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Dynamic Sensitivity Analysis in Lévy Process Driven Option Models

by

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London, March 2008

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Submitted to the London School of Economics and Political Science
for the degree of Doctor of Philosophy

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others.

Adrian Urs Gfeller

Acknowledgements

I would like to express my gratitude to my supervisor Professor Ragnar Norberg for his guidance and support throughout my doctorate. His advice and help have been invaluable.

My thanks also go to the present and former members of the Statistics Department of the London School of Economics, both faculty and fellow students, for the stimulating discussions and for the delightful working atmosphere.

Finally, I am so fortunate to have had the support and love of my friends and family. I am deeply indebted to my parents Ilse and Jürg, to my brother Martin, and to Tatiana.

Abstract

Option prices in the Black-Scholes model can usually be expressed as solutions of partial differential equations (PDE). In general exponential Lévy models an additional integral term has to be added and the prices can be expressed as solutions of partial integro-differential equations (PIDE). The sensitivity of a price function to changes in its arguments is given by its derivatives, in finance known as greeks. The greeks can be obtained as a solution to a PDE or PIDE which is obtained by differentiating the equation and side conditions of the price function. We call the method of simultaneously solving the equations for the price function and the greeks the dynamic partial (-integro) differential approach. So far this approach has been analysed for a few contracts in the Black-Scholes model and in a Markov Chain model.

In this thesis, we extend the use of the dynamic approach in the Black-Scholes model and apply it to a financial market where the underlying stock prices are driven by Lévy processes. We derive and solve systems of equations that determine the price and the greeks both for vanilla and for exotic options. In particular we are interested in options whose prices depend only on time and one state variable. Furthermore, we calculate sensitivities of option prices with respect to changes in the stochastic model of the underlying price process. Such sensitivities can again be expressed as solutions to PIDE. The occurring systems of PIDE are solved numerically via a finite difference approach and the results are compared with simulation and numerical integration methods to compute prices and sensitivities. We show that the dynamic approach in many cases outperforms its competitors. Finally, we investigate the smoothness of the price functions and give conditions for the existence of solutions of the PIDE.

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Chapter 1

Introduction and summary

1.1 Introduction

In order to be able to manage the risk of a financial contract it is crucial to know how sensitive the contract's price function is with respect to changes of the underlying asset price, with respect to changes in parameters of a chosen model, and with respect to changes of the model altogether. For option pricing the benchmark model is the one proposed by Black and Scholes [6], where the stock price is driven by a geometric Brownian motion. This model has gained great success mainly because it gives closed form solutions for a wide range of options. However, it has become clear that option pricing in the Black-Scholes framework is inconsistent with prices seen in the market. Empirical distributions of market returns tend to be skewed and have much heavier tails than returns generated by the Black-Scholes model. The two main lines of extensions of the Black-Scholes model aiming to accommodate such features are stochastic volatility models and models with jumps. Stochastic volatility models with heavy tails, though, are obtained at the price of a unrealistically high variation of the volatility. Models with jumps allow for more realistic representations of price dynamics. This is not too surprising as one can actually observe jumps in market data. A very popular type of option pricing models with jumps are exponential Lévy models, where the underlying asset price S_t is modelled as the exponential of a Lévy process X_t. Depending on the financial contract and the market under consideration, various Lévy processes have been put forward that can adequately represent price dynamics. In exponential Lévy models option prices can be expressed as solutions to partial integro-differential equations for which we will throughout use the abbreviation PIDE.

We are interested in how sensitive prices of financial derivatives are to changes in the model parameters in a given model and to changes of the stochastic nature of the model. Knowledge of the sensitivities, in finance known as the greeks, is crucial to management of the risk of the financial contract. Most textbooks on mathematical fiance such as Bingham and Kiesel [4], Björk [5], Hull [24], and Musiela and Rutkowski [32] have chapters on sensitivity analysis and the greeks. However, they only treat the case where the price function is given in a closed form. This is possible for a wide range of options in the Black-Scholes model. In exponential Lévy models closed form solutions do not exist in general. When closed form expressions are out of reach one has to resort to numerical methods in order to obtain option prices and their sensitivities. Several numerical methods have been put forward to price options. The most prominent are simulation and (integro-) differential equation methods, but also Fourier transform methods and numerical integration are being used. To obtain sensitivities simulation is most widely used.

1.2 Summary

In this thesis, we extend the use of a dynamic partial (integro-) differential equation approach to obtain greeks for a wide range of models. The greeks are computed through their governing equations which in turn are obtained by differentiating the equations of the price function. The price determining equations and the equations determining the values of the greeks are simultaneously solved as a system of equations. This approach is presented in Tavella and Randall [38] for the Black-Scholes model, proposed by Kalashinikov and Norberg [27] for the reserve in life insurance, and used by Norberg [33] for option prices, both in the Black-Scholes and in a Markov chain model. Following this latter approach we first extend its use to a range of options in the Black-Scholes model and then apply this method to exponential Lévy models where we solve a system of PIDE in order to obtain option prices and greeks. The dynamic PIDE approach is further extended to calculate sensitivities of option prices with respect to changes in the stochastic model of the underlying price process.

We start in Chapter 2 with an overview of sensitivity analysis in option pricing. Three different ways of calculating the prices and their sensitivities can are presented: the closed form approach, the dynamic approach, and Monte Carlo simulation. Next, we introduce the dynamic approach in the case of the Black-Scholes model in Chapter 3. For European vanilla and barrier options we follow Norberg [33] and then extend the technique to various types of path-dependent options. In Chapter 4 we review some facts about Lévy processes as described in Sato [35], Applebaum [2], or Kyprianou [28], and about exponential Lévy models as presented in Cont and Tankov [14]. Exponential Lévy models have become very popular in mathematical finance over the past few years as they can capture the observed prices of financial products very well and are still tractable. In particular we look at three different models, the jump-diffusion model introduced by Merton [30], the variance gamma model by Madan, Carr, and Chang [29], and the Carr Geman Madan Yor model [9]. In Chapter 5 we apply the dynamic sensitivity approach to a range of options in the three previously introduced exponential Lévy models. To start with we show that if the price function is sufficiently differentiable, it can be represented as a solution of a partial integro-differential equation. The PIDE is obtained by a standard martingale technique outlined as follows: The discounted option price is a martingale with respect to some risk neutral measure. Hence, we apply Itô's lemma to the discounted option price to obtain its dynamics. As we are dealing with a martingale, the drift term must vanish almost surely. Setting the drift term to zero we obtain the PIDE. We apply this method to European vanilla options, lookback options, Asian options, and exchange options. We are particularly interested in some lookback, Asian, and exchange options where the state space can be reduced and one has to solve a PIDE in time and one space direction only. For European vanilla and barrier options as well as for American vanilla options the PIDE method is presented in Cont and Tankov [14]. In Vecer and Xu [39] a PIDE for Asian options in a general semimartingale framework is derived. The application of this method to exchange options and lookback options combined with a state space reduction is original to this thesis. Upon differentiating the integro-differential equation with respect to some model parameters, we obtain the integro-differential equations for the sensitivities. Thus, to obtain both the price and the sensitivities, the system of corresponding partial integro-differential equations is solved simultaneously. The application of the dynamic sensitivity approach to exponential Lévy models is also original to this thesis. In Chapter 6 we introduce novel greeks as we investigate the sensitivity of an option price with respect to changes in the stochastic model of the underlying stock price. First, we investigate a model with two dependent stock prices and investigate how the option price changes when one process is gradually replaced by the other. Then, we propose a model where the price process is given as an exponential mixing of two Lévy processes and evaluate sensitivities in that model. After having derived a system of PIDE we want to solve it numerically. In Chapter 7 we extend the explicit-implicit finite difference scheme presented in Cont and Voltchkova [15] to numerically solve, not only the price determining PIDE, but simultaneously also the PIDE for the greeks. In the case of the vanilla option in the jump-diffusion model the price of the option can also be expressed as a sum of Black-Scholes prices. In the variance gamma model the density function of the process taken at some fixed time is known and the vanilla option price can alternatively be obtained using numerical integration. Prices of exotic options can as well be obtained using simulation. We perform numerical tests and compare our results with alternative ways to compute the prices and the sensitivities. We shall see that, when one is interested in the price and the sensitivities of a financial contract for a whole range of strikes and maturities, the dynamic approach not only outperforms simulation but may provide the superior algorithm for numerical computation even for contracts where closed form expressions exist. In the Black-Scholes framework, when we are dealing with a PDE, the price function is smooth and all the derivatives of the equation are well defined. In exponential Lévy models we are not guaranteed that the price function is smooth enough to ensure that the PIDE has a classical solution. In Chapter 8 we further investigate the smoothness of the price function. We find that the smoothness of the Lévy measure combined with some integrability conditions guarantees the existence of the derivatives.

Chapter 2

Sensitivity analysis in finance: the greeks

The *greeks* of a price function tell us how much the price function changes if there are changes in the price of the underlying asset, changes in model parameters, or potentially in changes across families of parametric models. At first glance sensitivities with respect to changes in model parameters seem self-contradictory, since a model parameter is by definition a constant, and thus cannot change within a given model. The greeks with respect to changes in model parameters are therefore sensitivities with respect to misspecifications of the model parameters. Greeks with respect to changes across families of parametric models are sensitivities with respect to misspecifications of the parametric model.

Whereas prices are observable in the market, the greeks are not, and hence accurate calculation of sensitivities is arguably even more important than calculation of prices. In this chapter we present three ways of calculating prices and sensitivities. These are the closed form approach, as presented in most textbooks on mathematical finance, the dynamic approach, and the Monte Carlo approach.

2.1 The Black-Scholes market and the greeks

We consider the *Black-Scholes market* [6] with two basic assets, a risk-free one (typically a bond or a money market account) and a risky one (typically a stock). Under the risk neutral measure the prices of the two assets are given by

$$\begin{aligned} B_t &= e^{r\,t}, \\ S_t &= S_0\, e^{\left(r-\frac{\sigma^2}{2}\right)\,t+\sigma\, W_t}, \end{aligned}$$

where r is the risk-free interest rate, σ the volatility of the stock, and W_t is a Brownian motion. Their dynamics under the risk neutral measure are given by the stochastic differential equations

$$\begin{aligned} dB_t &= B_t\, r\, dt, \\ dS_t &= S_t\,(r\, dt + \sigma\, dW_t). \end{aligned}$$

The unique price $c(t, S_t)$ at time t of a *European style option* with payoff h and maturity T is given by the risk-neutral valuation formula

$$c(t, S_t) = \mathbb{E}[e^{-r\,(T-t)}\, h(S_T)|\, \mathcal{F}_t], \tag{2.1}$$

where $\mathbb{E}$ is the expectation under the risk neutral measure and $(\mathcal{F}_\tau)_{0\le\tau\le t}$ is the filtration generated by the Brownian motion. Using Itô's formula it can be shown that the *price function* $c(t, s) = \mathbb{E}[e^{-r\,(T-t)}h(S_T)|S_t = s]$ satisfies the *Black-Scholes equation*

$$c_t(t, s) = r\, c(t, s) - rs\, c_s(t, s) - \frac{1}{2}\sigma^2 s^2\, c_{ss}(t, s), \tag{2.2}$$

for all $(t, s) \in [0, T) \times (0, \infty)$, subject to the terminal condition

$$c(T, s) = h(s), \quad s > 0,$$

where we have used subscripts to denote derivatives. For a *European call option* the terminal condition is

$$c(T,s) = \max(s-K,0), \quad s > 0.$$

For computational purposes it may be useful to add auxiliary boundary conditions, which for a European call option are

$$\begin{aligned} &c(t,0) = 0, \\ &c(t,s) \sim s - Ke^{-r(T-t)}, \quad s \to \infty. \end{aligned}$$

While deriving differential equations we will often encounter the quadratic covariation $[X,Y]_t$ of two processes X_t and Y_t. For all processes we consider, the quadratic variation exists and is defined as

$$[X,Y]_t = \lim_{|\Pi|\to 0} \sum_{t_i \in \Pi} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \quad \text{in probability,} \tag{2.3}$$

where we sum products of increments along the partition Π of the time interval $[0,t]$, letting the grid size go to zero. In case $X_t = Y_t$ we call it the quadratic variation process.

The greeks take their name from the fact that they are denoted by Greek letters. The most common ones are Δ measuring the sensitivity of the price function c with respect to the underlying asset price, Γ measuring the sensitivity of Δ with respect to changes in the underlying asset price, ρ the sensitivity of c with respect to the interest rate, $\mathcal{V}$ the sensitivity of c with respect to the volatility, and Θ the sensitivity of c with respect to time:

$$\Delta = \frac{\partial c}{\partial s},$$
$$\Gamma = \frac{\partial^2 c}{\partial s^2},$$
$$\rho = \frac{\partial c}{\partial r},$$
$$\mathcal{V} = \frac{\partial c}{\partial \sigma},$$
$$\Theta = \frac{\partial c}{\partial t},$$

where $c = c(t, s, \sigma, r)$ is a function of the initial stock price, time, and the model parameters. We now discuss several ways of calculating the greeks.

2.2 Classical approach based on closed from expressions

The greeks of options that have a closed form price function can be obtained by simply differentiating the price function with respect to the underlying stock price value, time, and the model parameters. Solving the Black-Scholes equation (2.2) for an European call with strike K and maturity T yields

$$c(s,t) = s\,N(d_1) - K\,e^{-r\,(T-t)}\,N(d_2),$$

where

$$d_{1,2}(s) = \frac{\ln\left(\frac{s}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

The greeks for this option are

$$\begin{aligned}
\Delta &= N(d_1), \\
\Gamma &= \frac{N'(d_1)}{s\,\sigma\,\sqrt{T-t}}, \\
\rho &= K\,(T-t)\,e^{-r\,(T-t)}\,N(d_2), \\
\mathcal{V} &= s\,\sqrt{T-t}\,N'(d_1), \\
\Theta &= -\frac{s\,\sigma\,N'(d_1)}{2\sqrt{T-t}} - K\,r\,e^{-r\,(T-t)}\,N(d_2),
\end{aligned}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution and $N'(\cdot)$ its density. This approach works for many options in the Black-Scholes framework such as for barrier options, lookback options, and exchange options, where the price function can be expressed in closed form. For Asian options this approach is not straightforward. The price of an Asian option can be expressed as a triple integral which is difficult to evaluate numerically, see Yor [42].

2.3 Dynamic approach

Upon differentiating the PDE for the price function with respect to some parameter in the model one obtains a PDE for the derivative. Solving the system of the two PDE one obtains solutions for both the price and the sensitivity. This dynamic approach has been in the air for a while. It is outlined in the Black-Scholes model by Wilmott [40] and by Tavella and Randall [38]. It is proposed and investigated by Kalashnikov and Norberg [27] in the context of life insurance mathematics. In Norberg [33] it is analysed in the Black-Scholes model for vanilla and barrier options and in a Markov Chain market for general European options where also the existence of derivatives is investigated. However, so far the powers of the method have not been widely recognised, and it is not widely used. We will investigate the potential of this approach in the next chapters.

2.4 Monte Carlo approach

Simulation has proved to be a valuable tool for estimating security prices for which simple closed form solutions do not exist. The estimation of sensitivities presents both theoretical and practical challenges to Monte Carlo simulation. Following Broadie and Glasserman [8] and Glasserman [21] we present the most important methods for obtaining derivatives of security prices using simulation. In many cases the parameter with respect to which we want to calculate the price sensitivity can be seen as either a parameter of the payoff or a parameter of the probability measure. We illustrate this with a European vanilla call option. If all the parameters are assumed to be in the payoff function, the price function at time t is written as

$$c(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max\left(s\, e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)-\sigma\sqrt{T-t}\,x} - K, 0 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

If all the parameters are put in the probability measure, the price is written as

$$c(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(x-K,0) \frac{1}{x\,\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2(x)}{2}}\, dx,$$

where

$$d(x) = \frac{\ln\left(\frac{x}{s}\right) - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

In the *finite difference method* the parameter of interest is assumed to be in the payoff function and the probability measure is fixed. This method goes as follows: Firstly, an initial simulation is run to determine the price of an asset. Secondly, the parameter of interest is perturbed and another simulation is run to determine the perturbed price. The estimate of the derivative is the difference in the simulated prices divided by the parameter perturbation. This method is easy to understand and implement, but since it involves simulating at two values of the parameter of interest it is computationally not very efficient. Moreover it produces biased estimates [21]. Consider a

European style option whose price depends on a parameter θ and is given by

$$c(\theta) = \mathbb{E}[h(X,\theta)] = \int_{-\infty}^{\infty} h(x;\theta)g(x)dx,$$

where $h(\theta)$ is the discounted payoff. To estimate $\frac{dc(\theta)}{d\theta}$ the derivative of $c(\theta)$ with respect to θ we simulate n independent replicates $h_1(\theta), \ldots, h_n(\theta)$ at the parameter value θ and n additional replicates $h_1(\theta + d), \ldots, h_n(\theta + d)$ at the parameter value $\theta + d$ for some $d > 0$. Then, we average each set of replicates to obtain $\bar{h}(\theta) = \frac{1}{n}\sum_{i=1}^{n} h_i(\theta)$ and $\bar{h}(\theta + d) = \frac{1}{n}\sum_{i=1}^{n} h_i(\theta + d)$. The form of the forward difference estimator of the sensitivity is then

$$\hat{\Delta}_F = \frac{\bar{h}(\theta + d) - \bar{h}(\theta)}{d}.$$

The bias of the forward difference estimator is

$$\text{Bias}(\hat{\Delta}_F) = \mathbb{E}\left[\hat{\Delta}_F - \frac{dc(X,\theta)}{d\theta}\right] = \mathcal{O}(d),$$

where we used Landau's notation $f = \mathcal{O}(g(d))$, meaning

$$\limsup_{d \to 0} \frac{f(d)}{g(d)} < \infty.$$

By simulating at $\theta + d$ and $\theta - d$, we can form a central difference estimator

$$\hat{\Delta}_C = \frac{\bar{h}(\theta + d) - \bar{h}(\theta - d)}{2d}.$$

It has a bias of

$$\text{Bias}(\hat{\Delta}_C) = \mathbb{E}\left[\hat{\Delta}_C - \frac{dc(X,\theta)}{d\theta}\right] = \mathcal{O}(d^2).$$

The *pathwise method* is also designed for situations where the parameter of interest is in the payoff function and the probability measure is fixed. It involves simulation at only one parameter value and produces unbiased estimates. The idea of this method is that if the differentiation and the expectation operator can be interchanged, one can write the sensitivity as

$$\frac{d}{d\theta}E[h(X,\theta)] = E\left[\frac{d}{d\theta}h(X,\theta)\right] = \int_{-\infty}^{\infty} \frac{d}{d\theta}h(x;\theta)g(x)dx,$$

where h is the discounted payoff function and θ is a parameter in the payoff. From the dominated convergence theorem we know that the interchange of differentiation and integration is allowed if the derivative $\frac{d}{d\theta}h(\theta)$ exists almost everywhere and there is an integrable function $k(x)$ such that $\left|\frac{1}{d}(h(x;\theta+d) - h(x;\theta))g(x)\right| < k(x)$ for all x and d small enough. This is typically only true if the payoff function is uniformly continuous in the parameter of differentiation θ. The name of the method stems from the fact that the expression $\frac{d}{d\theta}h(\theta)$ is called the pathwise derivative of h at θ.

The *likelihood method*, like the pathwise method, involves simulation at only one parameter value and produces unbiased estimates. It puts the dependence of the parameter of interest in the underlying probability measure rather than in the payoff function and hence does not require smoothness in the discounted payoff. We consider a discounted payoff h as a function of a random variable X and suppose that X has probability density $g(x,\theta)$ where θ is a parameter of that density taking values in $\mathbb{R}$. To derive a derivative estimator, we suppose that the order of differentiation and integration can be interchanged and we obtain for the sensitivity

$$\begin{aligned}\frac{d}{d\theta}\mathbb{E}_\theta[h(X)] &= \int h(x)\,\frac{d}{d\theta}g(x;\theta)\,dx \\ &= \mathbb{E}_\theta\left[h(X)\,\frac{\dot{g}(X;\theta)}{g(X;\theta)}\right],\end{aligned}$$

where we have written $\dot{g}(x;\theta)$ for $\frac{dg(x;\theta)}{d\theta}$. Just as in the pathwise method this method is valid if the differentiation and integration can be interchanged. This is true if the derivative $\frac{d}{d\theta}g(x;\theta)$ almost everywhere exists and the function $\left|h(x)\frac{1}{d}(g(x;\theta+d) - g(x;\theta))\right|$ can be bounded by an integrable function $k(x)$ for small enough x and d. As probability densities are typically smooth functions this is usually satisfied.

The fact that payoff functions are typically not smooth and sometimes not even continuous limits the scope of the pathwise method. In contrast, smoothness conditions are usually satisfied by the probability densities arising in applications of the likelihood ratio method. The main drawback of the likelihood ratio method is that it requires the explicit knowledge of the probability densities and that its estimates tend to have a large variance [21].

With the *Malliavin method* several authors [19], [3], [31] have extended the likelihood method. By means of Malliavin calculus, one can calculate unbiased estimators for sensitivities without having to differentiate the payoff function even when the density function $g(x;\theta)$ is not known in closed form. One has to calculate the derivative as an expectation of the payoff times a weight function. The Malliavin method is for example suitable to simulate sensitivities of options whose payoff depends on the time average of a geometric Brownian motion such as Asian options.

Chapter 3

Dynamic sensitivity analysis in the Black-Scholes market

3.1 European vanilla options

Throughout we will denote the interest rate by r, the strike price by K, and the volatility of the Brownian motion by σ. Recall the Black-Scholes equation (2.2) for a European call option

$$c_t(t,s) = r\,c(t,s) - rs\,c_s(t,s) - \frac{1}{2}\sigma^2 s^2\,c_{ss}(t,s), \tag{3.1}$$

for $(t,s) \in [0,T) \times (0,\infty)$, subject to the terminal condition

$$c(T,s) = \max(s-K,0), \quad s>0, \tag{3.2}$$

and the boundary conditions

$$c(t,0) = 0, \tag{3.3}$$

$$c(t,s) \sim s - K\,e^{-r\,(T-t)}, \quad s \to \infty, \tag{3.4}$$

which are added for computational convenience. Differentiating (3.1), (3.2), (3.3), and (3.4) with respect to the interest rate r, we obtain the differential equation for the sensitivity $\rho = \frac{\partial c}{\partial r}$:

$$\rho_t(t,s) = r\,\rho(t,s) + c(t,s) - rs\,\rho_s(t,s) - s\,c_s(t,s) - \frac{1}{2}\sigma^2 s^2\,\rho_{ss}(t,s), \quad (3.5)$$

for $(t,s) \in [0,T) \times (0,\infty)$, with the terminal condition

$$\rho(T,s) = 0, \quad s > 0, \quad (3.6)$$

and the auxiliary side conditions

$$\begin{aligned} &\rho(t,0) = 0, \\ &\rho(t,s) = (T-t)\,e^{-r\,(T-t)}\,K, \quad s \to \infty. \end{aligned}$$

Similarly, differentiating with respect to the volatility σ we obtain the PDE for $\mathcal{V} = \frac{\partial c}{\partial \sigma}$:

$$\mathcal{V}_t(t,s) = r\,\mathcal{V}(t,s) - rs\,\mathcal{V}_s(t,s) - \frac{1}{2}\sigma^2 s^2\,\mathcal{V}_{ss} - \sigma s^2\,c_{ss}(t,s), \quad (3.7)$$

for $(t,s) \in [0,T) \times (0,\infty)$, the terminal condition

$$\mathcal{V}(T,s) = 0, \quad s > 0, \quad (3.8)$$

and the auxiliary boundary conditions

$$\begin{aligned} &\mathcal{V}(t,0) = 0, \\ &\mathcal{V}(t,s) = 0, \quad s \to \infty. \end{aligned}$$

To determine the price function and its greeks, the system of PDE (3.1), (3.5), and (3.7) has to be solved subject to its terminal conditions (3.2), (3.6), and (3.8). One starts with the boundary condition at time $t = T$, specifying the known terminal values for the price function and the greeks in the state interval one has chosen to work in, and then works backwards in time calculating at each time step the option prices and the sensitivities. Doing so, one obtains the price and the sensitivities for a whole range of strikes and times to maturity. This method is called the dynamic approach.

3.2 Lookback options

Lookback options provide investors with the possibility to look back in time and exercise an option at the ideal time. Obviously this opportunity has its price and invariably lookback options are more expensive than their European counterparts. We write S_t for the stock price process, $M_t = \sup_{0 \leq \tau \leq t} S_\tau$ for its running maximum, and $m_t = \inf_{0 \leq \tau \leq t} S_\tau$ for its running minimum over the interval $[0, t]$. There are four basic types of lookback options. The two *floating strike lookback options* are, firstly, the call option with payoff

$$h(S_T, m_T) = S_T - m_T,$$

giving the right to buy at the low over $[0, T]$, and secondly, the put with payoff

$$h(S_T, M_T) = M_T - S_T,$$

giving the right to sell at the high over $[0, T]$. Floating strike lookback options are not options in a strict sense as they will always be exercised and hence the pricing reduces to finding the expectations of the running maximum $\mathbb{E}[M_T]$ and the running minimum $\mathbb{E}[m_T]$ under the risk neutral measure. The two *fixed strike lookback options* are the call, with payoff

$$h(M_T) = \max(M_T - K, 0),$$

and the put, with payoff

$$h(m_T) = \max(K - m_T, 0).$$

The fixed strike prices are special cases of the functional $\mathbb{E}[h(m_T)]$ and $\mathbb{E}[h(M_T)]$. PDE approaches to lookback options can be found for example in Wilmott [18] or Zhu [43]. In the Black-Scholes framework closed form solutions exist for the four standard lookback options presented above. The closed form solutions for the floating strike lookback options have been derived in Goldman et al. [22]. The closed form solutions for the fixed strike lookback options have been derived in Conze and Viswanathan [17].

3.2.1 Floating strike lookback put

We take the *floating strike lookback put* as an example and undertake to derive PDE governing its price and the greeks. We show two different ways how the state space can be reduced and a PDE in time and only one more variable can be obtained. The price of a floating strike lookback put option at time t is

$$c(t, S_t, M_t) = e^{-r\,(T-t)}\,\mathbb{E}[M_T - S_T|\mathcal{F}_t],$$

where $\mathcal{F}_t$ is the σ-algebra generated by the Markov process (S_t, M_t). To derive a PDE for the price of the lookback option we apply Itô's lemma to the discounted option price $e^{-r\,t}\,c(t, S_t, M_t)$. Following Shreve [37] we obtain

$$\begin{aligned}
d(e^{-r\,t}\,c(t, S_t, M_t)) &= e^{-r\,t}\Big(- r\,c(t, S_t, M_t)\,dt + c_t(t, S_t, M_t)\,dt \\
&\quad + c_s(t, S_t, M_t)\,dS_t + \frac{1}{2}c_{ss}(t, S_t, M_t)d[S, S]_t \\
&\quad + c_m(t, S_t, M_t)dM_t\Big) \\
&= e^{-r\,t}\Big(\Big(- r\,c(t, S_t, M_t) + c_t(t, S_t, M_t) \\
&\quad + r\,S_t\,c_s(t, S_t, M_t) + \frac{1}{2}\sigma^2\,S_t^2\,c_{ss}(t, S_t, M_t)\Big)\,dt \\
&\quad + \sigma S_t\,c_s(t, S_t, M_t)\,dW_t + c_m(t, S_t, M_t)\,dM_t\Big),
\end{aligned}$$

where we have written $[S, S]_t$ for the quadratic variation of the process S_t. The discounted option price is a martingale. As the stock price process attains its running maxima on a set with Lebesgue measure zero, the term involving dM_t cannot be cancelled by the drift term, and both the drift term and $c_m(t, S_t, M_t)\,dM_t$ must be zero. Setting the drift term to zero one obtains the PDE

$$c_t(t,s,m) + \frac{1}{2}\sigma^2 s^2\, c_{ss}(t,s,m) + rs\, c_s(t,s,m) - r\, c(t,s,m) = 0, \qquad (3.9)$$

for $t \in [0,T)$ and $0 < s \leq m < \infty$, subject to the terminal condition

$$c(T,s,m) = m - s. \qquad (3.10)$$

For the first auxiliary boundary condition we investigate the option price when the stock price is close to zero and obtain

$$\lim_{s\to 0} c(t,s,m) = m\, e^{-r\,(T-t)}. \qquad (3.11)$$

To obtain the second auxiliary boundary condition we use the fact that $c_m(t,S_t,M_t)dM_t$ must be zero. The term dM_t is zero when $S_t < M_t$. However, at the times when M_t increases, this is when the stock price is equal to its running maximum, $c_m(t,S_t,M_t)$ must be zero for $c_m(t,S_t,M_t)\, dM_t$ to vanish. This gives us

$$c_m(t,s,m)|_{s=m} = 0. \qquad (3.12)$$

Due to the special form of the payoff function (3.10), equation (3.9) and its side conditions can be transformed and written in new coordinates using $c(t,s,m) = m\, w(t,\xi)$ where $\xi = \frac{s}{m}$. Doing so, both the PDE and the side conditions only depend on time and the state variable $\xi = \frac{s}{m}$. The derivatives of the price function in the transformed coordinates are

$$\begin{aligned} c_t(t,s,m) &= m\, w_t(t,\xi), \\ c_s(t,s,m) &= w_\xi(t,\xi), \\ c_{ss}(t,s,m) &= \frac{1}{m} w_{\xi\xi}(t,\xi). \end{aligned}$$

In the transformed coordinates the PDE is

$$w_t(t,\xi) + \frac{1}{2}\sigma^2\xi^2\, w_{\xi\xi}(t,\xi) + \xi r\, w_\xi(t,\xi) - r\, w(t,\xi) = 0, \qquad (3.13)$$

for $(t,\xi) \in [0,T)\times(0,1)$. The boundary conditions for $w(t,\xi)$ can be obtained from the boundary conditions (3.10), (3.11), and (3.12) of $c(t,s,m)$. In particular,

$$m - s = c(T,s,m) = m\,w(T,\xi)$$

implies

$$w(T,\xi) = 1-\xi.$$

Furthermore,

$$m\,e^{-r\,(T-t)} = c(t,0,m) = m\,w(t,0)$$

implies

$$w(t,0) = e^{-r\,(T-t)},$$

and finally

$$0 = c_m(t,s,m)|_{s=m} = w(t,1) - w_\xi(t,\xi)|_{\xi=1}$$

implies

$$w_\xi(t,\xi)|_{\xi=1} = w(t,1).$$

We now calculate the greeks using the dynamic approach. As the PDE for the lookback option and the vanilla option differ only in the side condition, the same goes for the PDE for the greeks. Differentiating equation (3.13) with respect to the interest rate we obtain a PDE for $\varrho = \frac{\partial w}{\partial r}$:

$$\varrho_t(t,\xi)+\frac{1}{2}\sigma^2\xi^2\,\varrho_{\xi\xi}(t,\xi)+\xi r\,\varrho_\xi(t,\xi)+\xi\,w_\xi(t,\xi)-r\,\varrho(t,\xi)-w(t,\xi)=0, \quad (3.14)$$

for $(t,\xi) \in [0,T)\times(0,1)$ with terminal condition

$$\varrho(T,\xi) = 0,$$

and auxiliary side conditions

$$\begin{aligned} \varrho(t,0) &= -(T-t)\, e^{-r\,(T-t)}, \\ \varrho_\xi(t,\xi)|_{\xi=1} &= \varrho(t,1). \end{aligned}$$

Differentiating (3.13) with respect to the volatility, we obtain a differential equation for $v = \frac{\partial w}{\partial \sigma}$:

$$v_t(t,\xi) + \frac{1}{2}\sigma^2\xi^2\, v_{\xi\xi}(t,\xi) + \sigma\xi^2\, w_{\xi\xi}(t,\xi) + \xi r\, v_\xi(t,\xi) - r\, v(t,\xi) = 0, \quad (3.15)$$

for $(t,\xi) \in [0,T) \times (0,1)$ with terminal condition

$$v(T,\xi) = 0,$$

and auxiliary side conditions

$$\begin{aligned} v(t,0) &= 0, \\ v_\xi(t,\xi)|_{\xi=1} &= v(t,1). \end{aligned}$$

Solving the system of equations (3.13), (3.14), and (3.15) with the appropriate side conditions and transforming the variables w, ϱ, and v back to c, ρ, and $\mathcal{V}$, using

$$\begin{aligned} c(t,s,m) &= m\, w(t,\xi), \\ \rho(t,s,m) &= m\, \varrho(t,\xi), \\ \mathcal{V}(t,s,m) &= m\, v(t,\xi), \end{aligned}$$

we obtain the price and the greeks for the lookback put.

