Excursions of Lévy Processes and Applications in Mathematical Finance and Insurance

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Contents

Declaration

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Abstract

The excursion time of a Lévy process measures the time it spends continuously below or above a given barrier. This thesis contains five papers dealing with the excursions of different Lévy processes and their applications in mathematical finance and insurance. Each of the five papers is presented in one of the chapters of this thesis starting from Chapter 2.

In Chapter 2 the excursions of a Brownian motion with drift below or above a given barrier are studied by using a two-state semi-Markov model. Based on the results *single barrier two-sided Parisian options* are studied and the explicit expressions for the Laplace transforms of their price formulae are given.

In Chapter 3 the excursion time of a Brownian motion with drift outside a corridor is considered by using a four-state semi-Markov model. The results are used to obtain the explicit expressions for the Laplace transforms of the prices of the double barrier Parisian options.

In Chapter 4 Parisian corridor options are introduced and priced by using the results of the excursion time of a Brownian motion with drift inside a corridor.

In Chapter 5 the main focus is the excursions of a Lévy process with negative exponential jumps below a given barrier. Based on the results, a Parisian option whose underlying asset price follows this process is priced, as well as a Parisian type digital option. This is the first ever attempt to price Parisian options involving jump processes. Furthermore, the concept of ruin in risk theory is extended to the Parisian type of ruin.

In Chapter 6 the excursions of a risk surplus process with a more general claim distribution are considered. For the processes without initial reserve, the Parisian ruin probability in an infinite time horizon is calculated. For the positive initial reserve case, only the asymptotic form can be obtained for very large initial reserve and small claim distributions.

Chapter 1

Introduction

The excursion time measures the time a process spends above or below a given barrier. More precisely, the excursion time below (above) a barrier starts counting from zero each time the process crosses the barrier from above (below) and stops counting when the process crosses the barrier from below (above). Mathematically, for a continuous process S the excursions with respect to the barrier L can be defined as follows:

$$g^S_{L,t} = \sup\{s \le t \mid S_s = L\}, \qquad d^S_{L,t} = \inf\{s \ge t \mid S_s = L\}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. The trajectory between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion of process S above or below L, which straddles time t. Assuming $d_1 > 0$, $d_2 > 0$, we now define

$$\tau^S_{1,L} = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > L\}}\left(t - g^S_{L,t}\right) \ge d_1\right\},$$

$$\tau_{2,L}^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t < L\}}\left(t - g_{L,t}^S\right) \geq d_2\right\},$$

$$\tau_L^S = \tau_{1,L}^S \wedge \tau_{2,L}^S.$$

$\tau_{1,L}^S$ is therefore the first time that the length of the excursion of the process S above the barrier L reaches given level d_1; $\tau_{2,L}^S$ corresponds to the one below L; and τ_L^S is the smaller of $\tau_{1,L}^S$ and $\tau_{2,L}^S$. For a jump process, similar definitions are given in Chapters 5 and 6.

The excursion time has very important applications in both mathematical finance and insurance. In mathematical finance, it is the key to price a type of path dependent options, Parisian options.

The Parisian option was first introduced by Chesney, Jeanblanc-Picqué and Yor [13]. Its payoff does not only depend on the final price of the underlying asset, but also its price trajectory during the whole life span of the option. A Parisian option will be either initiated or terminated upon the price reaching a predetermined barrier level L and staying above or below the barrier for a predetermined time D before the maturity date T. Here are two examples. The owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D. For a *Parisian down-and-in option*, the same event gives the owner the right to exercise the option. Now assume S is the price of the underlying asset following a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad S_0 = x, \quad x > 0, \tag{1.1}$$

where W_t with $W_0 = 0$ is a standard Brownian motion under a risk neutral measure Q. Also assume r is the risk-free rate, T is the term of the option, K is the strike price. The price of a *Parisian down-and-out call option* with the barrier L can be expressed as:

$$P_{down-out-call} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau_{2,L}^S > T\}} \left(S_T - K \right)^+ \right);$$

and the price of a *Parisian down-and-in put option* is:

$$P_{down-in-put} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau_{2,L}^S < T\}} \left(K - S_T \right)^+ \right).$$

One advantage of Parisian options is that the cost is lower than the corresponding barrier options and for the knock-out options the owner can keep the right to exercise the options longer. Furthermore, to a certain degree, Parisian options protect the holders from deliberate action taken by the writers. One example is the down-and-out options. For a *barrier down-and-out option*, when the price of its underlying asset is approaching the barrier, an influential agent who has written the option could try to push the price below this barrier, even momentarily, to make the holder lose the right to exercise it and benefit from the elimination of liabilities. In the case of Parisian options, however, this action might prove more difficult or more expensive. For more details see [13].

There are many works concerning the pricing of Parisian options. See for example [13], [38], [46] and [37]. From (1.1) it is clear that in order to study the excursions of the asset price S we just need to study the excursions of the Brownian motion

W, on which S depends. In all works mentioned above the pricing problem was reduced to finding the Laplace transforms of the distribution density functions of the first time the length of the excursion of W reaches level D, i.e. $\tau_{i,L}^{W}$, $i = 1, 2$ and the position of the process W at time $\tau_{i,L}^{W}$, $i = 1, 2$. These were obtained by using the Brownian meander and the Azéma martingale (see [5]). A restriction of this technique is that it relies heavily on the properties of standard Brownian motions; therefore the result cannot be extended to other processes easily. It is also hard to see how it can be used for the pricing of the more complicated options that we will introduce.

In Chapters 2, 3, 4 and 5, a different approach is adopted. For the single barrier Parisian options studied in Chapter 2, a two-state semi-Markov model is considered. This model, however, cannot be applied to Brownian motions directly due to the peculiar properties of the sample paths of Brownian motions. A major problem is the occurrence of an infinite number of very small excursions. In order to solve these problems a new process, *perturbed Brownian motion*, $X^{(\epsilon)}$, where $\epsilon > 0$ is introduced as follows. Assume $L = 0$ and W^{μ} is a Brownian motion with non-negative drift and it starts from zero. Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0, \\ \sigma_n &= \inf \{ t > \delta_n \mid W_t^{\mu} = -\epsilon \}, \\ \delta_{n+1} &= \inf \{ t > \sigma_n \mid W_t^{\mu} = 0 \}, \end{aligned}$$

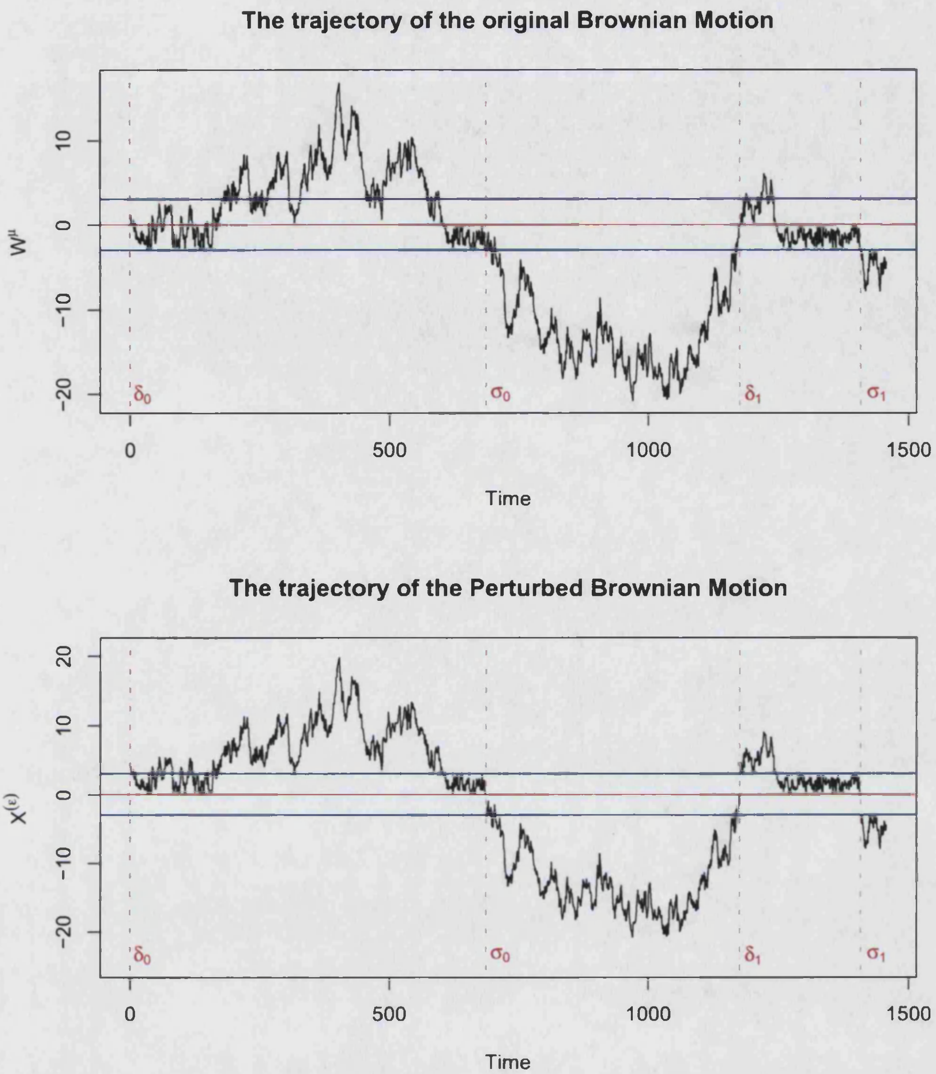

Figure 1.1: A Sample Path of $X^{(\epsilon)}$

where $n = 0, 1, \cdots$. Now define

$$X_t^{(\epsilon)} = \begin{cases} W_t^\mu + \epsilon, & \text{if} \quad \delta_n \le t < \sigma_n \\ W_t^\mu, & \text{if} \quad \sigma_n \le t < \delta_{n+1} \end{cases}, \text{ (see Figure 1.1).}$$

By introducing the jumps to the original Brownian motion, we get the new process $X^{(\epsilon)}$ which has a very clear structure of excursions above and below zero, i.e. the excursions above and below zero alternate with the length of each excursion greater than zero. It will be proved in Chapter 2 that the Laplace transforms of the variables defined based on $X^{(\epsilon)}$ converge to those based on W^{μ} as ϵ goes to 0. As a result, we can obtain the results for W^{μ} by carrying out the calculations for $X^{(\epsilon)}$ and taking the limit $\epsilon \longrightarrow 0$. Hence we will focus on studying the excursions of $X^{(\epsilon)}$ and introduce a two-state semi-Markov model based on it. Set

$$Z_t^X = \begin{cases} 1, & \text{if } X_t^{(\epsilon)} > L \\ 2, & \text{if } X_t^{(\epsilon)} < L \end{cases}.$$

We can now express the variables defined above in terms of Z^X:

$$g_{L,t}^X = \sup\left\{s \leq t \mid Z_s^X \neq Z_t^X\right\},$$

$$d_{L,t}^X = \inf\left\{s \geq t \mid Z_s^X \neq Z_t^X\right\},$$

$$\tau_{1,L}^X = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^X=1\}}\left(t - g_{L,t}^X\right) \geq d_1\right\},$$

$$\tau_{2,L}^X = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^X=2\}}\left(t - g_{L,t}^X\right) \geq d_2\right\},$$

$$\tau_L^X = \tau_{1,L}^X \wedge \tau_{2,L}^X.$$

We then define

$$V_t^X = t - g_{L,t}^X,$$

the time Z^X has spent in its current state. It is easy to see that $\left(Z^X, V^X\right)$ is a Markov process. Z^X is therefore a semi-Markov process with the state space $\{1, 2\}$, where 1 stands for the state when Z^X is above the barrier and 2 corresponds to the state below the barrier.

Furthermore, we set $U^X_{i,k}$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time Z^X spends in state i when it visits i for the kth time. And we have, for each given i and k,

$$U^X_{i,k} = V^X_{d^X_{L,t}} = d^X_{L,t} - g^X_{L,t}, \quad \text{for some } t.$$

Notice that given i, $U^X_{i,k}$, $k = 1, 2, \cdots$, are i.i.d. We therefore define the transition densities for Z^X:

$$p_{ij}(t) = \lim_{\Delta t \to 0} \frac{P\left(t < U^X_{i,k} < t + \Delta t\right)}{\Delta t}.$$

More precisely, according to the definition of Z^X, we actually have the transition densities for Z^X as follows:

$$p_{12}(s) = \frac{\epsilon}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\epsilon + \mu s)^2}{2s}\right\},$$
$$p_{21}(s) = \frac{\epsilon}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\epsilon - \mu s)^2}{2s}\right\}.$$

Based on the countable and alternating structure of the excursions above and below the barrier, together with the transition densities we have obtained there, the Laplace transforms of $\tau^X_{1,L}$, $\tau^X_{2,L}$ and τ^X_L can be easily calculated. Taking the limit $\epsilon \to 0$ yields the Laplace transform of $\tau^{W^\mu}_{1,L}$, $\tau^{W^\mu}_{2,L}$ and $\tau^{W^\mu}_L$ where W^μ is a Brownian motion with drift.

For the double barrier case related to the double barrier Parisian options and the Parisian corridor options in Chapters 3 and 4, a *doubly perturbed Brownian motion* and the four-state semi-Markov model are introduced. In this case, however, the alternating structure of the excursions does not exist anymore. In order to calculate the Laplace transforms of the relevant stopping times, a more advanced technique using the generator of the process is needed.

One of the main differences between the approach in this thesis and the one in [13] in terms of pricing is that, instead of looking for the Laplace transform of the stopping time for W, e.g. $\tau_{2,L}^{W}$, and the position of W at the stopping time, e.g. $W_{\tau_{2,L}^{W}}$, the Laplace transform of the stopping time for a Brownian motion with drift W^{μ} is obtained, e.g $\tau_{2,L}^{W^{\mu}}$, using which the joint probability of the right to exercise the option with respect to an exponential time and the position of W at the exponential time, e.g. $P\left(\tau_{2,L}^{W} < \widetilde{T}, W_{\widetilde{T}} \in \mathrm{d}x\right)$ for a *Parisian down-and-in option*, is calculated, where $\widetilde{T}$, independent of W, is exponentially distributed. The explicit form of the Laplace transform of the option price then can be obtained using this joint probability. Even in the single barrier one-sided case, the formula derived in this thesis involved one integral less than the formula in [13].

Moreover, pricing Parisian options with a jump process has also been attempted in this thesis. A classical surplus process in continuous time $\{X_t\}_{t>0}$ is considered, which is defined by

$$X_t = u + ct - \sum_{k=0}^{N_t} Y_k,$$

where $u \geq 0$ is the initial reserve, c is a constant rate of premium payment per time unit, N_t is the number of claims up to time t which has a Poisson distribution with parameter λ, and Y_k, $k = 1, 2, ...$, are claim sizes which are independent and identically distributed non-negative random variables that are also independent of N_t. We also assume $c > \lambda E(Y_1)$ (the *net profit condition*). Our underlying asset price follows

$$S_t = \exp(X_t), \text{ with } S_0 = e^u.$$

Since the process itself has the countable and alternating structure of excursions above and below zero, a similar technique as that in Chapter 2 can be directly applied to obtain the Laplace transform of the stopping time we are interested in, so as the Laplace transform of the option price. The transition densities required to complete the calculation can be calculated by inverting their Laplace transforms, which can be obtained by applying the optimal sampling theory to certain martingales.

As mentioned at the beginning, another application of the excursion time is in insurance. According to the bankruptcy regulations in many counties, such as U.S., Japan and France, the defaulted firm is granted some "grace" period before liquidation, during which the firm is given the chance to reorganize and to put its finance back in order. As a result, instead of the classical ruin, it makes more sense to consider the risk of a Parisian type of ruin, for which to occur, the surplus process must fall below zero and stay negative for a continuous time interval of specified length.

Two cases are discussed here, one with zero initial reserve, i.e. $u = 0$ and one with positive initial reserve, i.e. $u > 0$. With zero initial reserve, the probability of a Parisian type of ruin ever occurring is calculated, which can not be done for a general $u > 0$ and a general claim distribution. An asymptotic form is obtained for large u and small claim size. For an exponential claim distribution, however, an explicit form for the Parisian ruin probability in the finite time horizon is calculated for a general $u \geq 0$.

In Chapter 2, the excursion time is studied in a more general framework using a simple semi-Markov model consisting of two states indicating whether the process is above or below a fixed level L. Based on these results, for the first time, the explicit form of the Laplace transforms of the prices of the single barrier one-sided Parisian options defined in [13] are given. One can then invert the Laplace transforms using techniques as in [38].

Furthermore, the *single-barrier two-sided Parisian options* are studied. In contrast to the Parisian options mentioned above, the excursions below and above the barrier should both be considered. The explicit forms of the Laplace transforms for the prices of this type of options are also obtained.

In Chapter 3 the excursion time outside a given corridor is studied using a semi-Markov model consisting of four states. Applying these results gives the explicit forms of the Laplace transforms for the prices of double barrier Parisian options.

In Chapter 4 the main focus is on the excursion time inside the corridor. By using the similar technique as in Chapter 3 the explicit forms of the Laplace transforms for the prices of Parisian corridor options are calculated.

In Chapter 5 the excursions of a classical surplus process with negative exponential jumps below a given level are studied. Based on the result, pricing a Parisian option and a Parisian type digital option, whose underlying asset prices follow this jump process is attempted for the first time. The Parisian type of ruin is introduced here and the explicit form for the Parisian ruin probability in the finite time horizon for exponential claims is calculated. Moreover, a diffusion approximation is carried out to obtain similar results for Brownian motions with drift.

In Chapter 6 the Parisian ruin probabilities are studied for a general claim distribution. The probability of ruin in the infinite horizon is obtained for the processes without initial reserve. For positive initial reserve case, only an asymptotic form for large initial reserve can be obtained for small claim distributions. It is shown that in the small claim case an asymptotic formula similar to Cramér's formula, i.e. Ce^{-ru} where u is the initial reserve, is true.

Each of Chapters 2, 3, 4, 5, and 6 are independent papers. To keep the papers as self-contained as possible, some definitions and preliminary results are repeated in each paper.

Chapter 2

Perturbed Brownian Motion and Its Application to Parisian Option Pricing

Abstract

In this paper, we study the excursion time of a Brownian motion with drift below and above a given level by using a simple two-state semi-Markov model. In mathematical finance, these results have an important application in the valuation of path dependent options such as Parisian options. Based on our results single barrier two-sided Parisian options are priced.

Keywords: excursion time, two-state semi-Markov model, path dependent options, Parisian options, Laplace transform.

2.1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picqué and Yor [13]. A Parisian option is a special case of path dependent options. Its payoff does not only depend on the final price of the underlying asset, but also its price trajectory during the whole life span of the option. More precisely, a Parisian option will be either initiated or terminated upon the price reaching a predetermined barrier level L and staying above or below the barrier for a predetermined time D before the maturity date T.

There are two different ways of measuring the time spent above or below the barrier, corresponding to the excursion time and the occupation time defined below. The excursion time below (above) the barrier starts counting from 0 each time the process crosses the barrier from above (below) and stops counting when the process crosses the barrier from below (above). The occupation time up to a specific time t adds up all the time the process spend below (above) the barrier; it is therefore the summation of all excursion time intervals before time t. In [13] the Parisian options related to the occupation time are called *cumulative Parisian options.*

The owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D. For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. The owner of a *cumulative Parisian down-and-out option* loses the option if the total time the underlying asset price S stays below L up to the end of the contract for longer than D. For details on the pricing of Parisian options see [13], [37], [38] and [46]. For cumulative Parisian options see [13] and [41] and since these are related to the occupation times and hence the quantiles of the process, also see [1], [17] and [42]. For American Parisian options, see [11]. In this paper, we focus on the Parisian option defined upon the excursion time.

From the description above, we can see that the key for pricing a Parisian option is the derivation of the distribution of the excursion time. As in [13] we first focus on finding the Laplace transform of the first time the length of the excursion reaches level D. In [13] this was obtained by using the Brownian meander and the Azéma martingale (see [5]). A restriction of this technique is that it relies heavily on the properties of standard Brownian motions; therefore the result cannot be extended to other processes easily. It is also hard to see how it can be used for the pricing of the more complicated options that we will introduce.

In this paper, we are going to study the excursion time in a more general framework using a simple semi-Markov model consisting of two states indicating whether the process is above or below a fixed level L . By applying this model, we can, for the first time, get the explicit forms of the Laplace transforms for the prices of the Parisian options defined in [13]. One can then invert the Laplace transforms using techniques as in [38] and [6].

Furthermore, we study the *single-barrier two-sided Parisian options.* In contrast to the Parisian options mentioned above, we consider the excursions both below and above the barrier. Let us look at two examples, depending on whether the condition is that the required excursions above and below the barrier have to both happen before the maturity date or that either one of them happens before the maturity.

In one example, the owner of a *Parisian Max Out option* loses the option if the underlying asset price S has both an excursion above the barrier for longer than d_1 and below the barrier for longer than d_2 before the maturity of the option. In another example, the owner of a *Parisian Min Out option* loses the right to exercise the option if there is either an excursion above the barrier for longer than d_1 or below the barrier for longer than d_2 before the maturity. For more details, see [12]. Later on, we will give the explicit forms of the Laplace transforms for the prices of this type of options.

In Section 2.2 we give the mathematical definitions and set out the model. We also introduce a new process, *perturbed Brownian motion*, which has the same behavior as a Brownian motion except that each time when it hits 0, it jumps towards the other side of 0 by size ϵ. In Section 2.3 we present an important lemma for the perturbed Brownian motion together with its proof, which will be used in the following sections. We give our main results for Brownian motions in Section 2.4, including the Laplace transforms for the stopping times we define for both Brownian motions with drift and standard Brownian motions, which are vital for the pricing. In Section 2.5 we focus on pricing our newly defined Parisian options by using the results in Section 2.4. As a special case, we also give the explicit forms of the Laplace transforms for the prices of the Parisian options studied in [13] for the first time. In [13] these were given in the form of double integrals. Using a different approach yields explicit results in our paper (see remark after Corollary 2.4.3.1 later).

2.2 Definitions

We are going to use the same definition for the excursion as in [13], [14] and [43]. Let L be the level of the barrier and assume S is the price of the underlying asset following a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad S_0 = x, \quad x > 0, \tag{2.1}$$

where W_t with $W_0 = 0$ is a standard Brownian motion under a risk neutral measure Q. As in [13], we define

$$g^S_{L,t} = \sup\{s \leq t \mid S_s = L\}, \qquad d^S_{L,t} = \inf\{s \geq t \mid S_s = L\} \tag{2.2}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. The trajectory between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion of process S, which straddles time t. Assuming $d_1 > 0$, $d_2 > 0$, we now define

$$\tau^S_{1,L} = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > L\}}\left(t - g^S_{L,t}\right) \geq d_1\right\}, \tag{2.3}$$

$$\tau^S_{2,L} = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t < L\}}\left(t - g^S_{L,t}\right) \geq d_2\right\}, \tag{2.4}$$

$$\tau^S_L = \tau^S_{1,L} \wedge \tau^S_{2,L}. \tag{2.5}$$

$\tau^S_{1,L}$ is therefore the first time that the length of the excursion of the process S above the barrier L reaches given level d_1; $\tau^S_{2,L}$ corresponds to the one below L; and τ^S_L is the smaller of $\tau^S_{1,L}$ and $\tau^S_{2,L}$.

Assume r is the risk-free rate, T is the term of the option, K is the strike price, S is the underlying asset price defined as above. If we have an up-out Parisian call option with the barrier L, its price can be expressed as:

$$P_{up-out-call} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau^S_{1,L} > T\}} (S_T - K)^+ \right);$$

and the price of a down-in Parisian put option with the barrier L is:

$$P_{down-in-put} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau^S_{2,L} < T\}} (K - S_T)^+ \right).$$

Without loss of generality, from now on, we assume $L = 0$. We simplify the expressions of $g^S_{0,t}$, $d^S_{0,t}$, τ^S_0, $\tau^S_{1,0}$ and $\tau^S_{2,0}$ by g^S_t, d^S_t, τ^S, τ^S_1 and τ^S_2.

