\gamma-1)\gamma}{2}\pi^2\sigma^2]s\big)ds$. Combining the growth rates of (6.26) and (6.27), we conclude that the uniqueness criterion is satisfied if and only if:

$$-\delta+(1-\phi)(r-l)+(1-\phi)\pi\mu+\frac{(\gamma-1)\gamma-\phi(\gamma-\phi)}{2}\pi^2\sigma^2<0, \tag{6.28}$$

which yields inequality (6.5c) for the candidate optimal control (π^*, l^*).

6.5 Proofs for Section 6.2

6.5.1 Proof of Theorem 6.2.1

Part 1. Simplifying equation (6.11a) leads to $g_1(y)^{\phi\theta} = \mathbb{E}\Big(\int_0^\infty \delta e^{-\delta t}r_t^{1-\gamma}dt\Big|Y_0=y\Big)$, whence we obtain the estimate $\inf_{y\in E} r(y)^{\frac{1-\phi}{\phi}} \le g_1(y) \le \|r\|_\infty^{\frac{1-\phi}{\phi}}$.

For $g_2(y)$, the integrand can be simplified as follows:

$$\begin{aligned}&\exp\Big(-\frac{\delta}{\phi}t+\frac{1-\phi}{\phi}\int_0^t r_s ds\Big)\mathcal{E}\Big(-\int_0^\cdot \mu(\sigma')^{-1}dW_s^\rho\Big)_t^{\frac{\phi-1}{\phi}}\\ &\quad= \exp\Big(-\frac{\delta}{\phi}t+\int_0^t\Big(\frac{1-\phi}{\phi}r_s+\frac{1-\phi}{2\phi^2}\mu\Sigma^{-1}\mu\Big)ds\Big)\mathcal{E}\Big(\frac{\phi-1}{\phi}\int_0^\cdot \mu'(\sigma')^{-1}dW_s^\rho\Big)_t.\end{aligned} \tag{6.29}$$

We can exchange expectation and integral in (6.11b) and evaluate the upper bound directly. The strengthened assumption ii. ensures that the integral below is finite:

$$
\begin{aligned}
g_2(y) &\leq \int_0^\infty \exp\Big(-\frac{\delta}{\phi}+\frac{1-\phi}{\phi}\|r\|_\infty+\frac{1-\phi}{2\phi^2}\|\mu\Sigma^{-1}\mu\|_\infty\Big)ds \\
&= \frac{\phi}{\delta-(1-\phi)\|r\|_\infty-\frac{1-\phi}{2\phi}\|\mu\Sigma^{-1}\mu\|_\infty}.
\end{aligned}
\tag{6.30}
$$

Part 2. We work with the exponential parametrisation of the HJB equation. As remarked in he proof of Theorem 5.1.2, the transform $u=\phi\theta\ln(g)$ satisfies equation (5.32). It is sufficient to show that ∇u is bounded uniformly in E. If it is, then so is $\frac{\nabla g}{g}$ and consequently π^*. We can achieve this with Theorem 15.5 [25], a result on the interior gradient bound of an elliptic PDE. Below, we will state and prove a preliminary result which facilitates the necessary structural requirement for applying Theorem 15.5 [25], and then an abbreviated version of this Theorem which is sufficient for our purpose.

Proposition 6.5.1. *Define the mapping $B(y,z,p): E\times\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}$ by, where all model coefficients have argument y:*

$$
\begin{aligned}
B(y,z,p) = \phi\theta\delta^{\psi}e^{-\frac{z}{\phi\theta}} &+ p'\Big(b+\frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\Big)+\frac{1}{2}p'\Big(A+\frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\Big)p \\
&+\Big(r+\frac{\delta}{\phi-1}+\frac{\mu'\Sigma^{-1}\mu}{2\gamma}\Big)(1-\gamma).
\end{aligned}
\tag{6.31}
$$