The values obtained can then be compared with the values one obtains from the closed form expression as derived in [32]:

$$c(s,t) = -\, s\, N(-d_1) + m\, e^{-r\,(T-t)}\, N(-d_2) + s\, \frac{\sigma^2}{2r}\, N(d_1) \\ -\, s\, e^{-r\,(T-t)}\, \frac{\sigma^2}{2r} \left(\frac{s}{m}\right)^{-\frac{2r}{\sigma^2}} N\left(d_1 - \frac{2r\sqrt{T-t}}{\sigma}\right), \tag{3.16}$$

where

$$d_{1,2} = \frac{\ln\left(\frac{s}{m}\right) + (r \pm \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}.$$

The greeks in the closed form approach are then obtained by differentiating (3.16).

3.2.2 A martingale method

We now give an alternative derivation of the option price defining PDE based on a martingale technique. Consider the martingale

$$\tilde{M}_t = \mathbb{E}[M_T - S_T | \mathcal{F}_t],$$

where S_t is the stock price process and $M_t = \sup_{0 \le \tau \le t} S_\tau$ is its running maximum. The martingale can be written as

$$\begin{aligned} \tilde{M}_t &= \mathbb{E}\left[\max(M_t, S_t \sup_{t \le \tau \le T} e^{\left(r - \frac{\sigma^2}{2}\right)(\tau - t) + \sigma\,(W_\tau - W_t)}) - S_t\, e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\,(W_T - W_t)} \right] \\ &= S_t\, f(t, Q_t), \end{aligned}$$

with

$$Q_t = \frac{M_t}{S_t}$$

and

$$f(t,q) = \mathbb{E}\left[\max\left(q, \sup_{0\le\tau\le T-t} e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma W_\tau} \right) - e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \right]. \tag{3.17}$$

Using Itô's lemma we obtain the dynamics of Q_t:

$$dQ_t = \frac{1}{S_t}\,dM_t - \frac{M_t}{S_t^2}\,dS_t + \frac{M_t}{S_t^3}\,d[S,S]_t,$$

which can be written as

$$dQ_t = Q_t(-r\,dt - \sigma\,dW_t + \sigma^2\,dt), \quad Q_t > 1.$$

The dynamics of the martingale $\tilde{M}_t$ are

$$\begin{aligned} d\tilde{M}_t &= dS_t\,f(t,Q_t) + S_t\left(f_t(t,Q_t)\,dt + f_q(t,Q_t)dQ_t \right. \\ &\quad \left. + \frac{1}{2} f_{qq}(t,Q_t)d[Q,Q]_t \right) + dS_t\,f_q(t,Q_t)dQ_t, \\ &= S_t\left(f_t(t,Q_t) + r\,f(t,Q_t) + \frac{1}{2}\sigma^2 Q_t^2\,f_{qq}(t,Q_t) - r\,Q_t\,f_q(t,Q_t) \right)\,dt \\ &\quad + S_t\,\sigma f(t,Q_t)\,dW_t, \end{aligned} \tag{3.18}$$

where we used $d[Q,Q]_t = Q_t^2\,\sigma^2\,dt$. Setting the drift term in equation (3.18) to zero one obtains the PDE

$$f_t(t,q) + r\,f(t,q) + \frac{1}{2}\sigma^2 q^2\,f_{qq}(t,q) - rq\,f_q(t,q) = 0,$$

valid for $(t,q) \in [0,T) \times (1,\infty)$. The terminal condition in this parametrisation is

$$f(T,q) = q - 1.$$

If the stock price goes to zero then we obtain the first auxiliary boundary condition

$$f(t,q) = q - e^{r\,(T-t)}, \quad q \to \infty.$$

We obtain the second auxiliary boundary condition by inspecting equation (3.17). If the stock price is at its maximum then $q = 1$. Because of the diffuse nature of Brownian motion the second term in the maximum function in (3.17) is always bigger than 1. Therefore, $f(t,q)$ does not explicitly depend on q at $q = 1$. This leads to the second auxiliary boundary condition,

$$f_q(t,q)|_{q=1} = 0.$$

The price function of the option is then

$$p(s,t) = e^{-r\,(T-t)}\, s\, f(t,q).$$

3.3 Asian options

Asian option is a generic name for the class of options whose terminal payoff depends on the average value of the underlying asset during some period of the option's lifetime. In contrast to standard options, Asian options are more robust against manipulation near their expiry dates. Asian options are widely used in practice, for instance, for commodities and in foreign exchange markets. The two major types of Asian options are *floating strike Asian options* and *fixed strike Asian options*. The payoff of a floating strike Asian call option is $\max(S_T - \frac{1}{T}A_T, 0)$ and the payoff of a floating strike Asian put option is $\max(\frac{1}{T}A_T - S_T, 0)$. The expression $\frac{1}{t}A_t$ is the average of the stock price over the time interval from 0 to t. There are various ways of forming an average of past values of the stock price. We will restrict ourselves to the continuously sampled arithmetic average which for which $A_t = \int_0^t S_\tau d\tau$. The payoff of a fixed strike Asian call option is $\max(\frac{1}{T}A_T - K, 0)$ and the one of a fixed strike Asian put option is $\max(K - \frac{1}{T}A_T, 0)$, where K is a fixed number called the strike price. There also exist more general Asian options with payoffs such as $\max(\frac{1}{T}A_T - K_1 S_T - K_2, 0)$, where K_1 and K_2 are two fixed numbers.

3.3.1 Floating strike Asian options

We consider a *floating strike Asian call option*. The price of such an option is given as the expected discounted payoff under the risk neutral measure

$$c(t, S_t, A_t) = e^{-r\,(T-t)}\, \mathbb{E}\left[\max\left(S_T - \frac{1}{T}A_T, 0\right) \Big| \mathcal{F}_t\right].$$

Using Itô's formula one can show that in the Black-Scholes model the price function

$$c(t, s, a) = e^{-r\,(T-t)}\, \mathbb{E}\left[\max\left(S_T - \frac{1}{T}A_T, 0\right) \Big| S_t = s, A_t = a\right]$$

is the solution to the PDE

$$c_t(t, s, a) + rs\, c_s(t, s, a) + \frac{1}{2}\sigma^2 s^2\, c_{ss}(t, s, a) + s\, c_a(t, s, a) - r\, c(t, s, a) = 0, \quad (3.19)$$

valid for $(t, s, a) \in [0, T) \times (0, \infty) \times [0, \infty)$ subject to the terminal condition

$$c(T, s, a) = \max\left(s - \frac{1}{T}a, 0\right).$$

This is just the Black-Scholes equation for a vanilla option with the additional term $c_a(t, s, a)$, taking into account that the option also depends on an average price of the underlying asset over a certain time period. PDE methods for a wide range of exotic options can be found in Zhu [43]. Following [25], [36], and [18] we show, that the state space of equation (3.19) can be reduced and a PDE in only two variables can be obtained. Alternatively one can use a martingale method to directly derive a price determining PDE that only depends on time and one further variable. We will present the martingale technique in the next section as it works for both fixed strike and floating strike Asian options.

To reduce the state space we use the ansatz $c(t,s,a) = s\,h(t,q)$ with $q = \frac{1}{T}\frac{a}{s}$. The derivatives of the price function in the new coordinates are

$$\begin{aligned} c_t(t,s,a) &= s\,h_t(t,q), \\ c_s(t,s,a) &= h(t,q) - q\,h_q(t,q), \\ c_{ss}(t,s,a) &= \frac{q^2}{s}\,h_{qq}(t,q), \\ c_a(t,s,a) &= \frac{1}{T}\,h_q(t,q). \end{aligned} \tag{3.20}$$

Inserting (3.20) into (3.19), we obtain

$$h_t(t,q) + \frac{1}{2}\sigma^2 q^2\,h_{qq}(t,q) + \left(\frac{1}{T} - rq\right)\,h_q(t,q) = 0, \tag{3.21}$$

valid for $(t,q) \in [0,\infty) \times [0,\infty)$ with the terminal condition

$$h(T,q) = \max(1-q, 0).$$

For computational convenience we introduce the auxiliary boundary conditions as outlined in [18]. When $Q_t = \frac{1}{T}\frac{A_t}{S_t}$ is very large the probability that the stock price at expiry is greater than the average over T goes to zero:

$$\lim_{q\to\infty} \mathbb{P}\left[S_T > \frac{1}{T}A_T \middle| Q_t = q\right] = 0,$$

and the option expires worthless. This gives us the first auxiliary boundary condition

$$h(t,q) = 0, \quad q \to \infty.$$

The second auxiliary boundary condition, when q goes to zero, can be stated as a differential equation. This equation can be derived from equation (3.21) as follows: First, we note that the term $q\,h_q(t,q)$ is negligible for small q as it is much smaller that the term $\frac{1}{T}h_q(t,q)$. Second, we show that $q^2\,h_{qq}(t,q)$ vanishes for $q \to 0$ by assuming the opposite and demonstrating that this leads to a contradiction. By Assumption, the term $q^2\, h_{qq}(t,q)$ is bounded:

$$h_{qq}(t,q) = \mathcal{O}\left(\frac{1}{q^2}\right).$$

Integrating twice we see that this means $h(t,q) = \mathcal{O}(\log q)$. For small q this contradicts the fact that $h(t,q)$ is bounded. We therefore conclude that

$$\lim_{q\to 0} q^2\, h_{qq}(t,q) = 0,$$

and equation (3.21) reduces for $q \to 0$ to the second boundary condition

$$h_t(t,0) + \frac{1}{T}\, h_q(t,q)|_{q=0} = 0.$$

Using the dynamic approach, the greeks can easily be calculated. Differentiating (3.21) with respect to the interest rate r, using $\rho = \frac{\partial c}{\partial r} = s\frac{\partial h}{\partial r}$ and $k = \frac{\partial h}{\partial r}$ one obtains the PDE for k,

$$k_t(t,q) + \frac{1}{2}\sigma^2 q^2\, k_{qq}(t,q) + \left(\frac{1}{T} - rq\right)\, k_q(t,q) - q\, h_q(t,q) = 0, \qquad (3.22)$$

valid for $(t,q) \in [0,\infty) \times [0,\infty)$ with the terminal condition

$$k(T,q) = 0,$$

and auxiliary side conditions

$$k(t,q) = 0, \quad q \to \infty,$$
$$k_t(t,0) + \frac{1}{T}\, k_q(t,q)|_{q=0} = 0.$$

Differentiating (3.21) with respect to the volatility σ, using $\mathcal{V} = \frac{\partial c}{\partial \sigma} = s\frac{\partial h}{\partial \sigma}$ and $l = \frac{\partial h}{\partial \sigma}$ one obtains the PDE for l

$$l_t(t,q) + \frac{1}{2}\sigma^2 q^2\, l_{qq}(t,q) + \left(\frac{1}{T} - rq\right)\, l_q(t,q) + \sigma q^2\, h_{qq}(t,q) = 0, \qquad (3.23)$$

valid for $(t, q) \in [0, \infty) \times [0, \infty)$ with terminal condition

$$l(T, q) = 0,$$

and auxiliary side conditions

$$l(t, q) = 0, \quad q \to \infty,$$
$$l_t(t, 0) + \frac{1}{T} l_q(t, q)|_{q=0} = 0.$$

Then the system of equations (3.21), (3.22), and (3.23) has to be solved with respect to the relevant terminal and side conditions. Finally, the variables h, k, and l have to to transformed back to c, ρ and σ.

3.3.2 Fixed strike Asian options

Rogers and Shi [34] showed that not only for floating strike but also for fixed strike Asian options the state space can be reduced. Hence, the problem reduces again to solving a parabolic PDE in two variables. We derive the PDE for the *fixed strike Asian call option* with payoff $\max(\frac{1}{T}A_T - K, 0)$ using a martingale technique. Consider the martingale

$$\begin{aligned} M_t &= \mathbb{E}\left[\max\left(\frac{1}{T}\int_0^T S_\tau \, d\tau - K, 0\right) \middle| \mathcal{F}_t\right] \\ &= S_t \, \mathbb{E}\left[\max\left(\frac{\frac{1}{T}\int_0^t S_\tau \, d\tau}{S_t} + \frac{\frac{1}{T}\int_t^T S_\tau \, d\tau}{S_t} - \frac{K}{S_t}, 0\right) \middle| \mathcal{F}_t\right] \\ &= S_t \, h(t, Q_t), \end{aligned}$$

where

$$Q_t = \frac{\frac{1}{T}\int_0^t S_\tau \, d\tau - K}{S_t}$$

and

$$h(t,q) = \mathbb{E}\left[\max\left(q + \frac{\frac{1}{T}\int_t^T S_\tau \, d\tau}{S_t}, 0\right)\right]. \tag{3.24}$$

We have essentially split the payoff into a part which only depends on the past and a part which only depends on the future. To see this we write

$$\frac{\int_t^T S_\tau d\tau}{S_t} = \int_t^T e^{\left(r-\frac{\sigma^2}{2}\right)(\tau-t)+\sigma(W_\tau - W_t)} d\tau$$

which does not involve anything from the sigma-algebra $\mathcal{F}_t$ and hence, because of the independence of increments, is independent of $\mathcal{F}_t$. Using Itô's formula we obtain the dynamics of Q_t:

$$dQ_t = Q_t(-r\,dt - \sigma\,dW_t + \sigma^2\,dt) + \frac{1}{T}dt.$$

We can then apply Itô's formula to the martingale M_t to obtain its dynamics

$$\begin{aligned} dM_t &= dS_t\,h(t,Q_t) + S_t\,dh(t,Q_t) + dS_t\,dh(t,Q_t) \\ &= S_t(r\,dt + \sigma\,dW_t)\,h(t,Q_t) \\ &\quad + S_t\left(h_t(t,Q_t)\,dt + h_q(t,Q_t)\left(Q_t(-r\,dt - \sigma\,dW_t + \sigma^2\,dt) + \frac{1}{T}dt\right)\right. \\ &\quad \left. + \frac{1}{2}Q_t^2\sigma^2 h_{qq}(t,Q_t)\,dt\right) - \sigma^2 S_t Q_t h_q(Q_t,t)\,dt. \end{aligned}$$

Setting the drift term to zero, we obtain a PDE for h:

$$h_t(t,q) + r\,h(t,q) + \frac{1}{2}\sigma^2 q^2\,h_{qq}(t,q) + \left(\frac{1}{T} - rq\right)\,h_q(t,q) = 0, \tag{3.25}$$

valid for $(t,q) \in [0,T) \times (-\infty,\infty)$. Equation (3.25) has to be solved subject to the terminal condition

$$h(T,q) = \max(q,0).$$

Introducing

$$f(t,q) = e^{-r(T-t)}h(t,q)$$

equation (3.25) can be simplified to

$$f_t(t,q) + \frac{1}{2}\sigma^2 q^2 f_{qq}(t,q) + \left(\frac{1}{T} - rq\right) f_q(t,q) = 0, \tag{3.26}$$

valid for $(t,q) \in [0,T) \times (-\infty,\infty)$ with the terminal condition

$$f(T,q) = \max(q,0),$$

and the auxiliary side conditions

$$\begin{aligned} f(t,q) &= 0, \quad q \to -\infty \\ f(t,q) &= e^{-r(T-t)} q + \frac{1}{T}\frac{1-e^{-r(T-t)}}{r}, \quad q > 0. \end{aligned}$$

To derive the first auxiliary side condition note that when $\frac{1}{T}\int_0^t S_\tau \, d\tau < K$ and the value of the stock price S_t goes to zero, the probability that the option will expire worthless goes to one. In this case Q_t goes to minus infinity and this gives us the first auxiliary boundary condition. To derive the second auxiliary boundary condition, note that when $q > 0$ the term in the maximum function of (3.24) will always be greater than 0. In this case the maximum function drops out. Using that $e^{-rt}S_t$ is a martingale, one obtains

$$\begin{aligned} \mathbb{E}\left[\frac{1}{T}\int_t^T \frac{S_\tau}{S_t} d\tau \middle| \mathcal{F}_t\right] &= \int_t^T \frac{e^{r\tau}}{S_t} \mathbb{E}\left[e^{-r\tau} S_\tau | \mathcal{F}_t\right] \\ &= \int_t^T \frac{e^{r\tau}}{S_t} e^{-rt} S_t \, d\tau \\ &= \frac{1}{T}\frac{e^{r(T-t)} - 1}{r} \end{aligned} \tag{3.27}$$

and one can explicitly calculate the price of the option. The option price function at time $t = 0$ can be written as $c = s\, f(0, -\frac{K}{s})$. The same martingale technique can be applied to a floating strike Asian option and one obtains the PDE (3.21). Note that apart form having different side conditions the PDE for the fixed strike Asian option (3.26) and the PDE for the floating strike option (3.21) are the same. To obtain the greeks we use the dynamic approach and derive their corresponding PDE. ρ is obtained via $\rho(0, s) = s\, g(0, -\frac{K}{s})$, where $g = \frac{\partial f}{\partial r}$ is given by the PDE

$$g_t(t,q) + \frac{1}{2}\sigma^2 q^2\, g_{qq}(t,q) + \left(\frac{1}{T} - q\right) g_q(t,q) - q\, f_q(t,q) = 0,$$

valid for $(t, q) \in [0, T) \times (-\infty, \infty)$ with terminal condition

$$g(T, q) = 0,$$

and the auxiliary boundary conditions

$$\begin{aligned}
g(t,q) &= 0, \quad q \to -\infty,\\
g(t,q) &= \frac{T-t}{Tr} e^{-r\,(T-t)} - (T-t)e^{-r\,(T-t)} q - \frac{1}{Tr^2}\left(1 - e^{-r\,(T-t)}\right), \quad q > 0.
\end{aligned}$$

The greek $\mathcal{V}$ is given by $\mathcal{V}(0, s) = s\, w(0, -\frac{K}{s})$, where $w = \frac{\partial f}{\partial \sigma}$ satisfies the PDE

$$w_t(t,q) + \frac{1}{2}\sigma^2 q^2\, w_{qq}(t,q) + \left(\frac{1}{T} - rq\right) w_q(t,q) + \sigma q^2\, f_{qq}(t,q) = 0,$$

valid for $(t, q) \in [0, T) \times (-\infty, \infty)$ with terminal condition

$$w(T, q) = 0,$$

and the auxiliary boundary conditions

$$\begin{aligned}
w(t,q) &= 0, \quad q \to -\infty,\\
w(t,q) &= 0, \quad q > 0.
\end{aligned}$$

3.4 Barrier options

Barrier options are options where the payoff depends on whether the price of the underlying asset reaches a certain barrier level B during a certain period of time. There are four basic types of barrier options. *Down-and-out options* are similar to vanilla options, but they have the additional feature that they cease to exist if the stock price hits a barrier level $B < S_0$ and nothing is paid to the holder of the contract. *Up-and-out options* cease to exist if a barrier level $B > S_0$ is reached. *Up-and-in options* only come into existence if a barrier $B > S_0$ is hit during the lifetime of the option. *Down-and-in options* only come into existence if a barrier $B < S_0$ is hit during the lifetime of the option. The prices of the 'in' options can be obtained by calculating the price of the corresponding vanilla option and the corresponding 'out' option and the using the fact that the price of the 'in' option must be equal to the price of the vanilla option minus the price of the 'out' option. Many other types of barrier options have been developed. These include moving-boundary options where the constant barrier is replaced by a stochastic process, Asian barrier options, and options where the barrier only applies to a certain part of the time-interval. For the standard barrier options closed form solutions for the price exist in the Black-Scholes model. Many of the more exotic barrier options can easily be implemented with the PDE approach, and since usually no closed form solutions exist, this is the only possible analytic approach.

3.4.1 Down-and-out call

As an example we explain how a *down-and-out call option* and its greeks can be priced. Both the dynamic and the closed form approach to calculate the greeks for a down-and-out option are explained in Norberg [33]. The price function of such a down-and-out call option at time t is

$$c_{DO}(t,s) = e^{-r\,(T-t)}\,\mathbb{E}[(S_T - K)\mathbf{1}_{S_T \geq K, m_T > B} | S_t = s],$$

where m_t is the running minimum of the stock price process in the interval from 0 to t. The option price can be given as the solution to the Black-Scholes

PDE

$$c_t(t,s) = r\,c(t,s) - rs\,c_s(t,s) - \frac{1}{2}\sigma^2 s^2\, c_{ss}(t,s), \tag{3.28}$$

for $t \in [0,T)$ and $0 \le s < B$, subject to the natural side conditions

$$\begin{aligned} c(t,B) &= 0, \\ c(T,s) &= \max(0, s - K), \end{aligned}$$

and the auxiliary side condition

$$c(t,s) \sim s - Ke^{-r\,(T-t)}, \quad s \to \infty.$$

The greeks can be obtained by differentiating the defining PDE (3.28) using the dynamic approach. Doing so, we obtain the same equations as for the vanilla call option except for the additional barrier condition that the payoff is zero once the barrier has been hit. There exists a closed form expression for the price of a down-and-out call [32]. If the barrier level B is smaller that the strike K then the option price is given by

$$c_{DO}(t,s,B) = c_{BS}(t,s) - \left(\frac{B}{s}\right)^{\frac{2r}{\sigma^2}-1} c_{BS}\left(t, \frac{B^2}{s}\right), \tag{3.29}$$

where

$$\begin{aligned} c_{BS}(t,s) &= s\,N(d_1(t,s)) - e^{-r\,(T-t)}\,K\,N(d_2(t,s)), \\ d_{1,2} &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{s}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right), \end{aligned}$$

and $N(\cdot)$ is the cumulative normal distribution. If the barrier level B is bigger that the strike then the option price is given by

$$\begin{aligned} c_{DO}(t,s) = {} & sN(x_1) - KN(x_2) \\ & - s\left(\frac{B}{s}\right)^{\frac{2r}{\sigma^2}+1} N(y_1) + Ke^{-r\,(T-t)}\left(\frac{B}{s}\right)^{\frac{2r}{\sigma^2}-1} N(y_2), \end{aligned} \tag{3.30}$$

where

$$x_{1,2} = \frac{\ln\left(\frac{s}{B}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$y_{1,2} = \frac{\ln\left(\frac{B}{s}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Differentiating equation (3.29) or (3.30) with respect to the parameters in question we obtain the corresponding greeks.

3.5 Numerical results

The systems of PDE of this chapter can efficiently be solved using finite differences. A finite difference solver for PIDE is explained in detail in chapter 7. As the finite difference solver for PDE is a special case of the finite difference solver for PIDE we refer to chapter 7 for the technicalities and only report the numerical results here.

We compare the results obtained with the dynamic approach with the results obtained using the closed form expressions. All computation are performed on a Pentium 4, 2.8 GHz computer. For options with a continuous payoff such as the vanilla options, Asian options, and lookback options considered in this chapter the numerical solution of the PDE converges very fast to the correct value (which can be obtained by evaluating the closed form expression). For vanilla options it takes less than one second to compute the price and the greeks on a grid with 1000 strikes times 1000 maturities with a maximum relative error around the strike of less than one per mille even for the greeks. For barrier options the payoff is discontinuous and the PDE solution converges considerably slower to the closed form solution. We have to run the computer program for about one minute to obtain a maximum relative error for the greeks of less than one per cent.

Chapter 4

Exponential Lévy models

4.1 Lévy processes and exponential Lévy models

As we want to apply, in the next section, the dynamic sensitivity method to models with jumps, in particular to exponential Lévy models, we first state some facts about Lévy processes and about exponential Lévy models. The properties about Lévy processes are drawn from Sato [35]. We begin with the definition of a Lévy process. A stochastic process X_t is a *Lévy process* if the following statements are satisfied:

1. X_t has *independent increments*: For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. X_t has *stationary increments*: The distribution of $X_{s+t} - X_t$ does not depend on t.

3. X_t is *stochastically continuous*: $\forall \epsilon > 0 \lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.

4. $X_0 = 0$ a.s.

Let X_t be a Lévy process. Its *jump measure* $\mu([t_1, t_2], A)$ is defined as the number of jumps of X_t occurring between the times $t = t_1$ and $t = t_2$ and whose size is in the Borel set $A \subset \mathbb{R}$ bounded away from zero

$$\mu([t_1, t_2], A) = \#\{t \in [t_1, t_2] : \Delta X_t \in A\}.$$

The *Lévy measure* $\nu(A)$ of X_t is defined as the expected number of jumps of X_t per unit time interval whose size is in the Borel set A bounded away from zero

$$\nu(A) = \mathbb{E}[\#\{t \in [0,1] : \Delta X_t \in A\}].$$

The Lévy measure ν is a positive measure in $\mathbb{R}$. Its density (if it exists) need not be integrable and may exhibit a singularity at 0. It has the following properties

1. $\nu(\Gamma) < \infty$ if Γ is a Borel set bounded away from zero.

2. $\int_{|x|\leq 1} x^2\, \nu(dx) < \infty$ and $\nu(\{0\}) = 0$.

The *Lévy-Itô decomposition* states that any Lévy process X_t can be written as

$$\begin{aligned} X_t = \gamma\, t + \sigma\, B_t &+ \int_{(0,t]\times(\mathbb{R}\setminus(-1,1))} x\, \mu(ds, dx) \\ &+ \lim_{\epsilon\to 0} \int_{(0,t]\times([-1,-\epsilon)\cup(\epsilon,1])} x\, (\mu(ds,dx) - \nu(dx)ds), \end{aligned}$$

which is the sum of a drift, a Brownian motion, a term comprising large jumps, and a term comprising compensated small jumps.

The characteristic function of a Lévy process X_t has the so called *Lévy-Khinchin representation*

$$\mathbb{E}[e^{izX_t}] = e^{t\psi(z)}, \quad \psi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx 1_{|x|\leq 1})\, \nu(dx), \tag{4.1}$$

where σ^2 is the variance of the Brownian motion part of the Lévy process, γ is the drift, and ν is the Lévy measure of the jumps. The Lévy process is characterised by the triplet (γ, σ^2, ν), hence called the *characteristic triplet.* A Lévy process X_t with characteristic triplet (γ, σ^2, ν) is said to be of

1. *type A* if $\sigma^2 = 0$ and $\nu(\mathbb{R}) < \infty$. The process X_t is then of compound Poisson type.

2. *type B* if $\sigma^2 = 0$, $\nu(\mathbb{R}) = \infty$, and $\int_{|x|\le 1} |x| \, \nu(dx) < \infty$. The process X_t is then of finite variation and infinite activity, which means that its jumping times are dense in $\mathbb{R}^+$.

3. *type C* if either $\sigma^2 > 0$ or $\int_{|x|\le 1} |x| \, \nu(dx) = \infty$. The process X_t is then of infinite activity and unbounded variation.

In *exponential Lévy models*, the stock prices are given as an exponential of a Lévy process X_t,

$$S_t = S_0 \, e^{r\,t + X_t}.$$

A good introduction to exponential Lévy models can be found in Cont and Tankov [14]. The discounted stock price process $S_0 \, e^{X_t}$ has to be a martingale under some risk neutral measure. This can be achieved by imposing the following conditions

$$\int_{|x|\ge 1} e^x \, \nu(dx) < \infty,$$
$$\gamma + \frac{\sigma^2}{2} + \int (e^x - 1 - x 1_{|x|\le 1}) \, \nu(dx) = 0.$$

The first condition ensures that $\mathbb{E}[e^{X_t}] < \infty$. The second condition is obtained by setting $z = -i$ in the Lévy-Khinchin representation 4.1. From a mathematical point of view the process X_t can be any arbitrary Lévy process. However, certain choices of X_t are more sensible than others when we want to capture the dynamics of a stock price. Infinite activity jump processes include both frequent small moves and rare large moves. Empirical studies [9] show that in many cases a diffusion component is superfluous when the jump component has infinity activity. Therefore most models either have a Brownian motion part and a finite activity jump part or a infinite activity jump part but no diffusion part. The former models are called jump-diffusion models. However, Brownian motion can be obtained as the limit of an infinite activity process and is thus essentially included in pure jump infinite activity models.

4.2 Jump-diffusion model

In jump-diffusion models the evolution of a stock price is given by the sum of a Brownian motion and a process that has a finite number of jumps in each finite time interval, representing rare events such as crashes. We restrict ourselves to the *Merton jump-diffusion model* [30], where the stock price is driven by the sum of a Brownian motion W_t and a compound poisson process where N_t is the Poisson process counting the jumps of X_t and Y_i are normally distributed random variables representing the jump sizes:

$$\begin{aligned} X_t &= \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \\ S_t &= S_0 \, e^{r t + X_t}, \\ Y_i &\sim N(\mu, \delta^2). \end{aligned}$$

The drift parameter γ is chosen such that the discounted stock price process $\hat{S}_t = e^{-rt} S_t = e^{X_t}$ is a martingale under some risk neutral measure. In this model the Lévy measure is given by

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}} dx. \tag{4.2}$$

The parameter λ is the intensity of the occurrences of jumps, and μ and δ are the mean and the standard deviation of the jump size distribution, respectively.

4.3 Variance gamma model

The *variance gamma model* [29] is a pure jump model with infinite activity and finite variation. This model permits a good description of the volatility smile observed in option pricing at all maturities and for a wide variety of underlying assets. It has therefore become very popular within the finance community. The model is based on the variance gamma process

$$\begin{aligned} Y_t &= \theta\, Z_t + \sigma\, W_{Z_t}, \\ Z_t &\sim \Gamma\left(\frac{t}{\kappa}, \frac{1}{\kappa}\right), \end{aligned} \tag{4.3}$$

which is obtained by subordinating a Brownian motion with drift to a Gamma process Z_t. Here σ is the volatility of the Brownian motion and θ is the drift of the Brownian motion. The density function f of the gamma distribution $\Gamma(\alpha, \beta)$ is

$$f(x; \alpha, \beta) = x^{\alpha-1} \frac{\beta^{\alpha} e^{-\beta x}}{\Gamma(\alpha)}, \quad \text{for} \quad x > 0,$$

where Γ is the Gamma function, α is called the shape parameter, and β^{-1} is a scale parameter. We choose the parameters $\alpha = \frac{t}{\kappa}$ and $\beta = \frac{1}{\kappa}$ to obtain a gamma process with mean t and variance $\kappa\, t$. The variance gamma process has the particularly simple characteristic function

$$\phi_{vg}(z; t, \theta, \sigma, \kappa) = \mathbb{E}(e^{izY_t}) = \left(\frac{1}{1 - i\theta\kappa z + \sigma^2 \kappa z^2/2}\right)^{\frac{t}{\kappa}}.$$

The stock price process is given by

$$S_t = S_0\, e^{r\,t + X_t} = S_0\, e^{r\,t + Y_t + \omega\, t},$$

where the drift term

$$\omega = \frac{1}{\kappa} \ln\left(1 - \theta\kappa - \frac{1}{2}\sigma^2\kappa\right)$$

is such that e^{X_t} is a martingale under some risk neutral measure. The random variable X_t has the density

$$p(x,t) = e^{Ax}\, C(t)\, |x|^{\frac{t}{\kappa}-\frac{1}{2}}\, K_{\frac{t}{\kappa}-\frac{1}{2}}(B|x|),$$

where $K_n(x)$ is the modified Bessel function of the second kind,

$$\begin{aligned} A &= \frac{\theta}{\sigma^2}, \\ B &= \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}, \\ C(t) &= 2\big(\sqrt{2\pi}\kappa^{t/\kappa}\sigma\Gamma(t/\kappa)\big)^{-1}(2\sigma^2/\kappa + \theta^2)^{\frac{1}{4}-\frac{t}{2\kappa}}. \end{aligned}$$

The modified Bessel function of the second kind $K_n(x)$ is one of the solutions to the PDE

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} - (x^2+n^2)y = 0,$$

and can be expressed as the integral

$$K_n(x) = \frac{\Gamma(n+\frac{1}{2})(2x)^n}{\sqrt{n}} \int_0^\infty \frac{\cos(t)}{(t^2+x^2)^{n+\frac{1}{2}}}dt.$$

More on Bessel function can be found in Abroamovitz [1]. The fact that the density of the process X_t at time t is known as a function of a Bessel function will allow us to compute the price function of an option in the variance gamma model using numerical integration and to compare this results with the ones obtained using the dynamic PIDE approach. The Lévy measure of this model is

$$\nu(dx) = \frac{1}{\kappa|x|}e^{Ax-B|x|}dx. \tag{4.4}$$

In the next section we present a generalisation of the variance gamma model.