From (2.1) we can see that in order to study the excursion of the asset price S we just need to study the excursion of the Brownian motion W. However, the peculiar properties of the sample paths of Brownian motions result in many difficulties. A major problem is the occurrence of an infinite number of very small excursions. In order to solve these problems we introduce a new process, *perturbed Brownian motion*, $X^{(\epsilon)}$, where $\epsilon > 0$ as follows. Assume W^μ is a Brownian motion with non-negative drift and it starts from 0. Define a sequence of stopping times

$$\begin{aligned}
\delta_0 &= 0, \\
\sigma_n &= \inf\{t > \delta_n \mid W^\mu_t = -\epsilon\}, \\
\delta_{n+1} &= \inf\{t > \sigma_n \mid W^\mu_t = 0\},
\end{aligned}$$

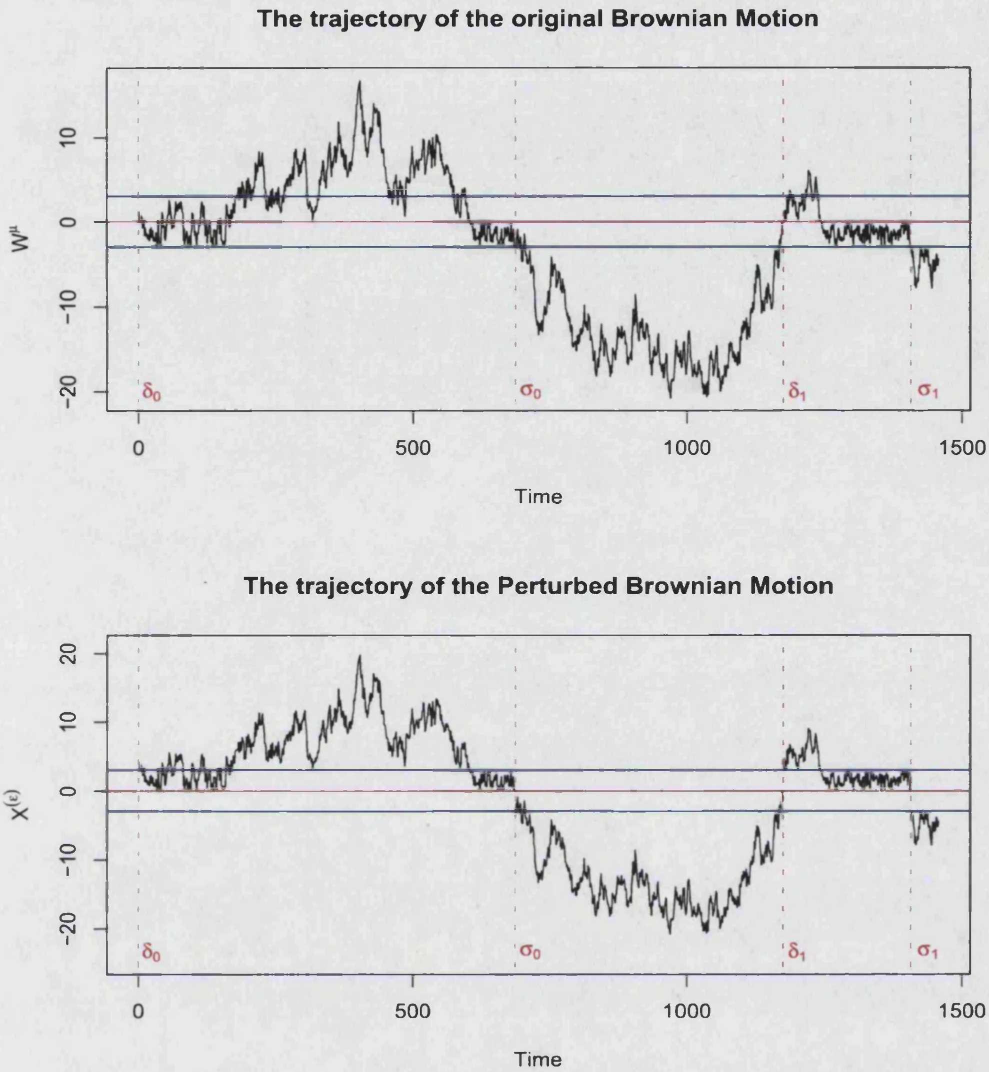

Figure 2.1: A Sample Path of $X^{(\epsilon)}$

where $n = 0, 1, \cdots$. Now define

$$X_t^{(\epsilon)} = \begin{cases} W_t^{\mu} + \epsilon, & \text{if} \quad \delta_n \le t < \sigma_n \\ W_t^{\mu}, & \text{if} \quad \sigma_n \le t < \delta_{n+1} \end{cases}, \text{ (see Figure 2.1).}$$

By introducing the jumps to the original Brownian motion, we get this new process $X^{(\epsilon)}$ which has a very clear structure of excursions above and below 0, i.e. the excursions above and below 0 alternate with the length of each excursion greater than 0. In the later section we prove that the Laplace transforms of the variables defined based on $X^{(\epsilon)}$ converge to those based on W^{μ} as ϵ goes to 0. As a result, we can obtain the results for W^{μ} by carrying out the calculations for $X^{(\epsilon)}$ and taking the limit $\epsilon \longrightarrow 0$; for more details see Theorem 2.4.1. Hence we will focus on studying the excursions of $X^{(\epsilon)}$ in the rest of this section and next section.

From the description of the excursion above, it is clear that we are actually considering two states, the state when the process is above the barrier and the state when it is below. For each state, we are interested in the time the process spends in it. We introduce a new process based on $X^{(\epsilon)}$.

$$Z_t^X = \begin{cases} 1, & \text{if } X_t^{(\epsilon)} > 0 \\ 2, & \text{if } X_t^{(\epsilon)} < 0 \end{cases} .$$

In this definition, we deliberately ignore the situation when $Z_t^X = 0$. It is because process Z^X satisfies

$$\int_0^t \mathbf{1}_{\{Z_u^X = 0\}} \mathrm{d}u = 0.$$

We can now express the variables defined above in terms of Z^X:

$$g_t^X = \sup \{ s \leq t \mid Z_s^X \neq Z_t^X \}, \tag{2.6}$$

$$d_t^X = \inf \{ s \geq t \mid Z_s^X \neq Z_t^X \}, \tag{2.7}$$

$$\tau_1^X = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^X = 1\}} \left(t - g_t^X \right) \geq d_1 \right\}, \tag{2.8}$$

$$\tau_2^X = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^X = 2\}} \left(t - g_t^X \right) \geq d_2 \right\}, \tag{2.9}$$

$$\tau^X = \tau_1^X \wedge \tau_2^X. \tag{2.10}$$

We then define

$$V_t^X = t - g_t^X,$$

the time Z^X has spent in its current state. It is easy to see that $\left(Z^X, V^X\right)$ is a Markov process. Z^X is therefore a semi-Markov processes with the state space $\{1, 2\}$, where 1 stands for the state when Z^X is above zero and 2 corresponds to the state below zero.

Furthermore, we set $U_{i,k}^X$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time Z^X spends in state i when it visits i for the kth time. And we have, for each given i and k,

$$U_{i,k}^X = V_{d_{L,t}^X}^X = d_t^X - g_t^X, \quad \text{for some } t.$$

Notice that given i, $U_{i,k}^X$, $k = 1, 2, \cdots$, are i.i.d. We therefore define the transition densities for Z^X:

$$p_{ij}(t) = \lim_{\Delta t \to 0} \frac{P\left(t < U_{i,k}^X < t + \Delta t\right)}{\Delta t},$$

$$P_{ij}(t) = P\left(U_{i,k}^X < t\right), \quad \bar{P}_{ij}(t) = P\left(U_{i,k}^X > t\right).$$

We have

$$P_{ij}(t) = \int_0^t p_{ij}(s) \mathrm{d}s = 1 - \bar{P}_{ij}(t),$$

which is actually the probability that the process will stay in state i for no more than time t. More precisely, according to the definition of Z^X, we actually have the transition densities for Z^X as follows:

$$p_{12}(s) = \frac{\epsilon}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\epsilon+\mu s)^2}{2s}\right\}, \tag{2.11}$$

$$p_{21}(s) = \frac{\epsilon}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\epsilon-\mu s)^2}{2s}\right\}, \tag{2.12}$$

where $p_{12}(s)$ is actually the density of the first time that a Brownian motion with drift started from ϵ hits 0 and $p_{21}(s)$ is the density of the first time that a Brownian motion with drift started from $-\epsilon$ hits 0.

2.3 An Important Lemma

In this section, we will present an important lemma for $X^{(\epsilon)}$ together with its proof.

Lemma 2.3.1 *For the perturbed Brownian motion $X^{(\epsilon)}$, we have the following results:*

$$E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\} \mathbf{1}_{\{\tau_1^X<\tau_2^X\}}\right) \tag{2.13}$$
$$= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{21}(d_2)\int_{d_1}^{\infty} e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s}{G(d_1,d_2)},$$

$$E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\} \mathbf{1}_{\{\tau_1^X>\tau_2^X\}}\right) \tag{2.14}$$
$$= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{12}(d_1)\int_{d_2}^{\infty} e^{-\alpha_1 s}p_{21}(s)\mathrm{d}s\int_0^{d_1} e^{-(\alpha_1+\alpha_2)s}p_{12}(s)\mathrm{d}s}{G(d_1,d_2)},$$

where

$$\begin{aligned} G(d_1,d_2) &= \left\{1-\int_0^{d_1} e^{-(\alpha_1+\alpha_2)s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-(\alpha_1+\alpha_2)s}p_{21}(s)\mathrm{d}s\right\} \\ &\quad \left\{1-\int_0^{\infty} e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\alpha_2 s}p_{21}(s)\mathrm{d}s\right\}. \end{aligned}$$

Proof: Let A_j^i denotes the event that the first time the length of the excursion above zero reaches d_1 happens during the ith excursion above zero, and the first time the length of the excursion below zero reaches d_2 happens during the jth excursion below zero. So we have,

$$E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\mathbf{1}_{\{\tau_1^X<\tau_2^X\}}\right)$$
$$=\sum_{j=1}^{\infty}\sum_{i=1}^{j} E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\Big|A_j^i\right)P\left(A_j^i\right),$$

and

$$E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\mathbf{1}_{\{\tau_1^X>\tau_2^X\}}\right)$$
$$=\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\Big|A_j^i\right)P\left(A_j^i\right).$$

Since excursions above and below alternate, given event A_j^i, τ_1^X is comprised of $i-1$ full excursions below zero with the length less than d_2, $i-1$ full excursions above zero with the length less than d_1 and the last one with the length d_1. We have

$$\tau_1^X\Big|A_j^i = U_{1,1}^X + U_{1,2}^X + \cdots + U_{1,i-1}^X + U_{2,1}^X + U_{2,2}^X + \cdots + U_{2,i-1}^X + d_1,$$

where $U_{1,k}^X < d_1$ for $k = 1, \cdots, i-1$, $U_{2,k}^X < d_2$ for $k = 1, \cdots, j-1$, $U_{1,i}^X \geq d_1$ and $U_{2,j}^X \geq d_2$. For simplicity, we denote the above condition of $U_{n,k}^X$'s by C. Similarly, for τ_2^X, we have

$$\tau_2^X\Big|A_j^i = U_{1,1}^X + U_{1,2}^X + \cdots + U_{1,j}^X + U_{2,1}^X + U_{2,2}^X + \cdots + U_{2,j-1}^X + d_2,$$

where $U_{n,k}^X$'s satisfy the condition C.

More importantly, due to the Markov property of $X^{(\epsilon)}$, these excursions are independent of each other. $U_{1,n}^X$'s have distribution P_{12}; $U_{2,n}^X$'s have distribution P_{21}. As a result, when $i \leq j$,

$$
\begin{aligned}
& E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\} \middle| A_j^i\right) \\
= \; & E\left(\exp\left\{-\alpha_1\left\{\sum_{k=1}^{i-1}\left(U_{1,k}^X + U_{2,k}^X\right) + d_1\right\}\right.\right. \\
& \left.\left. -\alpha_2\left\{\sum_{k=1}^{j-1}\left(U_{1,k}^X + U_{2,k}^X\right) + U_{1,j}^X + d_2\right\}\right\} \middle| C\right) \\
= \; & e^{-\alpha_1 d_1 - \alpha_2 d_2}\left\{\int_0^{d_1} e^{-(\alpha_1+\alpha_2)s}\frac{p_{12}(s)}{P_{12}(d_1)}\mathrm{d}s\right\}^{i-1}\left\{\int_{d_1}^{\infty} e^{-\alpha_2 s}\frac{p_{12}(s)}{\bar{P}_{12}(d_1)}\mathrm{d}s\right\} \\
& \left\{\int_0^{\infty} e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s\right\}^{j-i}\left\{\int_0^{d_2} e^{-\alpha_2 s}\frac{p_{21}(s)}{P_{21}(d_2)}\mathrm{d}s\right\}^{j-i} \\
& \left\{\int_0^{d_2} e^{-(\alpha_1+\alpha_2)s}\frac{p_{21}(s)}{P_{21}(d_2)}\mathrm{d}s\right\}^{i-1},
\end{aligned}
$$

and

$$
P(A_j^i) = P_{12}(d_1)^{i-1}P_{21}(d_2)^{j-1}\bar{P}_{12}(d_1)\bar{P}_{21}(d_2).
$$

We have therefore

$$
\begin{aligned}
& E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\mathbf{1}_{\{\tau_1^X < \tau_2^X\}}\right) \\
= \; & \sum_{j=1}^{\infty}\sum_{i=1}^{j} E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\} \middle| A_j^i\right) P\left(A_j^i\right) \\
= \; & \frac{e^{-\alpha_1 d_1 - \alpha_2 d_2}\bar{P}_{21}(d_2)\int_{d_1}^{\infty} e^{-\alpha_2 u}p_{12}(s)\mathrm{d}s}{G(d_1, d_2)}.
\end{aligned}
$$

The proof of the case when $\tau_1^X > \tau_2^X$ follows the same steps.

□

Remark: We can get $E\left(\exp\left\{-\alpha_1\tau_1^X - \alpha_2\tau_2^X\right\}\right)$ by adding up (2.13) and (2.14).

2.4 Main Results

In this section we show how to obtain results for Brownian motions through $X^{(\epsilon)}$.

In order to simplify the expressions, we define

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2},$$

where $\mathcal{N}(.)$ is the cumulative distribution function for the standard Normal distribution.

Theorem 2.4.1 *For a Brownian motion W^μ with $W_0^\mu = 0$, $\mu \geq 0$, $\tau_1^{W^\mu}$, $\tau_2^{W^\mu}$ and τ^{W^μ} defined as in (2.3), (2.4) and (2.5) with $S = W^\mu$, we have following Laplace transforms:*

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\left\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\right\}}\right) = \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}, \tag{2.15}$$

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\left\{\tau_1^{W^\mu}>\tau_2^{W^\mu}\right\}}\right) = \frac{e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}, \tag{2.16}$$

$$\begin{aligned} &E\left(e^{-\beta\tau^{W^\mu}}\right) \qquad (2.17)\\ &= \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}. \end{aligned}$$

For a standard Brownian motion, the special case when $\mu = 0$, we have

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W<\tau_2^W\}}\right) = \frac{\sqrt{d_2}e^{-\beta d_1}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}, \tag{2.18}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W>\tau_2^W\}}\right) = \frac{\sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}, \tag{2.19}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{\sqrt{d_2}e^{-\beta d_1}+\sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}. \tag{2.20}$$

Proof: We prove in the appendix that

$$E\left(e^{-\beta\tau_1^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) = \lim_{\epsilon\to 0} E\left(e^{-\beta\tau_1^X}\mathbf{1}_{\{\tau_1^X<\tau_2^X\}}\right).$$

We have therefore

$$\begin{aligned} E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) &= E\left(e^{-\beta\tau_1^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) \\ &= \lim_{\epsilon\to 0} E\left(e^{-\beta\tau_1^X}\mathbf{1}_{\{\tau_1^X<\tau_2^X\}}\right). \end{aligned}$$

Similarly, we can get

$$\begin{aligned} E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_2^{W^\mu}\}}\right) &= E\left(e^{-\beta\tau_2^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_2^{W^\mu}\}}\right) \\ &= \lim_{\epsilon\to 0} E\left(e^{-\beta\tau_2^X}\mathbf{1}_{\{\tau_1^X>\tau_2^X\}}\right). \end{aligned}$$

According to (2.13) and (2.14) of Lemma 2.3.1., we can actually calculate that

$$E\left(e^{-\beta\tau_1^X}\mathbf{1}_{\{\tau_1^X<\tau_2^X\}}\right) = \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)}{1-\int_0^{d_1} e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\beta s}p_{21}(s)\mathrm{d}s},$$

$$E\left(e^{-\beta\tau_2^X}\mathbf{1}_{\{\tau_1^X>\tau_2^X\}}\right)=\frac{e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{d_1}e^{-\beta s}p_{12}(s)\mathrm{d}s}{1-\int_0^{d_1}e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s},$$

where

$$\bar{P}_{12}(d_1)=1-e^{-2\epsilon\mu}\mathcal{N}\left(\mu\sqrt{d_1}-\frac{\epsilon}{\sqrt{d_1}}\right)-\mathcal{N}\left(-\mu\sqrt{d_1}-\frac{\epsilon}{\sqrt{d_1}}\right),$$

$$\bar{P}_{21}(d_2)=1-\mathcal{N}\left(\mu\sqrt{d_2}-\frac{\epsilon}{\sqrt{d_2}}\right)-e^{2\epsilon\mu}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{\epsilon}{\sqrt{d_2}}\right),$$

$$\begin{aligned}\int_0^{d_1}e^{-\beta u}p_{12}(u)\mathrm{d}u &= e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)\epsilon}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{\epsilon}{\sqrt{d_1}}\right)\\ &\quad+e^{\left(\sqrt{2\beta+\mu^2}-\mu\right)\epsilon}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{\epsilon}{\sqrt{d_1}}\right),\end{aligned}$$

$$\begin{aligned}\int_0^{d_2}e^{-\beta u}p_{21}(u)\mathrm{d}u &= e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)\epsilon}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_2}-\frac{\epsilon}{\sqrt{d_2}}\right)\\ &\quad+e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)\epsilon}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_2}-\frac{\epsilon}{\sqrt{d_2}}\right).\end{aligned}$$

By taking the limit $\epsilon\longrightarrow 0$, we obtain (2.15) and (2.16). Adding up (2.15) and (2.16) give (2.17).

□

Remark: A similar result for a standard Brownian motion, i.e. $\mu=0$ in the case when double barriers are considered can be found in [2].

If we let $\beta\longrightarrow 0$, we get the following remarkable results.

Corollary 2.4.1.1 *The probability that W^μ achieves an excursion above 0 with* *length as least d_1 before it achieves an excursion below 0 with length at least d_2 is*

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}}{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right)}. \tag{2.21}$$

Similarly, for a standard Brownian motion we have

$$P\left(\tau_1^{W} < \tau_2^{W}\right) = \frac{\sqrt{d_2}}{\sqrt{d_1}+\sqrt{d_2}}, \tag{2.22}$$

$$P\left(\tau_1^{W} > \tau_2^{W}\right) = \frac{\sqrt{d_1}}{\sqrt{d_1}+\sqrt{d_2}}. \tag{2.23}$$

Remark 1: The result stated by (2.22) has also been obtained in [2]. However, the result for Brownian motions with drift, (2.21) is presented here for the first time.

Remark 2: If we set $d_1 = d_2 = d$ in (2.22) and (2.23), we have for a standard Brownian motion

$$P\left(\tau_1^{W} < \tau_2^{W}\right) = P\left(\tau_1^{W} > \tau_2^{W}\right) = \frac{1}{2},$$

which can be explained by the symmetry of standard Brownian motions.

Remark 3: For a Brownian motion with positive drift, by setting $d_1 = d_2 = d$ in (2.21), we have

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{1}{2} + \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} > \frac{1}{2}, \quad P\left(\tau_1^{W^\mu} > \tau_2^{W^\mu}\right) = \frac{1}{2} - \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} < \frac{1}{2},$$

because it has a tendency to move upwards.

If we only consider the excursion below 0, we have the following results.

Corollary 2.4.1.2 *For a Brownian motion W^μ with $W_0^\mu = 0$ and $\tau_2^{W^\mu}$ defined as in (2.4) with $S = W^\mu$, we the have the Laplace transform for $\tau_2^{W^\mu}$:*

$$E\left(e^{-\beta\tau_2^{W^\mu}}\right) = \frac{e^{-\beta d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_2\pi}{2}}\right\}}{\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right) + \sqrt{\frac{(2\beta+\mu^2)d_2\pi}{2}}}. \tag{2.24}$$

When $\mu = 0$, we have the result for a standard Brownian motion:

$$E\left(e^{-\beta\tau_2^{W}}\right) = \frac{e^{-\beta d_2}}{\Psi\left(\sqrt{\beta d_2}\right) + \sqrt{\pi\beta d_2}}. \tag{2.25}$$

Proof: When $d_1 \longrightarrow \infty$, we have $\tau_1 \longrightarrow \infty$, therefore $\tau^S \longrightarrow \tau_2^S$.

As a result, we have

$$E\left(e^{-\beta\tau_2^S}\right) = \lim_{d_1\to\infty} E\left(e^{-\beta\tau^S}\right).$$

□

Remark: As one of the most important results, (2.25) has been obtained in [13]. But the result for Brownian motions with drift, (2.24) is presented here for the first time.

So far we have been considering the case when the process starts from 0 and the barrier level is set to be 0. In practice, however, the barrier is different from the starting point of the underlying asset price in most cases. Therefore, in order to price the options, we introduce the following theorems and corollaries.

Theorem 2.4.2 *For a Brownian motion W^μ with $W_0^\mu = 0$ and barrier $L = l$, the Laplace transform of $\tau_l^{W^\mu}$ is given by* *when* $l < 0$,

$$\begin{aligned} & E\left(e^{-\beta\tau_l^{W^\mu}}\right) \qquad (2.26) \\ = \;& e^{-\beta d_1}\left\{1 - e^{2\mu l}\mathcal{N}\left(\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right) - \mathcal{N}\left(-\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right)\right\} \\ & + \left\{e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1} + \frac{l}{\sqrt{d_1}}\right)\right. \\ & \left. + e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1} + \frac{l}{\sqrt{d_1}}\right)\right\} \\ & \frac{e^{-\beta d_1}\sqrt{d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\sqrt{d_1}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}; \end{aligned}$$

when $l > 0$,

$$\begin{aligned} & E\left(e^{-\beta\tau_l^{W^\mu}}\right) \qquad (2.27) \\ = \;& e^{-\beta d_2}\left\{1 - \mathcal{N}\left(\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right) - e^{2\mu l}\mathcal{N}\left(-\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} \\ & + \left\{e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_2} - \frac{l}{\sqrt{d_2}}\right)\right. \\ & \left. + e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} \\ & \frac{e^{-\beta d_1}\sqrt{d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\sqrt{d_1}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}. \end{aligned}$$

Proof: We only prove the case when $l < 0$. The same arguments apply to the case when $l > 0$. Define

$$T_l = \inf\{t \geq 0 \mid W_t^\mu = l\}.$$

The left hand side of (2.26) can be expressed as follows

$$E\left(e^{-\beta\tau_l^{W^\mu}}\right) = E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l \geq d_1\}}\right) + E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l < d_1\}}\right).$$

Moreover, we have

$$
\begin{aligned}
E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l\geq d_1\}}\right) &= e^{-\beta d_1}P\left(T_l \geq d_1\right) \\
&= e^{-\beta d_1}\left\{1 - e^{2\mu l}\mathcal{N}\left(\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right) - \mathcal{N}\left(-\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l<d_1\}}\right) &= E\left(e^{-\beta\left(T_l+\tau_l^{\widetilde{W}^\mu}\right)}\mathbf{1}_{\{T_l<d_1\}}\right) \\
&= E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right)E\left(e^{-\beta\tau_l^{\widetilde{W}^\mu}}\right) = E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right)E\left(e^{-\beta\tau^{W^\mu}}\right),
\end{aligned}
$$

where $\widetilde{W}^\mu$ stands for a Brownian motion starting from l. We have obtained $E\left(e^{-\beta\tau^{W^\mu}}\right)$ in Theorem 2.4.1. We also have that

$$
\begin{aligned}
& E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right) \\
= & \int_0^{d_1} e^{-\beta s}\frac{-l}{\sqrt{2\pi s^3}}\exp\left\{-\frac{(l-\mu s)^2}{2s}\right\}\mathrm{d}s \\
= & e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1} + \frac{l}{\sqrt{d_1}}\right) + e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1} + \frac{l}{\sqrt{d_1}}\right)
\end{aligned}
$$

We have therefore proved (2.26).

□

We will now extend Theorem 2.4.2 to obtain the join distribution of τ_l^W and W at an exponential time. This will be an application of (2.26), (2.27) and Girsanov's theorem.