Moreover, define the differential operators: $\partial_i \triangleq D_{p_i}, \bar{\delta}\triangleq\sum_i p_i\partial_i$ and $\tilde{\delta}\triangleq D_z+|p|^{-2}\sum_i p_i D_{y_i}$. Denote also by λ and Λ the minimum and maximum eigenvalue of A, respectively. Then, under assumption 6.2.1, the quasilinear PDE operator:

$$
\mathcal{G}^{\mathrm{EZ}}(y,u,\nabla u,D^2u|\phi,\gamma)=\frac{1}{2}\sum_{ij}A_{ij}(y)D_{ij}u+B(y,u,\nabla u) \tag{6.32}
$$

satisfies the following sets of structural conditions for $\vartheta\in(0,1)$:

$$
\Lambda,(\bar{\delta}+1)a^{ij},\tilde{\delta}a^{ij},|p|^{\vartheta}\partial_k a^{ij}=O(\lambda); \tag{6.33a}
$$

$$
B,|p|^{\vartheta}\partial_i B,\bar{\delta}B,\tilde{\delta}B=O(\lambda|p|^2). \tag{6.33b}
$$

Proof. The first set of structural conditions (6.33a) follow straightforwardly from the fact that λ is bounded away from zero, as assumed in 6.2.1, and that a^{ij} and its partial derivatives are bounded uniformly. In particular, we note that $\partial_k a^{ij} \equiv 0$.

The second set of structural conditions (6.33b) follows from the quadratic structure of B with respect to p and the boundedness of the model coefficients and their derivatives.

□

Theorem 6.5.2. *Let $u \in C^2(E)$ be a solution to the equation (5.32) and Ω be a subdomain of E. Under assumption 6.2.1, the following gradient bound for u holds:*

$$|Du(y)| \leq C(1 + \operatorname{dist}(y, \partial\Omega)^{-\frac{1}{\vartheta}}), \quad y \in \Omega, \tag{6.34}$$

where C depends on the quantities of in the structural conditions and $\sup_{y\in\Omega}|u(y)|$ *only. As a corollary,* $|Du(y)|$ *is bounded uniformly.*

Proof. In proposition 6.5.1, we have furnished sufficient regularity to apply Theorem 15.5 of [25], which immediately yields (6.34). For $z \in \mathbb{R}^k$ and $d > 0$, let us define $B_d(z)$ as the open balls centred around z with respect to the Euclidean norm: $\{y \in \mathbb{R}^k, \|y - z\| < d\}$. For any point $y \in \mathbb{R}^k$, we consider n sufficiently large so that $y \in B_n(0)$. We consider u as the solution of (5.32) in the larger domain $B_{n+1}(0)$. By construction, $B_{1/2}(y) \subseteq B_{n+1}(0)$. Combine this and inequality (6.34), we attain $|Du(y)| \leq C(1 + (1/2)^{-1/\vartheta})$, which concludes the result.

□

6.5.2 Proof of Proposition 6.2.2.

Part I. Integrability. Using the same argument as in Part I of the proof of Proposition 6.1.1, to verify the membership of $c^* = l^* X^{\pi^*, l^*}$ in C^∞, it is sufficient to show that $c^* \in L^p(\mathbb{R}^+ \times \Omega)$ for $p = 2(1 - \phi)$ and $2(\phi - \gamma)$. Since l^* is bounded above and away from zero, it is equivalent to confirm the integrability of X_t^*. For a generic exponent p, the form of $(X_t^*)^p$ can be written as:

$$\begin{aligned}(X_t^*)^p &= \exp\Big(\int_0^t \big(p(r-l^*) + p(\pi^*)\mu + \frac{p^2-p}{2}(\pi^*)^2\sigma^2\big)ds\Big)\mathcal{E}\Big(p\int_0^\cdot \pi^*\sigma\Big)_t \\ &= \exp\Big(\int_0^t \Big[p(r-l^*) + \Big[\frac{p}{\gamma} + \frac{p^2-p}{2\gamma^2}\Big]\mu\Sigma^{-1}\mu + \frac{\phi\theta}{\gamma}\Big(p + \frac{p^2-p}{\gamma}\Big)\frac{\nabla g'}{g}\Upsilon'\Sigma^{-1}\mu \\ &\quad + \frac{(p^2-p)\phi^2\theta^2}{2\gamma^2}\frac{\nabla g'}{g}\Upsilon'\Sigma^{-1}\Upsilon\frac{\nabla g}{g}\Big]ds\Big)\mathcal{E}\Big(p\int_0^\cdot \pi^*\sigma\Big)_t.\end{aligned} \tag{6.35}$$