4.4 Carr Geman Madan Yor (CGMY) model

In [9] a new model for asset returns is investigated. It is named the (CGMY) model after the authors Carr, Geman, Madan, and Yor and contains the variance gamma model as a special case. This model allows for the jump component of the asset return driving process to display finite or infinite activity and variation, depending on the choice of the parameters. The authors conclude in their empirical study that most equity prices are best described by pure jump processes of infinite activity and finite variation. The stock return driving process X_t is, not surprisingly, called the CGMY process. This process is a generalisation of the variance gamma process and a special case of the tempered stable process. Unlike in the variance gamma model, however, the density of the process X_t at time t is in general not known in terms of some special function of mathematics. The Lévy density of the CGMY process X_t is given by

$$\nu(dx) = \begin{cases} C\frac{e^{-G|x|}}{|x|^{1+Y}}dx & x \geq 0, \\ C\frac{e^{-M|x|}}{|x|^{1+Y}}dx & x < 0, \end{cases} \tag{4.5}$$

where $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$. The condition $Y < 2$ is induced by the requirement that the Lévy densities integrate x^2 in the neighbourhood of 0. In the case $Y = 0$ the CGMY model reduces to the variance gamma model. In the case $Y < 0$ the process X_t has finite activity. The parameter C may be viewed as a measure of the overall level of activity. The parameters G and M control the rate of exponential decay on the positive and the negative side respectively of the Lévy density. The parameter Y characterises the fine structure of the process and determines whether the process is of finite or infinite activity and variation. The stock price under the CGMY model is

$$S_t = S_0\, e^{(r+\omega)\, t + X_t},$$

where r is the interest rate and

$$\omega = C\Gamma(-Y)\big((M-1)^Y - M^Y + (G+1)^Y - G^Y\big).$$

is such that $e^{-r\,t}S_t$ is a martingale under some risk neutral measure. The characteristic function ϕ_{CGMY} of the CGMY process is

$$\phi_{CGMY} = E[e^{iuX_t}] = e^{t\,C\,\Gamma(-Y)\,((M-iu)^Y - M^Y + (G+iu)^Y - G^Y)}.$$

The variance gamma process can be constructed through subordination of simpler processes and in the next sections we will investigate sensitivities of option prices with respect to changes in the parameters of this simple processes. For the CGMY model we will investigate sensitivities with respect to changes in the parameters of the Lévy measure. This is the reason why, although the variance gamma model is a special case of the CGMY model, we presented them in separate sections.

Chapter 5

Option prices and greeks in exponential Lévy models

5.1 European vanilla options

5.1.1 Derivation of the PIDE

To start with we show how in exponential Lévy models the price of an option can be expressed as a solution to a PIDE. Throughout we assume that the underlying stock price is given by an exponential Lévy process which has the representation $S_t = S_0\, e^{r\,t+X_t}$, where X_t is a Lévy process with characteristic triplet (σ, γ, ν) and r stands for the interest rate. The dynamics of the stock price process are given by

$$dS_t = rS_{t-}\,dt + \sigma S_{t-}\,dW_t + S_{t-}\int_{-\infty}^{\infty}(e^x-1)\,(\mu(dt,dx)-\nu(dx)dt), \qquad (5.1)$$

where W_t is the Brownian motion part, $\mu(dt, dx)$ is the jump measure, and $\nu(dx)$ is the expected number of jumps in dx in a unit time interval. The integral term in (5.1) therefore represents a possible jump in the stock price process minus its expected value. The price of a European vanilla option at time t with payoff h on this stock is

$$c(t, S_t) = e^{-r\,(T-t)}\,\mathbb{E}[h(S_T)|\,\mathcal{F}_t]. \qquad (5.2)$$

Due to the independent increment property of Lévy processes the conditional expected value in equation (5.2) can be written as

$$\begin{aligned} \mathbb{E}[h(S_T)|\,\mathcal{F}_t] &= \mathbb{E}[h(S_t\, e^{r\,(T-t)+X_T-X_t})|\,\mathcal{F}_t] \\ &= K(t,S_t), \end{aligned}$$

where $K(t,u) = \mathbb{E}[h(u\,e^{r\,(T-t)+X_T-X_t})] = \mathbb{E}[h(u\,S_{T-t})]$. As S_t is the only relevant state variable, conditioning on the filtration $(\mathcal{F}_\tau)_{0\le\tau\le t}$ is the same as conditioning on S_t and we can write (5.2) as

$$c(t,S_t) = e^{-r\,(T-t)}\,\mathbb{E}[h(S_T)|\,S_t].$$

The discounted option price

$$\hat{c}(t,S_t) = e^{-r\,t}\,c(t,S_t) \tag{5.3}$$

is a martingale. We want to apply Itô's lemma to the discounted option price. The following condition assures that the option price function is smooth enough for Itô's formula to make sense, see [35] and [14]. If

$$\sigma > 0 \quad \text{or} \quad \exists\beta\in(0,2), \text{ s.t. } \liminf_{\epsilon\to 0}\epsilon^{-\beta}\int_{-\epsilon}^{\epsilon}|x|^2\nu(dx) > 0, \tag{5.4}$$

then for each $t > 0$, X_t has a smooth density with derivatives vanishing at infinity and therefore $c(t,s)$ is a smooth function in s. Differentiability in time can be shown by Fourier methods. Condition (5.4) is fulfilled for all jump-diffusion models with non-zero diffusion component as well as for Lévy densities behaving near zero as $\nu(x) \sim c/x^{1+\beta}$ for some constants c and $\beta > 0$ [16]. It is therefore true for the Merton model and for our choice of parameters for the CGMY model, but not for the variance gamma model. For the variance gamma model we show in Chapter 8 that the option price is continuously differentiable with respect to the stock price value and thus the use of Itô's lemma is justified. Note however, that in the variance gamma model the option price is not twice differentiable with respect to the stock price value s. From now on we assume that the option price function is sufficiently differentiable and we apply Itô's formula to equation (5.3) to obtain the dynamics of the discounted option price

$$\begin{aligned} d\hat{c}(t, S_t) &= -re^{-rt}c(t, S_{t-})\,dt + e^{-rt}\,dc(t, S_{t-}) \\ &= e^{-rt}\Big(-rc(t, S_{t-})\,dt + c_t(t, S_{t-})\,dt + c_s(t, S_{t-})\,dS_t \\ &\quad + \frac{1}{2}c_{ss}(t, S_{t-})\,d[S,S]_t^c \\ &\quad + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x) - c(t, S_{t-}) - (e^x - 1)S_{t-}c_s(t, S_{t-})\big)\mu(dt, dx)\Big), \end{aligned}$$

where $[S,S]_t^c$ is the continuous part of the quadratic variation $[S,S]_t$ defined in equation (2.3). Dividing the dynamics into a drift and a local martingale part and using the fact that $\mathbb{E}[\mu(dt, dx)] = \nu(dx)dt$, we obtain

$$\begin{aligned} d\hat{c}(t, S_t) &= e^{-rt}\Big(\Big(-rc(t, S_{t-}) + c_t(t, S_{t-}) + rS_{t-}c_s(t, S_{t-}) \\ &\quad + \frac{1}{2}S_{t-}^2\sigma^2 c_{ss}(t, S_{t-}) \\ &\quad + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x) - c(t, S_{t-}) - (e^x - 1)S_{t-}c_s(t, S_{t-})\big)\nu(dx)\Big)\,dt \\ &\quad + \sigma S_{t-}c_s(t, S_{t-})\,dW_t \\ &\quad + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x) - c(t, S_{t-})\big)\big(\mu(dt, dx) - \nu(dx)\,dt\big)\Big) \\ &= a(t)\,dt + dM_t. \end{aligned}$$

We now show that if

$$\int_{|x|>1} e^x \nu(dx) < \infty, \tag{5.5}$$

the local martingale part

$$\begin{aligned} dM_t &= \sigma\, S_{t-}\, c_s(t, S_{t-})\,dW_t \\ &\quad + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x) - c(t, S_{t-})\big)\big(\mu(dt, dx) - \nu(dx)\,dt\big) \end{aligned}$$

is actually a true martingale. From Sato [35] we know that

$$\int_{|x|>1} e^x \, \nu(dx) < \infty \Longleftrightarrow \mathbb{E}\left[e^{X_t}\right] < \infty,$$

which in turn implies that

$$\mathbb{E}\left[\int_0^t \int_{-\infty}^{\infty} (e^x - 1)(\mu(dx, d\tau) - \nu(dx)d\tau)\right] < \infty.$$

Since the payoff function is Lipschitz, the price function $c(t, s)$ is also Lipschitz with respect to the stock price value

$$|c(t, s\, e^x) - c(t, s)| \leq |s\, (e^x - 1)|,$$

which implies that

$$\mathbb{E}\left[\int_0^t \int_{-\infty}^{\infty} (c(t, s\, e^x) - c(t, s))(\mu(dx, d\tau) - \nu(dx)d\tau)\right] < \infty,$$

and that M_t is a true martingale. Thus, the coefficient in front of $a(t)$ has to be zero almost surely and it has to be zero for all possible values of S_t. This leaves us with the PIDE for the price function,

$$\begin{aligned} &c_t(t, s) + rs\, c_s(t, s) + \frac{1}{2}\sigma^2 s^2\, c_{ss}(t, s) - r\, c(t, s) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\big(c(t, se^x) - c(t, s) - (e^x - 1)s\, c_s(t, s)\big) = 0, \end{aligned} \tag{5.6}$$

valid for $(t, s) \in [0, T) \times (0, \infty)$, subject to some terminal condition. For a European vanilla call option the terminal condition is

$$c(T, s) = \max(s - K, 0), \tag{5.7}$$

where K is the strike price of the option. Equations (5.6) and (5.7) determine the price function $c(t, s)$ uniquely as a mathematical object. For computational purposes is necessary to add the boundary conditions

$$\begin{aligned} c(t,0) &= 0, \\ c(t,s) &\sim s - K\,e^{-r\,(T-t)}, \quad s \to \infty. \end{aligned}$$

In order to facilitate the numerical computations later on, we perform the change of variables as shown in Cont [16]:

$$\begin{aligned} \tau &= T - t, \\ y &= \ln\left(\frac{s}{K}\right) + r\,\tau, \end{aligned}$$

and define new functions $u(\tau, y)$ via

$$c(t,s) = e^{-r\,\tau}\,K\,u(\tau,y). \tag{5.8}$$

The derivatives of the price function in the new coordinates are

$$\begin{aligned} c_t(t,s) &= e^{-r\,\tau}\,K\,\big(r\,u(\tau,y) - u_\tau(\tau,y) - r\,u_y(\tau,y)\big), \\ c_s(t,s) &= e^{-r\,\tau}\,\frac{K}{s}\,u_y(\tau,y), \\ c_{ss}(t,s) &= e^{-r\,\tau}\,\frac{K}{s^2}\,(u_{yy}(\tau,y) - u_y(\tau,y)), \end{aligned} \tag{5.9}$$

and the shifted option price becomes

$$c(t, s\,e^x) = e^{-r\,\tau}\,K\,u(\tau, y + x). \tag{5.10}$$

Inserting the equations (5.8), (5.9), and (5.10) into equation (5.6) we obtain in equation (5.11) below a PIDE for the function $u(\tau, y)$. This PIDE has constant coefficients, linear arguments, and less terms than the original equation (5.6). The new equation is

$$\begin{aligned} u_\tau(\tau,y) = &\ \frac{\sigma^2}{2}\,(u_{yy}(\tau,y) - u_y(\tau,y)) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(u(\tau,y+x) - u(\tau,y) - (e^x - 1)u_y(\tau,y)\big), \end{aligned} \tag{5.11}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the initial condition

$$u(0, y) = \max(0, e^y - 1),$$

and the auxiliary boundary conditions

$$\begin{aligned} u(\tau, y) &= 0, \quad y \to -\infty, \\ u(\tau, y) &\sim e^y - 1, \quad y \to \infty. \end{aligned}$$

The fact that there are only linear functions in the argument of $u(\tau, y)$ will later on simplify the numerical computation of the PIDE as it will allow us to evaluate it with finite difference methods on a grid with constant grid size.

5.1.2 Greeks in the jump-diffusion model

In order to obtain the sensitivities we differentiate equation (5.11) with respect to the parameters in question. To start with we calculate $\mathcal{V}$, the sensitivity with respect to σ. In the transformed coordinates we will call this sensitivity v. We hence have

$$v = \frac{\partial u}{\partial \sigma}, \qquad \mathcal{V} = \frac{\partial c}{\partial \sigma} = K e^{-r\tau} v.$$

The equation for v is

$$\begin{aligned} v_\tau(\tau, y) = &\frac{\sigma^2}{2} (v_{yy}(\tau, y) - v_y(\tau, y)) + \sigma(u_{yy}(\tau, y) - u_y(\tau, y)) \\ &+ \int_{-\infty}^{\infty} \nu(dx) \big(v(\tau, y + x) - v(\tau, y) - (e^x - 1) v_y(\tau, y) \big), \end{aligned} \tag{5.12}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the initial condition

$$v(0, y) = 0,$$

and the auxiliary conditions

$$\begin{aligned} v(\tau, y) &= 0, \quad y \to -\infty, \\ v(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

To obtain $\mathcal{V}$, the solution of the system of equations is transformed back to the original coordinates.

Next, we calculate the sensitivity with respect to the volatility of the jump-size distribution δ and call this κ. The derivative of u with respect to δ is called k. This translates into

$$k = \frac{\partial u}{\partial \delta}, \qquad \kappa = \frac{\partial c}{\partial \delta} = K\, e^{-r\tau}\, k.$$

Recall from (4.2) that the Lévy density in the jump-diffusion model is

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}}\, dx.$$

The equation for k is

$$\begin{aligned} k_\tau(\tau, y) &= \frac{\sigma^2}{2}\big(k_{yy}(\tau, y) - k_y(\tau, y)\big) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\big(k(\tau, y+x) - k(\tau, y) - (e^x - 1)k_y(\tau, y)\big) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\frac{1}{\delta}\left(\frac{(x-\mu)^2}{\delta^2} - 1\right) \\ &\quad \times \big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned} \tag{5.13}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the initial condition

$$k(0, y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned} k(\tau, y) &= 0, \quad y \to -\infty, \\ k(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

The sensitivity with respect to the jump intensity λ will be named β and in the transformed coordinates we call it b. Thus

$$b = \frac{\partial u}{\partial \lambda}, \qquad \beta = \frac{\partial c}{\partial \lambda} = K\, e^{-r\tau}\, b.$$

The equation for b is

$$\begin{aligned} b_\tau(\tau, y) &= \frac{\sigma^2}{2}\big(b_{yy}(\tau, y) - b_y(\tau, y)\big) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\big(b(\tau, y+x) - b(\tau, y) - (e^x - 1) b_y(\tau, y)\big) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\frac{1}{\lambda}\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1) u_y(\tau, y)\big), \qquad (5.14) \end{aligned}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the initial condition

$$b(0, y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned} b(\tau, y) &= 0, \quad y \to -\infty, \\ b(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

The variable ρ is the sensitivity with respect to changes in the interest rate. Let ϱ be the sensitivity of the transformed variable u with respect to changes in the interest rate. The corresponding equations are

$$\varrho = \frac{\partial u}{\partial r}, \qquad \rho = \frac{\partial c}{\partial r} = K e^{-r\tau}(\varrho - \tau\, u).$$

The equation for ϱ is

$$\begin{aligned} \varrho_\tau(\tau, y) &= \frac{\sigma^2}{2}\big(\varrho_{yy}(\tau, y) - \varrho_y(\tau, y)\big) \\ &\quad + \int_{-\infty}^{\infty} \nu(dx)\big(\varrho(\tau, y+x) - \varrho(\tau, y) - (e^x - 1)\varrho_y(\tau, y)\big), \qquad (5.15) \end{aligned}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the initial condition

$$\varrho(0, y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned}
&\varrho(\tau, y) = 0, \quad y \to -\infty \\
&\varrho(\tau, y) = \frac{\partial}{\partial r}(e^y - 1) = \frac{\partial y}{\partial r}\frac{\partial}{\partial y}(e^y - 1) = \tau\, e^y, \quad y \to \infty.
\end{aligned}$$

Solving the system of PIDE for the price function (5.11) and the sensitivities (5.12), (5.13), (5.14), and (5.15), we obtain the price and the sensitivities in the transformed coordinates. Transforming back, we obtain the final results.

5.1.3 Greeks in the variance gamma model

The price of a European option in the variance gamma model (4.3), as in any pure jump model, can be given as the transformed solution of the PIDE

$$u_\tau(\tau, y) = \int_{-\infty}^{\infty} \nu(dx)\big(u(\tau, y + x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big),$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$. This is just equation (5.11) with all the terms stemming from the Brownian motion set to zero. Remember from (4.4) that the Lévy density in the variance gamma model is

$$\nu(dx) = \frac{1}{\kappa|x|} e^{\frac{\theta}{\sigma^2}x - \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}|x|} dx.$$

As, unlike in the Merton model, the measure has a singularity at zero one might be worried about the contribution of the small jumps. However, for small x the integrand

$$\big(u(\tau, y + x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big)$$

is of the order x^2 and the Lévy density is of the order $\frac{1}{x}$. Therefore the integrand goes faster to zero than the measure goes to infinity and the integral is well defined. In the numerical computations we will approximate the small jumps by a Brownian motion term. For sufficiently small jumps the jump term could also just be neglected, as in the variance gamma model the integral

$$\int_{-\epsilon}^{\epsilon} \nu(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big)$$

is of the order ϵ^2 for small ϵ.

Being interested in the greeks we first calculate vega, the sensitivity with respect to the volatility σ in the Lévy measure (4.4). The derivative of the transformed variable u with respect to σ is called v. Thus we have

$$v = \frac{\partial u}{\partial \sigma}, \qquad \mathcal{V} = \frac{\partial c}{\partial \sigma} = K\, e^{-r\tau}\, v.$$

The PIDE for v is

$$\begin{aligned} v_\tau(\tau, y) = &\int_{-\infty}^{\infty} \nu(dx)\big(v(\tau, y+x) - v(\tau, y) - (e^x - 1)v_y(\tau, y)\big) \\ &+ \int_{-\infty}^{\infty} \varsigma(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned}$$

where

$$\varsigma(dx) = \frac{\partial}{\partial \sigma}\nu(dx) = \left(-\frac{2\theta x}{\sigma^3} - \frac{2|x|}{\sqrt{\theta^2 + 2\sigma^2/\kappa}\,\sigma\kappa} + \frac{2\sqrt{\theta^2 + 2\sigma^2/\kappa}\,|x|}{\sigma^3}\right)\nu(dx),$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$, subject to the initial condition

$$v(0, y) = 0,$$

and the side conditions

$$\begin{aligned} v(\tau, y) &= 0, \quad y \to -\infty, \\ v(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

The derivative with respect to θ will be called ϕ, and in the transformed coordinates it will be called p, hence

$$p = \frac{\partial u}{\partial \theta}, \qquad \phi = \frac{\partial c}{\partial \theta} = K\, e^{-r\tau}\, p.$$

The PIDE for p is

$$\begin{aligned} p_\tau(\tau, y) = &\int_{-\infty}^{\infty} \nu(dx)\big(p(\tau, y+x) - p(\tau, y) - (e^x - 1)p_y(\tau, y)\big) \\ &+ \int_{-\infty}^{\infty} \rho(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned}$$

where

$$\rho(dx) = \frac{\partial}{\partial \theta}\nu(dx) = \left(\frac{x}{\sigma^2} - \frac{\theta}{\sqrt{\theta^2 + 2\sigma^2/\kappa}}\frac{|x|}{\sigma^2}\right)\nu(dx),$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$, subject to the initial condition

$$p(0, y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned} p(\tau, y) &= 0, \quad y \to -\infty, \\ p(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

Let ψ be the derivative of the price with respect to κ and q be the derivative in transformed coordinates

$$q = \frac{\partial u}{\partial \kappa}, \qquad \psi = \frac{\partial c}{\partial \kappa} = K e^{-r\tau} q.$$

The PIDE for q is

$$\begin{aligned} q_\tau(\tau, y) = &\int_{-\infty}^{\infty} \nu(dx)\big(q(\tau, y+x) - q(\tau, y) - (e^x - 1)q_y(\tau, y)\big) \\ &+ \int \varrho(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned}$$

where

$$\varrho(dx) = \frac{\partial}{\partial \kappa}\nu(dx) = \left(\frac{|x|}{\kappa^2\sqrt{\theta^2 + 2\sigma^2/\kappa}} - \frac{1}{\kappa} \right) \nu(dx),$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$, subject to the initial condition

$$q(0, y) = 0,$$

and the side conditions

$$\begin{aligned} q(\tau, 0) &= 0, \quad y \to -\infty, \\ q(\tau, y) &= 0, \quad y \to \infty. \end{aligned}$$

5.1.4 Greeks in the CGMY model

As before we start with the option price determining PIDE. The CGMY model defined in section (4.4) is a pure jump model and therefore all the terms in the PIDE coming from a Brownian motion are zero. In the transformed coordinates the PIDE for the price function therefore is

$$u_\tau(\tau, y) = \int_{-\infty}^{\infty} \nu(dx)\big(u(\tau, y + x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \tag{5.16}$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$. Recall from equation (4.5) that the Lévy measure in the CGMY model is

$$\nu(dx) = \begin{cases} C\frac{e^{-G|x|}}{|x|^{1+Y}}dx & x \geq 0, \\ C\frac{e^{-M|x|}}{|x|^{1+Y}}dx & x < 0. \end{cases}$$

Just as in the variance gamma model the measure $\nu(dx)$ has a singularity at zero. Therefore it can be difficult to numerically evaluate the contribution of the small jumps. This is most pronounced when Y is close to two and one is dealing with an infinite variation model. One way around this problem is again to approximate the small jumps by a diffusion term. We show how this is done in the section 7.1.2.

Starting from equation (5.16) we calculate the sensitivities with respect to the parameters C and Y. Other parameter sensitivities can be dealt with in a similar way. Note that here we calculate sensitivities with respect to the parameters in the Lévy measure whereas in the variance gamma model we calculated sensitivities with respect to the parameters θ, σ, and κ introduced in (4.3) which are parameters of the stock price driving process Y_t.

First, we derive the PIDE for the sensitivity of the option price with respect to the parameter C. Let

$$v = \frac{\partial u}{\partial C}, \qquad \mathcal{V} = \frac{\partial c}{\partial C} = Ke^{-r\tau}v.$$

We therefore differentiate equation (5.16) with respect to C to obtain the PIDE

$$\begin{aligned} v_\tau(\tau, y) = &\int_{-\infty}^{\infty} \nu(dx)\big(v(\tau, y+x) - v(\tau, y) - (e^x - 1)v_y(\tau, y)\big) \\ &+ \frac{1}{C}\int_{-\infty}^{\infty} \nu(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned}$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$, subject to the side conditions which are all zero. To obtain the sensitivity with respect to the parameter Y we differentiate equation (5.16) with respect to Y. We denote the derivative of c with respect to Y by ϕ, the derivative of u with respect to Y by q, and define the function

$$\xi(dx) = \frac{\partial \nu(dx)}{\partial Y} = \begin{cases} C\frac{e^{-G|x|}}{|x|^{1+Y}}(-\ln(x)) & x \geq 0 \\ C\frac{e^{-M|x|}}{|x|^{1+Y}}(-\ln(-x)) & x < 0. \end{cases}$$

Thus the PIDE for q reads

$$\begin{aligned} q_\tau(\tau, y) = &\int_{-\infty}^{\infty} \nu(dx)\big(q(\tau, y+x) - q(\tau, y) - (e^x - 1)q_y(\tau, y)\big) \\ &+ \int_{-\infty}^{\infty} \xi(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \end{aligned}$$

valid for $(\tau, y) \in (0, T] \times (0, \infty)$, subject to the side conditions which are all zero. Note for small ϵ the integral

$$\int_{-\epsilon}^{\epsilon} \xi(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big),$$

can be approximated by a constant times

$$\int_0^{\epsilon} x^{1-Y} \log(x)\, dx,$$

which is finite for $Y < 2$. Therefore we do not need any additional condition on the parameters.

5.2 Lookback options

5.2.1 Derivation of the PIDE

We will extend the results from section 3.2 to exponential Lévy models and derive the PIDE for a option whose payoff depends on the stock price $S_t = S_0\, e^{r\,t+X_t}$ and the running maximum $M_t = \sup_{0\leq\tau\leq t} S_\tau$ of the stock price realised over the lifetime of the option. The dynamics of the stock price are the same as in the previous section and are given by (5.1). To obtain the price of an option that depends on the minimum instead of the maximum one replaces the maximum process M_t with the minimum process $m_t = \inf_{0\leq\tau\leq t} S_\tau$ in a straightforward manner. In order for the discounted option price option price

$$\hat{c}(t, S_t, M_t) = e^{-r\,t}\, c(t, S_t, M_t), \tag{5.17}$$

to be a true martingale we have to impose the additional condition

$$\int_{|x|>1} e^{|x|}\nu(dx) < \infty. \tag{5.18}$$

Condition (5.18) implies that the running maximum process of the stock price is finite and therefore the option price process a true martingale:

$$\begin{aligned}\int_{|x|>1} e^{|x|}\nu(dx) < \infty &\Longleftrightarrow \mathbb{E}\left[e^{|X_t|}\right] < \infty \quad \text{for some } t > 0 \\ &\Longleftrightarrow \mathbb{E}\left[e^{\sup_{0<\tau<t}|X_\tau|}\right] < \infty \\ &\Longrightarrow \mathbb{E}\left[e^{\sup_{0<\tau<t} X_\tau}\right] < \infty \\ &\Longleftrightarrow \mathbb{E}\left[\sup_{0<\tau<t} e^{X_\tau}\right] < \infty\end{aligned}$$

The proofs of all these statements can be found in Sato [35]. The moment condition above is always true for the Merton model and implies the conditions $B > A + 1$ and $A + B > 1$ for the parameters in the variance gamma model and $G > 1$ and $M > 1$ for the parameters of the CGMY model.

The option price depends now on time, on the stock price process, and on the maximum process. The running maximum process of a Brownian motion is a strictly increasing process, hence of bounded variation. Therefore the continuous part of the quadratic variation of the maximum process $d[M, M]_t^c$ and the continuous part of the quadratic covariation of the stock price process and the maximum process $d[M, S]_t^c$ are zero. We want to apply Itô's formula to equation (5.17) and obtain the dynamics of the option price process. We shall see that the derivative of the option price function with respect to the value of the running maximum of the stock price disappears from our formula. Still we would have to verify sufficient differentiability of the option price function with respect to the stock price value and time for Itô's formula to make sense. This seems to be a very difficult task. We therefore assume sufficient differentiability of the option price function with respect to the relevant variables. The numerical results in section 7.6.2 will justify this assumption. Finally applying Itô's formula to equation (5.17) we obtain the dynamics

$$\begin{aligned}
d\hat{c}(t, S_t, M_t) = e^{-rt}\Big(& - rc(t, S_{t-}, M_{t-})\, dt + c_t(t, S_{t-}, M_{t-})\, dt \\
& + c_s(t, S_{t-}, M_{t-})\, dS_t + c_m(t, S_{t-}, M_{t-})\, dM_t \\
& + \frac{1}{2} c_{ss}(t, S_{t-}, M_{t-})\, d[S, S]_t^c \\
& + \big(c(t, S_t, M_t) - c(t, S_{t-}, M_{t-}) \\
& - \Delta S_t\, c_s(t, S_{t-}, M_{t-}) - \Delta M_t\, c_m(t, S_{t-}, M_{t-})\big)\Big), \qquad (5.19)
\end{aligned}$$

where we have written $\Delta S_t = S_t - S_{t-}$ for the jumps in the stock price process and $\Delta M_t = M_t - M_{t-}$ for the jumps in the maximum process. The parts in the brackets on the last two lines of 5.19 cannot necessarily be separated. However, as the running maximum process M_t is of bounded variation the terms $c(t, S_t, M_t) - c(t, S_{t-}, M_{t-}) - \Delta S_t\, c_s(t, S_{t-}, M_{t-})$ and $\Delta M_t\, c_m(t, S_{t-}, M_{t-})$ in equation (5.19) can be separated and one can combine $c_m(t, S_{t-}, M_{t-})\, dM_t$ and $-\Delta M_t\, c_m(t, S_{t-}, M_{t-})$ to $c_m(t, S_{t-}, M_{t-})\, dM_t^c$. Equation (5.19) therefore simplifies to

$$\begin{aligned}
d\hat{c}(t, S_t, M_t) = e^{-rt}\Big(& - rc(t, S_{t-}, M_{t-})\, dt + c_t(t, S_{t-}, M_{t-})\, dt \\
& + c_s(t, S_{t-}, M_{t-})\, dS_t + \frac{1}{2} c_{ss}(t, S_{t-}, M_{t-})\, d[S, S]_t^c \\
& + c_m(t, S_{t-}, M_{t-})\, dM_t^c \\
& + \int_{-\infty}^{\infty} \big(c(t, S_{t-} e^x, \max(M_{t-}, e^x S_{t-})) - c(t, S_{t-}, M_{t-}) \\
& - (e^x - 1) S_{t-}\, c_s(t, S_{t-}, M_{t-})\big)\mu(dt, dx)\Big). \qquad (5.20)
\end{aligned}$$

We now show that the term $c_m(t, S_{t-}, M_{t-})\, dM_t^c$ is zero. Plainly, when the stock price is not at its maximum then dM_t^c is zero. To obtain the result when the stock price is at the maximum we inspect the price function

$$c(t,s,m) = e^{-r(T-t)}\mathbb{E}\left[\max(m, s \sup_{t\le\tau\le T} e^{r(\tau-t)+X_\tau-X_t}) - S_T \middle| S_t = s, M_t = m\right]. \tag{5.21}$$

If the process S_t is of infinite variation or has a positive drift, then the price function does not depend on m at $s=m$, as the second term in the maximum function of (5.21) is always greater than m, and hence $c_m(t,s,m)|_{s=m} = 0$. If the stock price process has a negative drift, it attains its maxima on a countable set and therefore $c_m(t,s,m)\,dM_t^c$ must be zero for the discounted option price to be a martingale. Condition (5.18) guarantees that one can divide the dynamics of (5.20) into a drift and a true martingale part. Setting the drift term to zero one obtains the PIDE for the price function

$$\begin{aligned}&-r\,c(t,s,m) + c_t(t,s,m) + rs\,c_s(t,s,m) + \frac{1}{2}\sigma^2 s^2\, c_{ss}(t,s,m)\\&+\int_{-\infty}^{\infty}\nu(dx)\big(c(t,s\,e^x,\max(s\,e^x,m)) - c(t,s,m) - (e^x-1)s\,c_s(t,s,m)\big) = 0,\end{aligned} \tag{5.22}$$

valid for $t\in[0,T)$ and $0 < s \le m$, where s is the value of the stock price at time t and m is the value of the running maximum at time t.

As an example we consider the floating strike lookback put with terminal condition

$$c(T,s,m) = m - s.$$

For such a contract one would only need to find $\mathbb{E}[M_T]$, the expected value of the running maximum process at time T. However, the distribution of the maximum of a Lévy process is not always known and it makes perfect sense to work with the PIDE method to obtain the price and the sensitivities even for this simple contract. The auxiliary side conditions for this option which are used for computational convenience are

$$c_m(t,s,m)|_{s=m} = 0,$$
$$c(t,0,m) = m\,e^{-r\,(T-t)}.$$

These side conditions are the same as equations (3.11) and (3.12) used in the Brownian motion setting in section 3.2.1. If the stock price process has infinite variation or a positive drift then the first boundary condition follows from the derivation of the PIDE and we will restrict ourselves to this case. The reason behind the second auxiliary condition is that if the stock price approaches zero we have $\lim_{s\to 0} c(t,s,m) = m\,e^{-r\,(T-t)}$. The option price is then just the discounted value of the current maximum.