Theorem 2.4.3 *For a standard Brownian motion W with $W_0 = 0$, and τ_l^W defined* *as in (2.5) with $S = W$ and $L = l$, we have the following results:*

For the case $l \geq 0$, when $x \geq l$,

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau_l^W < \widetilde{T}\right) = \left\{a(d_2)e^{-\sqrt{2\gamma}(x-l)} + b_1 p(x-l, d_1, d_2)\right\} \mathrm{d}x; \tag{2.28}$$

when $x < l$

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau_l^W < \widetilde{T}\right) = \left\{a(d_2)e^{\sqrt{2\gamma}(x-l)} + b_1 p(l-x, d_2, d_1)\right\} \mathrm{d}x; \tag{2.29}$$

For the case $l < 0$, when $x \geq l$,

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau_l^W < \widetilde{T}\right) = \left\{a(d_1)e^{-\sqrt{2\gamma}(x-l)} + b_2 p(x-l, d_1, d_2)\right\} \mathrm{d}x; \tag{2.30}$$

when $x < l$

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau_l^W < \widetilde{T}\right) = \left\{a(d_1)e^{\sqrt{2\gamma}(x-l)} + b_2 p(l-x, d_2, d_1)\right\} \mathrm{d}x; \tag{2.31}$$

where $\widetilde{T}$ is a random variable independent of W, with an exponential distribution of parameter γ and

$$a(x) = \sqrt{\frac{2}{\gamma}}\left\{e^{-\sqrt{2\gamma}l}\mathcal{N}\left(-\frac{l}{\sqrt{x}} + \sqrt{2\gamma x}\right) - e^{\sqrt{2\gamma}l}\mathcal{N}\left(\frac{l}{\sqrt{x}} + \sqrt{2\gamma x}\right)\right\},$$

$$b_1 = e^{-\sqrt{2\gamma}l}\mathcal{N}\left(-\frac{l}{\sqrt{d_2}} + \sqrt{2\gamma d_2}\right) + e^{\sqrt{2\gamma}l}\mathcal{N}\left(-\frac{l}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right),$$

$$b_2 = e^{\sqrt{2\gamma} l}\mathcal{N}\left(\frac{l}{\sqrt{d_1}} + \sqrt{2\gamma d_1}\right) + e^{-\sqrt{2\gamma} l}\mathcal{N}\left(\frac{l}{\sqrt{d_2}} - \sqrt{2\gamma d_1}\right),$$

$$\begin{aligned} p(x,y,z) &= \frac{\gamma\sqrt{2\pi yz}e^{-\sqrt{2\gamma}(x-l)}}{\sqrt{z}\Psi\left(\sqrt{\gamma y}\right) + \sqrt{y}\Psi\left(\sqrt{\gamma z}\right)}\left\{\frac{e^{-\gamma y}}{2\sqrt{\pi\gamma y}} + \frac{e^{-\gamma z}}{2\sqrt{\pi\gamma z}} + \mathcal{N}\left(\frac{x-l}{\sqrt{y}} - \sqrt{2\gamma y}\right)\right. \\ &\quad \left. -\mathcal{N}\left(-\sqrt{2\gamma y}\right) - \mathcal{N}\left(-\sqrt{2\gamma z}\right) - e^{2\sqrt{2\gamma}(x-l)}\mathcal{N}\left(-\frac{x-l}{\sqrt{y}} - \sqrt{2\gamma y}\right)\right\}. \end{aligned}$$

Proof: see appendix.

□

Similarly, we can obtain the result when we only consider the excursion below the barrier by taking the limit $d_1 \to \infty$.

Corollary 2.4.3.1 *For a standard Brownian motion W with $W_0 = 0$ and $\tau_{2,l}^W$ defined as in (2.4) with $S_t = W_t$ and $L = l$, we have the following results:*

For the case $l \geq 0$, when $x \geq l$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau_{2,l}^W < \tilde{T}\right) = \left\{a_2' e^{-\sqrt{2\gamma}(x-l)} + b_1' q_1(x-l)\right\}\mathrm{d}x; \qquad (2.32)$$

when $x < l$

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau_{2,l}^W < \tilde{T}\right) = \left\{a_2' e^{\sqrt{2\gamma}(x-l)} + b_1' q_2(x-l)\right\}\mathrm{d}x; \qquad (2.33)$$

For the case $l < 0$, when $x \geq l$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau_{2,l}^W < \tilde{T}\right) = \left\{a_1' e^{-\sqrt{2\gamma}(x-l)} + b_2' q_1(x-l)\right\}\mathrm{d}x; \qquad (2.34)$$

when $x < l$

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau_{2,l}^{W} < \widetilde{T}\right) = \left\{a_1' e^{\sqrt{2\gamma}(x-l)} + b_2' q_2(x-l)\right\} \mathrm{d}x; \qquad (2.35)$$

where

$$a_1' = \frac{2}{\gamma}\left\{e^{-\sqrt{2\gamma}l} - e^{\sqrt{2\gamma}l}\right\}, \quad a_2' = a\left(d_2\right),$$

$$b_1' = b_1, \quad b_2' = e^{\sqrt{2\gamma}l},$$

$$q_1(x) = \sqrt{\frac{\gamma}{2}} e^{-\sqrt{2\gamma}x}\left(1 - \frac{2\sqrt{\pi\gamma d_2}}{2\sqrt{\pi\gamma d_2}\mathcal{N}\left(\sqrt{2\gamma d_2}\right) + e^{-\gamma d_2}}\right),$$

$$\begin{aligned} q_2(x) \;=\; & \frac{\gamma e^{\sqrt{2\gamma}x}\sqrt{2\pi d_2}}{2\sqrt{\pi\gamma d_2}\mathcal{N}\left(\sqrt{2\gamma d_2}\right) + e^{-\gamma d_2}} \left\{\frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}} + \mathcal{N}\left(-\frac{x}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right)\right. \\ & \left. -\mathcal{N}\left(-\sqrt{2\gamma d_2}\right) - e^{-2\sqrt{2\gamma}x}\mathcal{N}\left(\frac{x}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right)\right\}; \end{aligned}$$

and where $\widetilde{T}$ *is a random variable , independent of* W*, with an exponential distribution of parameter* γ*.*

Remark: By using this result, we can calculate the explicit form of the Laplace transform of the price of the Parisian option defined in [13]. This approach is different from [13], where they try to find the Laplace transform of $\tau_{2,l}^{W}$ and the density of $W_{\tau_{2,l}^{W}}$, and the Laplace transform is given in form of double integral. Our approach produces explicit expressions without integrals.

2.5 Pricing Parisian Options

The result presented by (2.25) has been obtained in [13] and used to price Parisian options which consider the excursions at only one side of the barrier. Here we want to introduce a new type of Parisian options, considering the excursions at both sides of the barrier.

For example, we want to price a Parisian call option, the owner of which will obtain the right to exercise it when either the length of an excursion above the barrier reaches d_1, or the length of an excursion below the barrier reaches d_2 before T. Its price formula is given by

$$P_{min-call-in} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau_L^S < T\}} \right),$$

where S is the underlying stock price, L is the barrier level, Q denotes the risk neutral measure. The subscript *min-call-in* means it is a Call option which will be triggered when the minimum of two stopping times, $\tau_{1,L}^S$ and $\tau_{2,L}^S$, is less than T, i.e. $\tau_L^S < T$. We assume S is a geometric Brownian motion defined as in (2.8). Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad l = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \quad Y_t = mt + W_t.$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\left\{\sigma(mt + W_t)\right\} = xe^{\sigma Y_t}.$$

By applying Girsanov's Theorem, we have

$$P_{min-call-in} = e^{-\left(r+\frac{1}{2}m^2\right)T} E_P \left[\left(xe^{\sigma Y_T} - K \right)^+ e^{mY_T} \mathbf{1}_{\{\tau_l^Y < T\}} \right],$$

where P is a new measure, under which Y_t is a standard Brownian motion with $Y_0 = 0$. And we define

$$P^*_{min-call-in} = e^{\left(r+\frac{1}{2}m^2\right)T} P_{min-call-in}.$$

We are going to show that we can obtain the Laplace transform of $P^*_{min-call-in}$ w.r.t T, denoted by $\mathscr{L}_T$.

First of all, we have

$$\begin{aligned}
& E_P \left[\left(xe^{\sigma Y_{\widetilde{T}}} - K \right)^+ e^{mY_{\widetilde{T}}} \mathbf{1}_{\{\tau_l^Y < \widetilde{T}\}} \right] \\
= & \int_b^\infty \left(xe^{\sigma y} - K \right) e^{my} P\left(Y_{\widetilde{T}} \in \mathrm{d}y, \tau_l^Y < \widetilde{T} \right) \\
= & \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty \left(xe^{\sigma y} - K \right) e^{my} P\left(Y_T \in \mathrm{d}y, \tau_l^Y < T \right) \mathrm{d}T \\
= & \gamma \int_0^\infty e^{-\gamma T} E_P \left[\left(xe^{\sigma Y_T} - K \right)^+ e^{mY_T} \mathbf{1}_{\{\tau_l^Y < T\}} \right] \mathrm{d}T \\
= & \gamma \mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T = \frac{1}{\gamma} \int_b^\infty \left(xe^{\sigma y} - K \right) e^{my} P\left(Y_{\widetilde{T}} \in \mathrm{d}y, \tau_l^Y < \widetilde{T} \right).$$

By using the results in Theorem 2.4.3, this Laplace transform can be calculated explicitly.

When $b \geq 0$, i.e. $L \geq x$, we have

$$\mathscr{L}_T = \frac{xf(\sigma+m) - Kf(m)}{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)},$$

where

$$\begin{aligned} f(x) \quad = \quad & \frac{\sqrt{2\pi d_1 d_2}e^{b\left(x-\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}-x}\left\{\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}} + \frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right. \\ & \left. +\mathscr{N}\left(\frac{b}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right) - \mathscr{N}\left(-\sqrt{2\gamma d_1}\right) - \mathscr{N}\left(-\sqrt{2\gamma d_2}\right)\right\} \\ & +\sqrt{2\pi d_1 d_2}\left\{\frac{e^{\left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}+x}\mathscr{N}\left(-\frac{b}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right)\right. \\ & \left. +\frac{2xe^{\frac{\left(x^2-2\gamma\right)d_1}{2}}}{2\gamma - x^2}\mathscr{N}\left(x\sqrt{d_1} - \frac{b}{\sqrt{d_1}}\right)\right\}; \end{aligned}$$

when $b < 0$, i.e. $L < x$, we have

$$\mathscr{L}_T = \frac{xg(\sigma+m) - Kg(m)}{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)},$$

where

$$
\begin{aligned}
g(x) &= \sqrt{2\pi d_1 d_2}\left\{\frac{e^{b\left(x+\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}+x}\left[\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)+\mathscr{N}\left(-\sqrt{2\gamma d_2}\right)\right.\right. \\
&\left. -\mathscr{N}\left(-\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)-\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}}-\frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right] \\
&-\frac{e^{\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x}\mathscr{N}\left(\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right) \\
&+\frac{2x}{2\gamma-x^2}\left[e^{\frac{\left(x^2-2\gamma\right)d_2}{2}}\left(\mathscr{N}\left(x\sqrt{d_2}-\frac{b}{\sqrt{d_2}}\right)-\mathscr{N}\left(x\sqrt{d_2}\right)\right)\right. \\
&\left.\left. -e^{\frac{\left(x^2-2\gamma\right)d_1}{2}}\mathscr{N}\left(x\sqrt{d_1}\right)\right]+\frac{2\sqrt{2\gamma}}{2\gamma-x^2}\left[\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}}+\frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right]\right\}.
\end{aligned}
$$

A special case is when we only consider the excursions below the barrier. The results can be calculated using the results in Corollary 2.4.3.1.

When $L \geq x$, we have

$$
\mathscr{L}_T = \left(\frac{1}{\sqrt{2\gamma}}-\frac{\sqrt{2\pi d_2}}{2\sqrt{\pi\gamma d_2}\mathscr{N}\left(\sqrt{2\gamma d_2}\right)+e^{-\gamma d_2}}\right)\left(\frac{xe^{\left(\sigma+m-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-\sigma-m}-\frac{Ke^{\left(m-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-m}\right);
$$

when $L < x$, we have

$$
\mathscr{L}_T = \frac{xh(\sigma+m)-Kh(m)}{2\sqrt{\pi\gamma d_2}\mathscr{N}\left(\sqrt{2\gamma d_2}\right)+e^{-\gamma d_2}},
$$

where

$$\begin{aligned} h(x) &= \frac{e^{b\left(x+\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}+x}\left\{\sqrt{2\pi d_2}\left[\mathcal{N}\left(-\sqrt{2\gamma d_2}\right)-\mathcal{N}\left(-\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)\right]-\frac{e^{-\gamma d_2}}{\sqrt{2\gamma}}\right\} \\ &+\frac{2e^{-\gamma d_2}}{2\gamma-x^2}-\sqrt{2\pi d_2}\left\{\frac{e^{\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x}\mathcal{N}\left(\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)\right. \\ &\left.+\frac{2xe^{\frac{\left(x^2-2\gamma\right)d_2}{2}}}{2\gamma-x^2}\left[\mathcal{N}\left(x\sqrt{d_2}-\frac{b}{\sqrt{d_2}}\right)-\mathcal{N}\left(x\sqrt{d_2}\right)\right]\right\}. \end{aligned}$$

Remark 1: It is the first time we manage to get the explicit expressions for the Laplace transforms of the option prices even for the one-sided excursion case. In [13] an expression involving double integrals is provided.

Remark 2: The prices can be calculated by numerical inversion of the Laplace transforms.

So far, we have shown how to obtain the Laplace transform of

$$P^*_{min-call-in} = e^{\left(r+\frac{1}{2}m^2\right)T} P_{min-call-in}.$$

For

$$P_{min-call-out} = e^{-rT} E_Q\left((S_T - K)^+ \mathbf{1}_{\{\tau_L^S > T\}}\right),$$

we can get the result from the relationship that

$$P_{min-call-out} = e^{-rT} E_Q\left\{(S_T - K)^+\right\} - P_{min-call-in}.$$

Furthermore, if we set

$$\tilde{\tau}_L^Y = \tau_{1,L}^Y \vee \tau_{2,L}^Y,$$

we can define another type of Parisian options by $\tilde{\tau}_L^Y$:

$$P_{max-call-in} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tilde{\tau}_L^S < T\}} \right).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\{\tilde{\tau}_L^S < T\}} = \mathbf{1}_{\{\tau_{1,L}^S < T\}} + \mathbf{1}_{\{\tau_{2,L}^S < T\}} - \mathbf{1}_{\{\tau_L^S < T\}}.$$

We have therefore

$$P_{max-call-in} = P_{up-in-call} + P_{down-in-call} - P_{min-call-in}.$$

Similarly, from

$$P_{max-call-out} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - P_{max-call-in},$$

we can work out $P_{max-call-out}$.

2.6 Appendix

2.6.1 Proof of the convergence

We show in this section that we can take limits of Laplace transforms when $\epsilon \to 0$ as we did earlier. First of all, we consider two processes W^μ and $W^{\mu,\epsilon} = W^\mu + \epsilon$.

According to the definitions, $X^{(\epsilon)}$ satisfies

$$\lim_{\epsilon\to 0} X_t^{(\epsilon)} = W_t^\mu, \ a.s. \text{ for all } t,$$

$$W_t^\mu \leq X_t^{(\epsilon)} \leq W_t^{\mu,\epsilon} \quad \text{for all } t,$$

and g_t^X always lies between $g_t^{W^\mu}$ and $g_t^{W^{\mu,\epsilon}}$. Since

$$\lim_{\epsilon\to 0} g_t^{W^{\mu,\epsilon}} = \lim_{\epsilon\to 0} g_{-\epsilon,t}^{W^\mu} = g_t^{W^\mu},$$

we have that

$$\lim_{\epsilon\to 0} g_t^X = g_t^{W^\mu}, \ a.s.$$

Since g_t^S is a right continuous function with respect to t, we have that

$$\lim_{\epsilon\to 0} g_t^X = g_t^{W^\mu}, \ a.s. \text{ for all } t$$

and therefore

$$\lim_{\epsilon\to 0} \mathbf{1}_{\left\{X_t^{(\epsilon)}>0\right\}} \left(t - g_t^X\right) = \mathbf{1}_{\left\{W_t^\mu>0\right\}} \left(t - g_t^{W^\mu}\right) \ a.s. \text{ for all } t.$$

From the definition of τ_1^S we have that

$$\begin{aligned} \left\{\tau_1^{W^\mu} < t\right\} &= \left\{\sup_{0\leq s\leq t} \left\{\mathbf{1}_{\left\{W_s^\mu>0\right\}} \left(s - g_s^{W^\mu}\right)\right\} \geq d_1\right\} \\ &= \lim_{\epsilon\to 0} \left\{\sup_{0\leq s\leq t} \left\{\mathbf{1}_{\left\{X_s^{(\epsilon)}>0\right\}} \left(t - g_s^X\right)\right\} \geq d_1\right\} = \lim_{\epsilon\to 0} \left\{\tau_1^X < t\right\}. \end{aligned}$$

Consequently,

$$\lim_{\epsilon \to 0} \tau_1^X = \tau_1^{W^\mu} \ a.s.$$

By the same argument, we can show that

$$\lim_{\epsilon \to 0} \tau_2^X = \tau_2^{W^\mu} \ a.s.$$

Since $\tau^S = \tau_1^S \wedge \tau_2^S$, we have

$$\lim_{\epsilon \to 0} \tau^X = \tau^{W^\mu} \ a.s.$$

The next step is to show the convergence of the stopping times leads to the convergence of their Laplace transforms. In order to simplify the notations, we define $R^S = \left(R_1^S, R_2^S, R_3^S\right) = \left(\tau_1^S, \tau_2^S, \tau^S\right)$. We have just shown that

$$\lim_{\epsilon \to 0} R_i^X \doteq R_i^{W^\mu} \ a.s.$$

Therefore for any given non-negative constants β_i, $i = 1, 2, 3$,

$$\lim_{\epsilon \to 0} \exp\left\{-\sum_{i=1}^{3} \beta_i R_i^X\right\} = \exp\left\{-\sum_{i=1}^{3} \beta_i R_i^{W^\mu}\right\} \ a.s.$$

Since $R_i^X \geq 0$, we also have,

$$\left|\exp\left\{-\sum_{i=1}^{3} \beta_i R_i^X\right\}\right| < 1.$$

By the Dominated Convergence Theorem,

$$\begin{aligned} E\left(\exp\left\{-\sum_{i=1}^{3}\beta_i R_i^{W^\mu}\right\}\right) &= E\left(\lim_{\epsilon\to 0}\exp\left\{-\sum_{i=1}^{3}\beta_i R_i^{X}\right\}\right) \\ &= \lim_{\epsilon\to 0} E\left(\exp\left\{-\sum_{i=1}^{3}\beta_i R_i^{X}\right\}\right). \end{aligned}$$

When $\mu = 0$, we can get the same conclusion for the standard Brownian motion by the above argument.

2.6.2 Proof of Theorem 2.4.3

We prove Theorem 2.4.3 in this section. Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x\mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta\tau_l^{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T}\mathbf{1}_{\{\tau_l^{W^\mu}<T\}}\mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta+\frac{1}{2}\mu^2)T}E\left(e^{\mu W_T}\mathbf{1}_{\{\tau_l^{W}<T\}}\right)\mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta\tau_l^{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2)e^{-\gamma T} E\left(e^{\mu W_T}\mathbf{1}_{\{\tau_l^W < T\}}\right) \mathrm{d}T \\ &= \frac{\gamma - \frac{1}{2}\mu^2}{\gamma} E\left(e^{\mu W_{\widetilde{T}}}\mathbf{1}_{\{\tau_l^W < \widetilde{T}\}}\right), \end{aligned}$$

where $\widetilde{T}$ is a random variable, independent of W, with an exponential distribution of parameter γ. Assume $\mu > 0$. We have therefore when $l \geq 0$

$$\begin{aligned} & E\left(e^{\mu W_{\widetilde{T}}}\mathbf{1}_{\{\tau_l^W < \widetilde{T}\}}\right) \\ = \; & \frac{\gamma}{\gamma - \frac{1}{2}\mu^2} E\left(e^{-\beta\tau_l^{W^\mu}}\right) \\ = \; & \frac{\gamma e^{-\gamma d_2}}{\gamma - \frac{1}{2}\mu^2} e^{\frac{d_2}{2}\mu^2} \left\{\mathcal{N}\left(-\mu\sqrt{d_2} + \frac{l}{\sqrt{d_2}}\right) - e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} \\ & + \frac{\gamma\left\{e^{-\sqrt{2\gamma}l}\mathcal{N}\left(\sqrt{2\gamma d_2} - \frac{l}{\sqrt{d_2}}\right) + e^{\sqrt{2\gamma}l}\mathcal{N}\left(-\sqrt{2\gamma d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} e^{\mu l}}{(\gamma - \frac{1}{2}\mu^2)\left\{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)\right\}} \\ & \left[e^{-(\gamma - \frac{\mu^2}{2})d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}\right. \\ & \left. + e^{-(\gamma - \frac{\mu^2}{2})d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}\right] \\ = \; & \frac{\gamma e^{-\gamma d_2}}{\gamma - \frac{1}{2}\mu^2} e^{\frac{d_2}{2}\mu^2} \left\{\mathcal{N}\left(-\mu\sqrt{d_2} + \frac{l}{\sqrt{d_2}}\right) - e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} \\ & + \frac{\gamma\left\{e^{-\sqrt{2\gamma}l}\mathcal{N}\left(\sqrt{2\gamma d_2} - \frac{l}{\sqrt{d_2}}\right) + e^{\sqrt{2\gamma}l}\mathcal{N}\left(-\sqrt{2\gamma d_2} - \frac{l}{\sqrt{d_2}}\right)\right\} e^{\mu l}}{(\gamma - \frac{1}{2}\mu^2)\left\{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)\right\}} \\ & \left[e^{-\gamma d_1}\left\{\sqrt{2\pi d_1 d_2}\mu e^{\frac{d_1}{2}\mu^2} + \sqrt{d_2}\left\{1 - \sqrt{\frac{d_1}{2}}\pi\mu e^{\frac{d_1}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_1}{2}}\mu\right)\right\}\right\}\right. \\ & \left. + e^{-\gamma d_2}\sqrt{d_1}\left\{1 - \sqrt{\frac{d_2}{2}}\pi\mu e^{\frac{d_2}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_2}{2}}\mu\right)\right\}\right]. \end{aligned}$$

We will now invert the moment generating function above. We have that

$$e^{\frac{d_2}{2}\mu^2}\mathcal{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right)=\int_l^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{x^2}{2d_2}}\mathrm{d}x,$$

$$e^{\frac{d_2}{2}\mu^2}e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right)=\int_l^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{(x-2l)^2}{2d_2}}\mathrm{d}x,$$

$$\frac{\mu}{\gamma-\frac{\mu^2}{2}}=\int_0^\infty e^{\mu x}e^{-\sqrt{2\gamma}x}\mathrm{d}x-\int_{-\infty}^0 e^{\mu x}e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$\frac{1}{\gamma-\frac{\mu^2}{2}}=\int_0^\infty e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}x}\mathrm{d}x+\int_{-\infty}^0 e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$e^{\frac{d_1}{2}\mu^2}=\int_{-\infty}^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_1}}\exp\left\{-\frac{x^2}{2d_1}\right\}\mathrm{d}x,$$

$$1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)=\int_{-\infty}^0 e^{\mu x}\frac{-x}{d_i}e^{-\frac{x^2}{2d_i}}\mathrm{d}x.$$

The inversion of $\frac{e^{\frac{d_2}{2}\mu^2}}{\gamma-\frac{\mu^2}{2}}\mathcal{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right)$ is given below.

For $x \geq l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{y^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathcal{N}\left(-\frac{l}{\sqrt{d_2}}+\sqrt{2\gamma d_2}\right);$$

for $x < l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{y^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathcal{N}\left(-\frac{l}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right).$$

The inversion of $\frac{e^{\frac{d_2}{2}\mu^2}e^{2l\mu}}{\gamma-\frac{\mu^2}{2}}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right)$ is given below.

For $x \geq l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{(y-2l)^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{2l\sqrt{2\gamma}}e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathcal{N}\left(\frac{l}{\sqrt{d_2}}+\sqrt{2\gamma d_2}\right);$$

for $x < l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}} e^{-\frac{(y-2l)^2}{2d_2}} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-y)} \mathrm{d}y = \frac{e^{\gamma d_2} e^{-2l\sqrt{2\gamma}} e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} \mathcal{N}\left(\frac{l}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right).$$

The inversion of $\frac{\mu e^{\frac{d_1}{2}\mu^2}}{\gamma - \frac{\mu^2}{2}}$ is

$$\begin{aligned} &\int_0^\infty e^{-\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} \mathrm{d}y - \int_{-\infty}^0 e^{\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} \mathrm{d}y \\ = \; & e^{\gamma d_1} \left\{ e^{-\sqrt{2\gamma}x} \mathcal{N}\left(\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right) - e^{\sqrt{2\gamma}x} \mathcal{N}\left(-\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right) \right\}. \end{aligned}$$

The inversion of $\frac{1 - \sqrt{\frac{d_i}{2}\pi}\mu e^{\frac{d_i}{2}\mu^2} \mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)}{\gamma - \frac{\mu^2}{2}}$ is given below.