For $p = 2(\phi - \gamma) < 0$:

$$\begin{aligned}(X_t^*)^{2(\phi-\gamma)} &= \exp\Big(\int_0^t \Big[2(\phi-\gamma)(r-l^*) + \frac{(\phi-\gamma)(2\phi-1)}{\gamma^2}\frac{\mu^2}{\sigma^2} \\ &\quad + \frac{2\phi\theta(\phi-\gamma)(2\phi-\gamma-1)}{\gamma^2}\frac{g'}{g}\frac{\mu a\rho}{\sigma} + \frac{(\phi-\gamma)(2\phi-2\gamma-1)\phi^2\theta^2}{\gamma^2}\Big(\frac{g'}{g}\Big)^2 a^2\rho^2\Big]ds\Big). \\ &\quad \times \mathcal{E}\Big(2(\phi-\gamma)\int_0^\cdot \pi^*\sigma\Big)_t.\end{aligned} \tag{6.36}$$

The specification $\phi < 1 < \gamma$ is not sufficient to determine the sign of the $\frac{(\phi-\gamma)(2\phi-1)}{\gamma^2}\frac{\mu^2}{\sigma^2}$ term. In the hypothesis of the theorem, we assumed that $\phi > \frac{1}{2}$, which is consistent with the empirically relevant values of ϕ. In this case:

$$\begin{aligned}(X_t^*)^{2(\phi-\gamma)} &\le \exp\Big(\Big\{2(\phi-\gamma)(\inf_{y\in E} r - \delta^\psi \underline{g}^{-1}) + \frac{(\phi-\gamma)(2\phi-1)}{\gamma^2}\inf_y\Big[\frac{\mu^2}{\sigma^2}\Big] \\ &\quad + \Big|\frac{2\phi\theta(\phi-\gamma)(2\phi-\gamma-1)\rho}{\gamma^2}\Big|\Big\|\frac{g'}{g}\Big\|_\infty \Big\|\frac{\mu a}{\sigma}\Big\|_\infty \\ &\quad + \Big|\frac{(\phi-\gamma)(2\phi-2\gamma-1)\phi^2\theta^2}{\gamma^2}\Big|\Big\|\frac{g'}{g}\Big\|_\infty^2 \|a\|_\infty^2\Big\}t\Big) \times \mathcal{E}\Big(2(\phi-\gamma)\int_0^\cdot \pi^*\sigma\Big)_t.\end{aligned} \tag{6.37}$$

For $p = 2(1 - \phi) > 0$: The form of the optimal wealth process is:

$$\begin{aligned}(X_t^*)^{2(1-\phi)} &= \exp\Big(\int_0^t 2(1-\phi)(r-l^*) + \frac{(1-\phi)(2\gamma-2\phi+1)}{\gamma^2}\frac{\mu^2}{\sigma^2} \\ &\quad + \frac{2\phi(1-\gamma)(\gamma+1-2\phi)}{\gamma^2}\frac{g'}{g}\frac{\mu a\rho}{\sigma} + \frac{\phi^2(1-\gamma)^2(1-2\phi)}{\gamma^2(1-\phi)}\Big(\frac{g'}{g}\Big)^2 a^2\rho^2\Big)\mathcal{E}\Big(2(1-\phi)\int_0^\cdot \pi^*\sigma\Big)_t.\end{aligned} \tag{6.38}$$