The floating strike lookback put can be recast in terms of only two variables, time and one state variable if the option price function is rewritten as

$$c(t,s,m) = m\,w(t,z),$$

with $z = \frac{s}{m}$. In the new coordinates the derivatives are

$$\begin{aligned} c_t(t,s,m) &= m\,w_t(t,z) \\ c_s(t,s,m) &= w_z(t,z), \\ c_{ss}(t,s,m) &= \frac{1}{m} w_{zz}(t,z), \\ c_m(t,s,m) &= w(t,z) - z\,w_z(t,z). \end{aligned} \tag{5.23}$$

The shifted option price becomes in the new coordinates

$$c(t, s\,e^x, \max(s\,e^x, m) = m\,\max(z\,e^x, 1)\,w(t, \min(1, z\,e^x)). \tag{5.24}$$

Replacing equations (5.23) and (5.24) into equation (5.22) one obtains a

PIDE that depends only on time and one further variable z

$$\begin{aligned}
&w_t(t,z) + rz\, w_z(t,z) + \frac{1}{2}\sigma^2 z^2\, w_{zz}(t,z) - r\, w(t,z) \\
&\quad + \int_{-\infty}^{\infty} \nu(dx)\big(\max(z\, e^x, 1)\, w(t, \min(z\, e^x, 1)) \\
&\quad - w(t,z) - (e^x - 1)z\, w_z(t,z)\big) = 0,
\end{aligned}$$

valid for $(t,z) \in [0,T) \times (0,1)$ subject to the terminal condition

$$w(T,z) = 1 - z,$$

and the auxiliary conditions

$$\begin{aligned}
w_z(t,z)|_{z=1} &= w(t,1), \\
w(t,0) &= e^{-r\,(T-t)}.
\end{aligned}$$

In order to have constant coefficients and a constant grid size, we perform a second transform of variables from $w(t,z)$ to $u(\tau,x)$:

$$\begin{aligned}
\tau &= T - t, \\
y &= \ln z, \\
w(t,z) &= e^{-r\,\tau} u(\tau,y), \\
w_t(t,z) &= e^{-r\,\tau}\,(r\, u(\tau,y) - u_\tau(\tau,y)) \\
w_z(t,z) &= e^{-r\,\tau}\frac{1}{z} u_y(\tau,y) \\
w_{zz}(t,z) &= e^{-r\,\tau}\frac{1}{z^2}(u_{yy}(\tau,y) - u_y(\tau,y)).
\end{aligned}$$

In the u coordinates we obtain the PIDE

$$\begin{aligned}
u_\tau(\tau,y) &= \left(r - \frac{\sigma^2}{2}\right) u_y(\tau,y) + \frac{\sigma^2}{2} u_{yy}(\tau,y) \\
&\quad + \int_{-\infty}^{\infty} \nu(dx)\big(\max(e^{y+x}, 1)\, u(\tau, \min(y+x, 0)) - u(\tau,y) \\
&\quad - (e^x - 1)u_y(\tau,y)\big), \qquad (5.25)
\end{aligned}$$

for $(\tau, y) \in (0, T] \times (-\infty, 0)$ subject to the initial condition

$$u(0, y) = 1 - e^y,$$

and the auxiliary side conditions

$$u(\tau, y) = 1, \quad y \to -\infty, \tag{5.26}$$

$$u_y(\tau, y)|_{y=0} = u(\tau, 0). \tag{5.27}$$

5.2.2 A martingale method

We now extend the alternative derivation of the option price defining PIDE from section 3.2.2 to a model where the underlying stock price is an exponential Lévy process. Consider the martingale

$$\tilde{M}_t = \mathbb{E}[M_T - S_T | \mathcal{F}_t],$$

where S_t is the stock price process and $M_t = \sup_{0 \le \tau \le t} S_\tau$ is the running maximum. The martingale can be written as

$$\begin{aligned}\tilde{M}_t &= \mathbb{E}\left[\max\left(M_t, S_t \sup_{t \le \tau \le T} e^{r(\tau - t) + X_\tau - X_t}\right) - S_t e^{r(T-t) + X_T - X_t} \middle| \mathcal{F}_t\right] \\ &= S_t f(t, Q_t),\end{aligned}$$

with

$$Q_t = \frac{M_t}{S_t},$$

and

$$f(t, q) = \mathbb{E}\left[\max\left(q, \sup_{0 \le \tau \le T-t} e^{r\tau + X_\tau}\right) - e^{r(T-t) + X_T - X_t}\right].$$

We have again separated the payoff into a part which only depends on the past and a part which only depends on the future. Using Itô's lemma, we obtain the dynamics of Q_t:

$$\begin{aligned} dQ_t = & \frac{1}{S_{t-}} dM_t - \frac{M_{t-}}{S_{t-}^2} dS_t + \frac{M_{t-}}{S_{t-}^3} d[S,S]_t^c \\ & + \left(\frac{M_t}{S_t} - \frac{M_{t-}}{S_{t-}} - \frac{1}{S_t} \Delta M_t + \frac{M_t}{S_t^2} \Delta S_t \right). \end{aligned} \tag{5.28}$$

The process M has finite variation. We assume that the jump part of the stock price process also has finite variation. This allows us to open the parentheses in equation (5.28) and to simplify the equation to

$$\begin{aligned} dQ_t = & \frac{1}{S_{t-}} dM_t^c - \frac{M_{t-}}{S_{t-}^2} dS_t^c + \frac{M_{t-}}{S_{t-}^3} d[S,S]_t^c \\ & + \left(\frac{M_t}{S_t} - \frac{M_{t-}}{S_{t-}} \right), \end{aligned}$$

which in turn can be written as

$$dQ_t = \frac{1}{S_{t-}} dM_t^c - \frac{Q_{t-}}{S_{t-}} dS_t^c + \frac{Q_{t-}}{S_{t-}^2} d[S,S]_t^c + \Delta Q_t, \tag{5.29}$$

where

$$\Delta Q_t = Q_t - Q_{t-} = \frac{M_t}{S_t} - \frac{M_{t-}}{S_{t-}}.$$

At jump times of S_t we have to express Q_t in terms of its value before the jump and the jump size:

$$Q_t = \frac{M_t}{S_t} = \frac{\max(M_{t-}, S_t)}{S_t} = \max\left(\frac{M_{t-}}{S_{t-} e^{\Delta X_t}}, 1 \right) = \max(Q_{t-} e^{-\Delta X_t}, 1).$$

The continuous part of the quadratic variation of Q_t is

$$d[Q,Q]_t^c = Q_{t-}^2 \sigma^2 \, dt. \tag{5.30}$$

Using equation (5.30), we proceed from (5.29):

$$dQ_t = -\frac{Q_{t-}}{S_{t-}}\, dS_t^c + Q_{t-}\sigma^2\, dt + \Delta Q_t, \quad Q_t > 1.$$

Now we work out the dynamics of the martingale $\tilde{M}$, still assuming the jump part of S_t has finite variation:

$$\begin{aligned}
d\tilde{M}_t &= dS_t^c\, f(t, Q_{t-}) + S_{t-}\, f_t(t, Q_{t-})\, dt \\
&\quad + S_{t-}\, f_q(t, Q_{t-})\, dQ_t^c + f_q(t, Q_{t-})\, d[S, Q]_t^c + \frac{1}{2} S_{t-} f_{qq}(t, Q_{t-})\, d[Q, Q]_t^c \\
&\quad + S_t\, f(t, Q_t) - S_{t-}\, f(t, Q_{t-}) \\
&= S_{t-} \left(r\, dt + \sigma\, dW_t - \int_{-\infty}^{\infty} (e^x - 1)\nu(dx)\, dt \right) f(t, Q_{t-}) \\
&\quad + S_{t-} f_t(t, Q_{t-})\, dt \\
&\quad + S_{t-}\, f_q(t, Q_{t-}) \Big(- Q_{t-} \left(r\, dt + \sigma\, dW_t - \int_{-\infty}^{\infty} (e^x - 1)\nu(dx)\, dt \right) \\
&\quad + Q_{t-}\sigma^2\, dt \Big) - f_q(t, Q_t) S_{t-} Q_{t-} \sigma^2\, dt + \frac{1}{2} S_{t-} f_{qq}(t, Q_{t-}) Q_{t-}^2 \sigma^2\, dt \\
&\quad + \int_{-\infty}^{\infty} \left(S_{t-}\, e^x\, f(t, \max(Q_{t-} e^{-x}, 1)) - S_{t-}\, f(t, Q_{t-}) \right) \mu(dt, dx). \qquad (5.31)
\end{aligned}$$

Setting the drift term in equation (5.31) to zero one obtains the PIDE

$$\begin{aligned}
&\left(r - \int_{-\infty}^{\infty} (e^x - 1)\nu(dx) \right) f(t, q) - q \left(r - \int_{-\infty}^{\infty} (e^x - 1)\nu(dx) \right) f_q(t, q) \\
&+ f_t(t, q) + \frac{1}{2}\sigma^2 q^2\, f_{qq}(t, q) + \int_{-\infty}^{\infty} \left(e^x (f(t, \max(q e^{-x}, 1)) - f(t, q)) \right) \nu(dx) = 0,
\end{aligned} \qquad (5.32)$$

valid for $(t, q) \in (0, T) \times (1, \infty)$. The terminal and the side conditions here are the same as the ones were the stock price process is a Brownian motion which where derived in section 3.2.1. For convenience we state them here as well. The terminal condition is

$$f(T, q) = q - 1, \quad q > 1,$$

and the side conditions are

$$f_q(t,q)|_{q=1} = 0,$$
$$f(t,q) = q - e^{r\,(T-t)}, \quad q \to \infty.$$

The price function of the option is

$$p(s,t) = e^{-r\,(T-t)}\, s\, f(t,q).$$

5.2.3 Greeks in the jump-diffusion model

To calculate the greeks we start from the price determining equation (5.25). Alternatively we could start from equation (5.32) derived with the martingale argument. To obtain vega, the sensitivity with respect to σ, we differentiate equation (5.25) with respect to σ. Using

$$v = \frac{\partial u}{\partial \sigma}, \qquad \mathcal{V} = \frac{\partial c}{\partial \sigma} = m\, e^{-r\tau}\, v$$

we obtain the PIDE for the sensitivity

$$\begin{aligned} v_\tau(\tau,y) &= \left(r - \frac{\sigma^2}{2}\right) v_y(\tau,y) + \frac{\sigma^2}{2} v_{yy}(\tau,y) + \sigma\big(u_{yy}(\tau,y) - u_y(\tau,y)\big) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(\max(e^{y+x},1)\, v(\tau,\min(y+x,0)) - v(\tau,y) \\ &- (e^x - 1) v_y(\tau,y)\big), \end{aligned}$$

valid for $(\tau, y) \in (0,T] \times (-\infty, 0)$, subject to the initial condition

$$v(0,y) = 0,$$

and the auxiliary side conditions

$$v(\tau,y) = 0, \quad y \to -\infty,$$
$$v_y(\tau,y)|_{y=0} = v(\tau,0).$$

We denote by κ the sensitivity of the option price with respect to changes in the volatility of the jump size distribution δ

$$k = \frac{\partial u}{\partial \delta}, \qquad \kappa = \frac{\partial c}{\partial \delta} = m\, e^{-r\tau}\, k.$$

Recall from (4.2) that the Lévy density in the jump-diffusion model is

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}}\, dx.$$

Differentiating equation (5.25) with respect to δ we obtain the PIDE for $k(\tau, y)$:

$$\begin{aligned} k_\tau(\tau, y) = {} & \left(r - \frac{\sigma^2}{2}\right) k_y(\tau, y) + \frac{\sigma^2}{2} k_{yy}(\tau, y) \\ & + \int_{-\infty}^{\infty} \nu(dx) \Bigg(\max(e^{y+x}, 1) k(\tau, \min(y+x, 0)) - k(\tau, y) \\ & - (e^x - 1) k_y(\tau, y) \\ & + \frac{1}{\delta}\left(\frac{(x-\mu)^2}{\delta^2} - 1\right) \big(\max(e^{y+x}, 1) u(\tau, \min(y+x, 0)) \\ & - u(\tau, y) - (e^x - 1) u_y(\tau, y) \big) \Bigg), \end{aligned}$$

valid for $(\tau, y) \in (0, T] \times (-\infty, 0)$, subject to the initial condition

$$k(0, y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned} k(\tau, y) &= 0, \quad y \to -\infty, \\ k_y(\tau, y)|_{y=0} &= k(\tau, 0). \end{aligned}$$

We denote by β the sensitivity of the option price with respect to changes in the jump intensity λ

$$b = \frac{\partial u}{\partial \lambda}, \qquad \beta = \frac{\partial c}{\partial \lambda} = m\, e^{-r\tau}\, b.$$

We differentiate equation (5.25) with respect to λ to obtain the PIDE for $b(\tau, y)$:

$$\begin{aligned} b_\tau(\tau,y) &= \left(r - \frac{\sigma^2}{2}\right) b_y(\tau,y) + \frac{\sigma^2}{2} b_{yy}(\tau,y) \\ &+ \int_{-\infty}^{\infty} \nu(dx) \Big(\max(e^{y+x},1) b(\tau, \min(y+x,0)) - b(\tau,y) \\ &- (e^x - 1) b_y(\tau,y) + \frac{1}{\lambda}\big(\max(e^{y+x},1) u(\tau,\min(y+x,0)) \\ &- u(\tau,y) - (e^x-1) u_y(\tau,y)\big) \Big), \end{aligned}$$

valid for $(\tau, y) \in (0,T] \times (-\infty, 0)$, subject to the initial condition

$$b(0,y) = 0,$$

and the auxiliary side conditions

$$\begin{aligned} b(\tau,y) &= 0, \quad y \to -\infty, \\ b_y(\tau,y)|_{y=0} &= b(\tau,0). \end{aligned}$$

5.2.4 Greeks in the variance gamma model

We start with the PIDE for the option price

$$\begin{aligned} u_\tau(\tau,y) &= r\, u_y(\tau,y) \\ &+ \int_{-\infty}^{\infty} \nu(dx) \big(\max(e^{y+x},1) u(\tau,\min(y+x,0)) - u(\tau,y) \\ &- (e^x-1) u_y(\tau,y)\big). \end{aligned} \tag{5.33}$$

This is the same equation as (5.25) except that the diffusion part is now zero and $\nu(dx)$ is the Lévy measure of the variance gamma model as defined in equation (4.4). First, we calculate the sensitivity of the lookback option price with respect to changes in σ, the volatility of the subordinated Brownian motion which occurs in the jump measure.

Differentiating equation (5.33) with respect to σ we obtain the PIDE for $v = \frac{\partial u}{\partial \sigma}$:

$$\begin{aligned} v_\tau(\tau, y) &= r\, v_y(\tau, y) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(\max(e^{y+x}, 1) v(\tau, \min(y + x, 0)) - v(\tau, y) \\ &- (e^x - 1) u_y(\tau, y)\big) \\ &+ \int_{-\infty}^{\infty} \varsigma(dx)\big(\max(e^{y+x}, 1) u(\tau, \min(y + x, 0)) \\ &- u(\tau, y) - (e^x - 1) u_y(\tau, y)\big), \end{aligned}$$

where

$$\varsigma(dx) = \frac{\partial}{\partial \sigma}\nu(dx) = \left(-\frac{2\theta x}{\sigma^3} - \frac{2|x|}{\sqrt{\theta^2 + 2\sigma^2/\kappa}\,\sigma\kappa} + \frac{2\sqrt{\theta^2 + 2\sigma^2/\kappa}\,|x|}{\sigma^3}\right)\nu(dx),$$

valid for $(\tau, y) \in (0, T] \times (-\infty, 0)$. The initial condition is

$$v(0, y) = 0,$$

and the auxiliary side conditions are

$$\begin{aligned} v(\tau, y) &= 0, \quad y \to -\infty, \\ v_y(\tau, y)|_{y=0} &= v(\tau, 0). \end{aligned}$$

To transform back to the original coordinates, we use the formula $\mathcal{V}(t, s, m) = m\, e^{-r\tau}\, v(\tau, y)$.

Next, we derive the PIDE for the sensitivity of the option price with respect to changes in θ, the drift of the subordinated Brownian motion as defined in equation (4.3). In the transformed coordinates we denote this sensitivity p. Consequently we have $p = \frac{\partial u}{\partial \theta}$ and $\frac{\partial c}{\partial \theta} = m\, e^{-r\tau}\, p$. The equation for p is

$$\begin{aligned}p_\tau(\tau, y) &= r\, p_y(\tau, y)\\&+ \int_{-\infty}^{\infty} \nu(dx)\big(\max(e^{y+x}, 1)\, p(\tau, \min(y+x, 0)) - p(\tau, y)\\&- (e^x - 1)p_y(\tau, y)\big)\\&+ \int_{-\infty}^{\infty} \rho(dx)\big(\max(e^{y+x}, 1)\, u(\tau, \min(y+x, 0))\\&- u(\tau, y) - (e^x - 1)u_y(\tau, y)\big),\end{aligned}$$

where

$$\rho(dx) = \frac{\partial}{\partial\theta}\nu(dx) = \left(\frac{x}{\sigma^2} - \frac{|x|\theta}{\sqrt{\theta^2 + \frac{2\sigma^2}{\kappa}\sigma^2}}\right)\nu(dx),$$

valid for $(\tau, y) \in (0, T] \times (-\infty, 0)$. The initial conditions is

$$p(0, y) = 0,$$

and the auxiliary side conditions are

$$\begin{aligned}p(\tau, y) &= 0, \quad y \to -\infty,\\p_y(\tau, y)|_{y=0} &= p(\tau, 0).\end{aligned}$$

Finally, we calculate the sensitivity of the option price with respect to κ, the variance of the subordinator. Using $q = \frac{\partial u}{\partial\kappa}$ and $\frac{\partial c}{\partial\kappa} = m\, e^{-r\tau}\, q$ one obtains the PIDE for q:

$$
\begin{aligned}
q_\tau(\tau,y) = {} & r\, q_y(\tau,y) \\
& + \int_{-\infty}^{\infty} \nu(dx)\big(\max(e^{y+x},1)\, q(\tau,\min(y+x,0)) - q(\tau,y) \\
& - (e^x - 1) q_y(\tau,y)\big) \\
& + \int_{-\infty}^{\infty} \varrho(dx)\big(\max(e^{y+x},1)\, u(\tau,\min(y+x,0)) - u(\tau,y) \\
& - (e^x - 1) u_y(\tau,y)\big),
\end{aligned}
$$

where

$$
\varrho(dx) = \frac{\partial}{\partial\kappa}\nu(dx) = \left(\frac{|x|}{\kappa^2\sqrt{\theta^2 + \frac{2\sigma^2}{\kappa}}} - \frac{1}{\kappa}\right)\nu(dx).
$$

The initial conditions is

$$
q(0,y) = 0,
$$

and the auxiliary side conditions are

$$
\begin{aligned}
q(\tau,y) &= 0, \quad y \to -\infty, \\
q_y(\tau,y)|_{y=0} &= q(\tau,0).
\end{aligned}
$$

5.3 Asian options

5.3.1 Derivation of the PIDE

We generalise the results from section 3.3 and derive the PIDE for a floating strike Asian call option when the underlying randomness stems from an exponential Lévy process. The payoff of such an option is

$$
\max\left(S_T - \frac{1}{T}\int_0^T S_\tau\, d\tau, 0\right).
$$

We denote the integral over the stock price process by $A_t = \int_0^t S_\tau d\tau$, which is a continuous process even when S_t is discontinuous. The discounted option price

$$\hat{c}(t, S_t, A_t) = e^{-rt} c(t, S_t, A_t)$$

is a martingale. Its dynamics are

$$\begin{aligned} d\hat{c}(t, S_t, A_t) = e^{-rt}\big(& - r\, c(t, S_{t-}, A_t)\, dt + c_t(t, S_{t-}, A_t)\, dt \\ & + c_s(t, S_{t-}, A_t)\, dS_t + \frac{1}{2} c_{ss}(t, S_{t-}, A_t)\, d[S, S]_t^c \\ & + c_a(t, S_{t-}, A_t)\, dA_t \\ & + \int_{-\infty}^{\infty} \big(c(t, S_{t-}e^x, A_t) - c(t, S_{t-}, A_t) \\ & - (e^x - 1)\, S_{t-}\, c_s(t, S_{t-}, A_t)\big)\mu(ds, dx)\big). \end{aligned} \tag{5.34}$$

We expect $c(t, s, a)$ to be differentiable in a whenever it is differentiable in s, that the condition (5.4) guarantees differentiability both in s and a, and thus the use of Itô's lemma is justified. Also condition (5.5) guarantees that the dynamics (5.34) can be split into a drift and a true martingale part. Setting the drift term to zero one obtains the PIDE for the price function

$$\begin{aligned} c_t(t, s, a) + rs\, c_s(t, s, a) + \frac{1}{2}\sigma^2 s^2\, c_{ss}(t, s, a) + s\, c_a(t, s, a) - r\, c(t, s, a) \\ + \int_{-\infty}^{\infty} \nu(dx)\, (c(t, se^x, a) - c(t, s, a) - (e^x - 1)s\, c_s(t, s, a)) = 0, \end{aligned} \tag{5.35}$$

valid for $(t, s, m) \in [0, T) \times (0, \infty) \times [0, \infty)$. The option price function can be written as the stock price times a function that depends only on time and the variable $z = \frac{1}{T}\frac{a}{s}$ which is the average value of the stock price over the lifetime of the option divided by the current value of the stock price. We write

$$c(t, s, a) = s\, w(\tau, z),$$

where $\tau = T - t$. The derivatives in the new variables w and the new parameters τ and z are

$$\begin{aligned} c_t(t,s,a) &= -s\,w_\tau(\tau,z), \\ c_s(t,s,a) &= w(\tau,z) - z\,w_z(\tau,z), \\ c_{ss}(t,s,a) &= \frac{z^2}{s}w_{zz}(\tau,z), \\ c_a(t,s,a) &= \frac{1}{T}w_z(\tau,z), \\ c(t,se^x,a) &= s\,w(\tau,ze^x). \end{aligned}$$

Inserting the equations above into equation (5.35) one obtains the PIDE

$$\begin{aligned} w_\tau(\tau,z) = {} & \frac{1}{2}\sigma^2 z^2\,w_{zz}(\tau,z) + \left(\frac{1}{T} - r\,z\right)\,w_z(\tau,z) \\ & + \int_{-\infty}^{\infty} \nu(dx)\big(w(\tau,z\,e^x) - w(\tau,z) - (e^x-1)(w(\tau,z) - z\,w_z(\tau,z))\big), \end{aligned} \tag{5.36}$$

valid for $(\tau,z) \in (0,T] \times [0,\infty)$ with the initial condition

$$w(0,z) = \max\left(1 - z, 0\right).$$

For numerical purposes we add the side conditions

$$w(\tau,z) = 0, \quad z \to \infty, \tag{5.37}$$

$$w_\tau(\tau,0) = \frac{1}{T}w_z(\tau,z)|_{z=0}. \tag{5.38}$$

One could perform an additional transformation of variables $w(\tau,z) =$

$f(\tau, y)$, where $y = \ln z$ to obtain the equation

$$\begin{aligned} f_\tau(\tau, y) &= \frac{1}{2}\sigma^2(f_{yy}(\tau, y) - f_y(\tau, y)) + \left(\frac{1}{T}e^{-y} - r\right) f_y(\tau, y) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(f(\tau, x+y) - f(\tau, y) - (e^x - 1)(f(\tau, y) - f_y(\tau, y))\big). \end{aligned} \tag{5.39}$$

On the one hand, this makes things easier as one can evaluate f on a constant grid. On the other hand, it is difficult to evaluate the above equation at the point $z = 0$ as the coefficient e^{-y} goes to infinity. The point $z = 0$, however, is interesting as it corresponds to an Asian option at $t = 0$ when the average so far is zero. For this reason we evaluate equation (5.36) instead of equation (5.39).

5.3.2 Greeks in the jump-diffusion model

To obtain the greeks we again differentiate equation (5.36) with respect to the parameters in question. First, we want to calculate vega, the sensitivity with respect to σ. We use

$$v = \frac{\partial w}{\partial \sigma}, \qquad \mathcal{V} = \frac{\partial c}{\partial \sigma} = s\, v$$

and obtain for the equation for v

$$\begin{aligned} v_\tau(\tau, z) &= \frac{1}{2}\sigma^2 z^2\, v_{zz}(\tau, z) + \left(\frac{1}{T} - rz\right) v_z(\tau, z) + \sigma z^2\, w_{zz}(\tau, z) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(v(\tau, z\, e^x) - v(\tau, z) - (e^x - 1)(v(\tau, z) - z\, v_z(\tau, z))\big), \end{aligned}$$

valid for $(\tau, z) \in (0, T] \times [0, \infty)$, subject to the initial condition

$$v(0, z) = 0,$$

and the auxiliary side conditions

$$v(\tau, z) = 0, \quad z \to \infty,$$
$$v_\tau(\tau, 0) = \frac{1}{T} v_z(\tau, z)|_{z=0}.$$

We call the derivative of the option price with respect to the jump intensity beta and use

$$b = \frac{\partial w}{\partial \lambda}, \qquad \beta = \frac{\partial c}{\partial \lambda} = s\, b.$$

The equation for $b(\tau, z)$ is

$$b_\tau(\tau, z) = \frac{1}{2}\sigma^2 z^2\, b_{zz}(\tau, z) + \left(\frac{1}{T} - rz\right) b_z(\tau, z)$$
$$+ \frac{1}{\lambda}\int_{-\infty}^{\infty} \nu(dx)\big(b(\tau, z\, e^x) - b(\tau, z) - (e^x - 1)(b(\tau, z) - z\, b_z(\tau, z))\big),$$

valid for $(\tau, z) \in (0, T] \times [0, \infty)$, with the initial condition

$$b(0, z) = 0,$$

and the auxiliary side conditions

$$b(\tau, z) = 0, \quad z \to \infty,$$
$$b_\tau(\tau, 0) = \frac{1}{T} b_z(\tau, z)|_{z=0}.$$

In the same manner one can derive the PIDE for other parameter sensitivities such as the sensitivity with respect to δ, the standard derivation of the jump distribution.

5.4 Exchange options

5.4.1 Derivation of the PIDE

Assume that the prices of two stocks are given under a martingale measure by the following dynamics

$$
\begin{aligned}
dS_t &= r\,S_{t-}\,dt + \sigma_1\,S_{t-}\,dW_t + S_{t-}\int_{-\infty}^{\infty}(e^x-1)(\mu(dt,dx)-\nu(dx)dt),\\
d\tilde{S}_t &= r\,\tilde{S}_{t-}\,dt + \sigma_2\,\tilde{S}_{t-}\,d\tilde{W}_t + \tilde{S}_{t-}\int_{-\infty}^{\infty}(e^x-1)(\tilde{\mu}(dt,dx)-\tilde{\nu}(dx)dt).
\end{aligned}
$$

The two Brownian motions are correlated with $\mathbb{E}[dW_t d\tilde{W}_t] = \rho\,dt$. For the joint jump measure, which counts the jumps both in S_t and in $\tilde{S}_t$, we write $\mu(dt,dx,dy)$ and for its expectation we write $\nu(dx,dy)\,dt = \mathbb{E}[\mu(dt,dx,dy)]$. We suppose that the stock prices are such that all the derivatives of the option price exist and thus the use of Itô's lemma in equation (5.40) can be justified. In particular we are will work in a jump-diffusion setting where no differentiability problems arise and sufficient moment conditions are in place such that the stock price process is a square integrable martingale. The price of an exchange option at time t is defined as

$$
\begin{aligned}
c(t,S_t,\tilde{S}_t) &= e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}[\max(S_T-\tilde{S}_T,0)|\mathcal{F}_t]\\
&= e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_T \max\left(\frac{S_T}{\tilde{S}_T}-1,0\right)\bigg|\mathcal{F}_t\right].
\end{aligned}
$$

In Vecer and Xu [39] change of measure techniques are used to obtain the price of a Asian option with a quite general payoff as the solution of a PIDE. We can use similar techniques to obtain a price determining PIDE for an exchange option which depends only on time and one additional state variable. To do so we introduce the process $Z_t = \frac{S_t}{\tilde{S}_t}$ and define a new measure $\tilde{\mathbb{Q}}$ by

$$
\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{\tilde{S}_t}{\tilde{S}_0\,e^{r\,t}}.
$$

Performing a change of measure, the option price can be written as

$$
\begin{aligned}
c(t, S_t, \tilde{S}_t) &= e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_T \max\left(\frac{S_T}{\tilde{S}_T} - 1, 0\right)\middle|\mathcal{F}_t\right] \\
&= e^{-r\,(T-t)}\frac{\tilde{S}_t}{\tilde{S}_0\, e^{r\,t}}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\tilde{S}_T \max\left(\frac{S_T}{\tilde{S}_T} - 1, 0\right)\frac{\tilde{S}_0\, e^{r\,T}}{\tilde{S}_T}\middle|\mathcal{F}_t\right] \\
&= \tilde{S}_t\,\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\max\left(\frac{S_T}{\tilde{S}_T} - 1, 0\right)\middle|\mathcal{F}_t\right] \\
&= \tilde{S}_t\,\mathbb{E}^{\tilde{\mathbb{Q}}}[\max(Z_T - 1, 0)|\mathcal{F}_t].
\end{aligned}
$$

We now show that the process Z_t is a local martingale under $\tilde{\mathbb{Q}}$. Using that the discounted stock price $e^{-r\,t}\,S_t$ is a $\mathbb{Q}$ martingale we obtain

$$
\mathbb{E}^{\tilde{\mathbb{Q}}}[Z_t|\mathcal{F}_u] = \frac{\tilde{S}_0\, e^{r\,u}}{\tilde{S}_u}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{\tilde{S}_t}\frac{\tilde{S}_t}{\tilde{S}_0\, e^{r\,t}}\middle|\mathcal{F}_u\right] = \frac{e^{r\,u}}{\tilde{S}_u}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\,t}\,S_t|\mathcal{F}_u\right] = \frac{S_u}{\tilde{S}_u} = Z_u.
$$

For notational convenience we introduce

$$
\begin{aligned}
r_1 &= r - \int_{-\infty}^{\infty}(e^x - 1)\nu(dx), \\
r_2 &= r - \int_{-\infty}^{\infty}(e^x - 1)\tilde{\nu}(dx).
\end{aligned}
$$

Let us write the process Z_t as a function f of S_t and $\tilde{S}_t$: $Z_t = f(S_t, \tilde{S}_t)$. Calculating the dynamics of Z_t one obtains

$$
\begin{aligned}
dZ_t = {} & f_s(t, S_{t-}, \tilde{S}_{t-})\, dS_t + f_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\, d\tilde{S}_t \\
& + \frac{1}{2} f_{\tilde{s}\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\, d[\tilde{S}, \tilde{S}]_t^c + f_{s\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\, d[S, \tilde{S}]_t^c \\
& + \frac{S_t}{\tilde{S}_t} - \frac{S_{t-}}{\tilde{S}_{t-}} - \Delta S_t f_s(t, S_{t-}, \tilde{S}_{t-}) - \Delta\tilde{S}_t f_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-}) \\
= {} & \frac{1}{\tilde{S}_{t-}}\, dS_t - \frac{S_{t-}}{(\tilde{S}_{t-})^2}\, d\tilde{S}_t + \frac{S_{t-}}{\tilde{S}_{t-}}\sigma_2^2\, dt - \frac{S_{t-}}{\tilde{S}_{t-}}\rho\sigma_1\sigma_2\, dt \\
& + \frac{S_t}{\tilde{S}_t} - \frac{S_{t-}}{\tilde{S}_{t-}} - \Delta S_t \frac{1}{\tilde{S}_{t-}} + \Delta\tilde{S}_t \frac{S_{t-}}{(\tilde{S}_{t-})^2}.
\end{aligned}
\tag{5.40}
$$

The jumps of the process Z_t can be expressed as

$$\frac{S_t}{\tilde{S}_t} - \frac{S_{t-}}{\tilde{S}_{t-}} = Z_{t-} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{x-y} - 1) \mu(dt, dx, dy). \tag{5.41}$$

Inserting equation (5.41) into the equation for the dynamics (5.40) one obtains

$$\begin{aligned} dZ_t = Z_{t-} \Big(& r_1\, dt + \sigma_1\, dW_t - r_2\, dt - \sigma_2\, d\tilde{W}_t + \sigma_2^2\, dt - \rho\sigma_1\sigma_2\, dt \\ & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{x-y} - 1) \mu(dt, dx, dy) \Big). \end{aligned} \tag{5.42}$$

By definition the value function $v(t, Z_t) = \mathbb{E}^{\tilde{\mathbb{Q}}}[\max(Z_T - 1, 0) | \mathcal{F}_t]$ is a martingale. Therefore we calculate its dynamics and set the drift term to zero to obtain the governing PIDE. For the dynamics of v we obtain

$$\begin{aligned} dv_t &= v_t(t, Z_{t-})\, dt + v_z(t, Z_{t-})\, dZ_t + \frac{1}{2} v_{zz}(t, Z_{t-})\, d[Z, Z]_t^c \\ &\quad + v(t, Z_t) - v(t, Z_{t-}) - \Delta Z_{t-}\, v_z(t, Z_{t-}) \\ &= v_t(t, Z_{t-})\, dt + v_z(t, Z_{t-})\, dZ_t + \frac{1}{2} v_{zz}(t, Z_{t-})(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) Z_{t-}^2\, dt \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \big(v(t, Z_{t-} e^{x-y}) - v(t, Z_{t-}) \\ &\quad - Z_{t-}(e^{x-y} - 1) v_z(t, Z_{t-}) \big) \mu(dt, dx, dy). \end{aligned}$$

To be able to reduce the dimension we have to specify the joint jump distribution of S_t and $\tilde{S}_t$. As an example, assume that the jump size distribution of S_t is normal $N(m_1, \delta_1)$, the jump size distribution of $\tilde{S}_t$ is normal $N(m_2, \delta_2)$, and their correlation coefficient is ϱ. The distribution of the difference of jumps in S_t and in $\tilde{S}_t$ is then also normal

$$\xi \sim N(m_1 - m_2, \delta_1^2 - 2\varrho\delta_1\delta_2 + \delta_2^2). \tag{5.43}$$

The dynamics of v can be written as

$$\begin{aligned} dv(t, Z_{t-}) = {} & v_t(t, Z_{t-})\,dt + v_z(t, Z_{t-})\,dZ_t \\ & + \frac{1}{2} v_{zz}(t, Z_{t-})(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) Z_{t-}^2\,dt \\ & + \int_{-\infty}^{\infty} \big(v(t, Z_{t-}\,e^{\xi}) - v(t, Z_{t-}) \\ & - Z_{t-}(e^{\xi} - 1) v_z(t, Z_{t-})\big)\mu(dt, d\xi). \end{aligned} \tag{5.44}$$