For $x \geqslant 0$,

$$\int_{-\infty}^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y = \frac{e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x} \sqrt{2\pi d_i} \mathcal{N}\left(-\sqrt{2\gamma d_i}\right);$$

for $x < 0$,

$$\begin{aligned} &\int_{-\infty}^x \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y + \int_x^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-y)} \mathrm{d}y \\ = \; & \frac{e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x} \sqrt{2\pi d_i} \mathcal{N}\left(\frac{x}{\sqrt{d_i}} - \sqrt{2\gamma d_i}\right) \\ & + e^{\gamma d_i + \sqrt{2\gamma}x} \sqrt{2\pi d_i} \left\{ \mathcal{N}\left(\sqrt{2\gamma d_i}\right) - \mathcal{N}\left(\frac{x}{\sqrt{d_i}} + \sqrt{2\gamma d_i}\right) \right\}. \end{aligned}$$

Consequently, we can get Theorem 2.4.3.

Chapter 3

Double Barrier Parisian Options

Abstract

In this paper, we study the excursion time of a Brownian motion with drift outside a corridor by using a four-state semi-Markov model. In mathematical finance, these results have an important application in the valuation of double barrier Parisian options. We subsequently obtain an explicit expression for the Laplace transform of its price.

Keywords: excursion time, four-state Semi-Markov model, double barrier Parisian options, Laplace transform.

3.1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picqué and Yor [13]. It is a special case of path dependent options. The owner of a Parisian option will either gain the right or lose the right to exercise the option upon the price reaching a predetermined barrier level L and staying above or below the level for a predetermined time d before the maturity date T.

More precisely, the owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than d. For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. For details on the pricing of Parisian options see [13], [37], [38] and [46].

Double barrier Parisian options are a version with two barriers of the standard Parisian options introduced in [13]. In contrast to the Parisian options mentioned above, we consider the excursions both below the lower barrier and above the upper barrier, i.e. outside a corridor formed by these two barriers. Let us look at two examples, depending on whether the condition is that the required excursions above the upper barrier and below the lower barrier have to both happen before the maturity date or that either one of them happens before the maturity. In one example, the owner of a *double barrier Parisian max-out option* loses the option if the underlying asset price process S has both an excursion above the upper barrier for longer than a continuous period d_1 and below lower the barrier for longer than d_2 before the maturity of the option. In the other example, the owner of a *double barrier Parisian min-out option* loses the right to exercise the option if either one of these two events happens before the maturity. For pricing double barrier Parisian options using excursion theory, see [39].

In this paper, we are going to use the same definition for the excursion as in [13] and [14]. Let S be a stochastic process and l_1, l_2, $l_1 > l_2$ be the levels of these two barriers. As in [13], we define

$$g^S_{l_i,t} = \sup\{s \leq t \mid S_s = l_i\}, \qquad d^S_{l_i,t} = \inf\{s \geq t \mid S_s = l_i\}, \qquad i = 1, 2, \tag{3.1}$$

with the usual conventions, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. Assuming $d_1 > 0$, $d_2 > 0$,we now define

$$\tau^S_1 = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g^S_{l_1,t}) \geq d_1\right\}, \tag{3.2}$$

$$\tau^S_2 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} > g^S_{l_2,t}\right\}}(t - g^S_{l_1,t}) \geq d_2\right\}, \tag{3.3}$$

$$\tau^S_3 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} < g^S_{l_2,t}\right\}}(t - g^S_{l_2,t}) \geq d_3\right\}, \tag{3.4}$$

$$\tau^S_4 = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g^S_{l_2,t}) \geq d_4\}, \tag{3.5}$$

$$\tau^S = \tau^S_1 \wedge \tau^S_4. \tag{3.6}$$

We can see that τ^S_1 is the first time that the length of the excursion of process S above the barrier l_1 reaches a given level d_1; τ^S_4 corresponds to the one below l_2 with required length d_4; and τ^S is the smaller of τ^S_1 and τ^S_4. We also see that τ^S_2 is the first time that the length of the excursion in the corridor reaches given level d_2, given that the excursion starts from the upper barrier l_1; τ^S_3 corresponds to the one in the corridor starting from the lower barrier l_2. Our aim is to study the excursions outside the corridor, therefore τ_2^S and τ_3^S are not of interest here. However we need to use these two stopping times to define our four-state semi-Markov model that will be the main tool used for calculation.

Now assume r is the risk-free rate, T is the term of the option, S is the price of its underlying asset, K is the strike price and Q is the risk neutral measure. If we have a double barrier Parisian min-out call option with the barrier l_1 and l_2, its price can be expressed as:

$$DP_{min-out-call} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau^S > T\}} \left(S_T - K \right)^+ \right);$$

and the price of a double barrier Parisian min-in put option is:

$$DP_{min-in-put} = e^{-rT} E_Q \left(\mathbf{1}_{\{\tau^S < T\}} \left(K - S_T \right)^+ \right).$$

In this paper, we are going to study the excursion time outside the corridor using a semi-Markov model consisting of four states. Based on the results, we can get the explicit form of the Laplace transform for the price of double barrier options. One can then invert using techniques as in [38].

In Section 3.2 we introduce the four-state semi-Markov model as well as a new process, doubly perturbed Brownian motion, which has the same behavior as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size ϵ. In Section 3.3 we obtain the martingale to which we can apply the optional sampling theorem and get the

Laplace transforms that we can use for pricing later. We give our main results applied to Brownian motions in Section 3.4, including the Laplace transforms for the stopping times defined by (3.2)-(3.6) for both a Brownian motion with drift, i.e. $S = W^{\mu}$, and a standard Brownian motion, i.e. $S = W$. In Section 3.5 we focus on pricing the double barrier Parisian options.

3.2 Definitions

From the description above, it is clear that we are actually considering four states, the state when the stochastic process is above the barrier l_1 the state when it is below l_2 and two states when it is between l_1 and l_2 depending on whether it comes into the corridor through l_1 or l_2. For each state, we are interested in the time the process spends in it. We introduce a new process

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > l_1 \\ 2, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S > g_{l_2,t}^S \\ 3, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S < g_{l_2,t}^S \\ 4, & \text{if } S_t < l_2 \end{cases} .$$

We can now express the variables defined above in terms of Z:

$$g_{l_i,t}^S = \sup\left\{s \le t \mid Z_s^S \ne Z_t^S\right\}, \tag{3.7}$$

$$d_{l_i,t}^S = \inf\left\{s \ge t \mid Z_s^S \ne Z_t^S\right\}, \tag{3.8}$$

$$\tau_1^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S = 1\}}\left(t - g_{l_1,t}^S\right) \ge d_1\right\}, \tag{3.9}$$

$$\tau_2^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}}\left(t - g_{l_1,t}^S\right) \geq d_2\right\}, \tag{3.10}$$

$$\tau_3^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=3\}}\left(t - g_{l_2,t}^S\right) \geq d_3\right\}, \tag{3.11}$$

$$\tau_4^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=4\}}\left(t - g_{l_2,t}^S\right) \geq d_4\right\}. \tag{3.12}$$

We then define

$$V_t^S = t - \max\left(g_{l_1,t}^S, g_{l_2,t}^S\right), \tag{3.13}$$

the time Z^S has spent in the current state. It is easy to see that $\left(Z^S, V^S\right)$ is a Markov process. Z^S is therefore a semi-Markov process with the state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the stochastic process S is above the barrier l_1; 4 corresponds to the state below the barrier l_2; 2 and 3 represent the state when S is in the corridor given that it comes in through l_1 and l_2 respectively.

For Z^S the transition intensities $\lambda_{ij}(u)$ satisfy

$$P\left(Z_{t+\Delta t}^S = j, i \neq j \mid Z_t^S = i, V_t^S = u\right) = \lambda_{ij}(u)\Delta t + o\left(\Delta t\right), \tag{3.14}$$

$$P\left(Z_{t+\Delta t}^S = i \mid Z_t^S = i, V_t^S = u\right) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o\left(\Delta t\right). \tag{3.15}$$

Define

$$\bar{P}_i(u) = \exp\left\{-\int_0^u \sum_{i \neq j} \lambda_{ij}(v)\mathrm{d}v\right\}, \quad p_{ij}(u) = \lambda_{ij}(u)\bar{P}_i(u).$$

Notice that

$$P_i(u) = 1 - \bar{P}_i(u)$$

is the distribution function of the excursion time in state i, which is a random variable U_i defined as

$$U_i = \inf_{s>0} \left\{ Z_s^S \neq i \mid Z_0^S = i, V_0^S = 0 \right\}.$$

Note that because the process is time homogeneous this has the same distribution as

$$\inf_{s>0} \left\{ Z_{t+s}^S \neq i \mid Z_t^S = i, V_t^S = 0 \right\}$$

for any time t. We have therefore

$$p_{ij}(u) = \lim_{\triangle u \to 0} \frac{P\left(U_i \in (u, u + \triangle u), Z_{U_i}^S = j\right)}{\triangle u}.$$

Moreover, in the definition of Z^S, we deliberately ignore the situation when $S_t = l_i$, $i = 1, 2$. The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u = l_i\}} \mathrm{d}u = 0, \ i = 1, 2.$$

Also, when l_1 and l_2 are the regular points of the process (see [8] for definition), we have to deal with the degeneration of p_{ij}. Let us take a Brownian Motion as an example. Assume $W_t^\mu = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian Motion. Setting x_0 to be its starting point, we know its density for the first hitting time of level l_i, $i = 1, 2$ is

$$p_{x_0}(t) = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(l_i - x_0 - \mu t)^2}{2t} \right\}$$

(see [9]). According to the definition of transition density, $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$ and $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$, for $t > 0$.

In Chapter 2 in order to solve the similar problem, we introduced the perturbed Brownian motion $X^{(\epsilon)}$ with the respect to the barrier we are interested in. We apply the same idea here, and construct a new process *doubly perturbed Brownian motion*, $Y^{(\epsilon)}$, $\epsilon > 0$, with the respect to barriers l_1 and l_2. Assume $W_0^{\mu} = l_1 + \epsilon$. Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0, \\ \sigma_n &= \inf\{t > \delta_n \mid W_t^{\mu} = l_1\}, \\ \delta_{n+1} &= \inf\{t > \sigma_n \mid W_t^{\mu} = l_1 + \epsilon\}, \end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 3.1). Now define

$$\begin{cases} X_t^{(\epsilon)} = W_t^{\mu} & \text{if} \quad \delta_n \leq t < \sigma_n \\ X_t^{(\epsilon)} = W_t^{\mu} - \epsilon & \text{if} \quad \sigma_n \leq t < \delta_{n+1} \end{cases}.$$

Similarly, we then define another sequence of stopping times with the respect to process $X^{(\epsilon)}$ and barrier l_2

$$\begin{aligned} \zeta_0 &= 0, \\ \eta_n &= \inf\{t > \zeta_n \,|\, X_t^{(\epsilon)} = l_2\}, \\ \zeta_{n+1} &= \inf\{t > \eta_n \,|\, X_t^{(\epsilon)} = l_2 + \epsilon\}, \end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 3.2). Then define

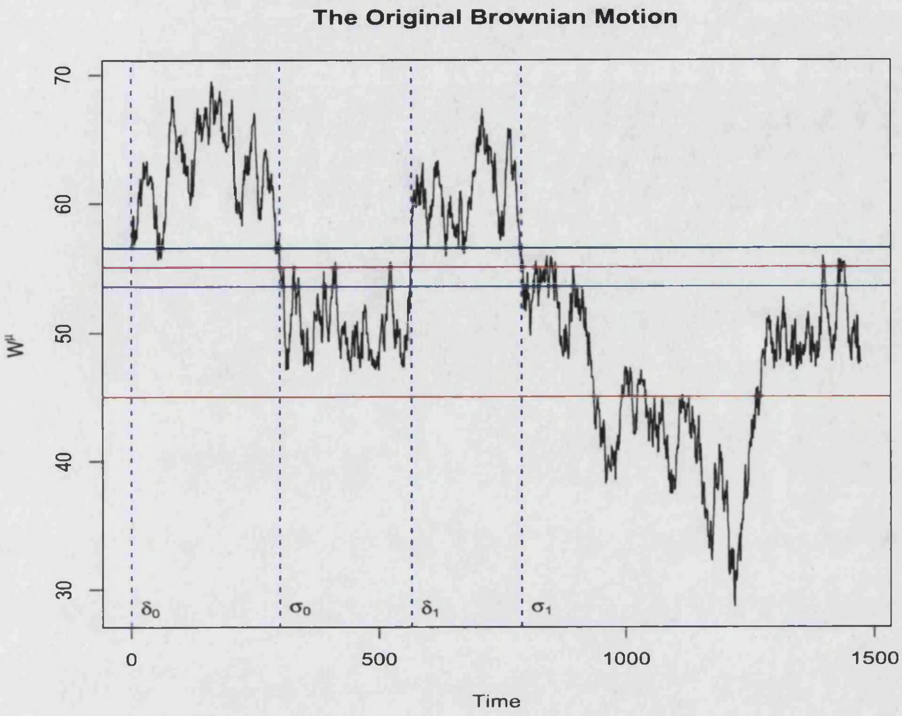

Figure 3.1: A Sample Path of W^μ

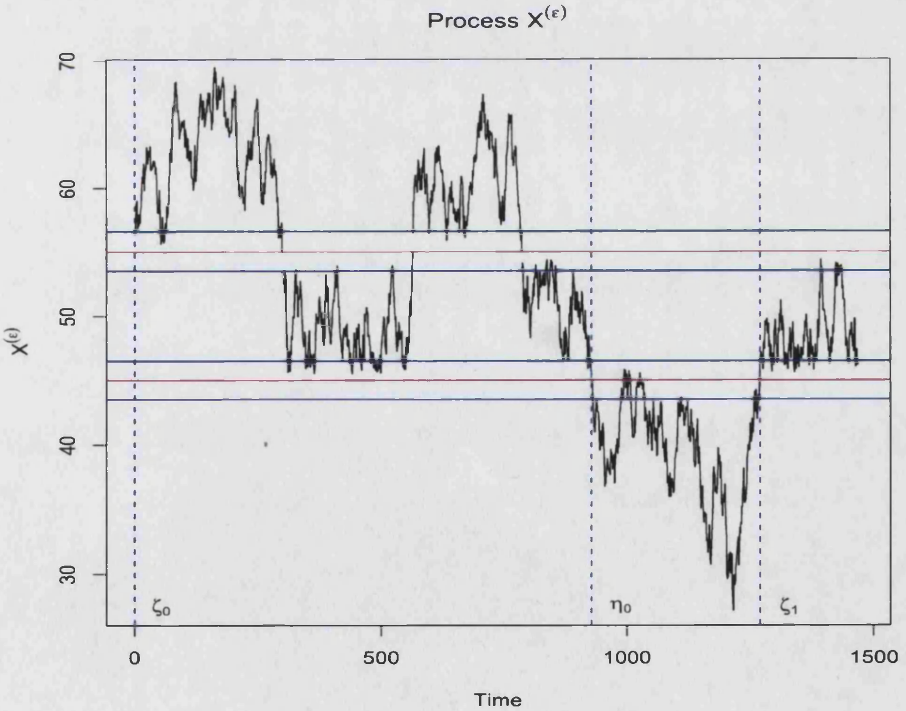

Figure 3.2: A Sample Path of $X^{(\epsilon)}$

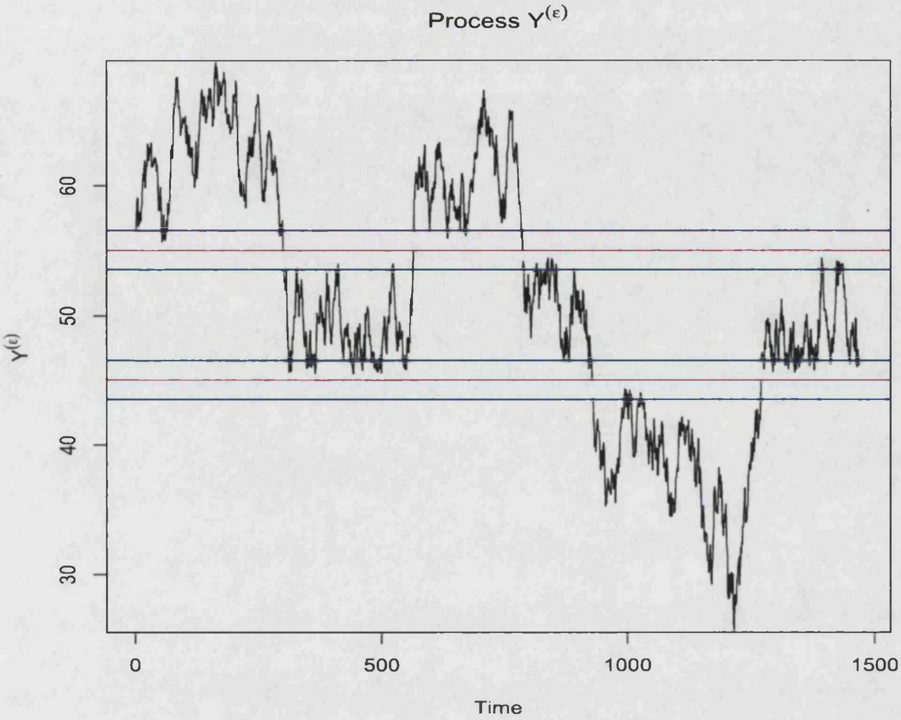

Figure 3.3: A Sample Path of $Y^{(\epsilon)}$

$$\begin{cases} Y_t^{(\epsilon)} = X_t^{(\epsilon)} & \text{if} \quad \zeta_n \leq t < \eta_n \\ Y_t^{(\epsilon)} = X_t^{(\epsilon)} - \epsilon & \text{if} \quad \eta_n \leq t < \zeta_{n+1} \end{cases}$$

It is actually a process which starts from $l_1 + \epsilon$ and has the same behavior as the related Brownian Motion expect that each time when it hits the barrier l_1 or l_2, it will have a jump towards the opposite side of the barrier with size ϵ (see Figure 3.3).

From the definition, it is clear that l_1 and l_2 become irregular points for $Y^{(\epsilon)}$. Furthermore, we prove later that the Laplace transforms of the variables defined based on $Y^{(\epsilon)}$ converge to those based on W^{μ}. As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for $Y^{(\epsilon)}$ and take the limit $\epsilon \to 0$.

For $Y^{(\epsilon)}$, we can define Z^Y, τ_1^Y, τ_2^Y, τ_3^Y, τ_4^Y and τ^Y as above (we suppress (ϵ) on the superscribe). For Z^Y, we have the transition densities (see [9])

$$\begin{aligned}
p_{12}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon+\mu t)^2}{2t}\right\}, && (3.16)\\
p_{21}(t) &= \exp\left\{\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), && (3.17)\\
p_{24}(t) &= \exp\left\{-\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), && (3.18)\\
p_{31}(t) &= \exp\left\{\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), && (3.19)\\
p_{34}(t) &= \exp\left\{-\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), && (3.20)\\
p_{43}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon-\mu t)^2}{2t}\right\}, && (3.21)
\end{aligned}$$

where

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{((2k+1)y-x)^2}{2t}\right\}.$$

Also we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \tag{3.22}$$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W^μ when $\mu = 0$.

3.3 Results for the semi-Markov model

In Section 3.2 we have introduced the Markov process $\left(Z^S, V^S\right)$. Now we apply the same definition to the doubly perturbed Brownian motion $Y^{(\epsilon)}$; therefore we have $\left(Z^Y, V^Y\right)$, where Z^Y is the current state of $Y^{(\epsilon)}$, taking value from state space $\{1,2,3,4\}$ and V^Y is the time $Y^{(\epsilon)}$ has spent in current state. V^Y is also a stochastic process. Now we consider a function of the form

$$f(u,i,t) = f_i(u,t),$$

where f_i, $i = 1,2,3,4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathscr{A}$ is defined as the operator such that

$$f\left(V_t^Y, Z_t^Y, t\right) - \int_0^t \mathscr{A} f\left(V_s^Y, Z_s^Y, s\right) \mathrm{d}s$$

is a martingale (see [18], chapter 2). Therefore solving

$$\mathscr{A} f = 0$$

subject to certain boundary conditions specified later will provide us with martingales of the form $f\left(V_t^Y, Z_t^Y, t\right)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\begin{cases} \mathscr{A} f_1(u,t) &= \dfrac{\partial f_1(u,t)}{\partial t} + \dfrac{\partial f_1(u,t)}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)) \\ \mathscr{A} f_2(u,t) &= \dfrac{\partial f_2(u,t)}{\partial t} + \dfrac{\partial f_2(u,t)}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t)) + \lambda_{24}(u)(f_4(0,t) - f_2(u,t)) \\ \mathscr{A} f_3(u,t) &= \dfrac{\partial f_3(u,t)}{\partial t} + \dfrac{\partial f_3(u,t)}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t)) + \lambda_{34}(u)(f_4(0,t) - f_3(u,t)) \\ \mathscr{A} f_4(u,t) &= \dfrac{\partial f_4(u,t)}{\partial t} + \dfrac{\partial f_4(u,t)}{\partial u} + \lambda_{43}(u)(f_4(0,t) - f_3(u,t)) \end{cases}$$

Assume f_i has the form

$$f_i(u,t) = e^{-\beta t} g_i(u).$$

By solving the equation $\mathscr{A} f = 0$, i.e. $\begin{cases} \mathscr{A} f_1 &= 0 \\ \mathscr{A} f_2 &= 0 \\ \mathscr{A} f_3 &= 0 \\ \mathscr{A} f_4 &= 0 \end{cases}$ subject to $\begin{cases} g_1(d_1) &= \alpha_1 \\ g_2(d_2) &= \alpha_2 \\ g_3(d_2) &= \alpha_3 \\ g_4(d_2) &= \alpha_4 \end{cases}$

we can get

$$\begin{aligned} g_i(u) &= \alpha_i \exp\left\{-\int_u^{d_i}\left(\beta + \sum_{j\neq i}\lambda_{ij}(v)\right)\mathrm{d}v\right\} \qquad (3.23) \\ &\quad + \sum_{j\neq i} g_j(0)\int_u^{d_i}\lambda_{ij}(s)\exp\left\{-\int_u^{s}\left(\beta + \sum_{k\neq i}\lambda_{ik}(v)\right)\mathrm{d}v\right\}\mathrm{d}s. \end{aligned}$$

In our case, we are only interested in the excursion outside the corridor. Hence, we set d_2 and d_3 to be ∞. Also $\lim_{d_2\to\infty} g_2(d_2) = \lim_{d_3\to\infty} g_3(d_3) = 0$ gives $\alpha_2 = \alpha_3 = 0$. Therefore, we have

$$g_1(0) = \alpha_1 e^{-\beta d_1}\bar{P}_1(d_1) + \left\{g_1(0)\hat{P}_{21}(\beta) + g_4(0)\hat{P}_{24}(\beta)\right\}\tilde{P}_{12}(\beta), \qquad (3.24)$$

$$g_4(0) = \alpha_4 e^{-\beta d_4}\bar{P}_4(d_4) + \left\{g_1(0)\hat{P}_{31}(\beta) + g_4(0)\hat{P}_{34}(\beta)\right\}\tilde{P}_{43}(\beta). \qquad (3.25)$$

Solving (3.24) and (3.25) gives

$$\begin{aligned} & g_1(0) \qquad (3.26) \\ = & \frac{\alpha_1 e^{-\beta d_1}\bar{P}_1(d_1)\left(1-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)\right) + \alpha_4 e^{-\beta d_4}\bar{P}_4(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}, \\ & g_4(0) \qquad (3.27) \\ = & \frac{\alpha_4 e^{-\beta d_4}\bar{P}_4(d_4)\left(1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\right) + \alpha_1 e^{-\beta d_1}\bar{P}_1(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}. \end{aligned}$$

where

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta s} p_{ij}(s) \mathrm{d}s, \tag{3.28}$$

$$\tilde{P}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} p_{ij}(s) \mathrm{d}s. \tag{3.29}$$

As a result, we have obtained the martingale

$$M_t = f\left(V_t^Y, t\right) = e^{-\beta t} g_{Z_t^Y}\left(V_t^Y\right), \quad i = 1, 2, 3, 4. \tag{3.30}$$

We now can apply the optional stopping theorem to M with the stopping time $\tau^Y \wedge t$, where τ^Y is the stopping time defined by (3.6):

$$E\left(M_{\tau^Y \wedge t}\right) = E\left(M_0\right). \tag{3.31}$$

The right hand side of (3.31) is

$$E\left(M_{\tau^Y \wedge t}\right) = E\left(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}\right) + E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right).$$

Furthermore,

$$\begin{aligned}
& E\left(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}\right) \\
= \;& E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right) \\
= \;& E\left(e^{-\beta \tau^Y} g_1\left(d_1\right) \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + E\left(e^{-\beta \tau^Y} g_4\left(d_4\right) \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right) \\
= \;& \alpha_1 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + \alpha_4 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right).
\end{aligned}$$

We also have

$$E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right) = e^{-\beta t} E\left(g_{Z_t^Y}\left(V_t^Y\right) \mathbf{1}_{\{\tau^Y > t\}}\right),$$

where Z_t^Y can take values $1, 2, 3$ or 4.