Again, the specification $\phi < 1 < \gamma$ is not sufficient to determine the sign of the quadratic term in $\frac{\nabla g}{g}$ in the above equation. It is negative under the extra assumption $\phi > \frac{1}{2}$, though. Thus:

$$\begin{aligned}(X_t^*)^{2(1-\phi)} \leq \exp\Big(\Big\{&2(1-\phi)(\sup_{y\in E} r - \delta^{\psi}\bar{g}^{-1}) + \frac{(1-\phi)(2\gamma-2\phi+1)}{\gamma^2}\sup_y\Big[\frac{\mu^2}{\sigma^2}\Big]\\ &+\Big|\frac{2\phi(1-\gamma)(\gamma+1-2\phi)\rho}{\gamma^2}\Big|\Big\|\frac{g'}{g}\Big\|_\infty\|\frac{\mu a}{\sigma}\|_\infty\Big\}t\Big)\mathcal{E}\Big(2(1-\phi)\int_0^\cdot \pi^*\sigma\Big)_t.\end{aligned} \tag{6.39}$$

For both exponents, $p = 2(\phi-\gamma)$ and $p = 2(1-\phi)$, the growth rate of $(X_t^*)^p$ take the form of $e^{Rt}M_t$, where $R \in \mathbb{R}$ and M is a martingale. Consequently, by taking expectation through the Lebesgue integral, $\mathbb{E}(\int_0^\infty \delta e^{-\delta t}(X^*)_t^p dt) = \int_0^\infty \delta e^{(-\delta+R)t}dt$, which is finite when $R < \delta$. By replacing R with the appropriate growth rates obtained in (6.37) & (6.39), we attain inequalities (6.12a) and (6.12b).

Part II. Uniqueness. We recall that, for c^* to satisfy the uniqueness criterion, we need to verify the limit condition:

$$\begin{aligned}&\lim_{T\to\infty} e^{-\delta\theta T}\mathbb{E}\Big[e^{\delta(1-\theta)\int_0^T -\Phi_s ds}\int_T^\infty \delta e^{-\delta s}(c_s^*)^{1-\gamma}ds\Big] = 0, \quad \text{where}\\ &\Phi_t = \mathbb{E}_t\Big(\int_t^\infty \delta e^{-\delta(s-t)}\Big[\frac{c_s^*}{c_t^*}\Big]^{1-\phi}\Big)^{-1}.\end{aligned} \tag{6.40}$$

Firstly, we begin by estimating $\Phi_t = (\mathbb{E}_t[\int_t^\infty \delta e^{-\delta(s-t)}(\frac{c_s^*}{c_t^*})^{1-\phi}ds])^{-1}$. Under assumption 6.2.1, it is possible to bound the ratio $(\frac{c_s^*}{c_t^*})^{1-\phi}$ from above with exponential growth rate, as follows:

$$\begin{aligned}\Big[\frac{c_s^*}{c_t^*}\Big]^{1-\phi} &= \frac{(g(Y_s)^{-1}X_s^*)^{1-\phi}}{(g(Y_t)^{-1}X_t^*)^{1-\phi}}\\ &\leq \Big[\frac{\bar{g}}{\underline{g}}\Big]^{1-\phi}\exp\Big(\int_t^s\Big[(1-\phi)(\sup_{y\in E} r - l^*) + \frac{(1-\phi)(2\gamma-\phi)}{2\gamma^2}\frac{\mu^2}{\sigma^2} + \frac{\phi(1-\gamma)(\gamma-\phi)}{\gamma^2}\frac{g'}{g}\frac{\mu a\rho}{\sigma}\\ &\quad - \frac{\phi^3(1-\gamma)^2}{2(1-\phi)\gamma^2}\Big(\frac{g'}{g}\Big)^2 a^2\rho^2\Big]du\Big)\mathcal{E}\Big(\int_0^\cdot (1-\phi)\pi^*\sigma dW_u^\rho\Big)_{t,s}\\ &\leq \Big[\frac{\bar{g}}{\underline{g}}\Big]^{1-\phi}\exp\Big(\int_t^s\Big[(1-\phi)(\sup_{y\in E} r - \delta^{\psi}\bar{g}^{-1}) + \frac{(1-\phi)(2\gamma-\phi)}{2\gamma^2}\Big\|\frac{\mu^2}{\sigma^2}\Big\|_\infty\\ &\quad + \Big|\frac{\phi(1-\gamma)(\gamma-\phi)\rho}{\gamma^2}\Big|\Big\|\frac{g'}{g}\Big\|_\infty\Big\|\frac{\mu a}{\sigma}\Big\|_\infty\Big]du\Big)\mathcal{E}\Big(\int_0^\cdot (1-\phi)\pi^*\sigma dW_u^\rho\Big)_{t,s}.\end{aligned} \tag{6.41}$$