The drift term in (5.44) has to be zero almost surely. Equating it to zero we obtain the PIDE

$$\begin{aligned} v_t(t, z) & + \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) z^2\, v_{zz}(t, z) \\ & + \int_{-\infty}^{\infty} \big(v(t, z\,e^{\xi}) - v(t, z) - z(e^{\xi} - 1)\big)\nu(d\xi) = 0, \end{aligned}$$

valid for $(t, z) \in [0, T) \times (0, \infty)$. The option price function at time t is then given by

$$c(t, s, \tilde{s}) = \tilde{s}\, v(t, z).$$

We perform the change of variables $y = \ln z$ to obtain constant coefficients and constant grid size. In addition we change the time to $\tau = T - t$. The terms in the PIDE then change to

$$\begin{aligned} v(t, z) &= u(\tau, y) \\ v(t, z\,e^{\xi}) &= u(\tau, y + \xi) \\ v_t(t, z) &= -u_\tau(\tau, y) \\ v_y(t, z) &= \frac{1}{y} u_y(\tau, y) \\ v_{yy}(t, z) &= \frac{1}{y^2}(u_{yy}(\tau, y) - u_y(\tau, y)). \end{aligned}$$

In the new coordinates the price determining PIDE is

$$u_\tau(\tau, y) = \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)(u_{yy}(\tau, y) - u_y(\tau, y)) + \int_{-\infty}^{\infty} \nu(d\xi)\big(u(\tau, y+\xi) - u(\tau, y) - (e^\xi - 1)u_y(\tau, y)\big), \quad (5.45)$$

valid for $(\tau, y) \in (0, T] \times (-\infty, \infty)$, subject to the terminal condition

$$u(T, y) = \max(e^y - 1, 0)$$

and the auxiliary boundary conditions

$$u(\tau, y) = 0, \quad y \to -\infty,$$
$$u(\tau, y) = e^y - 1, \quad y \to \infty.$$

5.4.2 A martingale method

Alternatively one can derive a PIDE whose solution is the discounted price of an exchange option using a martingale technique. Consider the martingale

$$\begin{aligned} M_t &= \mathbb{E}\left[\max\left(S_T - \tilde{S}_T, 0\right)\Big| \mathcal{F}_t\right] \\ &= \tilde{S}_t\,\mathbb{E}\left[\max\left(Z_t\frac{S_T}{S_t} - \frac{\tilde{S}_T}{\tilde{S}_t}, 0\right)\Big| \mathcal{F}_t\right] \\ &= \tilde{S}_t\, g(t, Z_t), \end{aligned}$$

where $Z_t = \frac{S_t}{\tilde{S}_t} = f(S_t, \tilde{S}_t)$. Applying Itô's lemma to Z_t we obtain its dynamics as in equation (5.42):

$$dZ_t = Z_{t-}\Big(r_1\,dt + \sigma_1\,dW_t - r_2\,dt - \sigma_2\,d\tilde{W}_t + \sigma_2^2\,dt - \rho\sigma_1\sigma_2\,dt + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{x-y} - 1)\mu(dt, dx, dy)\Big).$$

Applying Itô's lemma to the martingale M_t and assuming that S_t and $\tilde{S}_t$ have finite variation we obtain the dynamics

$$\begin{aligned}
dM_t = \tilde{S}_{t-}\Big((r_2\,dt + \sigma_2\,dW_t)g(t, Z_{t-}) + g_t(t, Z_{t-})\,dt \\
+ g_z(t, Z_{t-})\,Z_{t-}\,(r_1\,dt + \sigma_1\,dW_t - r_2\,dt - \sigma_2\,d\tilde{W}_t + \sigma_2^2\,dt - \rho\sigma_1\sigma_2\,dt) \\
+ \frac{1}{2}g_{zz}(t, Z_{t-})\,Z_{t-}^2(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)dt \\
+ (-\sigma_2^2 + \rho\sigma_1\sigma_2)Z_{t-}\,g_z(t, Z_{t-})\,dt\Big) + \tilde{S}_t\,g(t, Z_t) - \tilde{S}_{t-}\,g(t, Z_{t-}).
\end{aligned} \tag{5.46}$$

Setting the drift term in equation (5.46) to zero, one obtains the price determining PIDE:

$$\begin{aligned}
g_t(t,z) + r\,g(t,z) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)z^2\,g_{zz}(t,z) \\
+ \int_{-\infty}^{\infty}(e^y\,g(t, z\,e^{x-y}) - e^y\,g(t,z) + (e^x - e^y)z\,g_z(t,z))\nu(dx,dy) = 0,
\end{aligned}$$

valid for $t, z) \in [0,T) \times (0,\infty)$, subject to the terminal condition

$$g(T,z) = \max(z - 1, 0)$$

and the auxiliary boundary conditions

$$\begin{aligned}
g(t,0) &= 0 \\
g(t,z) &= z - 1, \quad z \to \infty.
\end{aligned}$$

5.4.3 Greeks in the jump-diffusion model

We are interested in the sensitivity of the option price with respect to changes in the correlation coefficients. First, we investigate how the price changes if the correlation coefficient between the two Brownian motions changes. Thus, denoting $R = \frac{\partial u}{\partial \rho}$ we differentiate equation (5.45) with respect to the correlation coefficient ρ to obtain the PIDE

$$\begin{aligned}
R_\tau(\tau,y) &= \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)(R_{yy}(\tau,y) - R_y(\tau,y)) \\
&\quad + \sigma_1\sigma_2(u_y(\tau,y) - u_{yy}(\tau,y)) \\
&\quad + \int_{-\infty}^{\infty} \nu(d\xi)\big(R(\tau,y+\xi) - R(\tau,y) - (e^\xi - 1)R_y(\tau,y)\big),
\end{aligned}$$

valid for $(\tau,y) \in (0,T] \times (-\infty,\infty)$, with $R = 0$ at all boundaries. The sensitivity of the option price $\frac{\partial c}{\partial \rho} = \tilde{s}R$ is then obtained by multiplying the value $\tilde{s}$ of the stock price $\tilde{S}_t$ times $R(\tau,y)$.

Secondly, we investigate the change of the option price with respect to changes of the correlation of the jumps of the two stocks. Recall from (4.2) the Lévy measure of the Merton model

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}} dx,$$

and from (5.43) that $\mu = m_1 - m_2$ and $\delta = \delta_1^2 - 2\varrho\delta_1\delta_2 + \delta_2^2$. We denote $Q = \frac{\partial u}{\partial \varrho}$. To obtain the PIDE for Q, we differentiate equation (5.45) with respect ϱ and obtain

$$\begin{aligned}
Q_\tau(\tau,y) &= \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)(Q_{yy}(\tau,y) - Q_y(\tau,y)) \\
&\quad + \int_{-\infty}^{\infty} \nu(d\xi)\big(Q(\tau,y+\xi) - Q(\tau,y) - (e^\xi - 1)Q_y(\tau,y)\big) \\
&\quad + \int_{-\infty}^{\infty} \nu(d\xi)\frac{2\delta_1\delta_2}{\delta}\left(1 - \frac{(\xi - m_1 + m_2)^2}{\delta^2}\right)\big(u(\tau,y+\xi) - u(\tau,y) \\
&\quad - (e^\xi - 1)u_y(\tau,y)\big),
\end{aligned}$$

valid for $(\tau,y) \in (0,T] \times (-\infty,\infty)$, with $Q = 0$ at all boundaries. The sensitivity $\frac{\partial c}{\partial \varrho} = \tilde{s}Q$ of the option price with respect to changes in ϱ is then obtained by multiplying the value $\tilde{s}$ of the stock price $\tilde{S}_t$ times $Q(\tau,x)$.

Chapter 6

Two factor models and model risk

6.1 Basket option

We are interested in finding the price of a basket option with payoff $\max(\alpha S_T + (1-\alpha)\tilde{S}_T - K, 0)$ on two stocks which are given as the exponential Lévy processes $S_t = e^{r\,t+X_t}$ and $\tilde{S}_t = e^{r\,t+\tilde{X}_t}$. Two dimensional option models driven by Lévy processes are considered in Clift and Forsyth [13]. The characteristic triplet of the process X_t is $(\sigma_1^2, \nu, \gamma_1)$ and the one of the process $\tilde{X}_t$ is $(\sigma_2^2, \tilde{\nu}, \gamma_2)$. The drift terms γ_1 and γ_2 are chosen such that the discounted stock prices are martingales. The parameter α takes a value in the unit interval. This basket option is an option on an asset which is the weighted sum of the two stock processes. We are interested in how the option price changes if one moves from one underlying process S_t to the other $\tilde{S}_t$ and therefore from $\alpha = 1$ to $\alpha = 0$. The dynamics of the two stock prices are

$$dS_t = r\,S_{t-}\,dt + \sigma_1\,S_{t-}\,dW_t + S_{t-}\int_{-\infty}^{\infty}(e^x - 1)(\mu(dt,dx) - \nu(dx)dt), \quad (6.1)$$

$$d\tilde{S}_t = r\,\tilde{S}_{t-}\,dt + \sigma_2\,\tilde{S}_{t-}\,d\tilde{W}_t + \tilde{S}_{t-}\int_{-\infty}^{\infty}(e^x - 1)(\tilde{\mu}(dt,dx) - \tilde{\nu}(dx)dt), \quad (6.2)$$

where r is the interest rate, $\mu(dt, dx)$ and $\tilde{\mu}(dt, dx)$ are the jump measures of the two stock prices, and the rest of the parameters come from the characteristic triplets of X_t and $\tilde{X}_t$. The two Brownian motions are correlated with $\rho\, dt = \mathbb{E}[dW_t\, d\tilde{W}_t]$. We introduce the joint jump measure $\mu(dt, dx, dy)$ which counts the jumps of size dx and dy in the time interval dt, and the two dimensional Lévy measure $\nu(dx, dy)\, dt = \mathbb{E}[\mu(dt, dx, dy)]$. We suppose that the stock prices are such that all the derivatives of the option price exist and thus the use of Itô's lemma can be justified. The price function of the basket option at time t is

$$c(t, s, \tilde{s}) = e^{-r(T-t)}\, \mathbb{E}\left[\max(\alpha S_T + (1-\alpha)\tilde{S}_T - K, 0) \Big| \, S_t = s, \tilde{S}_t = \tilde{s}\right],$$

The discounted option price

$$\hat{c}(t, S_t, \tilde{S}_t) = e^{-rt}\, c(t, S_t, \tilde{S}_t) \tag{6.3}$$

is a martingale. Applying Itô's lemma to equation (6.3) we obtain the dynamics of the discounted option price

$$\begin{aligned} d\hat{c}(t, S_{t-}, \tilde{S}_{t-}) =& - re^{-rt}c(t, S_{t-}, \tilde{S}_{t-})dt + e^{-rt}dc(t, S_{t-}, \tilde{S}_{t-}) \\ =& \, e^{-rt}\Big(- r\, c(t, S_{t-}, \tilde{S}_{t-})dt + c_t(t, S_{t-}, \tilde{S}_{t-})dt \\ &+ c_s(t, S_{t-}, \tilde{S}_{t-})dS_t + c_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})d\tilde{S}_t \\ &+ \frac{1}{2}c_{ss}(t, S_{t-}, \tilde{S}_{t-})d[S, S]_t^c + \frac{1}{2}c_{\tilde{s}\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})d[\tilde{S}, \tilde{S}]_t^c \\ &+ c_{s\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})d[S, \tilde{S}]_t^c + c(t, S_t, \tilde{S}_t) - c(t, S_{t-}, \tilde{S}_{t-}) \\ &- \Delta S_t\, c_s(t, S_{t-}, \tilde{S}_{t-}) - \Delta\tilde{S}_t\, c_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\Big), \end{aligned} \tag{6.4}$$

where $\Delta S_t = S_t - S_{t-}$ and $\Delta\tilde{S}_t = \tilde{S}_t - \tilde{S}_{t-}$. Inserting equation (6.1) and (6.2) into (6.4) and expressing the jumps as integrals over the jump measures, we can write the increment $d\hat{c}(t, S_t, \tilde{S}_t)$ in the discounted option price as

$$\begin{aligned}
d\hat{c}(t, S_{t-}, \tilde{S}_{t-}) = e^{-rt}\Big(& - r\, c(t, S_{t-}, \tilde{S}_{t-})dt + c_t(t, S_{t-}, \tilde{S}_{t-})dt \\
& + c_s(t, S_{t-}, \tilde{S}_{t-})\Big(rS_{t-}\, dt + \sigma_1 S_{t-}\, dW_t \\
& + S_{t-}\int_{-\infty}^{\infty}(e^x - 1)(\mu(dt, dx) - \nu(dx)dt)\Big) \\
& + c_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\Big(r\tilde{S}_{t-}\, dt + \sigma_2 S_{t-}\, d\tilde{W}_t \\
& + \tilde{S}_{t-}\int_{-\infty}^{\infty}(e^x - 1)(\tilde{\mu}(dt, dx) - \tilde{\nu}(dx)dt)\Big) \\
& + \frac{1}{2}c_{ss}(t, S_{t-}, \tilde{S}_{t-})d[S, S]_t^c + \frac{1}{2}c_{\tilde{s}\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})d[\tilde{S}, \tilde{S}]_t^c \\
& + c_{s\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})d[S, \tilde{S}]_t^c \\
& + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(c(t, S_{t-}e^x, \tilde{S}_{t-}e^y) - c(t, S_{t-}, \tilde{S}_{t-}) \\
& - S_{t-}(e^x - 1)\, c_s(t, S_{t-}, \tilde{S}_{t-}) \\
& - \tilde{S}_{t-}(e^y - 1)\, c_{\tilde{s}}(t, S_{t-}, \tilde{S}_{t-})\big)\mu(dt, dx, dy)\Big).
\end{aligned} \tag{6.5}$$

We split equation (6.5) into a drift and a martingale part and set the martingale part to zero and obtain the PIDE

$$\begin{aligned}
& c_t(t, s, \tilde{s}) - r\, c(t, s, \tilde{s}) + rs\, c_s(t, s, \tilde{s}) + r\tilde{s}\, c_{\tilde{s}}(t, s, \tilde{s}) \\
& + \frac{1}{2}\sigma_1^2 s^2\, c_{ss}(t, s, \tilde{s}) + \frac{1}{2}\sigma_2^2 \tilde{s}^2\, c_{\tilde{s}\tilde{s}}(t, s, \tilde{s}) + \rho\sigma_1\sigma_2 s\tilde{s}\, c_{s\tilde{s}}(t, s, \tilde{s}) \\
& + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(c(t, se^x, \tilde{s}e^y) - c(t, s, \tilde{s}) - s(e^x - 1)\, c_s(t, s, \tilde{s}) \\
& - \tilde{s}(e^y - 1)\, c_{\tilde{s}}(t, s, \tilde{s})\big)\nu(dx, dy) = 0,
\end{aligned} \tag{6.6}$$

for $(t, s, \tilde{s}) \in [0, T) \times (0, \infty) \times (0, \infty)$. The terminal condition accompanying (6.6) is

$$c(T, s, \tilde{s}) = \max(\alpha s + (1 - \alpha)\tilde{s} - K, 0).$$

In order to have a linear dependency in the arguments of the option price we perform the following transformations of variables

$$
\begin{aligned}
\tau &= T - t, \\
l &= \ln\left(\frac{s}{K}\right) + r\tau, \\
\tilde{l} &= \ln\left(\frac{\tilde{s}}{K}\right) + r\tau,
\end{aligned}
$$

and then introduce the transformed variables $u(\tau, l, \tilde{l})$ and its derivatives which are defined by

$$
\begin{aligned}
c(t, s, \tilde{s}) &= e^{-r\tau} K\, u(\tau, l, \tilde{l}), \\
c(t, se^{x}, \tilde{s}e^{y}) &= e^{-r\tau}\, K\, u(\tau, l + x, \tilde{l} + y), \\
c_t(t, s, \tilde{s}) &= e^{-r\tau} K (r\, u(\tau, l, \tilde{l}) - u_\tau(\tau, l, \tilde{l})), \\
c_s(t, s, \tilde{s}) &= e^{-r\tau} \frac{K}{s} u_l(\tau, l, \tilde{l}), \\
c_{ss}(t, s, \tilde{s}) &= e^{-r\tau} \frac{K}{s^2} (u_{ll}(\tau, l, \tilde{l}) - u_l(\tau, l, \tilde{l})), \\
c_{\tilde{s}}(t, s, \tilde{s}) &= e^{-r\tau} \frac{K}{\tilde{s}} u_{\tilde{l}}(\tau, l, \tilde{l}), \\
c_{\tilde{s}\tilde{s}}(t, s, \tilde{s}) &= e^{-r\tau} \frac{K}{\tilde{s}^2} (u_{\tilde{l}\tilde{l}}(\tau, l, \tilde{l}) - u_l(\tau, l, \tilde{l})), \\
c_{s\tilde{s}}(t, s, \tilde{s}) &= e^{-r\tau} \frac{K}{s\tilde{s}} u_{l\tilde{l}}(\tau, l, \tilde{l}).
\end{aligned}
$$

In the new variables the PIDE (6.6) turns into

$$
\begin{aligned}
&-u_\tau(\tau, l, \tilde{l}) + r\, u_l(\tau, l, \tilde{l}) + r\, u_{\tilde{l}}(\tau, l, \tilde{l}) + \frac{1}{2}\sigma_1^2 \big(u_{ll}(\tau, l, \tilde{l}) - u_l(\tau, l, \tilde{l})\big) \\
&\quad + \frac{1}{2}\sigma_2^2 \big(u_{\tilde{l}\tilde{l}}(\tau, l, \tilde{l}) - u_{\tilde{l}}(\tau, l, \tilde{l})\big) + \rho\sigma_1\sigma_2\, u_{l\tilde{l}}(\tau, l, \tilde{l}) \\
&\quad + \int_{-\infty}^{\infty} \big(u(\tau, l + x, \tilde{l} + y) - u(\tau, l, \tilde{l}) \\
&\quad - (e^{x} - 1)\, u_l(\tau, l, \tilde{l}) - (e^{y} - 1)\, u_{\tilde{l}}(\tau, l, \tilde{l})\big) \nu(dx, dy) = 0, \qquad (6.7)
\end{aligned}
$$

valid for $(t,l,\tilde{l}) \in (0,T]\times(-\infty,\infty)\times(-\infty,\infty)$, subject to the initial condition

$$u(0,l,\tilde{l}) = \max(\alpha e^l + (1-\alpha)e^{\tilde{l}} - 1, 0). \tag{6.8}$$

We are interested in how much the option price changes if one moves from one underlying process S_t to the other $\tilde{S}_t$, and thus how sensitive the option is with respect to changes in α. Therefore, we differentiate the PIDE (6.7) and the side condition (6.8) with respect to α and obtain a system of PIDE for the price and the sensitivity that can be solved simultaneously. Note that the parameter α only appears in the side conditions. Let $\phi = \frac{\partial u}{\partial \alpha}$ be the derivative of $u(\tau,l,\tilde{l})$ with respect to α. The PIDE for $\phi(\tau,l,\tilde{l})$ is then

$$\begin{aligned}
&-\phi_\tau(\tau,l,\tilde{l}) + r\,\phi_l(\tau,l,\tilde{l}) + r\,\phi_{\tilde{l}}(\tau,l,\tilde{l}) + \frac{1}{2}\sigma_1^2\big(\phi_{ll}(\tau,l,\tilde{l}) - \phi_l(\tau,l,\tilde{l})\big)\\
&+\frac{1}{2}\sigma_2^2\big(\phi_{\tilde{l}\tilde{l}}(\tau,l,\tilde{l}) - \phi_{\tilde{l}}(\tau,l,\tilde{l})\big) + \rho\sigma_1\sigma_2\,\phi_{l\tilde{l}}(\tau,l,\tilde{l})\\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(\phi(\tau,l+x,\tilde{l}+y) - \phi(\tau,l,\tilde{l}) - (e^x-1)\,\phi_l(\tau,l,\tilde{l})\\
&-(e^y-1)\,\phi_{\tilde{l}}(\tau,l,\tilde{l})\big)\nu(dx,dy) = 0,
\end{aligned}$$

valid for $(t,l,\tilde{l}) \in (0,T]\times(-\infty,\infty)\times(-\infty,\infty)$, subject to the initial condition

$$\phi(0,l,\tilde{l}) = (e^l - e^{\tilde{l}})\mathbf{1}_{\{\alpha e^l + (1-\alpha)e^{\tilde{l}} > 1\}}.$$

In general it is not straightforward to specify the joint jump measure and hence the Lévy measure $\nu(dx,dy)$ of the two processes S_t and $\tilde{S}_t$. One has to make some assumptions about the correlation between the jumps of S_t and $\tilde{S}_t$. One possible assumption which leads to a tractable model is that the jumps sizes are bivariate normally distributed. The Lévy measure is then

$$\nu(dx,dy) = \frac{\lambda}{2\pi\delta_1\delta_2\sqrt{1-\rho^2}}e^{-\frac{1}{2}Q(x,y)}dx\,dy,$$

with

$$Q(x,y) = \frac{1}{1-\rho^2}\left(\left(\frac{x-\mu_1}{\delta_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\delta_1}\right)\left(\frac{y-\mu_2}{\delta_2}\right) + \left(\frac{y-\mu_2}{\delta_2}\right)^2\right) dx\, dy.$$

6.2 Model sensitivity - exponential mixing

We are still interested in the sensitivity of an option price with respect to moving from one underlying process to another. Instead of looking at a model with two exponential Lévy processes we now investigate a model which has two sources of randomness in one exponent. To start with, we want to find the price of an option whose underlying stock price is the exponential of a mixture of the two independent Lévy processes X_t with characteristic triplet $(\sigma_1, \nu, \gamma_1)$ and $\tilde{X}_t$ with characteristic triplet $(\sigma_2, \tilde{\nu}, \gamma_2)$. The stock price in this model is given by

$$S_t = S_0\, e^{r\,t + \alpha\, X_t + (1-\alpha)\,\tilde{X}_t}.$$

The Brownian motion parts and the jump parts of the two driving processes X_t and $\tilde{X}_t$ do not explicitly depend on α. The drifts γ_1 and γ_2 however are chosen such that the processes $e^{\alpha\, X_t}$ and $e^{(1-\alpha)\,\tilde{X}_t}$ are martingales and thus usually depend on α. As X_t and $\tilde{X}_t$ are independent it then follows that also the discounted stock price process $e^{\alpha\, X_t + (1-\alpha)\,\tilde{X}_t}$ is a martingale. The drifts γ_1 and γ_2 can be obtained by solving the following equations

$$\alpha\gamma_1 + \frac{\alpha^2\sigma_1^2}{2} + \int_{-\infty}^{\infty} (e^{\alpha x} - 1 - \alpha x 1_{|x|<1})\nu(dx) = 0,$$

$$(1-\alpha)\gamma_2 + \frac{(1-\alpha)^2\sigma_2^2}{2} + \int_{-\infty}^{\infty} (e^{(1-\alpha)x} - 1 - (1-\alpha)x 1_{|x|<1})\tilde{\nu}(dx) = 0.$$

The dynamics of the stock price is

$$dS_t = S_{t-}\Big(r\,dt + \alpha\sigma_1\,dW_t + \int_{-\infty}^{\infty}(e^{\alpha x}-1)(\mu(dt,dx)-\nu(dx)dt)$$
$$+ (1-\alpha)\sigma_2\,d\tilde{W}_t + \int_{-\infty}^{\infty}(e^{(1-\alpha)x}-1)(\tilde{\mu}(dt,dx)-\tilde{\nu}(dx)dt)\Big), \quad (6.9)$$

where $\mu(dt,dx)$ and $\tilde{\mu}(dt,dx)$ are the jump measures of the processes X_t and $\tilde{X}_t$ respectively and the other parameters all occur in the characteristic triplets of the two processes. The price $c(t,S_t)$ of a European call option on a stock with a price process S_t is

$$c(t,S_t) = e^{-r\,(T-t)}\,\mathbb{E}[\max(S_T-K,0)|\mathcal{F}_t].$$

The discounted option price $\hat{c}(t,S_t) = e^{-r\,t}c(t,S_t)$ is a martingale and its dynamics is

$$d\hat{c}(t,S_t) = e^{-r\,t}\Big(-rc(t,S_{t-})\,dt + c_t(t,S_{t-})\,dt + c_s(t,S_{t-})\,dS_t$$
$$+\frac{1}{2}c_{ss}(t,S_{t-})\,d[S,S]^c_t + c(t,S_t) - c(t,S_{t-}) - \Delta S_t\,c_s(t,S_{t-})\Big), \quad (6.10)$$

where $\Delta S_t = S_t - S_{t-}$ are the jumps of the stock price process. As X_t and $\tilde{X}_t$ are independent we can reformulate equation (6.10) to

$$d\hat{c}(t,S_t) = e^{-rt}\Big(-rc(t,S_{t-})\,dt + c_t(t,S_{t-})\,dt + c_s(t,S_{t-})\,dS_t$$
$$+\frac{1}{2}c_{ss}(t,S_{t-})\,d[S,S]^c_t$$
$$+\int_{-\infty}^{\infty}(c(t,e^{\alpha x}S_{t-}) - c(t,S_{t-}) - S_{t-}(e^{\alpha x}-1)c_s(t,S_{t-}))\,\mu(dt,dx)$$
$$+\int_{-\infty}^{\infty}(c(t,e^{(1-\alpha)y}S_{t-}) - c(t,S_{t-})$$
$$- S_{t-}(e^{(1-\alpha)y}-1)c_s(t,S_{t-}))\,\tilde{\mu}(dt,dy)\Big). \quad (6.11)$$

Inserting equation (6.9) into equation (6.11) and using $d[S,S]^c_t = S^c_{t-}(\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2)\,dt$ we obtain

$$
\begin{aligned}
d\hat{c}(t,S_t) = e^{-rt}\Big(&- r\,c(t,S_{t-}) + c_t(t,S_{t-}) + c_s(t,S_{t-})S_{t-}\Big(r\,dt \\
&+ \alpha\sigma_1\,dW_t + \int_{-\infty}^{\infty}(e^{\alpha x}-1)(\mu(dt,dx)-\nu(dx)dt) \\
&+ (1-\alpha)\sigma_2\,d\tilde{W}_t + \int_{-\infty}^{\infty}(e^{(1-\alpha)x}-1)(\tilde{\mu}(dt,dx)-\tilde{\nu}(dx)dt)\Big) \\
&+ \frac{1}{2}c_{ss}(t,S_{t-})S_{t-}^2\,(\alpha^2\sigma_1^2+(1-\alpha)^2\sigma_2^2)\,dt \\
&+ \int_{-\infty}^{\infty}\big(c(t,e^{\alpha x}S_{t-}) - c(t,S_{t-}) - S_{t-}(e^{\alpha x}-1)\,c_s(t,S_{t-})\big)\mu(dt,dx) \\
&+ \int_{-\infty}^{\infty}\big(c(t,e^{(1-\alpha)y}S_{t-}) - c(t,S_{t-}) \\
&- S_{t-}(e^{(1-\alpha)y}-1)\,c_s(t,S_{t-})\big)\tilde{\mu}(dt,dy)\Big)
\end{aligned} \tag{6.12}
$$

Setting the drift term to zero, one obtains the PIDE

$$
\begin{aligned}
c_t(t,s) - r\,c(t,s) + rs\,c_s(t,s) &+ \big(\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2\big)s^2 c_{ss}(t,s) \\
&+ \int_{-\infty}^{\infty}\big(c(t,s\,e^{\alpha x}) - c(t,s) - s\,(e^{\alpha x}-1)\,c_s(t,s)\big)\nu(dx) \\
&+ \int_{-\infty}^{\infty}\big(c(t,s\,e^{(1-\alpha)y}) - c(t,s) - s\,(e^{(1-\alpha)y}-1)\,c_s(t,s)\big)\tilde{\nu}(dy) = 0,
\end{aligned} \tag{6.13}
$$

valid for $(t,s) \in [0,T) \times (0,\infty)$, subject to the side condition

$$c(T,s) = \max(s-K,0).$$

Differentiating (6.13) with respect to α and writing $\frac{\partial c}{\partial \alpha} = \phi$, we obtain the PIDE for the sensitivity:

$$\begin{aligned}
&\phi_t(t,s) - r\,\phi(t,s) + rs\,\phi_s(t,s) + (\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2)s^2\,\phi_{ss}(t,s) \\
&\quad + (2\alpha\sigma_1^2 + 2(\alpha-1)\sigma_2^2)s^2\,c_{ss}(t,s) \\
&\quad + \int_{-\infty}^{\infty} \big(\phi(t,s\,e^{\alpha x}) + x\,c(t,s\,e^{\alpha x}) - \phi(t,s) \\
&\quad - s\,(e^{\alpha x}-1)\phi_s(t,s) - s\,x\,e^{\alpha x}\,c_s(t,s)\big)\nu(dx) \\
&\quad + \int_{-\infty}^{\infty} \big(\phi(t,s\,e^{(1-\alpha)y}) - y\,c(t,s\,e^{(1-\alpha)y}) - \phi(t,s) \\
&\quad - s\,(e^{(1-\alpha)y}-1)\phi_s(t,s) + s\,y\,e^{(1-\alpha)y}\,c_s(t,s)\big)\tilde{\nu}(dy) = 0,
\end{aligned}$$

valid for $(t,s) \in [0,T) \times (0,\infty)$, subject to the side condition

$$\phi(T,s) = 0.$$

We want to compute the sensitivity in this framework for the very simple example where the first Lévy process is $X_t = \sigma\,W_t - \frac{\alpha\sigma^2}{2}t$ is driven by a Brownian motion W_t with volatility σ and the second process $\tilde{X}_t = N_t - \frac{\lambda\,(e^{(1-\alpha)}-1)}{1-\alpha}t$ is driven by a Poisson process N_t with intensity λ. The stock price process in this example is given by

$$dS_t = S_{t-}\,\left(r\,dt + \alpha\sigma\,dW_t + \left(e^{(1-\alpha)}-1\right)(dN_t - \lambda dt)\right).$$

As we are dealing with a Poisson process with a fixed jump size the PIDE reduces to a PDE as is given by

$$\begin{aligned}
&c_t(t,s) - r\,c(t,s) + rs\,c_s(t,s) + \frac{1}{2}\alpha^2\sigma^2 s^2\,c_{ss}(t,s) \\
&\quad + \lambda\left(c(t,s\,e^{(1-\alpha)}) - c(t,s) - s\,(e^{(1-\alpha)}-1)c_s(t,s)\right) = 0, \qquad (6.14)
\end{aligned}$$

valid for $(t,s) \in [0,T) \times (0,\infty)$, subject to the terminal condition

$$c(T,s) = \max(s-K,0).$$

Differentiating (6.14) with respect to the mixing parameter α, we obtain the PDE

$$\begin{aligned}\phi_t(t,s) - r\,\phi(t,s) + rs\,\phi_s(t,s) + \alpha^2\sigma_1^2\,s^2\,\phi_{ss}(t,s) + 2\alpha\sigma_1^2\,s^2\,c_{ss}(t,s)\\ + \lambda\big(\phi(t,s\,e^{(1-\alpha)}) - c(t,s\,e^{(1-\alpha)}) - \phi(t,s)\\ - s\,(e^{(1-\alpha)}-1)\phi_s(t,s) + s\,e^{(1-\alpha)}\,c_s(t,s)\big) = 0\end{aligned}$$

valid for $(t,s) \in [0,T) \times (0,\infty)$, subject to the terminal condition

$$\phi(T,s) = 0.$$

Chapter 7

Numerical solution of PIDE

7.1 Finite difference approximations for vanilla options

To numerically evaluate the systems of PIDE that we have derived in the previous chapters, we use finite difference methods. These methods are based on replacing derivatives with finite differences and replacing the integral with a sum. As outlined in Cont [14] there are three main steps in the construction of the finite difference approximations for PIDE: localisation, approximation of the small jumps, and discretisation.

7.1.1 Localisation to a bounded domain

The original PIDE for all the options we have been considering is defined on a unbounded domain. In order to be able to numerically solve the PIDE we localise the variables and the integral to bounded domains. We take the European vanilla call option as an example. The calculations for exotic options go along the same lines. Recall the PIDE for the European call from equation (5.11)

$$\begin{aligned} u_\tau(\tau,y) &= \frac{\sigma^2}{2}\left(u_{yy}(\tau,y) - u_y(\tau,y)\right) \\ &+ \int_{-\infty}^{\infty} \nu(dx)\big(u(\tau,y+x) - u(\tau,y) - (e^x-1)u_y(\tau,y)\big). \end{aligned}$$

To start with we truncate the domain of y from $\mathbb{R}$ to an interval $[a,b]$, where a and b are some finite numbers. We then need to assess $u(\tau,y)$ for $y \notin [a,b]$. Since the option price behaves asymptotically like the payoff, one sensible choice is to set the option price outside the domain to the payoff $h(y)$:

$$u(\tau,y) = h(y), \quad \forall y \notin [a,b].$$

It is shown in [15] that for bounded payoffs the error from localisation decreases exponentially with increasing domain size. Alternatively, we will show in section (7.1.3) that after discretisation one has to evaluate $u(\tau,x)$ only at a few points outside the interval $[a,b]$. Therefore, one can first calculate the Black-Scholes prices of the option at the relevant points outside the interval $[a,b]$ and then set the values of $u(\tau,y)$ outside the interval $[a,b]$ to the corresponding transformed Black-Scholes prices.