When $Z_t^Y = 1$ or 4, since $\tau^Y > t$, we have $0 \leq V_t^Y < d_1 \wedge d_4$. Since $g_i(\mu)$, $i = 1, 4$ are continuous functions, we have $g_1\left(V_t^Y\right)$ and $g_4\left(V_t^Y\right)$ are bounded.

When $Z_t^Y = 2$ or 3, since $\lim_{d_2 \to \infty} g_2\left(d_2\right) = \lim_{d_3 \to \infty} g_3\left(d_3\right) = 0$, we have that $g_2\left(V_t^Y\right)$ and $g_3\left(V_t^Y\right)$ are bounded.

Therefore

$$\lim_{t \to \infty} E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right) = 0.$$

Hence we have

$$\lim_{t \to \infty} E\left(M_{\tau^Y \wedge t}\right) = \alpha_1 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}\right) + \alpha_4 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}\right). \qquad (3.32)$$

The right hand side of (3.31) gives

$$\lim_{t \to \infty} E\left(M_0\right) = E\left(M_0\right) = \begin{cases} g_1(0), & Y_0^{(\epsilon)} = l_1 + \epsilon \\ g_4(0), & Y_0^{(\epsilon)} = l_2 - \epsilon \end{cases}.$$

By taking $\alpha_1 = 1$, $\alpha_4 = 0$ and $\alpha_1 = 0$, $\alpha_4 = 1$ we will have that when $Y_0^{(\epsilon)} = l_1 + \epsilon$

$$\begin{aligned} & E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y<\tau_4^Y\}}\right) && (3.33) \\ = & \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)\left(1-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)\right)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}, \\ & E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y>\tau_4^Y\}}\right) && (3.34) \\ = & \frac{e^{-\beta d_4}\bar{P}_{43}(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}; \end{aligned}$$

and when $Y_0^{(\epsilon)} = l_2 - \epsilon$

$$\begin{aligned} & E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y<\tau_4^Y\}}\right) && (3.35) \\ = & \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}, \\ & E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y>\tau_4^Y\}}\right) && (3.36) \\ = & \frac{e^{-\beta d_4}\bar{P}_{43}(d_4)\left(1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\right)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}. \end{aligned}$$

3.4 Main Results

In Section 3.2 we have stated that the main difficulty with Brownian Motion is that its origin point is regular, i.e. the probability that W^μ will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes $Y^{(\epsilon)}$ and (Z^Y, V^Y) with transition densities for Z^Y defined in (3.16) to (3.22).

In order to simplify the expressions, we define

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2},$$

where $\mathcal{N}(.)$ is the cumulative distribution function for the standard Normal Distribution.

Theorem 3.4.1 *For a Brownian Motion W^{μ}, $\tau_1^{W^{\mu}}$, $\tau_4^{W^{\mu}}$, $\tau^{W^{\mu}}$ defined as in (3.2), (3.5) and (3.6) with $S = W^{\mu}$, we have the following Laplace transforms:*

when $W_0^{\mu} = l_1$,

$$E\left(e^{-\beta\tau^{W^{\mu}}}\mathbf{1}_{\{\tau_1^{W^{\mu}}<\tau_4^{W^{\mu}}\}}\right) = \frac{G_1(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \tag{3.37}$$

$$E\left(e^{-\beta\tau^{W^{\mu}}}\mathbf{1}_{\{\tau_1^{W^{\mu}}>\tau_4^{W^{\mu}}\}}\right) = \frac{G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \tag{3.38}$$

$$E\left(e^{-\beta\tau^{W^{\mu}}}\right) = \frac{G_1(d_1, d_4, \mu) + G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \tag{3.39}$$

when $W_0^{\mu} = l_2$,

$$E\left(e^{-\beta\tau^{W^{\mu}}}\mathbf{1}_{\{\tau_1^{W^{\mu}}<\tau_4^{W^{\mu}}\}}\right) = \frac{G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \tag{3.40}$$

$$E\left(e^{-\beta\tau^{W^{\mu}}}\mathbf{1}_{\{\tau_1^{W^{\mu}}>\tau_4^{W^{\mu}}\}}\right) = \frac{G_1(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \tag{3.41}$$

$$E\left(e^{-\beta\tau^{W^{\mu}}}\right) = \frac{G_1(d_4, d_1, -\mu) + G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \tag{3.42}$$

where

$$\begin{aligned} G_1(x, y, z) = {} & e^{-2(l_1-l_2)\sqrt{2\beta+z^2}-\beta x}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi x y}{2}}\right\} \\ & + \frac{\left(1 - e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right)e^{-\beta x}}{2\sqrt{2\beta+z^2}}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi x}{2}}\right\} \\ & \left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right) + \sqrt{(2\beta+z^2)\,y}\right\}, \end{aligned} \tag{3.43}$$

$$G_2(x,y,z) = e^{-(l_1-l_2)\left(\sqrt{2\beta+z^2}-z\right)-\beta x}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi xy}{2}}\right\}, \qquad (3.44)$$

$$\begin{aligned} G(x,y,z) &= e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\left\{\sqrt{y}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,x}{2}}\right)+\sqrt{x}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right)\right\} \\ &+\frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right)}{2\sqrt{2\beta+z^2}}\left\{\Psi\left(\sqrt{\frac{(2\beta+z^2)\,x}{2}}\right)+\sqrt{\frac{(2\beta+z^2)\,\pi x}{2}}\right\} \\ &\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right)+\sqrt{(2\beta+z^2)\,y}\right\}. \qquad (3.45) \end{aligned}$$

Proof: We apply the transition densities in (3.16) to (3.22) to the results in (3.33) to (3.36) and take the limit as $\epsilon \to 0$. In order to show that we can take the limit, we consider two processes W^{μ} and $W^{\mu,-2\epsilon} = W^{\mu} - 2\epsilon$. According to the definitions, $Y^{(\epsilon)}$ satisfies

$$\lim_{\epsilon \to 0} Y_t^{(\epsilon)} = W_t^{\mu}, \ a.s. \text{ for all } t,$$

$$W_t^{\mu,-2\epsilon} \le Y_t^{(\epsilon)} \le W_t^{\mu} \quad \text{for all } t,$$

and $g_{l_1,t}^{Y}$ always lies between $g_{l_1,t}^{W^{\mu}}$ and $g_{l_1,t}^{W^{\mu,-2\epsilon}}$. Since

$$\lim_{\epsilon \to 0} g_{l_1,t}^{W^{\mu,-2\epsilon}} = \lim_{\epsilon \to 0} g_{l_1+2\epsilon,t}^{W^{\mu}} = g_{l_1,t}^{W^{\mu}},$$

we have that

$$\lim_{\epsilon \to 0} g_{l_1,t}^{Y} = g_{l_1,t}^{W^{\mu}}, \ a.s.$$

Since g_t^S is a right continuous function with respect to t, we have that

$$\lim_{\epsilon \to 0} g_{l_1,t}^{Y} = g_{l_1,t}^{W^{\mu}}, \ a.s. \text{ for all } t$$

and therefore

$$\lim_{\epsilon \to 0} \mathbf{1}_{\left\{Y_t^{(\epsilon)} > l_1\right\}} \left(t - g_{l_1,t}^{Y}\right) = \mathbf{1}_{\left\{W_t^{\mu} > l_1\right\}} \left(t - g_{l_1,t}^{W^\mu}\right) \ a.s. \text{ for all } t.$$

From the definition of τ_1^S we have that

$$\begin{aligned}\left\{\tau_1^{W^\mu} < t\right\} &= \left\{\sup_{0 \le s \le t} \left\{\mathbf{1}_{\left\{W_s^{\mu} > l_1\right\}} \left(s - g_{l_1,s}^{W^\mu}\right)\right\} \ge d_1\right\} \\ &= \lim_{\epsilon \to 0} \left\{\sup_{0 \le s \le t} \left\{\mathbf{1}_{\left\{Y_s^{(\epsilon)} > l_1\right\}} \left(s - g_{l_1,s}^{Y}\right)\right\} \ge d_1\right\} = \lim_{\epsilon \to 0} \left\{\tau_1^Y < t\right\}.\end{aligned}$$

Consequently,

$$\lim_{\epsilon \to 0} \tau_1^Y = \tau_1^{W^\mu} \ a.s.$$

By the same argument, we can show that

$$\lim_{\epsilon \to 0} \tau_4^Y = \tau_4^{W^\mu} \ a.s.$$

Since $\tau^S = \tau_1^S \wedge \tau_4^S$, we have

$$\lim_{\epsilon \to 0} \tau^Y = \tau^{W^\mu} \ a.s.$$

In Chapter 2 we have shown that the convergence of τ_1^Y, τ_4^Y and τ^Y to $\tau_1^{W^\mu}$, $\tau_4^{W^\mu}$ and τ^{W^μ} respectively leads to the convergence of their Laplace transforms. Therefore we will get the results shown by (3.37), (3.38), (3.40) and (3.41). We can then get

(3.39) and (3.42) by the fact that

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}\right) + E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right).$$

□

Corollary 3.4.1.1 *For a standard Brownian Motion ($\mu = 0$), we have*

when $W_0 = l_1$,

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W<\tau_4^W\}}\right) = \frac{G_1(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{3.46}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W>\tau_4^W\}}\right) = \frac{G_2(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{3.47}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_1,d_4,0)+G_2(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{3.48}$$

when $W_0 = l_2$,

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W<\tau_4^W\}}\right) = \frac{G_2(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{3.49}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W>\tau_4^W\}}\right) = \frac{G_1(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{3.50}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_4,d_1,0)+G_2(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{3.51}$$

where

$$G_1(x,y,0) = e^{-2(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y} + \frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta}}\right)e^{-\beta x}}{2\sqrt{2\beta}}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\beta y}\right)+\sqrt{2\beta y}\right\}, \tag{3.52}$$

$$G_2(x,y,0) = e^{-(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y}, \tag{3.53}$$

$$G(x,y,0) = e^{-2(l_1-l_2)\sqrt{2\beta}}\left\{\sqrt{y}\Psi\left(\sqrt{\beta x}\right)+\sqrt{x}\Psi\left(\sqrt{\beta y}\right)\right\} + \frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta}}\right)}{2\sqrt{2\beta}}\left\{\Psi\left(\sqrt{\beta x}\right)+\sqrt{\beta\pi x}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\beta y}\right)+\sqrt{2\beta y}\right\}. \tag{3.54}$$

Remark 1: By taking the limit $l_1 - l_2 \to 0$, we can get the result for the single barrier two-sided excursion case as in Chapter 2.

Remark 2: If we only want to consider the excursion above a barrier, we can let $l_2 \to -\infty$. Similarly, for the one below a barrier, we can let $l_1 \to +\infty$. These results have been shown in Chapter 2.

Corollary 3.4.1.2 *For a Brownian Motion W^μ, τ^{W^μ} defined as in (3.6) with $S = W^\mu$, we have the following Laplace transforms:*

when $W_0^\mu = x_0$, $x_0 > l_1$,

$$\begin{aligned} & E\left(e^{-\beta\tau^{W^\mu}}\right) \\ = & \left\{e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right. \\ & \left.+e^{-\left(\mu-\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right\}\frac{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)}{G(d_1,d_4,\mu)} \\ & +e^{-\beta d_1}\left\{1-e^{-(\mu+|\mu|)(x_0-l_1)}\mathcal{N}\left(|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right. \\ & \left.-e^{-(\mu-|\mu|)(x_0-l_1)}\mathcal{N}\left(-|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right\};. \end{aligned} \tag{3.55}$$

when $W_0^\mu = x_0,\ l_2 \le x_0 \le l_1,$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \qquad (3.56)\\
= & \frac{e^{(l_1-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(x_0-l_2)} - e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right\}\{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)\}}{\left\{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\right\}G(d_1,d_2,\mu)}\\
& +\frac{e^{(l_2-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right\}\{G_2(d_1,d_4,\mu)+G_1(d_4,d_1,-\mu)\}}{\left\{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\right\}G(d_1,d_2,\mu)};
\end{aligned}
$$

when $W_0^\mu = x_0,\ x_0 < l_2,$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \qquad (3.57)\\
= & \left\{e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)(l_2-x)}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_4} - \frac{l_2-x}{\sqrt{d_4}}\right)\right.\\
& \left.+e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)(l_2-x)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_4} - \frac{l_2-x}{\sqrt{d_4}}\right)\right\}\frac{G_1(d_4,d_1,-\mu)+G_2(d_1,d_4,\mu)}{G(d_1,d_4,\mu)}\\
& +e^{-\beta d_4}\left\{1 - e^{(\mu-|\mu|)(l_2-x)}\mathcal{N}\left(|\mu|\sqrt{d_4} - \frac{l_2-x}{\sqrt{d_4}}\right)\right.\\
& \left.-e^{(\mu+|\mu|)(l_2-x)}\mathcal{N}\left(-|\mu|\sqrt{d_4} - \frac{l_2-x}{\sqrt{d_4}}\right)\right\}.
\end{aligned}
$$

Proof: We will first prove the case when $x_0 > l_1$. Define $T = \inf\{t \mid W_t^\mu = l_1\}$, i.e. the first time W^μ hits l_1. By definition, we have $\tau^{W^\mu} = d_1$, if $T \ge d_1$; $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$, if $T < d_1$, where $\widetilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1.

As a result

$$\begin{aligned} & E\left(e^{-\beta\tau^{W^\mu}}\right) \\ = \; & E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T\geq d_1\}}\right)+E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d_1\}}\right) \\ = \; & e^{-\beta d_1}P\left(T\geq d_1\right)+E\left(e^{-\beta T}\mathbf{1}_{\{T<d_1\}}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right) \end{aligned}$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ has been calculated in Theorem 3.4.1 (see (3.39)). The density for T is given in [9] as

$$p_{x_0}=\frac{|l_1-x_0|}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(l_i-x_0-\mu t)^2}{2t}\right\}.$$

We can therefore calculate

$$\begin{aligned} P\left(T\geq d_1\right) &= 1-e^{-(\mu+|\mu|)(x_0-l_1)}\mathcal{N}\left(|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right) \\ &\quad -e^{-(\mu-|\mu|)(x_0-l_1)}\mathcal{N}\left(-|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right), \end{aligned}$$

$$\begin{aligned} E\left(e^{-\beta T}\mathbf{1}_{\{T<d_1\}}\right) &= e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right) \\ &\quad +e^{-\left(\mu-\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right). \end{aligned}$$

We therefore get the result in (3.55). For the case when $x_0<l_2$, we can apply the same argument.

When $l_2\leq x_0\leq l_1$, we define $\widetilde{T}=\inf\left(t\mid W_t^\mu\notin(l_2,l_1)\right)$. By definition, we have $\tau^{W^\mu}=T+\tau^{\widetilde{W}^\mu}$, if $W_{\widetilde{T}}^\mu=l_1$; $\tau^{W^\mu}=T+\tau^{\underline{W}^\mu}$, if $W_{\widetilde{T}}^\mu=l_2$, where $\underline{W}^\mu$ stands for a

Brownian motion with drift started from l_2. Consequently,

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \\
= \;& E\left(e^{-\beta T}e^{-\beta\tau^{\overline{W}^\mu}}\mathbf{1}_{\{T=l_1\}}\right) + E\left(e^{-\beta T}e^{-\beta\tau^{\underline{W}^\mu}}\mathbf{1}_{\{T=l_2\}}\right) \\
= \;& E\left(e^{-\beta T}\mathbf{1}_{\{T=l_1\}}\right)E\left(e^{-\beta\tau^{\overline{W}^\mu}}\right) + E\left(e^{-\beta T}\mathbf{1}_{\{T=l_2\}}\right)E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)
\end{aligned}
$$

The last equality is based on the independence of T and $\underline{W}^\mu$. $E\left(e^{-\beta\tau^{\overline{W}^\mu}}\right)$ and $E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)$ have been obtained by Theorem 3.4.1, (3.39) and (3.42). According to [9], we have

$$
E\left(e^{-\beta T}\mathbf{1}_{\{T=l_1\}}\right) = \frac{e^{(l_1-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(x_0-l_2)} - e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}},
$$

$$
E\left(e^{-\beta T}\mathbf{1}_{\{T=l_2\}}\right) = \frac{e^{(l_2-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}}.
$$

We have therefore obtained (3.56).

□

Theorem 3.4.2 *The probability that W^μ with $W_0^\mu = x_0$, $l_2 \le x_0 \le l_1$, achieves an excursion above l_1 with length as least d_1 before it achieves an excursion below l_2 with length at least d_4 is*

$$
\begin{aligned}
P\left(\tau_1^{W^\mu} < \tau_4^{W^\mu}\right) \;=\; & \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\right\}F_1(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)} \\
& + \frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\right\}F_2(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)},
\end{aligned}
\tag{3.58}
$$

$$P\left(\tau_1^{W^\mu} > \tau_4^{W^\mu}\right) = \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\right\} F_2(d_4, d_1, -\mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\} F(d_1, d_4, \mu)} + \frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\right\} F_1(d_4, d_1, -\mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\} F(d_1, d_4, \mu)}; \tag{3.59}$$

where

$$F_1(x,y,z) = e^{-2(l_1-l_2)|z|}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}}\right\} + \frac{\left(1-e^{-2(l_1-l_2)|z|}\right)}{2|z|}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi x}{2}}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right) + |z|\sqrt{y}\right\}, \tag{3.60}$$

$$F_2(x,y,z) = e^{-(l_1-l_2)(|z|-z)}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}}\right\}, \tag{3.61}$$

$$F(x,y,z) = e^{-2(l_1-l_2)|z|}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + \sqrt{x}\Psi\left(|z|\sqrt{\frac{y}{2}}\right)\right\} + \frac{\left(1-e^{-2(l_1-l_2)|z|}\right)}{2|z|}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + |z|\sqrt{\frac{\pi x}{2}}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right) + |z|\sqrt{y}\right\}. \tag{3.62}$$

Proof: From Theorem 3.4.1 and (3.56) in Corollary 3.4.1.2, we actually know that, when $W_0^\mu = x_0$, $l_2 \le x_0 \le l_1$,

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}}\right) = \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\right\} G_1(d_1, d_4, \mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\} G(d_1, d_4, \mu)} + \frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\right\} G_2(d_1, d_4, \mu)}{\left\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\right\} G(d_1, d_4, \mu)}, \tag{3.63}$$

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right)=\frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right\}G_2(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)}+\frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right\}G_1(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)}. \tag{3.64}$$

Setting $\beta = 0$ in (3.63) and (3.64) yields the results.

□

Theorem 3.4.2 leads to the following remarkable result.

Corollary 3.4.2.1 *The probability that a standard Brownian motion W with $W_0 = x_0$, $l_2 \le x_0 \le l_1$, we have*

$$P\left(\tau_1^W < \tau_4^W\right) = \frac{\sqrt{d_4} + (x_0 - l_2)\sqrt{\frac{2}{\pi}}}{\sqrt{d_1} + \sqrt{d_4} + (l_1 - l_2)\sqrt{\frac{2}{\pi}}}, \tag{3.65}$$

$$P\left(\tau_1^W > \tau_4^W\right) = \frac{\sqrt{d_1} + (l_1 - x_0)\sqrt{\frac{2}{\pi}}}{\sqrt{d_1} + \sqrt{d_4} + (l_1 - l_2)\sqrt{\frac{2}{\pi}}}. \tag{3.66}$$

Remark: When we take $l_1 \to 0$, $l_2 \to 0$, $x_0 \to 0$, we can get the results for the one barrier case as in Chapter 2.

We will now extent Corollary 3.4.1.2 to obtain the joint distribution of W and τ^W at an exponential time. This is an application of (3.56) and Girsanov's theorem.

Theorem 3.4.3 *For a standard Brownian Motion W with $W_0 = x_0$, $l_2 \le x_0 \le l_1$ and τ^W defined as in (3.4) with $S = W$, we have the following result:*

For the case $x > l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_1(x_0) h(x - l_1, d_1); \tag{3.67}$$

For the case $l_2 \leq x \leq l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4); \tag{3.68}$$

For the case $x < l_2$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_2(x_0) h(x - l_2, d_4); \tag{3.69}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W and

$$f(x, y) = \frac{e^{-\sqrt{2\gamma}|x|}}{\sqrt{2\gamma}} - e^{\gamma y - \sqrt{2\gamma}|x|} \sqrt{2\pi y} \mathcal{N}\left(-\sqrt{2\gamma y}\right), \tag{3.70}$$

$$h(x, y) = \sqrt{2\pi y} e^{\gamma y} \left\{ e^{-\sqrt{2\gamma}|x|} \mathcal{N}\left(\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right) - e^{\sqrt{2\gamma}|x|} \mathcal{N}\left(-\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right) \right\}, \tag{3.71}$$

$$a_1(x_0) = \frac{\gamma \left\{ e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)} \right\} b_1(d_1, d_4) + \gamma \left\{ e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)} \right\} b_2(d_1, d_4)}{G \left\{ e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)} \right\}}, \tag{3.72}$$

$$a_2(x_0) = \frac{\gamma \left\{ e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)} \right\} b_2(d_4, d_1) + \gamma \left\{ e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)} \right\} b_1(d_4, d_1)}{G \left\{ e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)} \right\}}, \tag{3.73}$$

$$b_1(x, y) = e^{-2(l_1 - l_2)\sqrt{2\gamma} - \gamma x} \sqrt{y} + \frac{1 - e^{-2\gamma\sqrt{2\gamma}}}{2\sqrt{2\gamma}} e^{-\gamma x} \left\{ \sqrt{\frac{2}{\pi}} \Psi\left(\sqrt{\gamma y}\right) + \sqrt{2\gamma y} \right\}, \tag{3.74}$$

$$b_2(x,y) = e^{-(l_1-l_2)\sqrt{2\gamma}-\gamma x}\sqrt{y}, \tag{3.75}$$

$$\begin{aligned} G &= e^{-2(l_1-l_2)\sqrt{2\gamma}}\left\{\sqrt{d_4}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_4}\right)\right\} \qquad (3.76) \\ &+\frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\gamma}}\right)}{2\sqrt{2\gamma}}\left\{\Psi\left(\sqrt{\gamma d_1}\right)+\sqrt{\gamma\pi d_1}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\gamma d_4}\right)+\sqrt{2\gamma d_4}\right\}. \end{aligned}$$

Proof: see appendix.

□

3.5 Pricing double barrier Parisian Options

We want to price a double barrier Parisian call option with the current price of its underlying asset to be x, $L_1 < x < L_2$, the owner of which will obtain the right to exercise it when either the length of the excursion above the barrier L_1 reaches d_1, or the length of the excursion below the barrier L_2 reaches d_2 before T. Its price formula is given by

$$DP_{min-in-call} = e^{-rT}E_Q\left((S_T-K)^+\mathbf{1}_{\{\tau^S<T\}}\right),$$

where S is the underlying stock price, Q denotes the risk neutral measure, τ^S is defined with the respect to barrier L_1 and L_2. The subscript *min-in-call* means it is a call option which will be triggered when the minimum of two stopping times, τ_1^S and τ_4^S, is less than T, i.e. $\tau^S < T$. We assume S is a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad S_0 = x,$$

where $L_1 < x < L_2$, r is the risk free rate, W_t with $W_0 = 0$ is a standard Brownian motion under Q. Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), \quad l_2 = \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right).$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\left\{\sigma(mt + W_t)\right\} = xe^{\sigma B_t}.$$

By applying Girsanov's Theorem, we have

$$DP_{min-in-call} = e^{-\left(r+\frac{1}{2}m^2\right)T} E_P\left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T}\mathbf{1}_{\{\tau^B < T\}}\right],$$

where P is a new measure, under which B_t is a standard Brownian motion with $B_0 = 0$, and τ^B is the stopping time defined with the respect to barrier l_1, l_2. And we define

$$DP^*_{min-in-call} = e^{\left(r+\frac{1}{2}m^2\right)T} DP_{min-in-call}.$$

We are going to show that we can obtain the Laplace transform of $DP^*_{min-in-call}$ w.r.t T, denoted by $\mathscr{L}_T$.