Let R be the constant defined in the statement of the theorem (equation (6.13)), then $\mathbb{E}_t\left([c_s^*/c_t^*]^{1-\phi}\right) \leq \left[\frac{\bar{g}}{\underline{g}}\right]^{1-\phi} \exp\left(R(s-t)\right)$. This gives a deterministic bound for Φ:

$$\Phi_t^{-1} = \mathbb{E}_t\Big(\int_t^\infty \delta e^{-\delta(s-t)}\left[\frac{c_s^*}{c_t^*}\right]^{1-\phi} ds\Big) \leq \left[\frac{\bar{g}}{\underline{g}}\right]^{1-\phi} \frac{\delta}{\delta - R}. \tag{6.42}$$

Secondly, we shall now estimate the integral involving $(c^*)^{1-\gamma}$ in the uniqueness criterion. By substituting the relation $\ln(g(Y_t)) - \ln(g(y)) = \int_0^t \left(\frac{\nabla g' b}{g} + \frac{\mathrm{tr}(AD^2 g)}{2g} - \frac{\nabla g' A \nabla g}{2g^2}\right)ds + \int_0^t \frac{\nabla g' a}{g} dW_s$ into the form of X^* in equation (6.35), we have:

$$\begin{aligned}(X_t^*)^{1-\gamma} = x^{1-\gamma}\left[\frac{g(y)}{g(Y_t)}\right]^{\phi\theta} &\exp\Big(\theta\int_0^t (\delta - \delta^\psi g(Y_s)^{-1})ds\Big) \\ &\times \mathcal{E}\Big((1-\gamma)\int_0^t (\pi^*)'\sigma dW_s^\rho + \phi\theta\int_0^t \frac{\nabla g' a}{g} dW_s\Big)_t.\end{aligned} \tag{6.43}$$

This allows for the following estimate for c^*, where K is a generic constant that may change between lines:

$$\begin{aligned}\mathbb{E}\Big(\int_T^\infty e^{-\delta t}(c_t^*)^{1-\gamma} dt\Big) &\leq K\mathbb{E}\Big(\int_T^\infty e^{-\delta t}(X_t^*)^{1-\gamma} dt\Big) \\ &\leq K\int_0^\infty e^{-\delta t + \delta\theta t - \theta\delta^\psi \underline{g}^{-1} t} dt \\ &= Ke^{-(\delta - \delta\theta + \theta\delta^\psi \underline{g}^{-1})T}\int_0^\infty e^{-(\delta - \delta\theta + \theta\delta^\psi \underline{g}^{-1})t} dt\end{aligned} \tag{6.44}$$

The final estimate for the uniqueness criterion is attained by combining estimates (6.42) and (6.44):

$$\begin{aligned}&\mathbb{E}\Big(e^{\delta(1-\theta)\int_0^T (1-\Phi_t)dt}\int_T^\infty \delta e^{-\delta s}(c_s^*)^{1-\gamma} ds\Big) \\ &\qquad \leq \exp\Big(\delta(1-\theta)\Big[\frac{\delta - (\bar{g}/\underline{g})^{\phi-1}(\delta - R)}{\delta}\Big]T\Big)\mathbb{E}\Big(\int_T^\infty \delta e^{-\delta s}(c_s^*)^{1-\gamma} ds\Big) \\ &\qquad \leq K\exp\Big((1-\theta)[\delta - (\bar{g}/\underline{g})^{\phi-1}(\delta - R)]T\Big)\exp\Big(-(\delta - \delta\theta + \theta\delta^\psi \underline{g}^{-1})T\Big),\end{aligned} \tag{6.45}$$

which vanishes at infinity if the relation (6.13) holds.

□

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