Secondly, we truncate the integral for values below a lower boundary B_l and above an upper boundary B_u. It is shown in [15] that the error occurring from truncating the integral decays exponentially with increasing upper and decreasing lower boundary. The truncated PIDE (5.11) is

$$\begin{aligned} u_\tau(\tau,y) &= \frac{\sigma^2}{2}\left(u_{yy}(\tau,y) - u_y(\tau,y)\right) \\ &+ \int_{B_l}^{B_u} \nu(dx)\big(u(\tau,y+x) - u(\tau,y) - (e^x-1)u_y(\tau,y)\big), \quad y \in [a,b]. \end{aligned} \tag{7.1}$$

7.1.2 Approximation of small jumps

Whenever we are working in an infinite activity model, the numerical evaluation of the integral around zero of the PIDE needs special attention. As we are dealing with models with infinitely many jumps, the integral over the Lévy measure $\int_{-\epsilon}^{\epsilon} \nu(dx)$ around zero goes to infinity. However, the integrand in equation (7.1) goes faster to zero than the Lévy measure goes to infinity and the integral is well defined. Still, the problem remains how one can numerically evaluate the integral close to zero. One way around this problem is to approximate the small jumps by a diffusion term. Doing so the stock price driving process X_t is replaced by the process X_t^ϵ which has the Lévy triplet $(\gamma(\epsilon), \sigma^2 + \sigma^2(\epsilon), \nu 1_{|x| \geq \epsilon})$ with

$$\begin{aligned} \sigma^2(\epsilon) &= \int_{-\epsilon}^{\epsilon} x^2 \nu(dx), \\ \gamma(\epsilon) &= -\frac{\sigma^2 + \sigma^2(\epsilon)}{2} - \int_{|x| \geq \epsilon} (e^x - 1 - x\, 1_{|x| \leq 1}) \nu(dx). \end{aligned}$$

Hence, for small jump sizes the term under the integral is replaced by

$$u(\tau, y + x) - u(\tau, y) - (e^x - 1) u_y(\tau, y) \to \frac{1}{2} x^2 \big(u_{yy}(\tau, y) - u_y(\tau, y) \big),$$

and the PIDE can be written as

$$\begin{aligned} u_\tau(\tau, y) = & \frac{\sigma^2 + \sigma^2(\epsilon)}{2} (u_{yy}(\tau, y) - u_y(\tau, y)) \\ & + \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \nu(dx) \big(u(\tau, y + x) - u(\tau, y) - (e^x - 1) u_y(\tau, y) \big). \end{aligned} \tag{7.2}$$

Starting from equation (7.2) we can derive the PIDE for the sensitivity with respect to any parameter ξ occurring in the Lévy measure. We use

$$v = \frac{\partial u}{\partial \xi}, \qquad \mathcal{V} = \frac{\partial c}{\partial \xi} = K e^{-r\tau} v.$$

and therefore differentiate equation (7.2) with respect to ξ to obtain the

PIDE

$$\begin{aligned}
v_\tau(\tau,y) = &\frac{\sigma^2+\sigma^2(\epsilon)}{2}\big(v_{yy}(\tau,y)-v_y(\tau,y)\big)\\
&+\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dy)\big(v(\tau,y+x)-v(\tau,y)-(e^x-1)v_y(\tau,y)\big)\\
&+\big(u_{yy}(\tau,y)-u_y(\tau,y)\big)\int_{-\epsilon}^{\epsilon}\mu(dx)x^2\\
&+\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\mu(dx)\big(u(\tau,y+x)-u(\tau,y)-(e^x-1)u_y(\tau,y)\big),
\end{aligned}$$

where

$$\mu(dx) = \frac{\partial}{\partial\xi}\nu(dx).$$

7.1.3 Discretisation

To numerically evaluate equation (7.2), we replace it by its finite difference approximation and solve the finite difference approximation on a uniform grid. The function $u(\tau,y)$ will not be evaluated on the continuous domain $[0,T]\times[a,b]$ but on a grid with uniform grid spacing in time τ and space y laid over the continuous domain. The grid points in the y direction are $y_i = a + i\cdot\Delta y$, $i = 0,\dots,N$. The grid size in the y direction is given by $\Delta y = \frac{b-a}{N}$. The continuous variable τ representing time is replaced by its discrete counterpart $\tau_n = n\cdot\Delta\tau$, $n = 0,\dots,M$. The discrete time increment is then given by $\Delta\tau = \frac{T}{M}$. For $u(\tau,y)$ evaluated at $\tau=\tau_n$ and $y = y_i$ we write u_i^n

$$u(\tau_n,y_i)\to u_i^n = u(n\cdot\Delta\tau, a+i\cdot\Delta y).$$

We denote by u^n the vector of all u_i^n for $i = 0, \dots, N$. The derivatives of $u(\tau, y)$ are replaced by their corresponding finite differences

$$\frac{\partial u(\tau,y)}{\partial \tau} \to \frac{u_i^{n+1} - u_i^n}{\Delta\tau},$$
$$\frac{\partial u(\tau,y)}{\partial y} \to \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta y},$$
$$\frac{\partial^2 u(\tau,y)}{\partial y^2} \to \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta y)^2},$$

where we used a forward discretisation in time and centred discretisations in the space direction. The integral in (7.2) over the large jumps with continuous jump size distribution is approximated by a sum over $K_l + K_u + 1$ jumps with fixed jump sizes $u_{i+j}^n - u_i^n$ weighted by the integral over the Lévy measure

$$\nu_j = \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} \nu(dx),$$

with the corresponding jump size. The approximation becomes

$$\int_{B_l}^{B_u} \nu(dx) u(\tau, y + x) \approx \sum_{j=K_l}^{K_u} \nu_j \, u_{i+j},$$

where K_l and K_u are integers such that

$$(K_l - \frac{1}{2})\Delta y \le B_l < (K_l + \frac{1}{2})\Delta y,$$
$$(K_u - \frac{1}{2})\Delta y \le B_u < (K_u + \frac{1}{2})\Delta y.$$

The diffusion part D and the jump part J of the finite difference approximation of the PIDE can be written as

$$(Du^{n+1})_i = \frac{\sigma^2 + \sigma^2(\epsilon)}{2} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta y)^2} - \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta y} \right),$$

$$(Ju^n)_i = \sum_{j=-K_l}^{K_u} \nu_j \left(u_{i+j}^n - u_i^n + (e^{j\Delta y} - 1) \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta y} \right).$$

To solve the PIDE, we use an explicit-implicit scheme as explained in [15] which means that when moving from time step n to $n+1$ we use an implicit scheme for the diffusion part D and evaluate the function $u(\tau, y)$ at the time point $n+1$, whereas for the jump part J we use an explicit scheme and evaluate the function $u(\tau, y)$ at the time point n. In an explicit scheme the time derivative of the option price at time τ_n, given by $\frac{u^{n+1}-u^n}{\Delta\tau}$, is the only term depending on the option price u^{n+1} at time τ_{n+1}. Therefore, one can explicitly solve the discretised equation for u^{n+1}. In an implicit scheme the time derivative of the option price is again $\frac{u^{n+1}-u^n}{\Delta\tau}$, however, all the other terms are evaluated at time τ_{n+1}. As u^{n+1} occurs in all the terms, the equation cannot be solved explicitly for u^{n+1}. One can obtain u^{n+1} by inverting a matrix as discussed below. The discretised PIDE can be written as

$$\frac{u^{n+1} - u^n}{\Delta t} = Du^{n+1} + Ju^n. \tag{7.3}$$

Using an explicit-implicit scheme, we do not have to invert the non-sparse matrix J coming from the jump part, and our scheme is stable, meaning that for bounded initial conditions we will have a bounded solution. To obtain the values of u for the next time step, equation (7.3) is solved recursively via

$$u^{n+1} = u^n + \Delta t\, Du^{n+1} + \Delta t\, Ju^n.$$

In order to do so first the pure diffusion equation

$$\tilde{u}^{n+1} = u^n + \Delta t D\, \tilde{u}^{n+1}$$

is solved for the auxiliary vector $\tilde{u}^{n+1}$. Inverting the matrix $I - \Delta t\, D$ one obtains

$$\tilde{u}^{n+1} = (I - \Delta t\, D)^{-1} u^n.$$

As the matrix $I - \Delta t\, D$ is tridiagonal, the inversion can be done rapidly [18]. The explicit jump part is added and one obtains the values of u at the next time step

$$u^{n+1} = (1 + \Delta t J)\tilde{u}^{n+1}.$$

When adding the jump part, we need a grid approximation of the Lévy measure. For the jump-diffusion model this is

$$\nu_j = \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} \nu(dx) = \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} \Delta y,$$

where $x_j = j \cdot \Delta y$. The grid approximation of the Variance Gamma Lévy measure is

$$\nu_j = \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} \nu(dx) = \frac{1}{\kappa |x_j|} e^{Ax_j - B|x_j|} \Delta y.$$

For the CGMY model, the grid approximation is

$$\nu_j = \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} \nu(dx) = \begin{cases} C\frac{e^{-G|x_j|}}{|x_j|^{1+Y}} \Delta y & x \geq 0, \\ C\frac{e^{-M|x_j|}}{|x_j|^{1+Y}} \Delta y & x < 0, \end{cases}$$

For the jump-diffusion model this approximation is unproblematic. For the variance gamma model and the CGMY model we have a singularity at $y = 0$. This problem can be solved by approximating the small jumps by a diffusion term as outlined in the section 7.1.2.

7.2 Finite difference approximation for exotic options

Most of the finite difference approximations carry over from the European vanilla options in a straightforward way to exotic options. Still, there are a few things to mention. For vanilla options we are mainly interested in values of the option price and the greeks around the strike. This means we are interested in values far away from the boundaries of the grid. For exotic options this is not always true and we might have to very accurately compute option prices and greeks at a boundary of the grid. Another difference is that for vanilla options we know the value of the price function and its derivatives at the boundaries, which is not always true for exotic options where the values of the price function at the boundaries might only be given as the solution to a differential equation.

7.2.1 Lookback option

For lookback options we are interested in the value of $u(\tau, y)$ close to $y = 0$ as this is where the value of the stock price is close to its maximum. Hence, we are interested in the function $u(\tau, y)$ close to the boundary of its domain at $y = 0$. Remember the auxiliary side conditions of the PIDE for the lookback option from equation (5.26):

$$\begin{aligned} u(\tau, y) &= 1, \qquad y \to -\infty, \\ u_y(\tau, y)|_{y=0} &= u(\tau, 0). \end{aligned}$$

To solve the PIDE for the lookback option (5.25) numerically we have to make approximations both at the lower and at the upper boundary. In order to obtain a good approximation for the lower boundary condition, we have to restrict the domain of computation from $[-\infty, 0]$ to $[a, 0]$, where a is such that $u(\tau, a)$ is close to one. At the upper boundary we approximate the derivative by a one sided difference

$$\frac{\partial u(\tau,y)}{\partial y} \to \frac{3u(\tau,y) - 4u(\tau,y-\Delta y) - u(\tau,y-2\Delta y)}{2\Delta y}.$$

Thus we obtain the value of $u(\tau,y)$ at $y=0$ by solving the equation

$$\frac{3u(\tau,y) - 4u(\tau,y-\Delta y) - u(y-2\Delta y)}{2\Delta y} = u(\tau,y)$$

for $u(\tau,y)$, and we obtain the value at the boundary

$$u(\tau,0) = \frac{4u(\tau,-\Delta y) + u(-2\Delta y)}{3 - 2\Delta y}.$$

7.2.2 Asian option

For the Asian option we have from equation (5.37) the boundary condition $w_\tau = w_z$ at $z = 0$. At the boundary $z = 0$ we cannot use the centred difference approximation as this would require the knowledge of $w(\tau,z)$ at negative values of z. Instead we use the one sided finite difference approximation

$$\frac{\partial w(\tau,z)}{\partial z} \to \frac{-3w(\tau,z) + 4w(\tau,z+\Delta z) - w(\tau,z+2\Delta z)}{2\Delta z}.$$

Using this one sided difference approximation the side condition at $z = 0$ and time $\tau = i\Delta\tau$ becomes

$$\frac{-3w(\tau,z) + 4w(\tau,z+\Delta z) - w(\tau,z+2\Delta z)}{2\Delta z} = \frac{w(\tau,0) - w(\tau-\Delta\tau,0)}{\Delta\tau}.$$

Another complication for the Asian option is that we have the term $w(t, z\,e^x)$ in the PIDE. If a jump occurs we have to evaluate $w(t, z\,e^x)$ at the point $z\,e^x$ which is not a grid point. Therefore we have to approximate $z\,e^x$ to the nearest grid point.

7.3 Splitting the integral

The integrand of the PIDE in our models has three parts. The value of a function, say u, before a jump occurred, the value of the function u after a jump occurred, and a term involving a derivative of u. For the vanilla option in transformed coordinates the integral part is

$$\int_{-\infty}^{\infty} \nu(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big).$$

The question arises whether these three parts can and potentially should be separated into more than one integral during the numerical analysis. In jump-diffusion models the three terms can be separated as each of them is integrable on its own. When we are working in a model with infinite activity but finite variation, the first and the second part of the integrand cannot be separated, as they are not integrable on their own. In a model with infinite variation none of the terms in the integrand can be separated. In the numerical evaluation of the integral in infinite activity models we replace the small jumps by a diffusion term. Using this trick, the integral can again be separated into three terms.

The numerical calculation of the integral is very time consuming compared with the calculation of the diffusion part since, at each point (τ_n, y_i) in the grid, one has to evaluate a sum over all the jump sizes. Thus it seems to be a good idea to approximate the small jumps with a diffusion term and to split the integral

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \nu(dx)\big(u(\tau, y+x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big). \tag{7.4}$$

into three parts. The PIDE for the option price then becomes

$$\begin{aligned}u_\tau(\tau,x) &= \frac{\sigma^2+\sigma^2(\epsilon)}{2}(u_{yy}(\tau,y)-u_y(\tau,y))\\&+\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)u(\tau,y+x)-u(\tau,y)\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)\\&-u_y(\tau,y)\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)(e^x-1)).\end{aligned}$$

In this form only the integral $\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)u(\tau,y+x)$ has to be evaluated at each grid point, whereas for the second and third part of the integral in equation (7.4) one has to calculate $\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)$ and $\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\nu(dx)(e^x-1)$ only once in the numerical procedure and then can just multiply them at each grid point with $u(\tau,y)$ and $u_y(\tau,y)$.

7.4 Consistency, stability, and convergence

A finite difference scheme is said to be *consistent* with a (integro-) differential equation if the difference of the finite difference operator and the (integro-) differential operator acting on any sufficiently smooth function goes to zero as the time and space increments go to zero. A scheme is *stable* if for a bounded initial condition a solution exists that is bounded independent of the size of the time increment Δt and the space increment Δy. For this to be true, the discretisation error, that is the difference between the exact solution and the numerical approximation, has to be damped with time. A numerical scheme is said to be *convergent* to the exact solution if the difference between the finite difference solution and the exact solution of the PIDE goes to zero when both the time and the space increment go to zero. It is show in [15] that the scheme we used is consistent and stable. For sufficiently smooth functions consistency and stability ensure convergence of the difference scheme to the exact solution. If the solution is not sufficiently smooth, then the finite difference approximation converges to a viscosity solution.

7.5 Higher order schemes

There exist many different finite difference schemes to solve the PDE that occurs in the Black-Scholes framework. There are mainly two things that have to be considered when choosing the scheme: the stability and the order of the error term. The simplest scheme is the explicit scheme, which is has error terms of $\mathcal{O}(\Delta t)$ in time and $\mathcal{O}((\Delta y)^2)$ in space and is only stable if $\frac{\Delta t}{(\Delta y)^2} \leq \frac{1}{2}$. Standard textbooks in numerical finance advocate the Crank-Nicolson scheme as it is unconditionally stable, which means that it is stable for all choices of Δt and Δy, and has error terms of the order $\mathcal{O}((\Delta t)^2)$ in time and $\mathcal{O}((\Delta y)^2)$ in space. Another popular scheme is the Douglas scheme which has a higher order error term in the space direction. It can be shown that these schemes are consistent and converge to the exact solution if they are stable. Modern finite difference solvers can have error terms of much higher order. In [12] a finite difference method is presented which is fourth order both in time and space. When we are leaving the Black-Scholes framework and move from a PDE to a PIDE the numerical analysis becomes much more involved. The jump term is generally calculated explicitly and hence we have error terms of the order $\mathcal{O}(\Delta t)$ in time and $\mathcal{O}((\Delta y)^2)$ in space. In [7] so-called Implicit-Explicit (IMEX) Runge-Kutta methods to solve PIDE in a jump-diffusion setting are proposed. The methods presented have error terms up to order three. We apply an implicit-explicit midpoint scheme which has error terms of order two both in time and space to the European vanilla option price and the its vega. Thus, we consider the system of equations

$$\begin{aligned}
u_\tau(\tau, y) &= \frac{\sigma^2}{2}(u_{yy}(\tau, y) - u_y(\tau, y)) \\
&\quad + \int_{-\infty}^{\infty} \nu(dx)\big(u(\tau, y + x) - u(\tau, y) - (e^x - 1)u_y(\tau, y)\big), \\
v_\tau(\tau, y) &= \frac{\sigma^2}{2}(v_{yy}(\tau, y) - v_y(\tau, y)) + \sigma(u_{yy}(\tau, y) - u_y(\tau, y))^{\cdot} \\
&\quad + \int_{-\infty}^{\infty} \nu(dx)\big(v(\tau, y + x) - v(\tau, y) - (e^x - 1)v_y(\tau, y)\big),
\end{aligned}$$

subject to the appropriate boundary conditions. The idea of the midpoint scheme is to calculate in each time step a midpoint u_i^m between the grid points u_i^n and u_i^{n+1}, where the upper scripts denote the time steps and the lower scripts denote space steps. In a second step the PIDE is then evaluated at this midpoint. In the first step of the midpoint scheme the diffusion component is calculated implicitly for reasons of stability whereas the other terms are calculated explicitly. The discretisation thus becomes

$$\begin{aligned} u_i^m = u_i^n + \frac{\Delta t}{2}\Bigg(\frac{\sigma^2}{2}\left(\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta y)^2} - \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta y}\right) \\ + \sum_j \nu_j \left(u_{i+j}^n - u_i^n - (e^{x_j} - 1)\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta y}\right)\Bigg), \end{aligned}$$

and the second step is

$$\begin{aligned} u_i^{n+1} = u_i^n + \Delta t\Bigg(\frac{\sigma^2}{2}\left(\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta y)^2} - \frac{u_{i+1}^m - u_{i-1}^m}{2\Delta y}\right) \\ + \sum_j \nu_j \left(u_{i+j}^m - u_i^m - (e^{x_j} - 1)\frac{u_{i+1}^m - u_{i-1}^m}{2\Delta y}\right)\Bigg), \end{aligned}$$

where $x_j = j\,\Delta y$. The grid approximations for the PIDE for vega are for the first step

$$\begin{aligned} v_i^m = v_i^n + \frac{\Delta t}{2}\Bigg(\frac{\sigma^2}{2}\left(\frac{v_{i+1}^m - 2v_i^m + v_{i-1}^m}{(\Delta y)^2} - \frac{v_{i+1}^n - v_{i-1}^n}{2\Delta y}\right) \\ + \sum_j \nu_j \left(v_{i+j}^n - v_i^n - (e^{x_j} - 1)\frac{v_{i+1}^m - v_{i-1}^m}{2\Delta y}\right)\Bigg), \end{aligned}$$

and the second step is

$$\begin{aligned} v_i^{n+1} = v_i^n + \Delta t\Bigg(\left(\frac{\sigma^2}{2}\frac{v_{i+1}^m - 2v_i^m + v_{i-1}^m}{(\Delta y)^2} - \frac{v_{i+1}^m - v_{i-1}^m}{2\Delta y}\right) \\ + \sum_j \nu_j \left(v_{i+j}^m - v_i^m - (e^{x_j} - 1)\frac{u_{i+1}^m - u_{i-1}^m}{2\Delta y}\right) \\ + \sigma\left(\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta y)^2} - \frac{u_{i+1}^m - u_{i-1}^m}{2\Delta y}\right)\Bigg). \end{aligned}$$

Comparing the midpoint scheme with the simple implicit-explicit scheme we find that the error term given equal computation time can be significantly reduced with the higher order scheme. Furthermore, this scheme could be applied not only in jump-diffusion models but generally in Lévy process driven models.

7.6 Numerical results

We compute the price and the Greeks both for vanilla options and for exotic options, and compare the performance of the dynamic approach with numerical integration and simulation. All computation are performed on a Pentium 4, 2.8 GHz computer.

7.6.1 Vanilla options

First, we illustrate the performance of the dynamic partial integro-differential method for a European call option in the jump-diffusion model from section 4.2 and solve numerically equation (5.11) and the corresponding equations for the greeks subject to their side conditions. The price of a call option in the Merton jump-diffusion model can also be expressed as a series expansion of Black-Scholes option prices. We compare the prices and sensitivities obtained with the two different methods. For the numerical computations we use the parameters $r = 0.1$, $K = 50$, $T = 1$, $\sigma = 0.1$, $\lambda = 0.1$, $\mu = 0$, and $\delta = 0.1$. The results can be seen as plots (A.1), (A.2), (A.3), (A.4), and (A.5) in the Appendix. For the price function, vega, rho, and beta, we obtain virtually the same numerical results for both methods. Using the dynamic approach, the calculations of the price and the greeks on a grid of 1000 maturities times 1000 stock prices takes only a few seconds when we approximate the continuous jump size distribution with a discrete jump size distribution that can take 200 different jump sizes. Using the series expansion approach it takes roughly one minute to calculate the price and the greeks for 1000 stock prices at just one maturity.

Next, we investigate how the dynamic partial integro-differential method performs for a European call option in the variance gamma model introduced in section 4.3. The price of a call option in the variance gamma model can also be expressed by means of Bessel functions and can be obtained via numerical integration. We compare the prices and sensitivities obtained with the two different methods. For the numerical computations we use the parameters $r = 0.1$, $K = 50$, $T = 1$, $\sigma = 0.1$, $\kappa = 3$, and $\theta = -0.001$. The results can be seen as plots (A.6), (A.7), and (A.8) in the Appendix. We use again a lattice of 1000 time steps times 1000 stock prices and approximate the continuous jump size distribution with a distribution with 200 different fixed different jump sizes. The calculations of the price and the greeks for these specifications take about fifteen seconds. Alternative calculations using integrals over a Bessel function take several minutes for just 100 strikes and only one maturity. Better results can be expected when pricing via fast Fourier methods. Still, when one is interested in prices and sensitivities for a range of initial stock prices and maturities, the dynamic approach clearly outperforms its competitors.

Despite the fact that most stock returns in the study [9] are of finite variation we choose for our investigation the parameters of the CGMY process from section 4.4 such that it has infinite variation, namely $Y = 1.5$, $C = 0.081$, $G = 25.04$, and $M = 25.04$. The reason for this is that some stocks are modelled by an infinity variation process and the infinite variation process is more difficult to tackle. Therefore we want to show that our methodology can also be applied to this situation. The results can be seen as plots (A.9) and (A.10) in the Appendix.

7.6.2 Exotic options

The distribution of the maximum or the average over a certain time period of a Lévy process is generally not know, nor is the distribution of the difference of two exponential Lévy processes. Therefore, we have to resort to simulation to obtain prices and sensitivities that we can compare with the results obtained from the dynamic PIDE method.

We compare numerical results obtained from the dynamic PIDE method with simulation results for the price and the greeks of a lookback option with price function

$$c(t,s,m) = e^{-r(T-t)}\,\mathbb{E}[M_T - S_T | S_t = s, M_t = m],$$

where S_t is the stock price at time t and M_t is the running maximum of the stock price. We assume that the stock price follows the jump-diffusion process as outlined in section (4.2). With the dynamic PIDE method outlined in section (5.2) we calculate the price and the greeks of a lookback option where the running maximum is measured continuously $M_t = \max_{0<\tau<t} S_\tau$. In the table the results from this approach are labelled PIDE. With the simulation method the maximum is measured at n discrete points along the path of the stock price. The maximum is calculated as $M_t = \max_{i=1\ldots n} S_{t_i}$, where $t_i = \frac{i}{n}t$, $i = 1\ldots n$. We choose three different values form the number of sampling points: n=10, n=100, and n=500. For all calculations we set the value of the stock price at time $t = 0$ to $s = 1$, the interest rate to $r = 0.1$ and the time to maturity to $T = 1$.

	SIM n=10	SIM n=100	SIM n=500	PIDE
c	0.027	0.039	0.042	0.044
$\frac{\partial c}{\partial \sigma}$	0.047	0.061	0.066	0.068
$\frac{\partial c}{\partial \lambda}$	0.014	0.018	0.018	0.019
$\frac{\partial c}{\partial \delta}$	0.023	0.025	0.027	0.029

Table 7.1: Price function and greeks for a lookback option with parameters $\lambda = 0.1$, $\sigma = 0.1$, $\delta = 0.1$.

	SIM n=10	SIM n=100	SIM n=500	PIDE
c	0.041	0.054	0.057	0.060
$\frac{\partial c}{\partial \sigma}$	0.44	0.58	0.62	0.63
$\frac{\partial c}{\partial \lambda}$	0.015	0.017	0.017	0.018
$\frac{\partial c}{\partial \delta}$	0.25	0.27	0.27	0.27

Table 7.2: Price function and greeks for a lookback option with parameters $\lambda = 1$, $\sigma = 0.1$, $\delta = 0.1$.

From the tables one clearly sees that, when the number of sampling points increases, the price and the greeks of the lookback option with discrete sampling converge to the price and the greeks of the lookback option with continuous sampling. This shows that the results obtained with the dynamic PIDE approach are consistent with the simulation results.

The computation of the price and the greeks with the dynamics PIDE approach on a grid of 1000 initial quotients of maximum over current stock price times 1000 time steps and with 100 possible jump sizes takes about two minutes. The simulation of the price and the greeks takes about five minutes for $n = 10$ and much longer for the $n = 100$, and $n = 500$. Moreover, with the simulation one only obtains a result for the case where the maximum is equal to the stock price at the initial time and only obtains the option price at time $t = 0$. The results of the PIDE method for this lookback option are also displayed in the figures (A.11), (A.12), (A.13), and (A.14) in the Appendix.

In the Appendix we furthermore plot the results for an Asian option in figures (A.19), (A.20), and (A.21) and for an exchange option in figure (A.22), (A.23), and (A.24).

Chapter 8

Existence of derivatives

8.1 Vanilla options - density known

We shall show that all the terms in the PIDE that determine the price and the greeks are well defined for vanilla options in the jump-diffusion and the variance gamma model.

8.1.1 Existence in the jump-diffusion model

The PIDE approach does only make sense if all the terms in the equation are well defined. In particular, we have to prove that the price function and the relevant derivatives thereof exist. The integral part does not cause any problems, as we are dealing with only a finite number of jumps in finite time intervals. The value of a European call option in the transformed coordinates can be expressed as

$$\begin{aligned} u(\tau, x) &= \mathbb{E}[\max(e^{x+X_\tau}, 0)] \\ &= \int_{-\infty}^{\infty} \max(e^{x+y} - 1, 0)\, p_\tau(y)\, dy \end{aligned} \tag{8.1}$$

where $p_\tau(y)$ is the density function of the Lévy process X_τ at time τ and $x = \ln(s/K) + r\tau$. In the Merton jump-diffusion model, the density $p_\tau(y)$

has the form

$$p_\tau(y) = e^{-\lambda\tau} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k \, e^{-\frac{(y-\gamma\tau-k\mu)^2}{2(\sigma^2\tau+k\delta^2)}}}{k! \sqrt{2\pi(\sigma^2\tau + k\delta^2)}}.$$

Inserting the density $p_\tau(y)$ into equation (8.1), a simple change of variables yields

$$u(\tau, x) = \int_{-\infty}^{\infty} \max(e^y - 1, 0) \left(e^{-\lambda\tau} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k \, e^{-\frac{(y-x-\gamma\tau-k\mu)^2}{2(\sigma^2\tau+k\delta^2)}}}{k! \sqrt{2\pi(\sigma^2\tau + k\delta^2)}} \right) dy. \quad (8.2)$$

The partial sum in equation (8.2) up to summand n can be seen as the limit of a bounded monotone sequence of functions. As all the summands are integrable and decay rapidly, $u(\tau, x)$ is finite and we can interchange the sum and the integral

$$u(\tau, x) = e^{-\lambda\tau} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} \int_{-\infty}^{\infty} \max(e^y - 1, 0) \frac{e^{-\frac{(y-x-\gamma\tau-k\mu)^2}{2(\sigma^2\tau+k\delta^2)}}}{\sqrt{2\pi(\sigma^2\tau + k\delta^2)}} dy. \quad (8.3)$$

For the price determining PIDE to be well defined, also the derivatives with respect to τ, x, and the second derivative with respect to x have to exist. For the PIDE that determines the value of the greeks to be well defined the derivatives of all terms in the price determining PIDE with respect to the parameter in question have to exist. The proof of existence of any of those derivatives relies on the dominated convergence theorem. We explicitly show that the time derivative of $u(\tau, x)$ is well defined. The proofs for all other derivatives go along the same lines. Writing the integral as

$$u(\tau, x) = \int_{-\infty}^{\infty} f(\tau, y) dy,$$

the time derivative of $u(\tau, x)$ exists if we can find an integrable g such that

$$\frac{1}{h} |f(\tau + h, y) - f(\tau, y)| \leq g(y),$$

and can show that the limit

$$\lim_{h \to 0} \frac{1}{h} |f(\tau + h, y) - f(\tau, y)|$$

exists. The difference is

$$\frac{1}{h}\Big| f(\tau + h, y) - f(\tau, y) \Big| \leq$$

$$\sup_{\tau^* \in (t, t+h)} \Bigg| \frac{(y - x - \gamma\tau^* - k\mu)^2 \sigma^2}{(\sigma^2\tau^* + k\delta^2)^{5/2}} \frac{e^{-\frac{(y-x-\gamma\tau^*-k\mu)^2}{2(\sigma^2\tau^*+k\delta^2)}}}{\sqrt{2\pi(\sigma^2\tau + k\delta^2)}} - \frac{\sigma^2}{2(\sigma^2\tau^* + k\delta^2)} \frac{e^{-\frac{(y-x-\gamma\tau^*-k\mu)^2}{2(\sigma^2\tau^*+k\delta^2)}}}{\sqrt{2\pi(\sigma^2\tau^* + k\delta^2)}} \Bigg|,$$

which is integrable. Clearly, the limit exists as the integrand is smooth in τ.