Firstly, assuming $\tilde{T}$, independent of W, is a random variable with an exponential distribution with parameter γ, we have

$$\begin{aligned}
& E_P\left[\left(xe^{\sigma B_{\tilde{T}}}-K\right)^+ e^{mB_{\tilde{T}}}\mathbf{1}_{\{\tau^B<\tilde{T}\}}\right] \\
= & \int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_{\tilde{T}}\in \mathrm{d}y,\tau^B<\tilde{T}\right) \\
= & \int_0^\infty \gamma e^{-\gamma T}\int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_T\in \mathrm{d}y,\tau^B<T\right)\mathrm{d}T \\
= & \gamma\int_0^\infty e^{-\gamma T}E_P\left[\left(xe^{\sigma B_T}-K\right)^+ e^{mB_T}\mathbf{1}_{\{\tau^B<T\}}\right]\mathrm{d}T \\
= & \gamma\mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T=\frac{1}{\gamma}\int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_{\tilde{T}}\in \mathrm{d}y,\tau^B<\tilde{T}\right).$$

By using the results in Theorem 3.4.3, this Laplace transform can be calculated explicitly.

When $b\geq l_1$, i.e. $K\geq L_1$, we have

$$\mathscr{L}_T=\frac{x}{\gamma}F_1(\sigma+m)-\frac{K}{\gamma}F_1(m),$$

where

$$
\begin{aligned}
F_1(x) &= a_1(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1}\sqrt{2\pi d_1}\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{\sqrt{2\gamma}l_1+\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
&+a_2(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4}\sqrt{2\pi d_4}\mathscr{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{\sqrt{2\gamma}l_2+\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
&+a_1(0)\sqrt{2\pi d_1}e^{\gamma d_1}\left\{\frac{2xe^{xl_1-rd_1+\frac{d_1x^2}{2}}\mathscr{N}\left(x\sqrt{d_1}-\frac{b-l_1}{\sqrt{d_1}}\right)}{2\gamma-x^2}\right. \\
&\left.+\frac{e^{\sqrt{2\gamma}l_1+\left(x-\sqrt{2\gamma}\right)b}\mathscr{N}\left(\frac{b-l_1}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)}{\sqrt{2\gamma}-x}+\frac{e^{-\sqrt{2\gamma}l_1+\left(x+\sqrt{2\gamma}\right)b}\mathscr{N}\left(-\frac{b-l_1}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)}{\sqrt{2\gamma}+x}\right\};
\end{aligned}
$$

when $l_2 < b < l_1$, i.e. $L_2 < K < L_1$, we have

$$
\mathscr{L}_T = \frac{x}{\gamma}F_2(\sigma+m) - \frac{K}{\gamma}F_2(m),
$$

where

$$
\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1x}}{2\gamma-x^2}\left\{1+x\sqrt{2\pi d_1}e^{\frac{d_1x^2}{2}}\mathscr{N}\left(x\sqrt{d_1}\right)\right\} \\
&-a_1(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1}\sqrt{2\pi d_1}\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_1+\left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
&+a_2(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4}\sqrt{2\pi d_4}\mathscr{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{\sqrt{2\gamma}l_2+\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x};
\end{aligned}
$$

when $b \leq l_2$, i.e. $K \leq L_2$, we have

$$
\mathscr{L}_T = \frac{x}{\gamma}F_3(\sigma+m) - \frac{K}{\gamma}F_3(m),
$$

where

$$
\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1 x}}{2\gamma - x^2}\left\{1 + x\sqrt{2\pi d_1}e^{\frac{d_1 x^2}{2}}\mathcal{N}\left(x\sqrt{d_1}\right)\right\} \\
&-a_1(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1}\sqrt{2\pi d_1}\mathcal{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_1+\left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
&+\frac{2a_2(0)e^{l_2 x}}{2\gamma - x^2}\left\{1 - 2\sqrt{\pi d_4 \gamma}e^{\frac{d_4 x^2}{2}}\mathcal{N}\left(x\sqrt{d_4}\right)\right\} \\
&-a_2(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4}\sqrt{2\pi d_4}\mathcal{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_2+\left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
&+a_2(0)\sqrt{2\pi d_4}e^{\gamma d_4}\left\{\frac{2\sqrt{2\gamma}e^{xl_2 - rd_4 + \frac{d_4 x^2}{2}}\mathcal{N}\left(x\sqrt{d_4} - \frac{b-l_2}{\sqrt{d_4}}\right)}{2\gamma - x^2}\right. \\
&\left. - \frac{e^{\sqrt{2\gamma}l_2+\left(x-\sqrt{2\gamma}\right)b}\mathcal{N}\left(\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma}-x} - \frac{e^{-\sqrt{2\gamma}l_2+\left(x+\sqrt{2\gamma}\right)b}\mathcal{N}\left(-\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma}+x}\right\}.
\end{aligned}
$$

Remark: The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$DP^*_{min-call-in} = e^{\left(r+\frac{1}{2}m^2\right)T}DP_{min-call-in}.$$

For

$$DP_{min-call-out} = e^{-rT}E_Q\left((S_T - K)^+\mathbf{1}_{\{\tau^S > T\}}\right),$$

we can get the result from the relationship that

$$DP_{min-call-out} = e^{-rT}E_Q\left\{(S_T - K)^+\right\} - DP_{min-call-in}.$$

Furthermore, if we set

$$\tilde{\tau}^S = \tau_1^S \vee \tau_2^S,$$

we can define another type of Parisian options by $\tilde{\tau}^Y$:

$$DP_{max-call-in} = e^{-rT} E_Q\left((S_T - K)^+ \mathbf{1}_{\{\tilde{\tau}^S < T\}}\right).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\{\tilde{\tau}^S < T\}} = \mathbf{1}_{\{\tau_1^S < T\}} + \mathbf{1}_{\{\tau_2^S < T\}} - \mathbf{1}_{\{\tau^S < T\}}.$$

We have therefore

$$DP_{max-call-in} = DP_{up-in-call} + P_{down-in-call} - DP_{min-call-in}.$$

Similarly, from

$$DP_{max-call-out} = e^{-rT} E_Q\left\{(S_T - K)^+\right\} - DP_{max-call-in},$$

we can work out $DP_{max-call-out}$.

3.6 Appendix

We prove Theorem 3.4.3 in this section. Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x\mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta\tau_{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T}\mathbf{1}_{\{\tau_{W^\mu}<T\}}\mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta+\frac{1}{2}\mu^2)T-\mu x_0} E\left(e^{\mu W_T}\mathbf{1}_{\{\tau_W<T\}}\right)\mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta\tau_{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2)e^{-\gamma T-\mu x_0} E\left(e^{\mu W_T}\mathbf{1}_{\{\tau_W<T\}}\right)\mathrm{d}T \\ &= \frac{\gamma-\frac{1}{2}\mu^2}{\gamma}e^{-\mu x_0}E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau_W<\tilde{T}\}}\right), \end{aligned}$$

where $\tilde{T}$, independent of W, is a random variable with an exponential distribution of parameter γ. Therefore we have

$$E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau_W<\tilde{T}\}}\right) = \frac{\gamma e^{\mu x_0}}{\gamma-\frac{1}{2}\mu^2}E\left(e^{-\beta\tau_{W^\mu}}\right)$$

In order to inverse the above moment generating function, we first need to inverse the following expressions:

$$\frac{\mu}{\gamma-\frac{\mu^2}{2}} = \int_0^\infty e^{\mu x} e^{-\sqrt{2\gamma}x} \mathrm{d}x - \int_{-\infty}^0 e^{\mu x} e^{\sqrt{2\gamma}x} \mathrm{d}x,$$

$$\frac{1}{\gamma-\frac{\mu^2}{2}} = \int_0^\infty e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}x} \mathrm{d}x + \int_{-\infty}^0 e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}x} \mathrm{d}x,$$

$$e^{\frac{d_1}{2}\mu^2} = \int_{-\infty}^\infty e^{\mu x} \frac{1}{\sqrt{2\pi d_1}} \exp\left\{-\frac{x^2}{2d_1}\right\} \mathrm{d}x,$$

$$1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right) = \int_{-\infty}^0 e^{\mu x} \frac{-x}{d_i} e^{-\frac{x^2}{2d_i}} \mathrm{d}x.$$

Therefore the inversion of $\frac{\mu e^{\frac{d_1}{2}\mu^2}}{\gamma-\frac{\mu^2}{2}}$ is

$$\begin{aligned} &\int_0^\infty e^{-\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} \mathrm{d}y - \int_{-\infty}^0 e^{\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} \mathrm{d}y \\ = \ & e^{\gamma d_1}\left\{e^{-\sqrt{2\gamma}x}\mathcal{N}\left(\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right) - e^{\sqrt{2\gamma}x}\mathcal{N}\left(-\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)\right\}. \end{aligned}$$

The inversion of $\frac{1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)}{\gamma-\frac{\mu^2}{2}}$ is given below.

For $x \geqslant 0$,

$$\int_{-\infty}^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y = \frac{e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x}\sqrt{2\pi d_i}\mathcal{N}\left(-\sqrt{2\gamma d_i}\right);$$

For $x < 0$,

$$\begin{aligned}
&\int_{-\infty}^{x} \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y + \int_{x}^{0} \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-y)} \mathrm{d}y \\
= \;& \frac{e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x} \sqrt{2\pi d_i} \mathscr{N}\left(\frac{x}{\sqrt{d_i}} - \sqrt{2\gamma d_i}\right) \\
& + e^{\gamma d_i + \sqrt{2\gamma}x} \sqrt{2\pi d_i} \left\{ \mathscr{N}\left(\sqrt{2\gamma d_i}\right) - \mathscr{N}\left(\frac{x}{\sqrt{d_i}} + \sqrt{2\gamma d_i}\right) \right\}.
\end{aligned}$$

Consequently, we can get Theorem 3.4.3.

Chapter 4

Parisian Corridor Options

Abstract

In this paper, we study the excursion time of a Brownian motion with drift inside a corridor by using a four-state semi-Markov model. In mathematical finance, these results have an important application in the valuation of options whose prices depend on the time their underlying assets prices spend between two different values. In this paper, we introduce the Parisian corridor option and obtain an explicit expression for the Laplace transform of its price formula.

Keywords: excursion time, four-state Semi-Markov model, Parisian corridor options, Laplace transform.

4.1 Introduction

The Parisian corridor options replace the barrier by a corridor. Instead of considering the excursion above or below a barrier, we consider the excursions inside a corridor. For example, the owner of a *Parisian corridor in option* gains the option if the underlying asset price process S has an excursion in the corridor for longer than d before the maturity of the option. For the pricing of the Parisian options whose prices depends on the excursion outside a corridor see Chapter 3. We will explain later in this section that these options can be used to take positions depending on volatility. We will also explain that they can be viewed as generalisations of certain types of double barrier options.

In this paper, we are going to use the same definition for the excursion as in [13] and [14]. Let S be a stochastic process and l_1, l_2, $l_1 > l_2$ be the level of two barriers forming the corridor. We define

$$g^S_{l_i,t} = \sup\{s \leq t \mid S_s = l_i\}, \qquad d^S_{l_i,t} = \inf\{s \geq t \mid S_s = l_i\}, \qquad i = 1, 2, \tag{4.1}$$

with the usual conventions, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. Assuming $d_i > 0$, $i = 1, 2, 3, 4$, we now define

$$\tau^S_1 = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g^S_{l_1,t}) \geq d_1\right\}, \tag{4.2}$$

$$\tau^S_2 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} > g^S_{l_2,t}\right\}}(t - g^S_{l_1,t}) \geq d_2\right\}, \tag{4.3}$$

$$\tau^S_3 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} < g^S_{l_2,t}\right\}}(t - g^S_{l_2,t}) \geq d_3\right\}, \tag{4.4}$$

$$\tau_4^S = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g_{l_2,t}^S) \geq d_4\}, \tag{4.5}$$

$$\tau^S = \tau_2^S \wedge \tau_3^S. \tag{4.6}$$

We can see that τ_2^S is the first time that the length of the excursion in the corridor reaches the given level d_2, given that this excursion starts from the upper barrier l_1; τ_3^S corresponds to the one in the corridor with the given level d_3 starting from the lower barrier l_2; and τ^S is the smaller of τ_2^S and τ_3^S. When we take $d_2 = d_3 = d$, τ^S is actually the the first time that the length of the excursion inside the corridor reaches given level d, which is what we want to study later on.

We can also see that τ_1^S is the first time that the length of the excursion of process S above the barrier l_1 reaches given level d_1; τ_4^S corresponds to the one below l_2 with required length d_4. Although τ_1^S and τ_4^S are not of our interest in this paper (see Chapter 3 for the pricing of the Parisian options depend on τ_1^S and τ_4^S), we need to use these two stopping times to define our four states semi-Markov model.

Now assume r is the risk-free rate, T is the term of the option, S is the price of its underlying asset, K is the strike price, Q is risk neutral measure. If we have a Parisian corridor out-call option with the barrier l_1 and l_2, its price can be expressed as:

$$PC_{out-call} = e^{-rT} E_Q\left(\mathbf{1}_{\{\tau^S > T\}}\left(S_T - K\right)^+\right);$$

and the price of a Parisian corridor in-put option is:

$$PC_{in-put} = e^{-rT} E_Q\left(\mathbf{1}_{\{\tau^S < T\}}\left(K - S_T\right)^+\right).$$

In-put and *in-call* Parisian corridor options can be viewed as options that are activated only when the price has gone through a low volatility period, demonstrated by the fact that it has stayed between two fixed values for a certain time interval. *Out-put* and *out-call* Parisian corridor options can be viewed the opposite way. Either way, the buyer and the seller of these derivatives take positions on volatility in the sense that they are betting on the ability of the price to stay within two values long enough.

As we said earlier, Parisian corridor options can also be viewed as generalisations of double barrier options. For example in the case where the starting price is inside the interval, they are generalisations of one-touch knock-out double barrier options. For more details on double barrier options and their pricing, see Chapter 2 and [39].

In this paper, we are going to study the excursion time inside the corridor using a semi-Markov model consisting of four states. By applying the model to a Brownian motion, we can get the explicit form of the Laplace transform for the price of Parisian corridor options. One can then invert using techniques as in [38].

In Section 4.2 we introduce the four-state semi-Markov model as well as a new process, doubly perturbed Brownian motion, which has the same behavior as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size ϵ. In Section 4.3 we obtain the martingale to which we can apply the optional sampling theorem and get the Laplace transform that we can use for pricing later. We give our main results applied to Brownian motions in Section 4.4, including the Laplace transforms for the stopping times we defined by (4.6) for both a Brownian motion with drift, i.e.

$S = W^{\mu}$, and a standard Brownian motion, i.e. $S = W$. In Section 4.5 we focus on pricing the Parisian corridor options.

4.2 Definitions

From the description above, it is clear that we are actually considering four states, the state when the stochastic process is above the barrier l_1 the state when it is below l_2 and two states when it is between l_1 and l_2 depending on whether it comes into the corridor through l_1 or l_2. For each state, we are interested in the time the process spends in it. We therefore introduce a new process

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > l_1 \\ 2, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S > g_{l_2,t}^S \\ 3, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S < g_{l_2,t}^S \\ 4, & \text{if } S_t < l_2 \end{cases}.$$

We can now express the variables defined above in terms of Z^S:

$$g_{l_i,t}^S = \sup\left\{s \leq t \mid Z_s^S \neq Z_t^S\right\}, \tag{4.7}$$

$$d_{l_i,t}^S = \inf\left\{s \geq t \mid Z_s^S \neq Z_t^S\right\}, \tag{4.8}$$

$$\tau_1^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=1\}}\left(t - g_{l_1,t}^S\right) \geq d_1\right\}, \tag{4.9}$$

$$\tau_2^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}}\left(t - g_{l_1,t}^S\right) \geq d_2\right\}, \tag{4.10}$$

$$\tau_3^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=3\}}\left(t - g_{l_2,t}^S\right) \geq d_3\right\}, \tag{4.11}$$

$$\tau_4^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=4\}}\left(t - g_{l_2,t}^S\right) \geq d_4\right\}. \tag{4.12}$$

We then define

$$V_t^S = t - \max\left(g_{l_1,t}^S, g_{l_2,t}^S\right), \tag{4.13}$$

the time Z^S has spent in the current state. It is easy to see that $\left(Z^S, V^S\right)$ is a Markov process. Z^S is therefore a semi-Markov process with the state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the stochastic process S is above the barrier l_1; 4 corresponds to the state below the barrier l_2; 2 and 3 represent the state when S is in the corridor given that it comes in through l_1 and l_2 respectively.

For Z^S the transition intensities $\lambda_{ij}(u)$ satisfy

$$P\left(Z_{t+\Delta t}^S = j, i \neq j \mid Z_t^S = i, V_t^S = u\right) = \lambda_{ij}(u)\Delta t + o\left(\Delta t\right), \tag{4.14}$$

$$P\left(Z_{t+\Delta t}^S = i \mid Z_t^S = i, V_t^S = u\right) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o\left(\Delta t\right). \tag{4.15}$$

Define

$$\bar{P}_i(u) = \exp\left\{-\int_0^u \sum_{i \neq j} \lambda_{ij}(v)\mathrm{d}v\right\}, \quad p_{ij}(u) = \lambda_{ij}(u)\bar{P}_i(u).$$

Notice that

$$P_i(u) = 1 - \bar{P}_i(u)$$

is the distribution function of the excursion time in state i, which is a random variable U_i defined as

$$U_i = \inf_{s>0}\left\{Z_s^S \neq i \mid Z_0^S = i, V_0^S = 0\right\}.$$

Note that because the process is time homogeneous this has the same distribution as

$$\inf_{s>0} \{Z^S_{t+s} \neq i \mid Z^S_t = i, V^S_t = 0\}$$

for any time t. We have therefore

$$p_{ij}(u) = \lim_{\triangle u \to 0} \frac{P\left(U_i \in (u, u+\triangle u), Z^S_{U_i} = j\right)}{\triangle u}.$$

Moreover, in the definition of Z^S, we deliberately ignore the situation when $S_t = l_i$, $i = 1, 2$. The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u = l_i\}} \mathrm{d}u = 0, \ i = 1, 2.$$

Also, when l_1 and l_2 are the regular points of the process (see [8] for definition), we have to deal with the degeneration of p_{ij}. Let us take a Brownian Motion as an example. Assume $W^\mu_t = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian Motion. Setting x_0 to be its starting point, we know its density for the first hitting time of level l_i, $i = 1, 2$ is

$$p_{x_0}(t) = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l_i - x_0 - \mu t)^2}{2t}\right\}$$

(see [9]). According to the definition of transition density, $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$ and $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$, for $t > 0$.

In Chapter 2, in order to solve the similar problem, we introduced the perturbed Brownian motion $X^{(\epsilon)}$ with respect to the barrier we are interested in. We apply

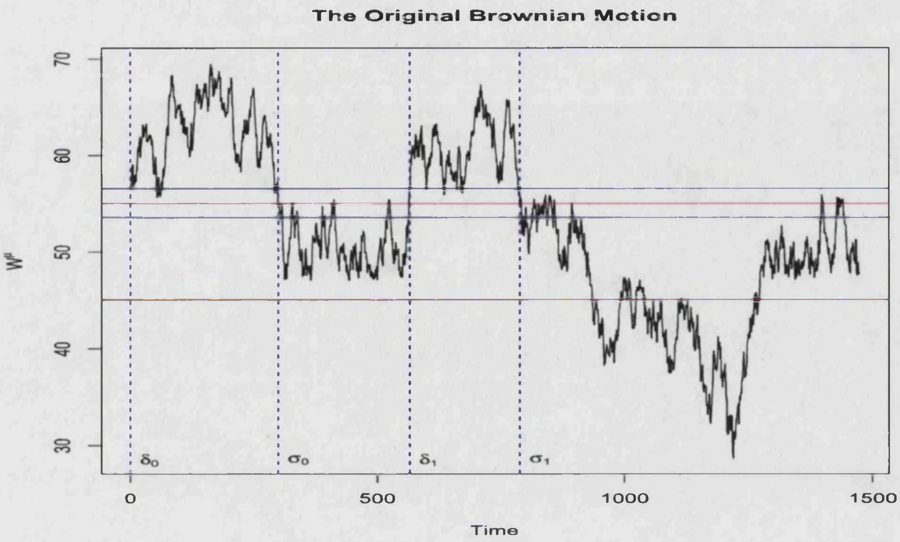

Figure 4.1: A Sample Path of W^{μ}

the same idea here, and construct a new process *doubly perturbed Brownian motion*, $Y^{(\epsilon)}$, $\epsilon > 0$, with respect to barriers l_1 and l_2. Assume $W_0^{\mu} = l_1 + \epsilon$. Define a sequence of stopping times

$$
\begin{aligned}
\delta_0 &= 0, \\
\sigma_n &= \inf\{t > \delta_n \mid W_t^{\mu} = l_1\}, \\
\delta_{n+1} &= \inf\{t > \sigma_n \mid W_t^{\mu} = l_1 + \epsilon\},
\end{aligned}
$$

where $n = 0, 1, \cdots$ (see Figure 4.1). Now define

$$
\begin{cases}
X_t^{(\epsilon)} = W_t^{\mu} & \text{if} \quad \delta_n \le t < \sigma_n \\
X_t^{(\epsilon)} = W_t^{\mu} - \epsilon & \text{if} \quad \sigma_n \le t < \delta_{n+1}
\end{cases}.
$$

Similarly, we then define another sequence of stopping times with respect to process

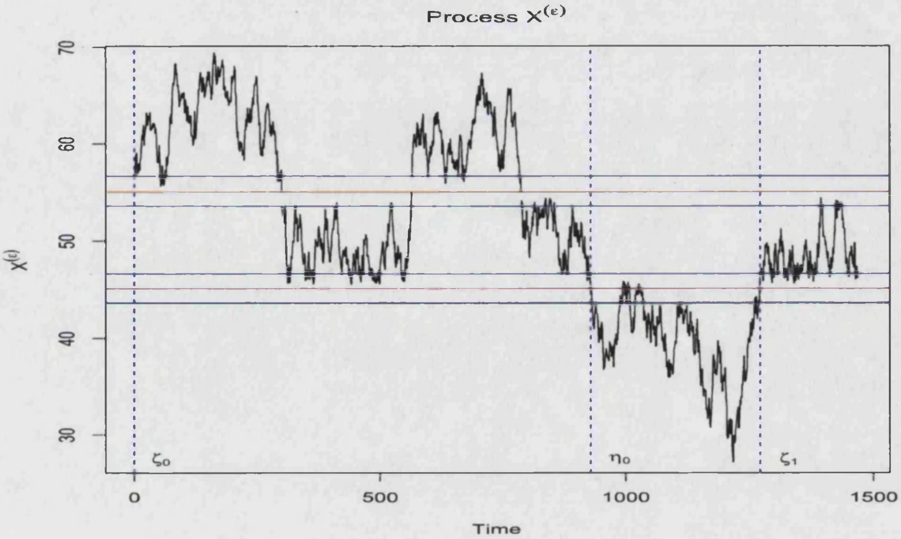

Figure 4.2: A Sample Path of $X^{(\epsilon)}$

$X^{(\epsilon)}$ and barrier l_2

$$\begin{aligned} \zeta_0 &= 0, \\ \eta_n &= \inf\{t > \zeta_n \mid X_t^{(\epsilon)} = l_2\}, \\ \zeta_{n+1} &= \inf\{t > \eta_n \mid X_t^{(\epsilon)} = l_2 + \epsilon\}, \end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 4.2). Then define

$$\begin{cases} Y_t^{(\epsilon)} = X_t^{(\epsilon)} & \text{if} \quad \zeta_n \le t < \eta_n \\ Y_t^{(\epsilon)} = X_t^{(\epsilon)} - \epsilon & \text{if} \quad \eta_n \le t < \zeta_{n+1} \end{cases}$$

The process $Y^{(\epsilon)}$ is actually a process which starts from $l_1 + \epsilon$ and has the same behavior as the related Brownian Motion expect that each time when it hits the barrier l_1 or l_2, it will have a jump towards the opposite side of the barrier with size ϵ (see Figure 4.3).

From the definition, it is clear that l_1 and l_2 become irregular points for $Y^{(\epsilon)}$. Also

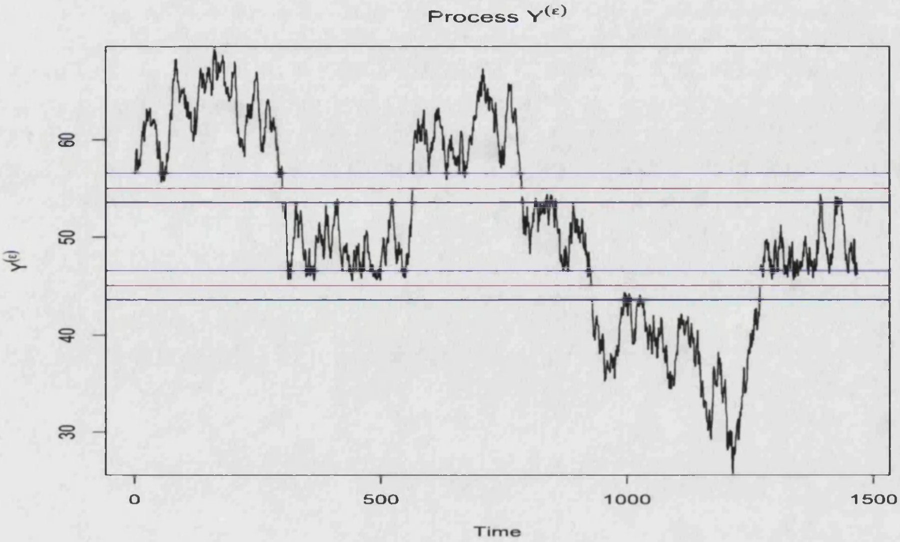

Figure 4.3: A Sample Path of $Y^{(\epsilon)}$

$Y^{(\epsilon)}$ converges to W^{μ} with $W_0^{\mu} = l_1$ almost surely for all t. Therefore as we prove in Chapter 2, the Laplace transforms of the variables defined based on $Y^{(\epsilon)}$ converge to those based on W^{μ}. As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for $Y^{(\epsilon)}$ and take the limit as $\epsilon \to 0$.

For $Y^{(\epsilon)}$, we can define Z^Y, τ_1^Y, τ_2^Y, τ_3^Y, τ_4^Y and τ^Y as above (we suppress (ϵ) on the superscript). For Z^Y, we have the transition densities (see [9])

$$p_{12}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon+\mu t)^2}{2t}\right\}, \tag{4.16}$$

$$p_{21}(t) = \exp\left\{\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), \tag{4.17}$$

$$p_{24}(t) = \exp\left\{-\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), \tag{4.18}$$

$$p_{31}(t) = \exp\left\{\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), \tag{4.19}$$

$$p_{34}(t) = \exp\left\{-\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), \tag{4.20}$$

$$p_{43}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon-\mu t)^2}{2t}\right\}, \tag{4.21}$$

where

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{((2k+1)y-x)^2}{2t} \right\}.$$

Also we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \tag{4.22}$$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W^μ when $\mu = 0$.

4.3 Results for the semi-Markov model

In Section 4.2 we have introduced the Markov process $\left(Z^S, V^S\right)$. Now we apply the same definition to the doubly perturbed Brownian motion $Y^{(\epsilon)}$; therefore we have $\left(Z^Y, V^Y\right)$, where Z^Y is the current state of $Y^{(\epsilon)}$, taking value from state space $\{1,2,3,4\}$ and V^Y is the time $Y^{(\epsilon)}$ has spent in current state. V^Y is also a stochastic process. Now we consider a function of the form

$$f(u,i,t) = f_i(u,t),$$

where f_i, $i = 1,2,3,4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathscr{A}$ is defined as an operator such that

$$f\left(V_t^Y, Z_t^Y, t\right) - \int_0^t \mathscr{A} f\left(V_s^Y, Z_s^Y, s\right) \mathrm{d}s$$

is a martingale (see [18], Chapter 3). Therefore solving

$$\mathscr{A} f = 0$$

subject to certain conditions will provide us with martingales of the form $f\left(V_t^Y, Z_t^Y, t\right)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\begin{cases} \mathscr{A} f_1(u,t) &= \dfrac{\partial f_1(u,t)}{\partial t} + \dfrac{\partial f_1(u,t)}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)) \\ \\ \mathscr{A} f_2(u,t) &= \dfrac{\partial f_2(u,t)}{\partial t} + \dfrac{\partial f_2(u,t)}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t)) + \lambda_{24}(u)(f_4(0,t) - f_2(u,t)) \\ \\ \mathscr{A} f_3(u,t) &= \dfrac{\partial f_3(u,t)}{\partial t} + \dfrac{\partial f_3(u,t)}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t)) + \lambda_{34}(u)(f_4(0,t) - f_3(u,t)) \\ \\ \mathscr{A} f_4(u,t) &= \dfrac{\partial f_4(u,t)}{\partial t} + \dfrac{\partial f_4(u,t)}{\partial u} + \lambda_{43}(u)(f_4(0,t) - f_3(u,t)) \end{cases}$$

Assume f_i has the form

$$f_i(u,t) = e^{-\beta t} g_i(u).$$

By solving the equation $\mathscr{A}f = 0$, i.e. $\begin{cases} \mathscr{A}f_1 &= 0 \\ \mathscr{A}f_2 &= 0 \\ \mathscr{A}f_3 &= 0 \\ \mathscr{A}f_4 &= 0 \end{cases}$ subject to $\begin{cases} g_1(d_1) &= \alpha_1 \\ g_2(d_2) &= \alpha_2 \\ g_3(d_2) &= \alpha_3 \\ g_4(d_2) &= \alpha_4 \end{cases}$

we can get

$$\begin{aligned} g_i(u) &= \alpha_i \exp\left\{ -\int_u^{d_i} \left(\beta + \sum_{j \neq i} \lambda_{ij}(v) \right) \mathrm{d}v \right\} \\ &\quad + \sum_{j \neq i} g_j(0) \int_u^{d_i} \lambda_{ij}(s) \exp\left\{ -\int_u^{s} \left(\beta + \sum_{k \neq i} \lambda_{ik}(v) \right) \mathrm{d}v \right\} \mathrm{d}s. \end{aligned} \tag{4.23}$$

In our case, we are only interested in the excursion inside the corridor. Hence, we set d_1 and d_4 to be ∞. Also $\lim_{d_1 \to \infty} g_1(d_1) = \lim_{d_4 \to \infty} g_4(d_4) = 0$ gives $\alpha_1 = \alpha_4 = 0$. Therefore, we have

$$g_2(0) = \alpha_2 e^{-\beta d_2} \bar{P}_2(d_2) + g_2(0) \hat{P}_{12}(\beta) \tilde{P}_{21}(\beta) + g_3(0) \hat{P}_{43}(\beta) \tilde{P}_{24}(\beta), \tag{4.24}$$

$$g_3(0) = \alpha_3 e^{-\beta d_3} \bar{P}_3(d_3) + g_2(0) \hat{P}_{12}(\beta) \tilde{P}_{31}(\beta) + g_3(0) \hat{P}_{43}(\beta) \tilde{P}_{34}(\beta). \tag{4.25}$$

Solving (4.24) and (4.25) gives

$$g_2(0) = \frac{\alpha_2 e^{-\beta d_2} \bar{P}_2(d_2) \left(1 - \hat{P}_{43}(\beta) \tilde{P}_{34}(\beta)\right) + \alpha_3 e^{-\beta d_3} \bar{P}_3(d_3) \hat{P}_{43}(\beta) \tilde{P}_{24}(\beta)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta) - \hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) + \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) - \hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}, \tag{4.26}$$

$$g_3(0) = \frac{\alpha_3 e^{-\beta d_3} \bar{P}_3(d_3) \left(1 - \hat{P}_{12}(\beta) \tilde{P}_{21}(\beta)\right) + \alpha_2 e^{-\beta d_2} \bar{P}_2(d_2) \hat{P}_{12}(\beta) \tilde{P}_{31}(\beta)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta) - \hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) + \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) - \hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}. \tag{4.27}$$

where

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta s} p_{ij}(s) \mathrm{d}s, \tag{4.28}$$

$$\tilde{P}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} p_{ij}(s) \mathrm{d}s. \tag{4.29}$$

As a result, we have obtained the martingale

$$M_t = f\left(V_t^Y, t\right) = e^{-\beta t} g_{Z_t^Y}\left(V_t^Y\right), \quad i = 1, 2, 3, 4. \tag{4.30}$$

We now can apply the optional stopping theorem to M_t with the stopping time $\tau^Y \wedge t$, where τ^Y is the stopping time defined by (4.6):

$$E\left(M_{\tau^Y \wedge t}\right) = E\left(M_0\right). \tag{4.31}$$

The right hand side of (4.31) is

$$E\left(M_{\tau^Y \wedge t}\right) = E\left(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}\right) + E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right).$$

Furthermore,

$$\begin{aligned} & E\left(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}\right) \\ = \; & E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_2^Y < \tau_3^Y\}} \mathbf{1}_{\{\tau_2^Y < t\}}\right) + E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_2^Y > \tau_3^Y\}} \mathbf{1}_{\{\tau_3^Y < t\}}\right) \\ = \; & E\left(e^{-\beta \tau^Y} g_2\left(d_2\right) \mathbf{1}_{\{\tau_2^Y < \tau_3^Y\}} \mathbf{1}_{\{\tau_2^Y < t\}}\right) + E\left(e^{-\beta \tau^Y} g_3\left(d_3\right) \mathbf{1}_{\{\tau_2^Y > \tau_3^Y\}} \mathbf{1}_{\{\tau_3^Y < t\}}\right) \\ = \; & \alpha_2 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_2^Y < \tau_3^Y\}} \mathbf{1}_{\{\tau_2^Y < t\}}\right) + \alpha_3 E\left(e^{-\beta \tau^Y} \mathbf{1}_{\{\tau_2^Y > \tau_3^Y\}} \mathbf{1}_{\{\tau_3^Y < t\}}\right). \end{aligned}$$

We also have

$$E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right) = e^{-\beta t} E\left(g_{Z_t^Y}\left(V_t^Y\right) \mathbf{1}_{\{\tau^Y > t\}}\right),$$

where Z_t^Y can take values $1, 2, 3$ or 4.

When $Z_t^Y = 2$ or 3, since $\tau^Y > t$, we have $0 \leq V_t^Y < d_2 \wedge d_3$. According to the definition of $g_i(\mu)$ in (4.23), we have $g_2\left(V_t^Y\right)$ and $g_3\left(V_t^Y\right)$ are bounded.

When $Z_t^Y = 1$ or 4, since $\lim_{d_1 \to \infty} g_1(d_1) = \lim_{d_4 \to \infty} g_4(d_4) = 0$ and looking at (4.23) with d_1 and d_4 replaced by ∞ we have that $g_1\left(V_t^Y\right)$ and $g_4\left(V_t^Y\right)$ are bounded.

Therefore

$$\lim_{t \to \infty} E\left(M_t \mathbf{1}_{\{\tau^Y > t\}}\right) = 0.$$

The right hand side of (4.31) gives

$$E(M_0) = \begin{cases} g_2(0), & Y_0^{(\epsilon)} = l_1 + \epsilon \\ g_3(0), & Y_0^{(\epsilon)} = l_2 - \epsilon \end{cases}.$$

By taking $\alpha_2 = \alpha_3 = 1$ and $d_2 = d_3 = d$, we will have when $Y_0^{(\epsilon)} = l_1 + \epsilon$

$$\begin{aligned} & E\left(e^{-\beta\tau^Y}\right) \\ = & \frac{e^{-\beta d}\bar{P}_2(d)\left(1 - \hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)\right) + e^{-\beta d}\bar{P}_3(d)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta) - \hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) + \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) - \hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}, \end{aligned} \tag{4.32}$$

when $Y_0^{(\epsilon)} = l_2 - \epsilon$

$$\begin{aligned} & E\left(e^{-\beta\tau^Y}\right) \\ = & \frac{e^{-\beta d}\bar{P}_2(d)\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta) + e^{-\beta d}\bar{P}_3(d)\left(1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\right)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta) - \hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) + \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta) - \hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}. \end{aligned} \tag{4.33}$$

4.4 Main Results

In Section 4.2 we have stated that the main difficulty with Brownian Motions is that its origin point is regular, i.e. the probability that W^μ will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes $Y^{(\epsilon)}$ and (Z^Y, V^Y) with transition densities for Z^Y defined in (4.16) to (4.22).

Theorem 4.4.1 *For a Brownian Motion W^μ, τ^{W^μ} defined as in (4.6) with $S = W^\mu$, we have following Laplace transforms:*

when $W_0^\mu = l_1$,

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = e^{-\beta d}\frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right) - F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right) - G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \tag{4.34}$$

when $W_0^\mu = l_2$,

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = e^{-\beta d}\frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right) - F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right) - G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \tag{4.35}$$

where

$$l = l_1 - l_2; \tag{4.36}$$

$$\begin{aligned} F_1(x) = & \sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}} - |x|\sqrt{d}\right)^2\right\}\right. \\ & \left. -e^{-(|x|+x)l}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}} - |x|\sqrt{d}\right)^2\right\}\right\} \\ & +2|x|\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{e^{-(|x|+x)l}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}} - |x|\sqrt{d}\right) - \mathcal{N}\left(\frac{2lk}{\sqrt{d}} - |x|\sqrt{d}\right)\right\}, \end{aligned} \tag{4.37}$$

$$F_2(x) = \sqrt{\frac{2}{\pi d}} \sum_{k=-\infty}^{\infty} e^{-2|x|lk} \left\{ \exp\left\{ -\frac{1}{2}\left(\frac{2lk}{\sqrt{d}} - |x|\sqrt{d}\right)^2 \right\} \right. \tag{4.38}$$

$$\left. -e^{-(|x|-x)l} \exp\left\{ -\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}} - |x|\sqrt{d}\right)^2 \right\} \right\}$$

$$+2|x| \sum_{k=-\infty}^{\infty} e^{-2|x|lk} \left\{ e^{-(|x|-x)l} \mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}} - |x|\sqrt{d}\right) - \mathcal{N}\left(\frac{2lk}{\sqrt{d}} - |x|\sqrt{d}\right) \right\},$$

$$G_1(x) = \frac{-2\sqrt{2x}}{1-e^{-2l\sqrt{2x}}} + 2\sqrt{2x} \sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2x}k} \mathcal{N}\left(\frac{2lk}{\sqrt{d}} - \sqrt{2xd}\right) \tag{4.39}$$

$$-\sqrt{\frac{2}{\pi d}} \sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2x}k} \exp\left\{ -\frac{1}{2}\left(\frac{2lk}{\sqrt{d}} - \sqrt{2xd}\right)^2 \right\},$$

$$G_2(x) = \frac{2\sqrt{2x}e^{-l\sqrt{2x}}}{1-e^{-2l\sqrt{2x}}} - 2\sqrt{2x} \sum_{k=-\infty}^{\infty} e^{-l\sqrt{2x}(2k+1)} \mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}} - \sqrt{2xd}\right)$$

$$+\sqrt{\frac{2}{\pi d}} \sum_{k=-\infty}^{\infty} e^{-l\sqrt{2x}(2k+1)} \exp\left\{ -\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}} - \sqrt{2xd}\right)^2 \right\}. \tag{4.40}$$

Proof: We apply the transition densities in (4.16) to (4.22) to the results in (4.32) and (4.33) and taking the limit $\epsilon \to 0$. We now show that we can take the limit. We have shown in Chapter 3 that

$$\lim_{\epsilon \to 0} g^{Y}_{l_1,t} = g^{W^\mu}_{l_1,t}, \ a.s. \text{ for all } t.$$

Therefore we have that

$$\lim_{\epsilon \to 0} \mathbf{1}_{\left\{l_2 < Y^{(\epsilon)}_t < l_1\right\}} \mathbf{1}_{\left\{g^Y_{l_1,t} > g^Y_{l_2,t}\right\}} \left(t - g^Y_{l_1,t}\right) = \mathbf{1}_{\left\{l_2 < W^\mu_t < l_1\right\}} \mathbf{1}_{\left\{g^{W^\mu}_{l_1,t} > g^{W^\mu}_{l_2,t}\right\}} \left(t - g^{W^\mu}_{l_1,t}\right), \ a.s.$$

From the definition of τ_2^S we have that

$$\begin{aligned}\{\tau_2^{W^\mu} < t\} &= \left\{\sup_{0\leq s\leq t}\left\{\mathbf{1}_{\{l_2<W_s^\mu<l_1\}}\mathbf{1}_{\{g_{l_1,s}^{W^\mu}>g_{l_2,s}^{W^\mu}\}}\left(s-g_{l_1,s}^{W^\mu}\right)\right\}\geq d_2\right\}\\ &= \lim_{\epsilon\to 0}\left\{\sup_{0\leq s\leq t}\left\{\mathbf{1}_{\{l_2<Y_s^{(\epsilon)}<l_1\}}\mathbf{1}_{\{g_{l_1,s}^{Y}>g_{l_2,s}^{Y}\}}\left(t-g_{l_1,s}^{Y}\right)\right\}\geq d_2\right\}\\ &= \lim_{\epsilon\to 0}\{\tau_2^Y < t\}.\end{aligned}$$

Consequently,

$$\lim_{\epsilon\to 0}\tau_2^Y = \tau_2^{W^\mu} \ a.s.$$

By the same argument, we can show that

$$\lim_{\epsilon\to 0}\tau_3^Y = \tau_3^{W^\mu} \ a.s.$$

Since $\tau^S = \tau_2^S \wedge \tau_3^S$, we have

$$\lim_{\epsilon\to 0}\tau^Y = \tau^{W^\mu} \ a.s.$$

In Chapter 2 we have shown that the convergence of τ^Y to τ^{W^μ} leads to the convergence of their Laplace transforms, i.e.

$$\lim_{\epsilon\to 0}E\left(\exp\left\{-\beta\tau^Y\right\}\right) = E\left(\exp\left\{-\beta\tau^{W^\mu}\right\}\right) \ a.s.$$

Therefore we get the results shown by (4.34) and (4.35).

□

Corollary 4.4.1.1 *For a standard Brownian Motion W ($\mu = 0$), we have for both* *cases (i.e. when* $W_0 = l_1$ *and when* $W_0 = l_2$*)*

$$E\left(e^{-\beta\tau^{W}}\right) = e^{-\beta d}\frac{h(0)}{h(\beta)}; \tag{4.41}$$

where

$$\begin{aligned} h(\beta) = & \sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2\beta}k}\left\{e^{-l\sqrt{2\beta}}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2\beta d}\right)^2\right\}\right. \\ & \left.-\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2\beta d}\right)^2\right\}\right\} - \frac{2\sqrt{2\beta}}{1+e^{-l\sqrt{2\beta}}} \\ & +2\sqrt{2\beta}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2\beta}k}\left\{\mathcal{N}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2\beta d}\right)-e^{-l\sqrt{2\beta}}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2\beta d}\right)\right\} \end{aligned} \tag{4.42}$$

We are also interested in the cases when a Brownian Motion starts from the point other than l_1 and l_2. The results are shown in the following corollary.

Corollary 4.4.1.2 *For a Brownian Motion* W^{μ}, $\tau^{W^{\mu}}$ *defined as in (4.6) with* $S = W^{\mu}$*, we have the following Laplace transforms:*

when $W_0^{\mu} = x_0$, $x_0 \geq l_1$,

$$\begin{aligned} E\left(e^{-\beta\tau^{W^{\mu}}}\right) = & \exp\left\{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)-\beta d\right\} \\ & \frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \end{aligned} \tag{4.43}$$

when $W_0^\mu = x_0,\ x_0 \le l_2$,

$$E\left(e^{-\beta\tau_{W^\mu}}\right) = \exp\left\{\left(\mu - \sqrt{2\beta+\mu^2}\right)(l_2 - x_0) - \beta d\right\} \frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right) - F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right) - G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \qquad (4.44)$$

when $W_0^\mu = x_0,\ l_2 < x_0 < l_1$,

$$\begin{aligned}
E\left(e^{-\beta\tau_{W^\mu}}\right) = {} & e^{\mu(l_2-x_0)-\beta d}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl+x_0-l_2)}\mathcal{N}\left(-|\mu|\sqrt{d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right. \qquad (4.45)\\
& \left.-e^{|\mu|(2kl+x_0-l_2)}\mathcal{N}\left(-|\mu|\sqrt{d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\}\\
& +e^{\mu(l_1-x_0)-\beta d}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl-x_0+l_1)}\mathcal{N}\left(-|\mu|\sqrt{d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right.\\
& \left.-e^{|\mu|(2kl-x_0+l_1)}\mathcal{N}\left(-|\mu|\sqrt{d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}+e^{-\beta d}\\
& -\frac{e^{-|\mu|l-\beta d}\left\{e^{\mu(l_2-x_0)}\left(e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right)+e^{\mu(l_1-x_0)}\left(e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right)\right\}}{1-e^{-2|\mu|l}}\\
& +\left[\frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_2-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}\right.\\
& +e^{\mu(l_2-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right.\\
& \left.\left.-e^{-\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\}\right]\\
& \frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}\\
& +\left[\frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_1-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}\right.\\
& +e^{\mu(l_1-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right.\\
& \left.\left.-e^{-\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}\right]\\
& \frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}.
\end{aligned}$$

Proof: We will prove the case when $x_0 \geq l_1$ at first. Defined $T = \inf\{t \mid W_t^\mu = l_1\}$, i.e. the first time W_t^μ hits l_1. By definition, we have $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$, where $\widetilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1. By the strong Markov property of the Brownian motion, we therefore have

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = E\left(e^{-\beta T}\right) E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right).$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ has been calculate in Theorem 4.4.1 (4.34). According to [9], we have

$$E\left(e^{-\beta T}\right) = \exp\left\{-\left(\mu + \sqrt{2\beta + \mu^2}\right)(x_0 - l_1)\right\}.$$

For the case when $x_0 \leq l_2$, we can apply the same argument.

When $l_2 < x_0 < l_1$, by definition, we have $\tau^{W^\mu} = d$, if $T \geq d$; $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$, if $T < d$, and $W_T^\mu = l_1$ where $\widetilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1; $\tau^{W^\mu} = T + \tau^{\underline{W}^\mu}$, if $T < d$, and $W_T^\mu = l_2$ where $\underline{W}^\mu$ here stands for a Brownian motion with drift started from l_2. As a result

$$\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \\
= \;& E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T\geq d\}}\right) + E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_1\}}\right) + E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right) \\
= \;& e^{-\beta d}P(T\geq d) + E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_1\}}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right) \\
& + E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right)E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right).
\end{aligned}$$

$E\left(e^{-\beta_T \widetilde{W}^\mu}\right)$ and $E\left(e^{-\beta_T \underline{W}^\mu}\right)$ have been calculated in Theorem 4.4.1 (see (4.34) and (4.35)). The density for T is given in [9] as

$$p_{x_0}(t) = e^{\mu(l_2-x_0)-\frac{\mu^2 t}{2}} ss_t\left(l_1 - x_0, l\right) + e^{\mu(l_1-x_0)-\frac{\mu^2 t}{2}} ss_t\left(x_0 - l_2, l\right).$$

We can therefore calculate

$$\begin{aligned}
& P\left(T \geq d\right) \\
= \ & 1 - \frac{e^{-|\mu|l}\left\{e^{\mu(l_2-x_0)}\left(e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\right) + e^{\mu(l_1-x_0)}\left(e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\right)\right\}}{1-e^{-2|\mu|l}} \\
& + e^{\mu(l_2-x_0)} \sum_{k=-\infty}^{\infty} \left\{ e^{-|\mu|(2kl+x_0-l_2)} \mathcal{N}\left(-|\mu|\sqrt{d} + \frac{2kl+x_0-l_2}{\sqrt{d}}\right) \right. \\
& \left. - e^{|\mu|(2kl+x_0-l_2)} \mathcal{N}\left(-|\mu|\sqrt{d} - \frac{2kl+x_0-l_2}{\sqrt{d}}\right) \right\} \\
& + e^{\mu(l_1-x_0)} \sum_{k=-\infty}^{\infty} \left\{ e^{-|\mu|(2kl-x_0+l_1)} \mathcal{N}\left(-|\mu|\sqrt{d} + \frac{2kl-x_0+l_1}{\sqrt{d}}\right) \right. \\
& \left. - e^{|\mu|(2kl-x_0+l_1)} \mathcal{N}\left(-|\mu|\sqrt{d} - \frac{2kl-x_0+l_1}{\sqrt{d}}\right) \right\},
\end{aligned}$$

$$\begin{aligned}
& E\left(e^{-\beta T} \mathbf{1}_{\{T<d\}} \mathbf{1}_{\{W_T^\mu = l_1\}}\right) \\
= \ & \frac{e^{-\sqrt{2\beta+\mu^2}l} e^{\mu(l_2-x_0)} \left(e^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}} \\
& + e^{\mu(l_2-x_0)} \sum_{k=-\infty}^{\infty} \left\{ e^{\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)} \mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d} - \frac{2kl+x_0-l_2}{\sqrt{d}}\right) \right. \\
& \left. - e^{-\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)} \mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d} + \frac{2kl+x_0-l_2}{\sqrt{d}}\right) \right\},
\end{aligned}$$

$$
\begin{aligned}
& E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right) \\
= \; & \frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_1-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}} \\
& +e^{\mu(l_1-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right. \\
& \left.-e^{-\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}.
\end{aligned}
$$

We therefore get the result in (4.45).

□

Notice that for a Brownian motion with drift, it is possible that τ^{W^μ} will never be achieved. Take the case when $\mu > 0$ and $x_0 > l_1$ as an example. We obtain the following result by taking $\beta = 0$ in (4.43).