8.1.2 Existence in the variance gamma model

The PIDE approach in the variance gamma model makes sense if all the terms in the PIDE, namely the price function, its first derivatives with respect to time and space, and the integral part are well defined. Note, that the second space derivative has a pole. This rules out models that would combine the variance gamma model with an additional Brownian motion term, as they would require the existence of the second derivative. The value of a call option in the variance gamma model can be written, in transformed coordinates, as

$$u(\tau, x) = \int_{-\infty}^{\infty} \max(e^{x+y+\omega} - 1, 0)\, C(\tau)\, |y|^{\frac{\tau}{\kappa} - \frac{1}{2}}\, e^{Ay}\, K_{\frac{\tau}{\kappa} - \frac{1}{2}}(B|y|)\, dy, \qquad (8.4)$$

where

$$A = \frac{\theta}{\sigma^2},$$

$$B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2},$$

$$C(\tau) = 2(\sqrt{2\pi}\kappa^{\tau/\kappa}\sigma\Gamma(\tau/\kappa))^{-1}\, (2\sigma^2/\kappa + \theta^2)^{\frac{1}{4} - \frac{\tau}{2\kappa}},$$

K is the modified Bessel function of the second kind and

$$\omega = \frac{1}{\kappa}\ln(1-\theta\kappa-\frac{1}{2}\sigma^2\kappa).$$

We can rewrite (8.4) as

$$u(\tau,x) = \int_{-\infty}^{\infty} \max(e^{x-y+\omega}-1,0)\, C(\tau)\, |y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\, e^{-Ay}\, K_{\frac{\tau}{\kappa}-\frac{1}{2}}(B|y|)\, dy. \quad (8.5)$$

First, we show that the integral (8.5) converges. For fixed n and large y the Bessel function $K_n(y)$ can be approximated by

$$K_n(y) \approx \sqrt{\frac{\pi}{2y}}e^{-y}. \quad (8.6)$$

The integral converges if the term from the Bessel function approximation (8.6) dominates $e^{-Ay}\,e^{-y}$ for large negative y. Hence, the integral is convergent if $A+1 < B$. For small values of y the Bessel function can be approximated by

$$K_n(y) \approx \frac{1}{2}\Gamma(n)\left(\frac{y}{2}\right)^{-|n|},$$

and the integral (8.5) is approximated around zero by

$$u(\tau,x) \approx \int_{-\epsilon}^{\epsilon} \max(e^{x-y+\omega}-1,0)\, C(\tau)\, |y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\, e^{-Ay} \times \frac{1}{2}\Gamma\left(\frac{\tau}{\kappa}-\frac{1}{2}\right)\left(\frac{1}{2}B|y|\right)^{-|\frac{\tau}{\kappa}-\frac{1}{2}|} dy,$$

for a small ϵ. This integral is convergent if $B \neq 0$ and $\tau/\kappa > 0$. Since B is strictly positive and κ, being a variance, is also bigger than zero, the only requirement for the integral to converge is $A+1 < B$. Next, we show that the derivative with respect to x exists. We want to differentiate (8.5) under the integral sign. The integrand is differentiable in x and is

$$1_{\{y<x+\omega\}}e^{x-y+\omega}\, C(\tau)\, |y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\, e^{-Ay}\, K_{\frac{\tau}{\kappa}-\frac{1}{2}}(B|y|).$$

The difference operator is bounded by an integrable function g

$$
\begin{aligned}
&\frac{1}{h}\left|f(\tau,x+h,y)-f(\tau,x,y)\right| = \\
&C(\tau)\,|y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\,e^{-Ay}\,K_{\frac{\tau}{\kappa}-\frac{1}{2}}(B|y|)\,\frac{1}{h}\left|\max(e^{x+h-y+\omega}-1,0)-\max(e^{x-y+\omega}-1,0)\right| \\
&< C(\tau)\,|y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\,e^{-Ay}\,K_{\frac{\tau}{\kappa}-\frac{1}{2}}(B|y|)\sup_{x^*\in[x,x+h]}|e^{x^*-y+\omega}| = g.
\end{aligned}
$$

We know that the function g is integrable as it is essentially of the same form as the integrand in (8.4). Hence, due to dominated convergence we can interchange the integration and the limit and the derivative is given by

$$
\partial_x u(\tau,x) = \int_{-\infty}^{x+\omega} e^{x-y+\omega}\,C(\tau)\,|y|^{\frac{\tau}{\kappa}-\frac{1}{2}}\,e^{-Ay}\,K_{\frac{\tau}{\kappa}-\frac{1}{2}}(B|y|)\,dy.
$$

The existence of the time derivative of $u(\tau,x)$ and the derivatives of $u(\tau,x)$ with respect to model parameters can be shown with the same method based on the dominated convergence theorem. The integral term in the PIDE exists, as by definition of the Lévy process

$$
\int_{|y|\le\epsilon} \nu(dy)y^2 \quad < \quad \infty
$$

and there can only be a finite number of large jumps in a finite time interval.

8.2 Vanilla options - density not known

The method above works only when the density function of the price process fixed at time τ is known. We consider a European vanilla option in an exponential Lévy model where the density is not known explicitly. The price of this option can be expressed as an integral over a functional depending on the characteristic function of the driving Lévy process. The option price as a function of the log strike k is not an integrable function. Therefore we have either to multiply it with a damping function as in [10] or subtract and later on add a term which then makes the new function integrable as presented in [14]. We will follow the latter approach and compute the Fourier transform of the modified time value $z_T(k)$ of the option

$$\begin{aligned} z_T(k) &= e^{-rT}\,\mathbb{E}[\max(e^{rT+X_T} - e^k, 0)] - \max(1 - e^{k-rT}, 0) \\ &= e^{-rT}\int_{-\infty}^{\infty} \rho_T(dx)(e^{rT+x} - e^k)(1_{k\le x+rT} - 1_{k\le rT}), \end{aligned}$$

here $\rho_T(x)$ is the density of the process X_t at time T. We used that the discounted stock price is a martingale and thus

$$\int_{-\infty}^{\infty} dx\,\rho_T(x)\,e^x = 1.$$

Let $\zeta_T(v)$ be the Fourier transform of the time value.

$$\begin{aligned} \zeta_T(v) =& \int_{-\infty}^{\infty} e^{ivk}\,z_T(k)\,dk \\ =& e^{-rT}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx\,\rho_T(x) e^{ivk}(e^{rT+x} - e^k)(1_{k\le x+rT} - 1_{k\le rT}) \\ =& e^{-rT}\int_{-\infty}^{\infty} dx\,\rho_T(x)\int_{rT}^{x+rT} dk\,e^{ivk}(e^{rT+x} - e^k) \\ =& e^{-rT}\int_{-\infty}^{\infty} dx\,\rho_T(x)\left(\frac{e^{(iv+1)rT} - e^{(iv+1)(x+rT)}}{iv+1}\right. \\ &- \left.\frac{e^{ivrT+rT+x} - e^{iv(x+rT)+x+rT}}{iv}\right) \\ =& e^{-rT}\int_{-\infty}^{\infty} dx\rho_T(x)\left(\frac{e^{ivrT}(1-e^x)}{iv+1} + \frac{e^{ivrT+(iv+1)x} - e^{ivrT+x}}{iv(iv+1)}\right). \end{aligned} \tag{8.7}$$

Using

$$\int_{-\infty}^{\infty} dx\,\rho_T(x)\,(e^x - 1) = 0,$$

we can simplify the equation (8.7) to

$$\begin{aligned} \zeta_T(v) =& \frac{e^{ivrT}}{iv(iv+1)}\left(\int_{-\infty}^{\infty} dx\,\rho_T(x)\,e^{ix(v-i)} - 1\right) \\ =& e^{ivrT}\,\frac{\phi_T(v-i) - 1}{iv(1+iv)}. \end{aligned}$$

The option price can then be written as

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\zeta_T(v)dv + \max(1 - e^{k-rT}, 0). \tag{8.8}$$

Our goal is to prove that the option price function (8.8) is differentiable with respect to some parameters. To do so we have to show that we can switch differentiation and integration, then check that the characteristic function is differentiable with respect to the parameter, and finally show that the functional we are dealing with is integrable. Given a specific Lévy process all these steps can be checked.

8.3 Brownian motion case

It is rather easy to check differentiability of option prices with respect to some parameters when the stock price driving process is a Brownian motion. The only model parameters we are dealing with are the variance of the Brownian motion σ^2, the interest rate r, and time t. Furthermore we are obviously interested in the option's delta and gamma. For many options in the Black-Scholes model such as the European vanilla options, barrier options, lookback options, and exchange options the price of the option can be expressed as a functional of the cumulative normal density. One can then simply differentiate this so called closed form solution and one easily sees if there are any problems. For a vanilla call option in the Black-Scholes model both the theta, the derivative of the option price with respect to time, and the gamma go to infinity if the option is at the money and the time to maturity goes to zero. This is seen from the formulas for theta and gamma

$$\Gamma = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}}{s\sigma\sqrt{T-t}},$$

$$\Theta = -\frac{s\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2),$$

where $N(x)$ is the normal probability distribution and

$$d_{1,2} = \frac{\ln \frac{s}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Now, if $s = K$ the equations for d_1 and d_2 reduce to

$$d_{1,2} = \frac{\sqrt{T-t}\left(r \pm \frac{1}{2}\sigma\right)}{\sigma^2},$$

which is zero as t goes to T and hence the gamma and theta go to infinity.

The price of an Asian option can be expressed as a triple integral [41]. The numerical evaluation of this integral seems to be very difficult and hence other methods, in particular Laplace inversion techniques [20], [11], have been put forward to obtain an option price.

8.4 A Girsanov transform technique

The methods presented so far to prove the differentiability of the price function and hence the existence of the greeks all have some disadvantages. Either they are only applicable to vanilla options or they are only applicable in a Brownian motion setup. In Norberg [33] existence results of sensitivities in a Markov chain market were established by a martingale and change of measure technique. In this section we present and expand this method to investigate the existence of sensitivities of contingent claims with respect to changes in model parameters in a Lévy process driven market. We show that for the most part Norberg's approach can be applied to a Lévy process driven market and obtain the corresponding results.

Consider a contingent claim whose price process under the measure $\mathbb{P}_\theta$ is a functional of a Lévy process

$$X_t = \gamma_\theta \, t + \sigma_\theta \, W_t + \int_{(0,t]\times(\mathbb{R}\setminus(-1,1))} x \, \mu(ds, dx) \\ + \lim_{\epsilon\to 0} \int_{(0,t]\times([-1,-\epsilon)\cup(\epsilon,1])} x(\mu(ds,dx) - \nu_\theta(dx)ds).$$

Examples of such contingent claims are options in exponential Lévy markets where the stock price is $S_t = S_0 \, e^{X_t}$. Let $(\mathcal{F}_\tau)_{0\le\tau\le t}$ be the filtration generated by X_t. The price function of such a contingent claim is in general an expected value $\mathbb{E}_\theta[Y]$, where Y is an integrable $\mathcal{F}_T$ measurable random variable. The sensitivity of the contingent claim with respect to changes in the model parameter θ is

$$\lim_{\vartheta\to 0} \frac{1}{\vartheta}(\mathbb{E}_{\theta+\vartheta}[Y|\,\mathcal{F}_t] - \mathbb{E}_\theta[Y|\,\mathcal{F}_t]). \tag{8.9}$$

We now investigate under what conditions this limit is well defined. Due to the weak predictable representation property as described in Jacod [26] or He [23] the random variable Y can be written as

$$Y = \mathbb{E}_\theta[Y] + \int_0^T \alpha_\theta(\tau)dW_\tau + \int_0^T \int_{\mathbb{R}} \beta_\theta(\tau,x)(\mu(dx,d\tau) - \nu_\theta(dx)d\tau), \tag{8.10}$$

where $\alpha_\theta(\tau)$ is a square integrable predictable process $\alpha : \Omega\times[0,T] \to \mathbb{R}$ and $\beta_\theta(\tau,x)$ is a square integrable predictable process $\beta : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$.

In order to calculate the sensitivities by the martingale method, we have to perform a change of measure. We first introduce the model space $\{\mathbb{P}_\theta : \theta\in\Theta\}$, where $\Theta\subset\mathbb{R}^d$. We fix two points θ and $\theta+\vartheta$ in Θ, where θ is our reference point and ϑ represents a small deviation from that. We aim at forming differentials. To do so we use the following two lemmas drawn form Cont [14] and Sato [35]. Let $(X_t, \mathbb{P}_\theta)$ and $(X_t, \mathbb{P}_{\theta+\vartheta})$ be two Lévy processes on $\mathbb{R}$ with characteristic triplets $(\sigma_\theta^2, \nu_\theta, \gamma_\theta)$ and $(\sigma_{\theta+\vartheta}^2, \nu_{\theta+\vartheta}, \gamma_{\theta+\vartheta})$. This can be interpreted as investigating X_t at two points of our model space. We say the measures $\mathbb{P}_\theta|_{\mathcal{F}_t}$ and $\mathbb{P}_{\theta+\vartheta}|_{\mathcal{F}_t}$ are equivalent for all t if $\mathbb{P}_\theta[B]=0$ implies $\mathbb{P}_{\theta+\vartheta}[B]=0$ and vice versa for all $B\in\mathcal{F}_t$.

Lemma 1. The measures $\mathbb{P}_\theta$ and $\mathbb{P}_{\theta+\vartheta}$ are equivalent if and only if:

1. $\sigma_\theta = \sigma_{\theta+\vartheta}$.

2. The Lévy measures are equivalent, with

$$\int_{-\infty}^{\infty} (e^{\phi(x)/2} - 1)^2 \nu_\theta(dx) < \infty,$$

where $\phi(x) = \ln(\frac{d\nu_{\theta+\vartheta}(x)}{d\nu_\theta(x)})$.

3. If $\sigma_\theta = 0$, we must in addition have

$$\gamma_\theta - \gamma_{\theta+\vartheta} = \int_{-1}^{1} x(\nu_{\theta+\vartheta} - \nu_\theta)(dx).$$

Lemma 2. When the two measures are equivalent, the likelihood process $L_{\theta,\vartheta}(t)$ is

$$\frac{d\mathbb{P}_{\theta+\vartheta}|_{\mathcal{F}_t}}{d\mathbb{P}_\theta|_{\mathcal{F}_t}} = L_{\theta,\vartheta}(t) = e^{U_{\theta,\vartheta}(t)}$$

with

$$U_{\theta,\vartheta}(t) = \eta X_t^c - \frac{\eta^2\sigma_\theta^2 t}{2} - \eta\gamma_\theta t$$
$$+ \lim_{\epsilon\to 0}\left(\sum_{s\le t, |\Delta X_s|>\epsilon} \phi(\Delta X_s) - t\int_{|x|>\epsilon} (e^{\phi(x)} - 1)\nu_\theta(dx)\right).$$

Here X_t^c is the continuous part of X_t, and η is such that

$$\gamma_{\theta+\vartheta} - \gamma_\theta - \int_{-1}^{1} x(\nu_{\theta+\vartheta} - \nu_\theta)(dx) = \sigma_\theta^2\eta$$

if $\sigma_\theta > 0$ and zero if $\sigma_\theta = 0$. Applying Itô's lemma to $L_{\theta,\vartheta}(t)$ and using

$X_t^c = \sigma_\theta W_t + \gamma_\theta t$ we obtain

$$
\begin{aligned}
dL_{\theta,\vartheta}(t) &= \eta\sigma_\theta L_{\theta,\vartheta}(t)dW_t \\
&\quad + L_{\theta,\vartheta}(t)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}} \frac{\nu_{\theta+\vartheta}(dx)-\nu_\theta(dx)}{\nu_\theta(dx)}(\mu(dx,dt)-\nu_\theta(dx)dt).
\end{aligned}
$$

The likelihood process can therefore be written as

$$
\begin{aligned}
L_{\theta,\vartheta}(t) = 1 &+ \int_0^t \eta\sigma L_{\theta,\vartheta}(\tau-)dW_\tau \\
&+ \int_0^t L_{\theta,\vartheta}(\tau-)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}} \frac{\nu_{\theta+\vartheta}(dx)-\nu_\theta(dx)}{\nu_\theta(dx)} \\
&\times (\mu(dx,d\tau)-\nu_\theta(dx)d\tau).
\end{aligned} \tag{8.11}
$$

We are going to use the general relationship

$$
\mathbb{E}_{\theta+\vartheta}[Y|\,\mathcal{F}_t] = \frac{\mathbb{E}_\theta[Y L_{\theta,\vartheta}(T)|\,\mathcal{F}_t]}{\mathbb{E}_\theta[L_{\theta,\vartheta}(T)|\,\mathcal{F}_t]}
$$

together with equations (8.10) and (8.11) to evaluate the differential (8.9). As both $Y_t = \mathbb{E}_\theta[Y|\,\mathcal{F}_t]$ and the likelihood process $L_{\theta,\vartheta}(t)$ are martingales most of the terms in our calculation vanish. There are only two non-zero terms, which can be evaluated by using the two subsequent formulae,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T \alpha_\theta(\tau)dW_\tau\Big)\Big(\int_t^T \eta\sigma_\theta L_{\theta,\vartheta}(\tau)dW_\tau\Big)\,\Big|\,\mathcal{F}_t\Big] \\
&\quad = \mathbb{E}\left[\int_t^T \alpha_\theta(\tau)\eta\sigma_\theta L_{\theta,\vartheta}(\tau)d\tau\,\middle|\,\mathcal{F}_t\right],
\end{aligned}
$$

and, as $\mu(dx,dt)$ is a pure counting measure

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T\int_{\mathbb{R}} \beta(x,\tau)(\mu(dx,d\tau)-\nu_\theta(dx)d\tau)\Big) \\
&\quad\times\Big(\int_t^T\int_{\mathbb{R}} \frac{\nu_{\theta+\vartheta}(dx)-\nu_\theta(dx)}{\nu_\theta(dx)}(\mu(dx,d\tau)-\nu_\theta(dx)d\tau)\Big)\,\Big|\,\mathcal{F}_t\Big] \\
&\quad = \mathbb{E}\left[\int_t^T\int_{\mathbb{R}} \beta(x,\tau)\frac{\nu_{\theta+\vartheta}(dx)-\nu_\theta(dx)}{\nu_\theta(dx)}\nu_\theta(dx)d\tau\,\middle|\,\mathcal{F}_t\right].
\end{aligned}
$$

Hence, we obtain for the differential

$$
\begin{aligned}
&\frac{1}{\vartheta}(\mathbb{E}_{\theta+\vartheta}[Y|\,\mathcal{F}_t] - \mathbb{E}_\theta[Y|\,\mathcal{F}_t]) \\
&= \frac{1}{\vartheta}\left(\frac{\mathbb{E}_\theta[Y L_{\theta,\vartheta}(T)|\,\mathcal{F}_t]}{\mathbb{E}_\theta[L_{\theta,\vartheta}(T)|\,\mathcal{F}_t]} - \mathbb{E}_\theta[Y|\,\mathcal{F}_t]\right) \\
&= \frac{1}{\vartheta}\frac{1}{L_{\theta,\vartheta}(t)}\mathbb{C}ov[Y, L_{\theta,\vartheta}(T)|\,\mathcal{F}_t] \\
&= \frac{1}{\vartheta}\frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}_\theta\Bigg[\left(\int_t^T \alpha_\theta(\tau)dW_\tau + \int_t^T\int_{\mathbb{R}} \beta_\theta(\tau,x)(\mu(dx,d\tau) - \nu_\theta(dx)d\tau)\right) \\
&\quad \times \Bigg(\int_t^T \eta\sigma_\theta L_{\theta,\vartheta}(\tau)dW_\tau \\
&\quad + \int_t^T \lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}} \frac{\nu_{\theta+\vartheta}(dx) - \nu_\theta(dx)}{\nu_\theta(dx)}(\mu(dx,d\tau) - \nu_\theta(dx)d\tau)\Bigg)\Bigg|\,\mathcal{F}_t\Bigg] \\
&= \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}\Bigg[\int_t^T L_{\theta,\vartheta}(\tau)\alpha_\theta(\tau)\frac{1}{\sigma_\theta} \\
&\quad \times \left(\frac{\gamma_{\theta+\vartheta} - \gamma_\theta}{\vartheta} - \int_{-1}^1 x\left(\frac{\nu_{\theta+\vartheta}(dx) - \nu_\theta(dx)}{\vartheta}\right)\right)d\tau \\
&\quad + \int_t^T L_{\theta,\vartheta}(\tau)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}} \beta_\theta(\tau,x)\frac{\nu_{\theta+\vartheta}(dx) - \nu_\theta(dx)}{\vartheta}d\tau\Bigg|\,\mathcal{F}_t\Bigg].
\end{aligned}
$$

Lemma. If the Lévy measure $\nu_\theta(dx)$ and the drift γ_θ are differentiable functions of θ and the processes

$$L_{\theta,\vartheta}(\tau)\alpha_\theta(\tau)\frac{1}{\sigma_\theta}\left(\frac{\gamma_{\theta+\vartheta} - \gamma_\theta}{\vartheta} - \int_{-1}^1 x\left(\frac{\nu_{\theta+\vartheta}(dx) - \nu_\theta(dx)}{\vartheta}\right)\right)$$

and

$$L_{\theta,\vartheta}(\tau)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}} \beta_\theta(\tau,x)\frac{\nu_{\theta+\vartheta}(dx) - \nu_\theta(dx)}{\vartheta}$$

can be dominated by integrable processes, then the derivative of $\mathbb{E}_\theta[Y|\,\mathcal{F}_t]$

exists and is given by

$$\frac{d}{d\theta}\mathbb{E}_\theta[Y|\,\mathcal{F}_t] = \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}\Bigg[\frac{1}{\sigma_\theta}\left(\frac{d}{d\theta}\gamma_\theta - \int_{-1}^{1} x\frac{d}{d\theta}\nu_\theta(dx)\right)\int_t^T L_{\theta,\vartheta}(\tau)\alpha_\theta(\tau)d\tau \\ + \int_t^T L_{\theta,\vartheta}(\tau)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}}\beta_\theta(\tau,x)\frac{d}{d\theta}\nu_\theta(dx)d\tau\,\Bigg|\,\mathcal{F}_t\Bigg].$$

Example 1. As a special case of the above lemma, we investigate the existence of a parameter sensitivity in a model driven by a compound Poisson process. The likelihood density is then

$$L_{\theta,\vartheta}(t) = \exp\left(t(\lambda_\theta - \lambda_{\theta+\vartheta}) + \sum_{\tau<t}\ln\left(\frac{d\nu_{\theta+\vartheta}}{d\nu_\theta}(\Delta X_\tau)\right)\right), \tag{8.12}$$

where $\lambda_\theta = \nu_\theta(\mathbb{R})$ and $\lambda_{\theta+\vartheta} = \nu_{\theta+\vartheta}(\mathbb{R})$ are the jump intensities of the two processes. Applying Itô's lemma to equation (8.12), one obtains the dynamics

$$dL_{\theta,\vartheta}(t) = L_{\theta,\vartheta}(t-)\left((\lambda_\theta - \lambda_{\theta+\vartheta})dt + \int_{\mathbb{R}}\left(\frac{\lambda_{\theta+\vartheta}f_{\theta+\vartheta}(x)}{\lambda_\theta f_\theta(x)} - 1\right)\mu(dx,dt)\right),$$

where f is the density of the jump size distribution. The likelihood function can thus be written as

$$L_{\theta,\vartheta}(t) = \\ 1 + \int_0^t L_{\theta,\vartheta}(\tau-)\left((\lambda_\theta - \lambda_{\theta+\vartheta})d\tau + \int_{\mathbb{R}}\left(\frac{\lambda_{\theta+\vartheta}f_{\theta+\vartheta}(x)}{\lambda_\theta f_\theta(x)} - 1\right)\mu(dx,d\tau)\right).$$

The random variable Y, where $\mathbb{E}_\theta[Y]$ is again the price function of a contingent claim, can be written as

$$Y = \mathbb{E}_\theta[Y] + \int_0^T\int_{\mathbb{R}}\beta_\theta(\tau,x)(\mu(dx,d\tau) - \nu_\theta(dx)d\tau).$$

For the sensitivity we then obtain

$$\frac{1}{\vartheta}(\mathbb{E}_{\theta+\vartheta}[Y|\mathcal{F}_t] - \mathbb{E}_{\theta}[Y|\mathcal{F}_t])$$
$$= \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}_{\theta}\left[\int_t^T L_{\theta,\vartheta}(\tau)\beta(x,\tau)\left(\frac{\lambda_{\theta+\vartheta}f_{\theta+\vartheta} - \lambda_{\theta}f_{\theta}}{\vartheta}\right)d\tau\right|\mathcal{F}_t\right].$$

If $\frac{d}{d\theta}\lambda_{\theta}f_{\theta}$ exists and the expression

$$L_{\theta,\vartheta}(\tau)\beta(x,\tau)\left(\frac{\lambda_{\theta+\vartheta}f_{\theta+\vartheta} - \lambda_{\theta}f_{\theta}}{\vartheta}\right)$$

can be bounded by an integrable function, the sensitivity of the contingent claim with respect to the parameter θ exists and is given by

$$\frac{d}{d\theta}\mathbb{E}_{\theta}[Y|\mathcal{F}_t] = \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}_{\theta}\left[\int_t^T L_{\theta,\vartheta}(\tau)\beta(x,\tau)\frac{d}{d\theta}(\lambda_{\theta}f_{\theta})d\tau\right|\mathcal{F}_t\right].$$

Example 2. Another interesting special case of the general Lévy process case is obtained if one only looks at the continuous part of the Lévy process, that is if one considers a Brownian motion with drift.

$$X_t = \gamma_{\theta}t + \sigma_{\theta}W_t.$$

The likelihood process is then

$$L_{\theta,\vartheta}(t) = \exp\left(\frac{\gamma_{\theta} - \gamma_{\theta+\vartheta}}{\sigma_{\theta}}W_t - \frac{1}{2}\frac{(\gamma_{\theta} - \gamma_{\theta+\vartheta})^2}{\sigma_{\theta}^2}t\right).$$

This can be written as

$$L_{\theta,\vartheta}(t) = 1 + \int_0^t L_{\theta,\vartheta}(\tau-)\frac{\gamma_{\theta} - \gamma_{\theta+\vartheta}}{\sigma_{\theta}}dW_{\tau}.$$

The random variable Y can be written as

$$Y = \mathbb{E}[Y] + \int_0^T \alpha_{\theta}(\tau)dW_{\tau}.$$

If the drift μ_θ is a differentiable functions of θ and the processes

$$L_{\theta,\vartheta}(\tau)\alpha_\theta(\tau)\frac{1}{\sigma_\theta}\left(\frac{\gamma_{\theta+\vartheta}-\gamma_\theta}{\vartheta}\right)$$

can be dominated by integrable processes, then the derivative of $\mathbb{E}_\theta[Y|\,\mathcal{F}_t]$ exists and is given by

$$\frac{d}{d\theta}\mathbb{E}_\theta[Y|\,\mathcal{F}_t] = \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}\left[\int_t^T L_{\theta,\vartheta}(\tau)\frac{\alpha_\theta}{\sigma_\theta}\left(\frac{d}{d\theta}\gamma_\theta\right)d\tau\middle|\,\mathcal{F}_t\right].$$

For a simple t-claim with price process $c(t, S_t)$ the process $\alpha_\theta(\tau)$ is

$$\alpha_\theta(\tau) = c_s(\tau, S_\tau)\sigma\hat{S}_\tau.$$

Example 3. The discounted option price $\hat{c}(t, S_t)$ of a European vanilla option in the variance gamma model can be written as

$$\hat{c}(t,S_t) = \hat{c}(0,S_0) + \int_0^t\int_{\mathbb{R}}(\hat{c}(\tau,S_{\tau-}e^x) - \hat{c}(\tau,S_{\tau-}))(\mu(d\tau,dx) - \nu_\theta(dx)d\tau)$$

Identifying

$$\begin{aligned} Y &= \hat{c}(T,S_T),\\ \mathbb{E}[Y] &= \hat{c}(0,S_0),\\ \alpha_\theta(\tau) &= 0,\\ \beta_\theta(\tau,x) &= \hat{c}(\tau,S_{\tau-}e^x) - \hat{c}(\tau,S_{\tau-}), \end{aligned}$$

the sensitivity $\frac{d}{d\theta}\mathbb{E}_\theta[Y|\mathcal{F}_t]$ becomes

$$\begin{aligned}\frac{d}{d\theta}\mathbb{E}_\theta[Y|\mathcal{F}_t] &= \frac{1}{L_{\theta,\vartheta}(t)}\\ &\times \mathbb{E}_\theta\left[\int_t^T L_{\theta,\vartheta}(\tau)\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus\{-\epsilon,\epsilon\}}(\hat{c}(\tau,S_{\tau-}e^x) - \hat{c}(\tau,S_{\tau-}))\frac{d}{d\theta}\nu_\theta(dx)d\tau\middle|\mathcal{F}_t\right].\end{aligned}$$

As an example we calculate the sensitivity with respect to θ, the drift of the

Brownian motion in the variance gamma model. This sensitivity is

$$\frac{d}{d\theta}\mathbb{E}_\theta[Y|\mathcal{F}_t] = \frac{1}{L_{\theta,\vartheta}(t)}\mathbb{E}_\theta\left[\int_t^T L_{\theta,\vartheta}(\tau)\int_{\mathbb{R}}(\hat{c}(\tau,S_{\tau-}e^x)-\hat{c}(\tau,S_{\tau-}))\right.$$
$$\left.\left(\frac{x}{\sigma^2}-\frac{|x|\theta}{\sqrt{\theta^2+\frac{2\sigma^2}{\kappa}\sigma^2}}\right)\nu_\theta(dx)d\tau\middle|\mathcal{F}_t\right].$$

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Appendix A

Plots

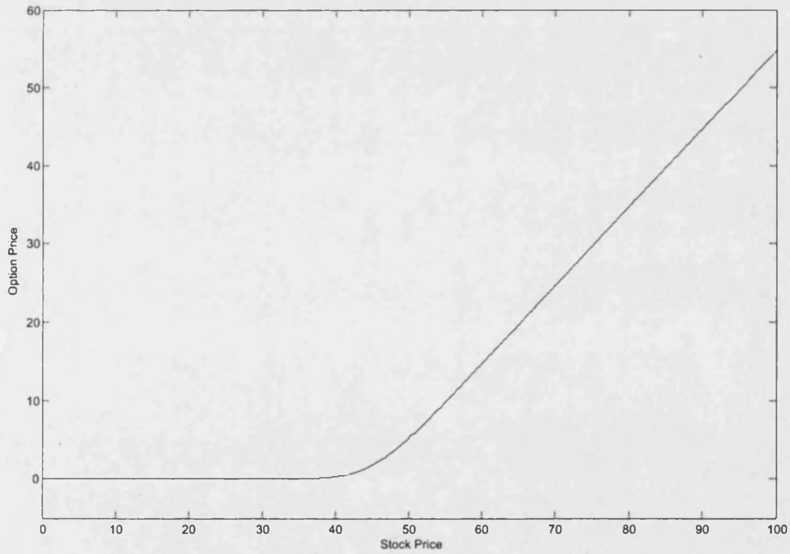

Figure A.1: Price of a call option with strike $K = 50$ in the Merton model

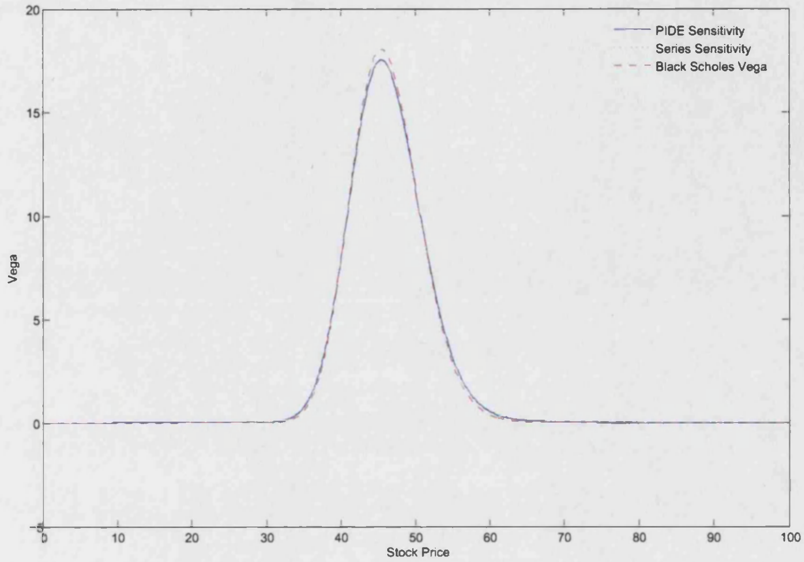

Figure A.2: Vega of a vanilla call option in the Merton model

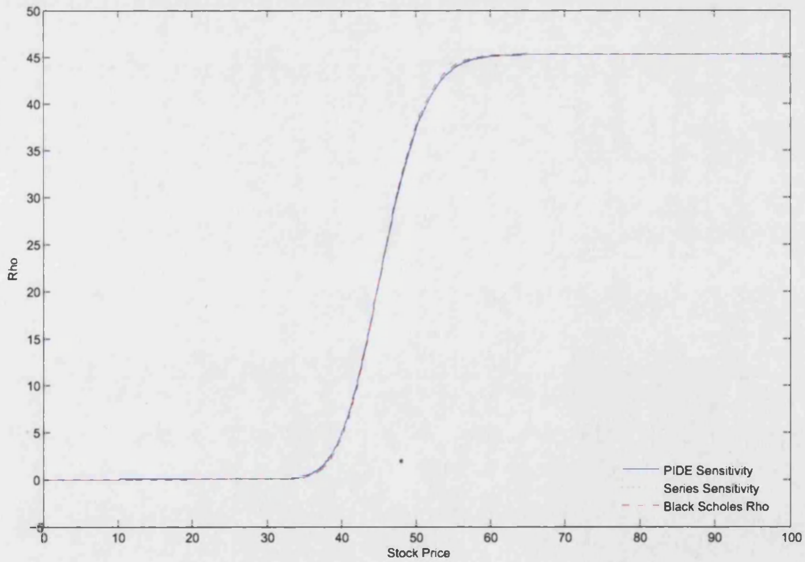

Figure A.3: Rho of a vanilla call option in the Merton Model

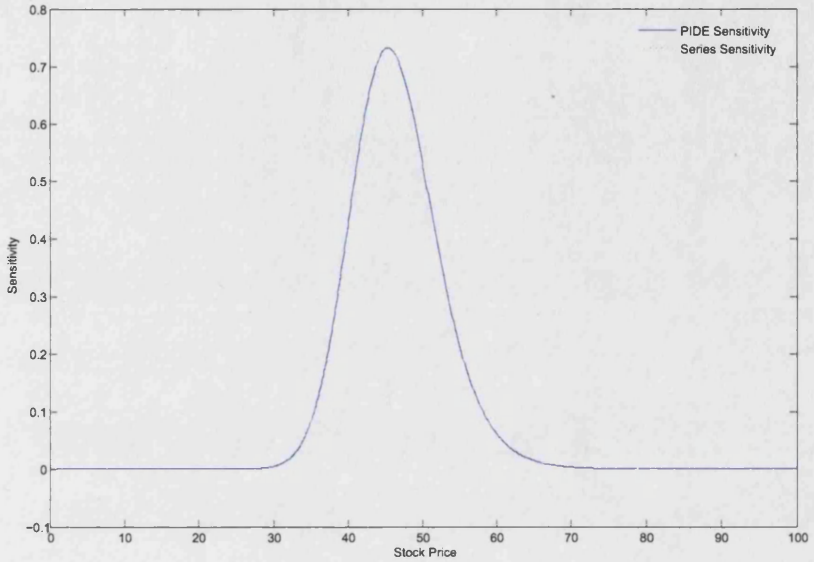

Figure A.4: Sensitivity with respect to changes in the jump intensity λ of a vanilla call option in the Merton model

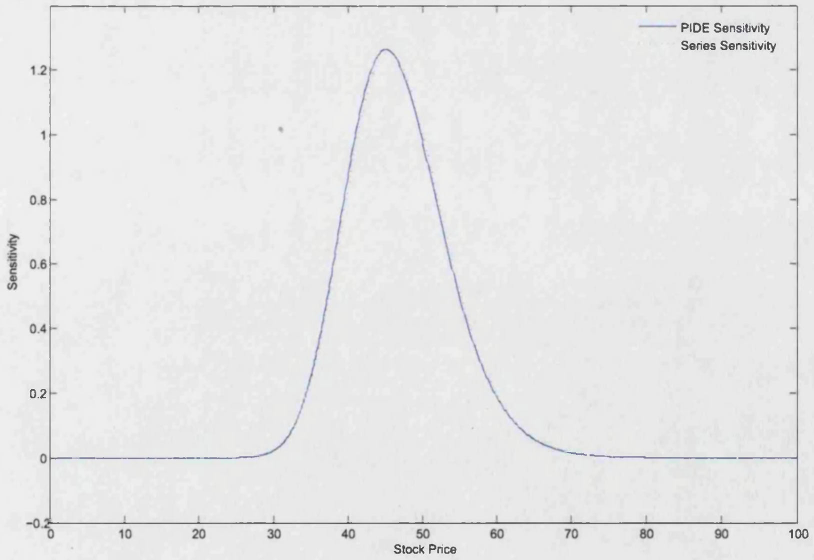

Figure A.5: Sensitivity with respect to changes in the standard deviation δ of the jump size distribution of a vanilla call option in the Merton model

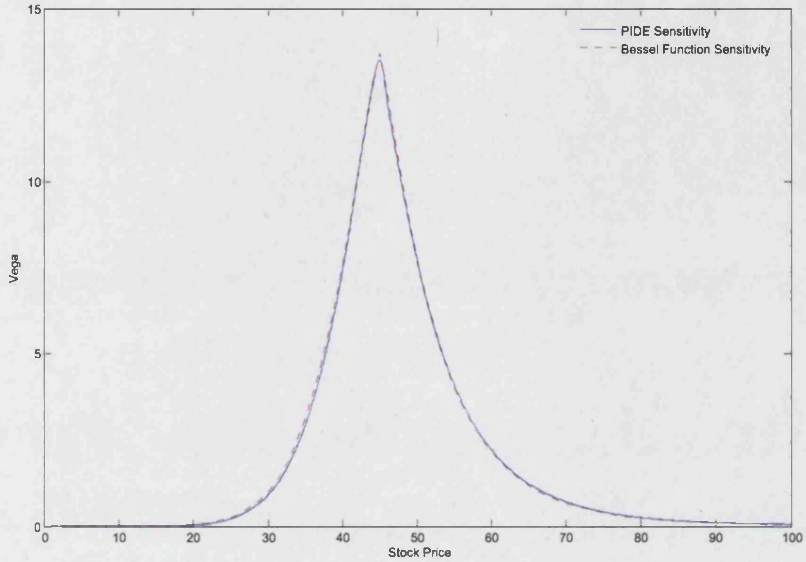

Figure A.6: Sensitivity with respect to changes in σ of a vanilla call option in the variance gamma model

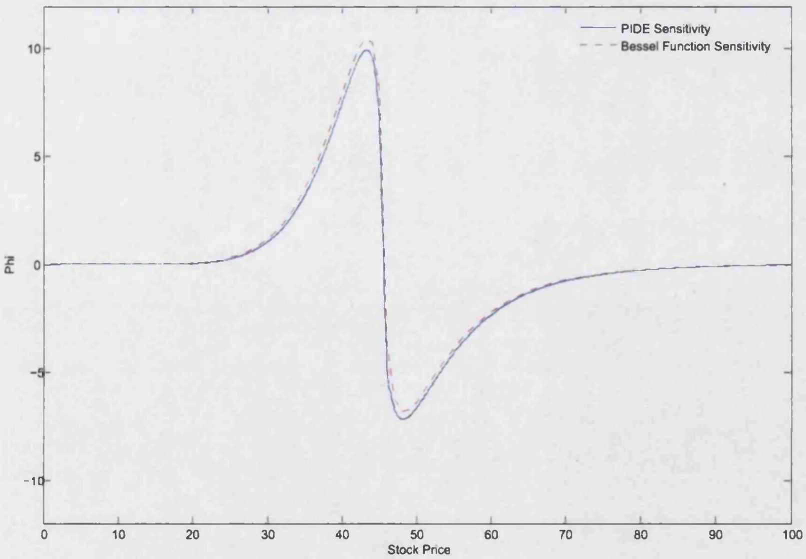

Figure A.7: Sensitivity with respect to changes in θ of a vanilla call option in the variance gamma model

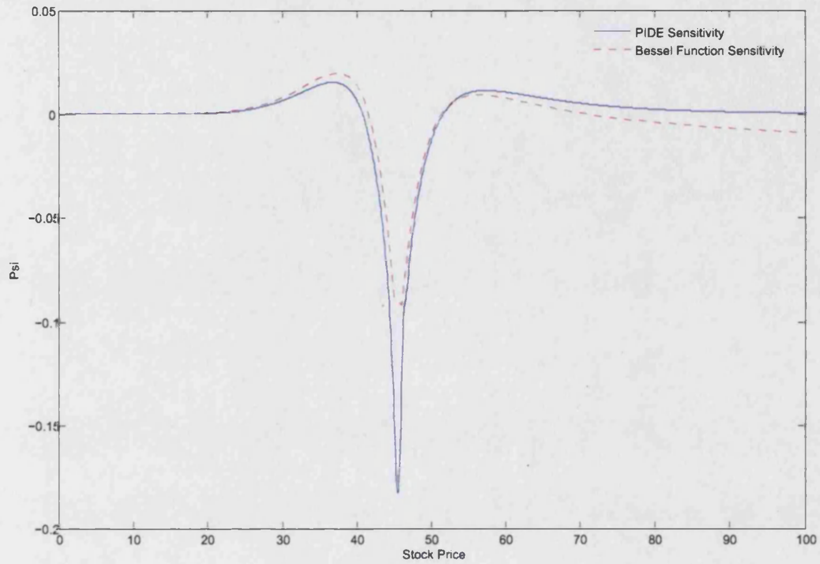

Figure A.8: Sensitivity with respect to changes in κ of a vanilla call option in the variance gamma model

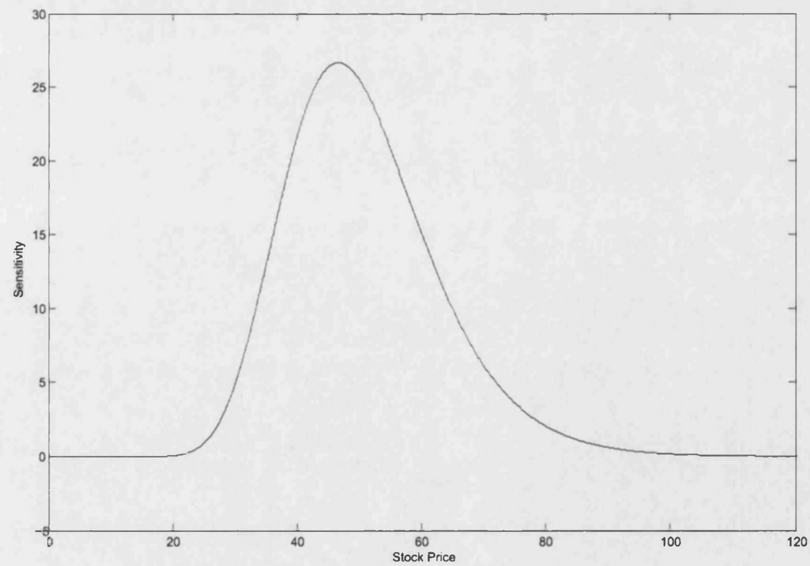

Figure A.9: Sensitivity with respect to changes in C of a vanilla call option in the CGMY model

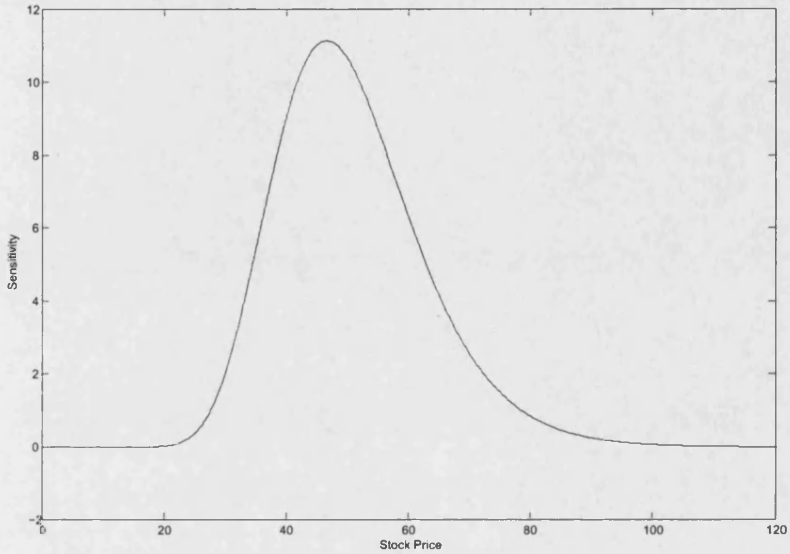

Figure A.10: Sensitivity with respect to changes in Y of a vanilla call option in the CGMY model

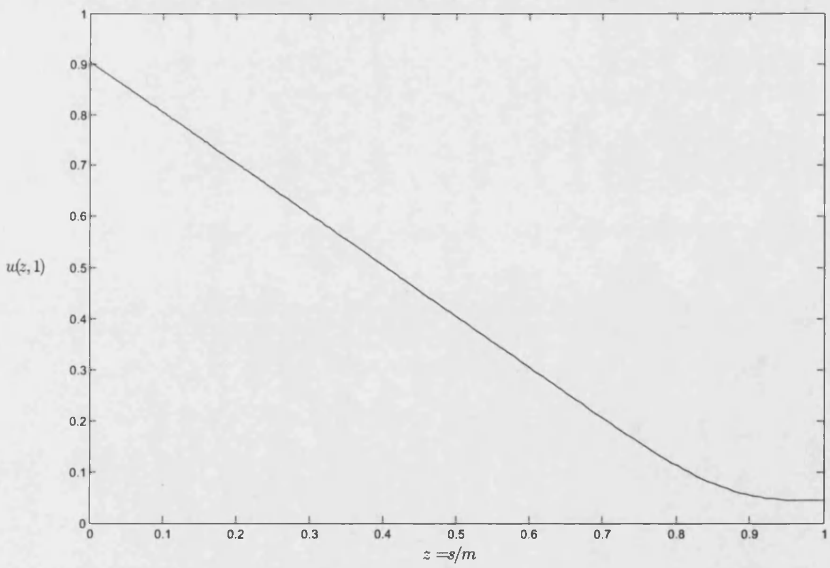

Figure A.11: Lookback option in the Merton model $w(z,t) = \frac{1}{M}c(1,s,M)$

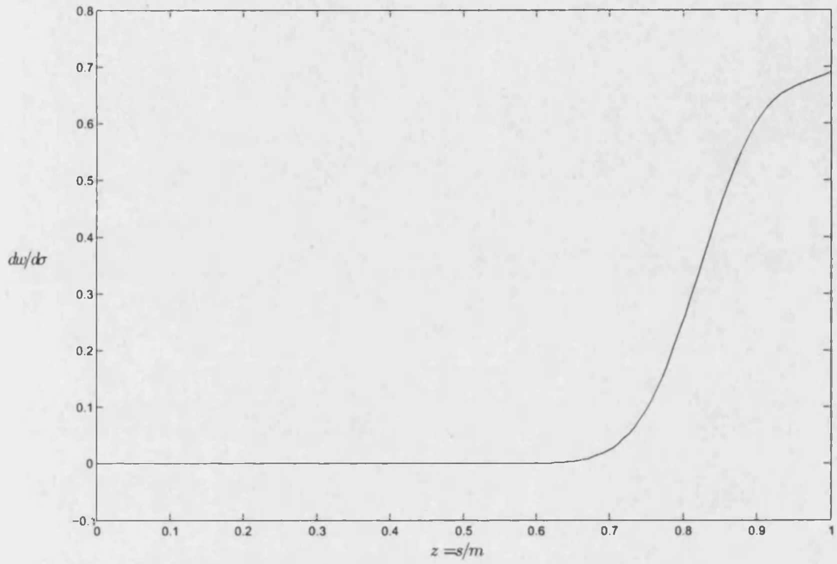

Figure A.12: Sensitivity with respect to changes in σ of a lookback option in the Merton Model

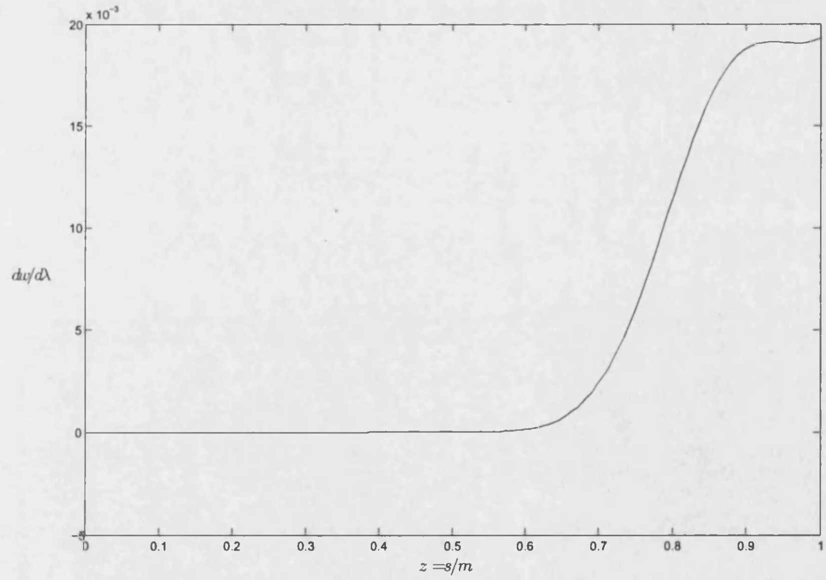

Figure A.13: Sensitivity with respect to changes in λ of a lookback option in the Merton model

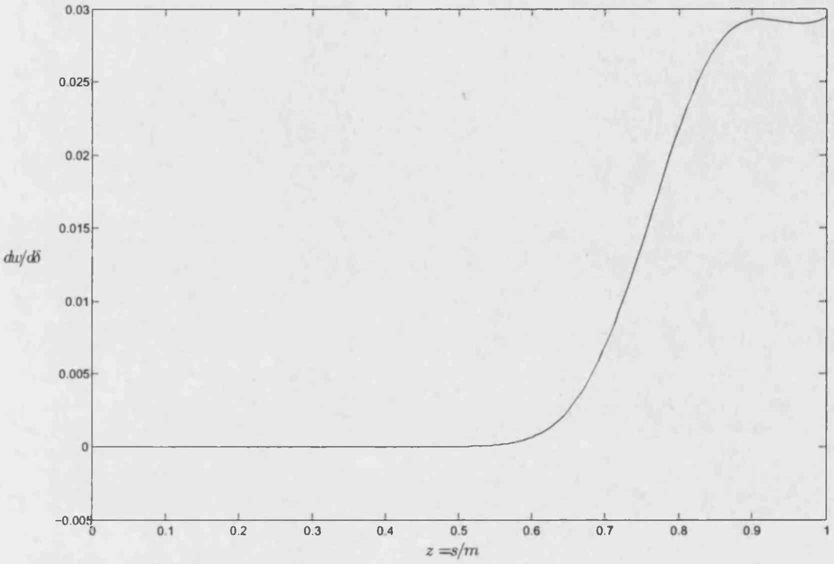

Figure A.14: Sensitivity with respect to changes in δ of a lookback option in the Merton model

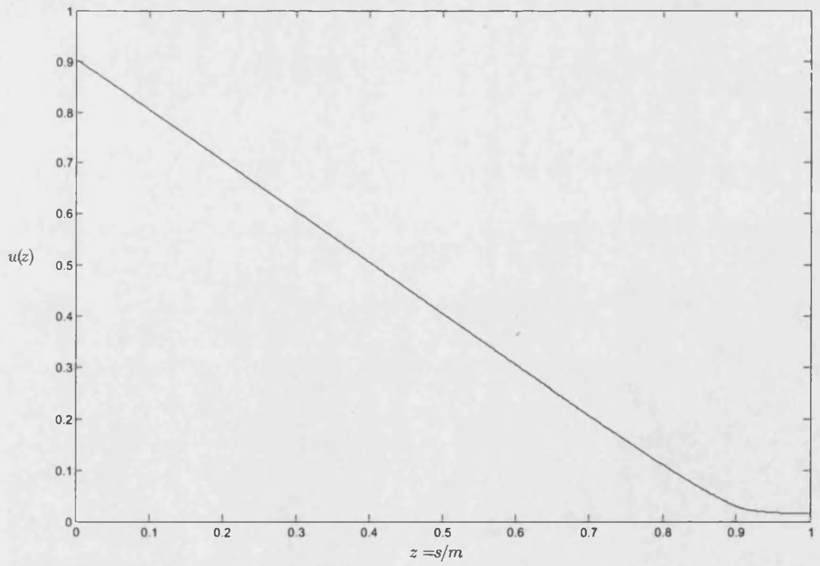

Figure A.15: Lookback option in the variance gamma model
$w(z,t) = \frac{1}{M}\, c(1, s, M)$

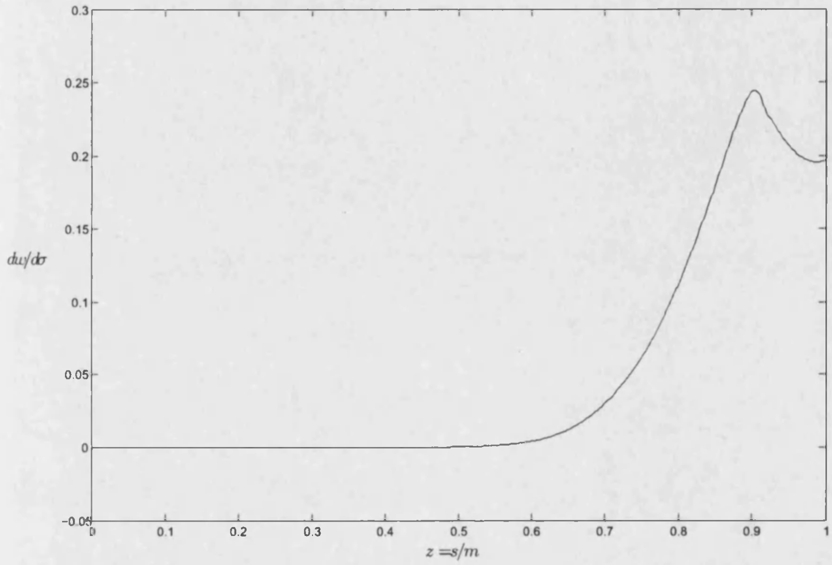

Figure A.16: Sensitivity with respect to changes in σ of a lookback option in the variance gamma model

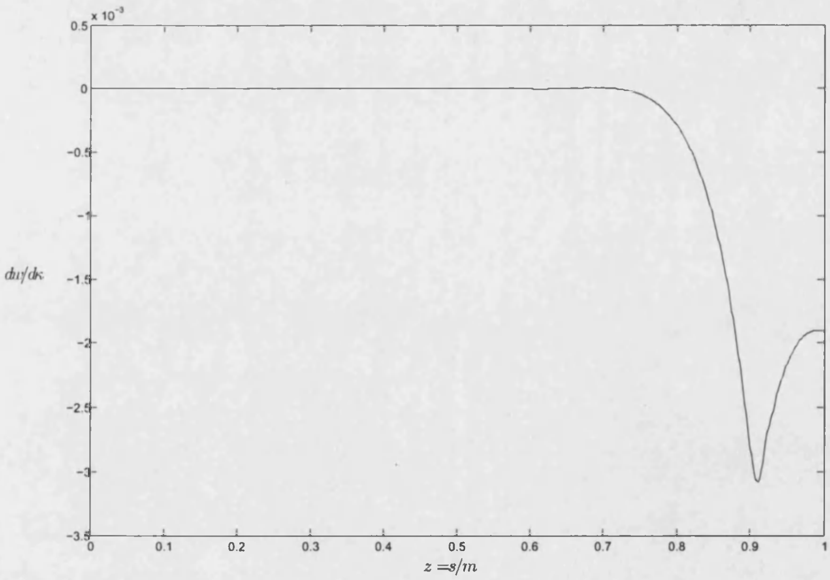

Figure A.17: Sensitivity with respect to changes in κ of a lookback option in the variance gamma model

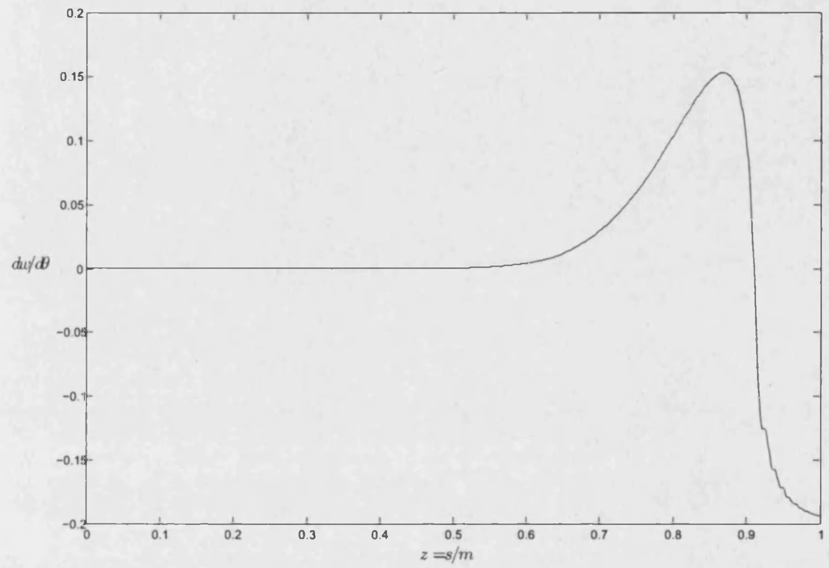

Figure A.18: Sensitivity with respect to changes in θ of a lookback option in the variance gamma model

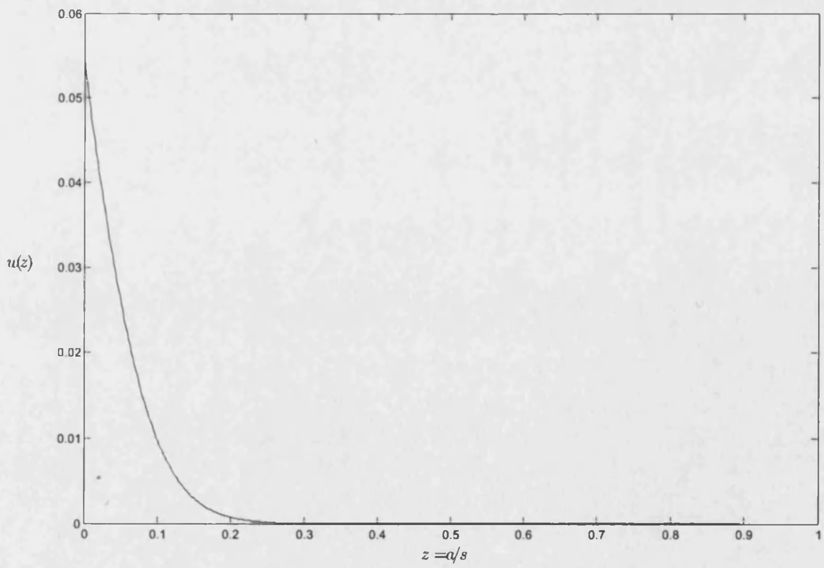

Figure A.19: Asian option in the Merton model $w(t,z) = \frac{1}{s}\, c(t,s,a)$

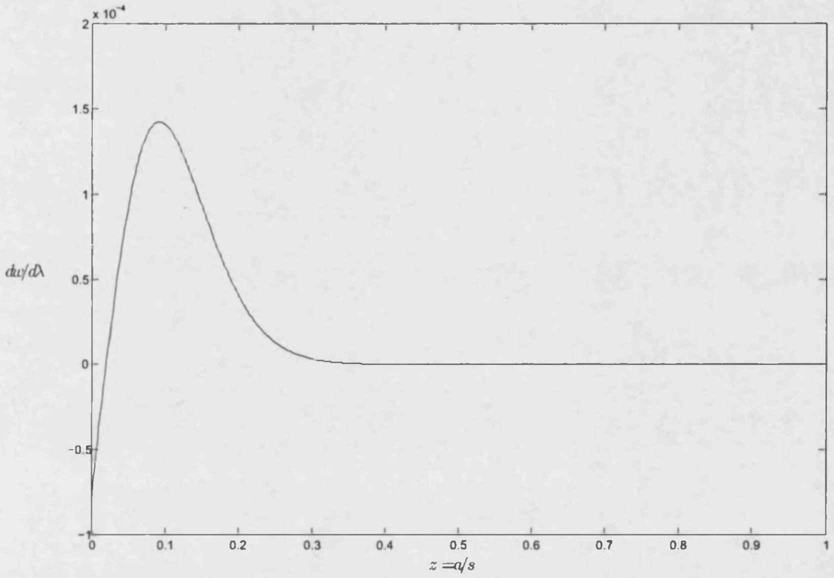

Figure A.20: Sensitivity with respect to changes in λ of an Asian option in the Merton model

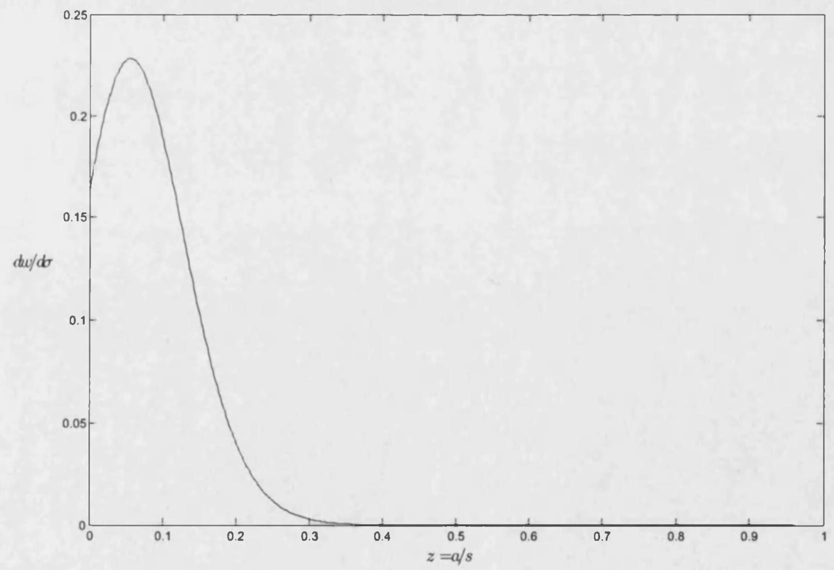

Figure A.21: Sensitivity with respect to changes in σ of an Asian option in the Merton model

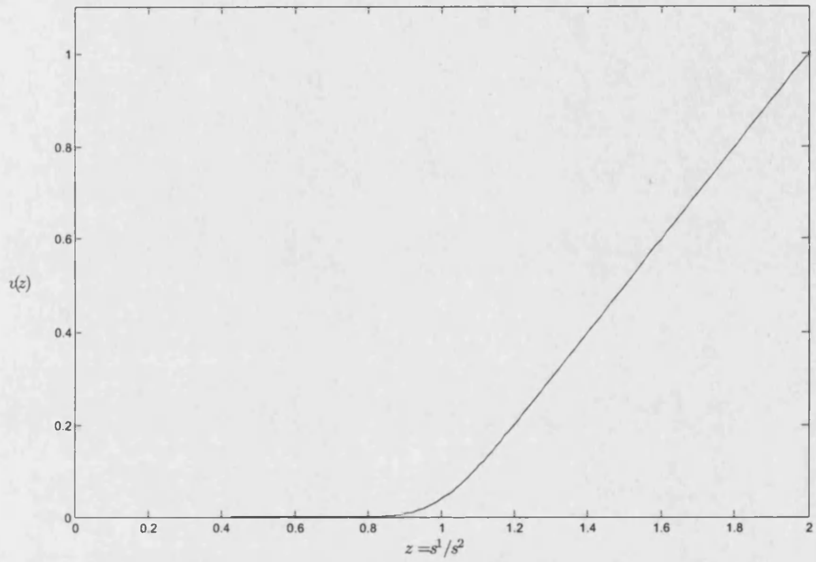

Figure A.22: Exchange option $v(t,z) = \frac{1}{\tilde{s}}\, c(t,s,\tilde{s})$

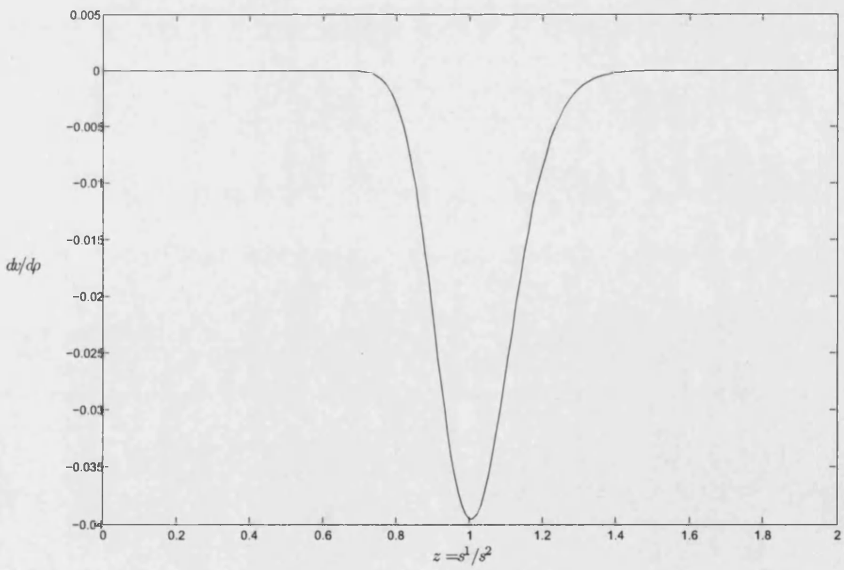

Figure A.23: Sensitivity with respect to changes in ρ of an exchange option in the Merton model

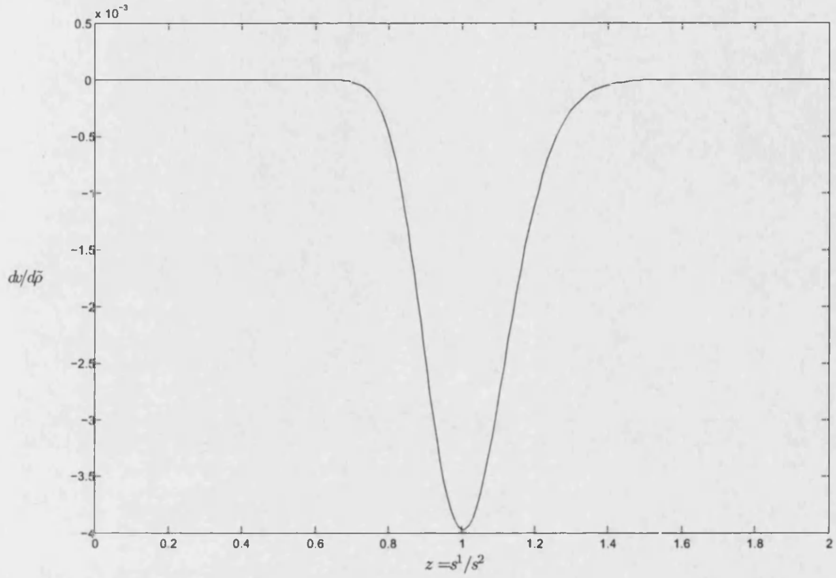

Figure A.24: Sensitivity with respect to changes in ϱ of an exchange option in the Merton model