Corollary 4.4.1.3 *For a Brownian motion with positive drift, W^μ with $\mu > 0$ and $x_0 > l_1$ we have that,*

$$
P\left(\tau^{W^\mu}<\infty\right)=\exp\left\{-2\mu\left(x_0-l_1\right)\right\}\frac{e^{-\mu l}F_2(\mu)G_2\left(\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right)-G_2^2\left(\frac{\mu^2}{2}\right)}. \qquad (4.46)
$$

Remark 1: As a result, for a Brownian motion with positive drift and $x_0 > l_1$, with probability

$$
1-\exp\left\{-2\mu\left(x_0-l_1\right)\right\}\frac{e^{-\mu l}F_2(\mu)G_2\left(\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right)-G_2^2\left(\frac{\mu^2}{2}\right)}
$$

that it will never achieved a excursion in the corridor (l_2, l_1) with length equal or greater than d.

Remark 2: For a Brownian motion with negative drift and $x_0 > l_1$, taking $\beta = 0$ in (4.43) gives that with probability

$$1 - \frac{e^{-\mu l} F_2(\mu) G_2\left(\frac{\mu^2}{2}\right) - F_1(\mu) G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right) - G_2^2\left(\frac{\mu^2}{2}\right)}$$

that it will never achieved a excursion in the corridor (l_2, l_1) with length equal or greater than d.

Remark 3: For a standard Brownian motion, we can carry out a similar calculation to (4.41), from which we can easily that the result that

$$P\left(\tau^W < \infty\right) = 1.$$

We will now extent Corollary 4.4.1.2 to obtain the joint distribution of W and τ^W at an exponential time. This is an application of (4.43), (4.44) and Girsanov's theorem.

Theorem 4.4.2 *For a standard Brownian Motion W with $W_0 = x_0$ and τ^W defined as in (4.6) with $S = W$, we have the following result:*

For the case $x_0 \geq l_1$ and $x \geq l_1$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \qquad (4.47) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\left(u_1\left(x - l_1\right) - u_2(x - l_2)\right) - G_1(\gamma)\left(u_1\left(x - l_2\right) - u_2(x - l_1)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x_0 \geq l_1$ and $l_2 < x < l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{4.48}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\,(u_3\,(x-l_1) - u_2(x-l_2)) - G_1(\gamma)\,(u_1\,(x-l_2) - u_4(x-l_1))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

for the case $x_0 \geq l_1$ and $x \leq l_2$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{4.49}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\,(u_3\,(x-l_1) - u_4(x-l_2)) - G_1(\gamma)\,(u_3\,(x-l_2) - u_4(x-l_1))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

for the case $x \leq l_2$ and $x \geq l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{4.50}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\,(u_1\,(x-l_2) - u_2(x-l_1)) - G_1(\gamma)\,(u_1\,(x-l_1) - u_2(x-l_2))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

for the case $x \leq l_2$ and $l_2 < x < l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{4.51}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\,(u_1\,(x-l_2) - u_4(x-l_1)) - G_1(\gamma)\,(u_3\,(x-l_1) - u_2(x-l_2))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

for the case $x \leq l_2$ and $x \leq l_2$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{4.52}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\,(u_3\,(x-l_2) - u_4(x-l_1)) - G_1(\gamma)\,(u_3\,(x-l_1) - u_4(x-l_2))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

where $\tilde{T}$, independent of W, is a random variable with an exponential distribution *of parameter γ and*

$$u_1(x) = e^{-\sqrt{2\gamma}x} a_1\left(-\sqrt{2\gamma}\right), \tag{4.53}$$

$$u_2(x) = e^{-\sqrt{2\gamma}x} a_2\left(-\sqrt{2\gamma}\right), \tag{4.54}$$

$$\begin{aligned} u_3(x) &= 2\sum_{k=-\infty}^{\infty}\left[\exp\left\{-\sqrt{2\gamma}\,((2k+1)l+x)\right\}\mathcal{N}\left(\frac{x+(2k+1)l-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &\quad \left.+\exp\left\{\sqrt{2\gamma}\,((2k+1)l+x)\right\}\mathcal{N}\left(\frac{x+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right] - e^{\sqrt{2\gamma}x}a_1\left(\sqrt{2\gamma}\right), \end{aligned} \tag{4.55}$$

$$\begin{aligned} u_4(x) &= 2\sum_{k=-\infty}^{\infty}\left[\exp\left\{-\sqrt{2\gamma}\,(2kl+x)\right\}\mathcal{N}\left(\frac{x+2kl-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &\quad \left.+\exp\left\{\sqrt{2\gamma}\,(2kl+x)\right\}\mathcal{N}\left(\frac{x+2kl+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right] - e^{\sqrt{2\gamma}x}a_2\left(\sqrt{2\gamma}\right), \end{aligned} \tag{4.56}$$

$$a_1(x) = 2\sum_{k=-\infty}^{\infty}\exp\{x(2k+1)l\}\,\mathcal{N}\left(\frac{(2k+1)l+xd}{\sqrt{d}}\right)+\frac{e^{-\gamma d}}{x}\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{(2k+1)^2l^2}{2d}\right\}, \tag{4.57}$$

$$a_2(x) = 2\sum_{k=-\infty}^{\infty}\exp\{2xkl\}\,\mathcal{N}\left(\frac{2kl+xd}{\sqrt{d}}\right)+\frac{e^{-\gamma d}}{x}\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{2k^2l^2}{d}\right\}. \tag{4.58}$$

Proof: see appendix.

□

4.5 Pricing Parisian Corridor Options

We want to price a Parisian corridor in-call option with the current price of its underlying asset to be x, $x > L_1$, the owner of which will obtain the right to exercise it when the length of the excursion inside the corridor formed by the barriers L_1 and L_2 ($L_1 > L_2$) reaches d before T. Its price formula is given by

$$PC_{in-call} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau^S < T\}} \right),$$

where S is the underlying stock price, Q denotes the risk neutral measure, τ^S is defined with the respect to L_1 and L_2. We assume S is a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = x,$$

where $x > L_1$, r is the risk free rate, W_t with $W_0 = 0$ is a standard Brownian motion under Q. Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), \quad l_2 = \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right).$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\left\{\sigma(mt + W_t)\right\} = xe^{\sigma B_t}.$$

By applying Girsanov's Theorem, we have

$$PC_{in-call} = e^{-\left(r+\frac{1}{2}m^2\right)T} E_P \left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T} \mathbf{1}_{\{\tau^B < T\}} \right],$$

where P is a new measure, under which B_t is a standard Brownian motion with $B_0 = 0$, and τ^B is the stopping time defined with respect to barrier l_1, l_2. And we define

$$PC^*_{in-call} = e^{\left(r+\frac{1}{2}m^2\right)T} PC_{in-call}.$$

We are going to show that we can obtain the Laplace transform of $PC^*_{in-call}$ w.r.t T, denoted by $\mathscr{L}_T$.

Firstly, assuming $\tilde{T}$, independent of W, is a random variable with an exponential distribution of parameter γ, we have

$$\begin{aligned}
& E_P\left[\left(xe^{\sigma B_{\tilde{T}}} - K\right)^+ e^{mB_{\tilde{T}}} \mathbf{1}_{\{\tau^B<\tilde{T}\}}\right] \\
= & \int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_{\tilde{T}} \in \mathrm{d}y, \tau^B < \tilde{T}\right) \\
= & \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_T \in \mathrm{d}y, \tau^B < T\right) \mathrm{d}T \\
= & \gamma \int_0^\infty e^{-\gamma T} E_P\left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T} \mathbf{1}_{\{\tau^B<T\}}\right] \mathrm{d}T \\
= & \gamma \mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T = \frac{1}{\gamma} \int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_{\tilde{T}} \in \mathrm{d}y, \tau^B < \tilde{T}\right).$$

By using the results in Theorem 4.4.2, this Laplace transform can be calculated explicitly.

When $b \geq l_1$, i.e. $K \geq L_1$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_1(\sigma + m) - \frac{K}{\gamma} F_1(m), \tag{4.59}$$

when $l_2 < b < l_1$, i.e. $L_2 < K < L_1$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_2(\sigma + m) - \frac{K}{\gamma} F_2(m), \tag{4.60}$$

when $b \leq l_2$, i.e. $K \leq L_2$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_3(\sigma + m) - \frac{K}{\gamma} F_3(m), \tag{4.61}$$

where

$$\begin{aligned} F_1(x) &= \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma)\left\{q_1(x, b, l_1) - q_2(x, b, l_2)\right\}\right. \\ &\left. -G_1(\gamma)\left\{q_1(x, b, l_2) - q_2(x, b, l_1)\right\}\right], \end{aligned} \tag{4.62}$$

$$\begin{aligned} F_2(x) &= \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma)\left\{q_1(x, l_1, l_1) - q_2(x, b, l_2) + q_3(x, l_1, l_1) - q_3(x, b, l_1)\right\}\right. \\ &\left. -G_1(\gamma)\left\{q_1(x, b, l_2) - q_2(x, l_1, l_1) - q_4(x, l_1, l_1) + q_4(x, b, l_1)\right\}\right], \end{aligned} \tag{4.63}$$

$$\begin{aligned} F_3(x) &= \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma)\left\{q_1(x, l_1, l_1) - q_2(x, l_2, l_2) + q_3(x, l_1, l_1) - q_3(x, b, l_1)\right.\right. \\ &\left. + q_4(x, b, l_2) - q_4(x, l_2, l_2)\right\} - G_1(\gamma)\left\{q_1(x, l_2, l_2) - q_2(x, l_1, l_1) + q_3(x, l_2, l_2)\right. \\ &\left.\left. -q_3(x, b, l_2) - q_4(x, l_1, l_1) + q_4(x, b, l_1)\right\}\right], \end{aligned} \tag{4.64}$$

$$q_1(x,y,z) = \frac{e^{\left(x-\sqrt{2\gamma}\right)y+\sqrt{2\gamma}z}}{\sqrt{2\gamma}-x} a_1\left(-\sqrt{2\gamma}\right), \tag{4.65}$$

$$q_2(x,y,z) = \frac{e^{\left(x-\sqrt{2\gamma}\right)y+\sqrt{2\gamma}z}}{\sqrt{2\gamma}-x} a_2\left(-\sqrt{2\gamma}\right), \tag{4.66}$$

$$\begin{aligned} q_3(x,y,z) &= 2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{-\sqrt{2\gamma}((2k+1)l-z)\right\}}{x-\sqrt{2\gamma}} \left[e^{(x-\sqrt{2\gamma})y}\mathcal{N}\left(\frac{y-z+(2k+1)l-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &- \exp\left\{\left(x-\sqrt{2\gamma}\right)\left(z-(2k+1)l+\sqrt{2\gamma}d\right)+\frac{\left(x-\sqrt{2\gamma}\right)^2 d}{2}\right\} \\ &\left.\mathcal{N}\left(\frac{y-z+(2k+1)l-xd}{\sqrt{d}}\right)\right] \\ &+2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{\sqrt{2\gamma}((2k+1)l-z)\right\}}{x+\sqrt{2\gamma}} \left[e^{(x+\sqrt{2\gamma})y}\mathcal{N}\left(\frac{y-z+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &- \exp\left\{\left(x+\sqrt{2\gamma}\right)\left(z-(2k+1)l-\sqrt{2\gamma}d\right)+\frac{\left(x+\sqrt{2\gamma}\right)^2 d}{2}\right\} \\ &\left.\mathcal{N}\left(\frac{y-z+(2k+1)l-xd}{\sqrt{d}}\right)\right] - \frac{e^{\left(\sqrt{2\gamma}+x\right)y-\sqrt{2\gamma}z}}{x+\sqrt{2\gamma}} a_1\left(\sqrt{2\gamma}\right), \end{aligned} \tag{4.67}$$

$$\begin{aligned} q_4(x,y,z) &= 2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{-\sqrt{2\gamma}(2kl-z)\right\}}{x-\sqrt{2\gamma}} \left[e^{(x-\sqrt{2\gamma})y}\mathcal{N}\left(\frac{y-z+2kl-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &\left.- \exp\left\{\left(x-\sqrt{2\gamma}\right)\left(z-2kl+\sqrt{2\gamma}d\right)+\frac{\left(x-\sqrt{2\gamma}\right)^2 d}{2}\right\}\mathcal{N}\left(\frac{y-z+2kl-xd}{\sqrt{d}}\right)\right] \\ &+2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{\sqrt{2\gamma}(2kl-z)\right\}}{x+\sqrt{2\gamma}} \left[e^{(x+\sqrt{2\gamma})y}\mathcal{N}\left(\frac{y-z+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \\ &\left.- \exp\left\{\left(x+\sqrt{2\gamma}\right)\left(z-2kl-\sqrt{2\gamma}d\right)+\frac{\left(x+\sqrt{2\gamma}\right)^2 d}{2}\right\}\mathcal{N}\left(\frac{y-z+2kl-xd}{\sqrt{d}}\right)\right] \\ &-\frac{e^{\left(\sqrt{2\gamma}+x\right)y-\sqrt{2\gamma}z}}{x+\sqrt{2\gamma}} a_2\left(\sqrt{2\gamma}\right). \end{aligned} \tag{4.68}$$

Remark: The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$PC^*_{in-call} = e^{\left(r+\frac{1}{2}m^2\right)T} P_{in-call}.$$

For

$$PC_{out-call} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau^S > T\}} \right),$$

we can get the result from the relationship that

$$PC_{out-call} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - PC_{in-call}.$$

4.6 Appendix

We prove Theorem 4.4.2 in this section. Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi} x \mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x \mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta \tau^{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T} \mathbf{1}_{\{\tau^{W^\mu} < T\}} \mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta + \frac{1}{2}\mu^2)T - \mu x_0} E\left(e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta\tau^{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2)e^{-\gamma T - \mu x_0} E\left(e^{\mu W_T}\mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T \\ &= \frac{\gamma - \frac{1}{2}\mu^2}{\gamma} e^{-\mu x_0} E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W < \tilde{T}\}}\right), \end{aligned}$$

where $\tilde{T}$, independent of W, is a random variable with an exponential distribution of parameter γ. Therefore we have

$$E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W < \tilde{T}\}}\right) = \frac{\gamma e^{\mu x_0}}{\gamma - \frac{1}{2}\mu^2} E\left(e^{-\beta\tau^{W^\mu}}\right)$$

In order to inverse the above moment generating function, we just need to inverse the following expressions:

$$\frac{\mu}{\gamma - \frac{\mu^2}{2}} = \int_0^\infty e^{\mu x} e^{-\sqrt{2\gamma}x}\mathrm{d}x - \int_{-\infty}^0 e^{\mu x} e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$\frac{e^{\mu l_i}}{\gamma - \frac{\mu^2}{2}} = \int_{l_i}^\infty e^{\mu x}\frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-l_i)}\mathrm{d}x + \int_{-\infty}^{l_i} e^{\mu x}\frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-l_i)}\mathrm{d}x,$$

$$e^{-n\mu l} e^{\frac{\mu^2 d}{2} + \mu l_i} \mathcal{N}\left(\frac{nl}{\sqrt{d}} - \mu\sqrt{d}\right) = \int_{-\infty}^{l_i} e^{\mu x}\frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(x + nl - l_i)^2}{2d}\right\}.$$

Therefor the inversion of $\frac{\mu e^{-n\mu l} e^{\frac{\mu^2 d}{2} + \mu l_i} \mathcal{N}\left(\frac{nl}{\sqrt{d}} - \mu\sqrt{d}\right)}{\gamma - \frac{\mu^2}{2}}$ is as follows:

for $x \geq l_i$,

$$\begin{aligned}
&\int_{-\infty}^{l_i} \frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y \\
= \;& \exp\left\{\gamma d - \sqrt{2\gamma}\,(nl - l_i + x)\right\} \mathscr{N}\left(\frac{nl - \sqrt{2\gamma}d}{\sqrt{d}}\right);
\end{aligned}$$

for $x < l_i$,

$$\begin{aligned}
&\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y \\
&- \int_{x}^{l_i} \frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\} e^{\sqrt{2\gamma}(x-y)} \mathrm{d}y \\
= \;& \exp\left\{\gamma d - \sqrt{2\gamma}\,(nl - l_i + x)\right\} \mathscr{N}\left(\frac{x + nl - l_i - \sqrt{2\gamma}d}{\sqrt{d}}\right) \\
&- \exp\left\{\gamma d + \sqrt{2\gamma}\,(nl - l_i + x)\right\} \left\{\mathscr{N}\left(\frac{nl + \sqrt{2\gamma}d}{\sqrt{d}}\right) - \mathscr{N}\left(\frac{x + nl - l_i + \sqrt{2\gamma}d}{\sqrt{d}}\right)\right\}.
\end{aligned}$$

Consequently, we can get Theorem 4.4.2.

Chapter 5

Parisian Options and Parisian Type Ruin Probabilities with Exponential Jump Size

Abstract

In this paper, we study the excursions of a Lévy process with negative exponential jump size below a given level by using a simple two-state semi-Markov model. Based on this result, we price a Parisian option whose underlying asset price follows this process. To our knowledge this is the first ever attempt to price Parisian options involving processes with jumps. We also price the Parisian type digital options and extend the concept of ruin in risk theory to the Parisian type of ruin. Moreover, we consider a diffusion approximation and use it to obtain similar results for the Brownian motion with drift.

Keywords: Parisian type of ruin, Parisian option, risk process, Laplace transform, ruin probability.

5.1 Introduction

A Parisian option is a special case of path dependent options. It will be either initiated or terminated upon the price reaching a predetermined barrier level L and staying above or below the level for a predetermined time d before the maturity date T. An example is a *Parisian down-and-in option*, the owner of which gains the right to exercise the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than d . For details on the pricing, see [13], [37], [38], Chapters 2, 3 and 4.

Under the Black-Scholes framework, one of the basic assumptions is that the underlying asset price follows a geometric Brownian motion. However the features of the price trajectory violate the continuity and the scale-invariance properties of Brownian motions and therefore pricing models based on jump processes have become more and more popular. In this paper, we try price the Parisian options with an underlying asset price with jumps for the first time. Although, the model is rather simplistic, it could be a starting point for further results. From risk theory, the classical surplus process in continuous time $\{X_t\}_{t>0}$ is defined by

$$X_t = x + ct - \sum_{k=0}^{N_t} Y_k, \tag{5.1}$$

where $x \geq 0$ is the initial reserve, c is a constant rate of premium payment per time unit, and $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ representing the numbers of claims up to time t. The sequence $\{Y_k\}$, $k = 1, 2, ...$, are claim sizes which are independent and identically distributed non-negative random variables that are also independent of the number of claims. We also assume $c > \lambda E(Y_1)$ (the *net profit condition*). Our underlying asset price follows

$$S_t = \exp(X_t), \text{ with } S_0 = e^x. \tag{5.2}$$

We assume that this is the behaviour of the underlying assets under an equivalent martingale measure. As the market is not complete, this would not be unique but the calculations are valid under any equivalent martingale measure that preserves the structure defined by (5.1) and only changes the values of the parameters. On later sections, we show that we can obtain the Laplace transform of the option price.

Moreover, the classical surplus process X defined in (5.1) has been widely used in risk theory. Motivated by the idea of Parisian option, we extend the concept of ruin to the Parisian type of ruin. Define the stopping time

$$T_x = \inf\{t > 0 \mid X_t < 0\}. \tag{5.3}$$

In risk theory, the event of ruin in infinite time horizon can be expressed as $\{T_x < \infty\}$. The density of T_x and the probability of ruin have been widely studied. See for example [19], [20], [21], [22], [27], [28], [32] and [40].

Parisian type ruin will occur if the surplus falls below zero and stays below zero for a continuous time interval of length d. More practically, this level can be any level greater than 0. In some respects, this gives a useful measure to monitor the financial situation of a company as it gives the office some time to put its finances in order.

In order to introduce the above concepts mathematically, we will first define the excursion. Set

$$g_{L,t}^{S} = \sup\{s < t \mid \text{sign}\,(S_s - L)\,\text{sign}\,(S_t - L) \leq 0\}, \tag{5.4}$$

$$d_{L,t}^{S} = \inf\{s > t \mid \text{sign}\,(S_s - L)\,\text{sign}\,(S_t - L) \leq 0\}, \tag{5.5}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$, where

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

The trajectory between $g_{L,t}^{S}$ and $d_{L,t}^{S}$ is the excursion of process S which straddles time t. Assuming $d > 0$, we now define

$$\tau_{L,d}^{S} = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < L\}}(t - g_{L,t}^{S}) \geq d\}. \tag{5.6}$$

We can see that $\tau_{L,d}^{S}$ is the first time that the length of the excursion of process X below L reaches given level d. The price for a Parisian down-in-call option can therefore be expressed as

$$P_{down-in-call} = e^{-rT} E\left((S_T - K)^{+} \mathbf{1}_{\{\tau_{L,d}^{S} < T\}} \right), \tag{5.7}$$

where r is the risk-free rate, K is the strike price and S is the underlying stock price satisfying (5.2).

When $L = 0$, we simplify the notation to g_t^S, d_t^S and τ_d^S. We define the events $\{\tau_d^X \leq t\}$ and $\{\tau_d^X < \infty\}$ to be the Parisian type ruin in the finite and infinite horizons. We are interested in the corresponding probabilities

$$P\left(\tau_d^X \leq t\right)$$

and

$$P\left(\tau_d^X < \infty\right).$$

We will restrict ourselves here to claim sizes that are exponentially distributed as this is a case where explicit results can be obtained. We therefore assume that the claim sizes have density $\alpha e^{-\alpha x}$, where $x > 0$. From the net profit condition above, we also have that $c > \frac{\lambda}{\alpha}$.

In Section 5.2 we provide results on hitting times that will be used in Section 5.3 to give the Laplace transform of the stopping time τ_d^X, together with price of a Parisian type digital option and the Parisian type ruin probability in the infinite horizon as its immediate results. In Section 5.4, we focus on pricing the Parisian options. In Section 5.5 we introduce a diffusion approximation and thus obtain results for the Brownian motion.

5.2 Definitions

Set $l = \ln L$. Since X is translation invariant, without losing the generality we simply study the case when $l = 0$. We consider the X with $x = 0$ at first. In this section we are going to introduce a semi-Markov model consisting of two states, the state when the process is above the 0 and the state when it is below. Therefore we define

$$Z_t^X = \begin{cases} 1, & \text{if } X_t > 0 \\ 2, & \text{if } X_t < 0 \end{cases}.$$

We can now express the variables defined above in terms of Z^X:

$$g_t^X = \sup\{s < t \mid Z_s^X \neq Z_t^X\}, \tag{5.8}$$

$$d_t^X = \inf\{s > t \mid Z_s^X \neq Z_t^X\}, \tag{5.9}$$

$$\tau_d^X = \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^X = 2\}}(t - g_t^X) \geq d\}. \tag{5.10}$$

We then define

$$V_t^X = t - g_t^X,$$

the time Z^X has spent in the current state. It is easy to prove that (Z^X, V^X) is a Markov process. Z^X is therefore a semi-Markov process with the state space $\{1, 2\}$, where 1 stands for the state when the stochastic process X is above 0 and 2 corresponds to the state below 0.

Furthermore, we set $U_{i,k}^X$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time Z^X spends in state i when it visits i for the kth time. And we have, for each given i and k there exist some t satisfying that

$$U_{i,k}^X = V_{d_t^X}^X = d_t^X - g_t^X.$$

Notice that assuming that the jump size Y_k is exponentially distributed, it is a well-known result that the size of the overshoots are also exponentially distributed with the same parameter (the *memoryless property*). Therefore the excursions above 0 and below 0 are independent. Consequently, we have that $U_{1,k}^X$, $k = 1, 2, \cdots$, are independent and identically distributed, so as for $U_{2,k}^X$, $k = 1, 2, \cdots$, and $U_{i,k}^X$, $i = 1, 2$, $k = 1, 2, \cdots$, are independent with each other. We therefore define the transition density for Z^X:

$$p_{ij}(t) = \lim_{\Delta t \to 0} \frac{P\left(t < U_{i,k}^X < t + \Delta t\right)}{\Delta t},$$

$$P_{ij}(t) = P\left(U_{i,k}^X < t\right), \quad \bar{P}_{ij}(t) = P\left(U_{i,k}^X > t\right).$$

We have

$$P_{ij}(t) = \int_0^t p_{ij}(s) \mathrm{d}s = 1 - \bar{P}_{ij}(t),$$

which is actually the probability that the process will stay in state i for no more than time t. We will see in the later discussion that the condition $c > \frac{\lambda}{\alpha}$ results in $P\left(U_{1,k}^X = \infty\right) > 0$ for all k (we adopt the convention $U_{i,k}^X = \infty$ if the process never leaves state i at its kth excursion); therefore $\int_0^{+\infty} p_{12}(s)\mathrm{d}s < 1$, i.e. with a positive probability, the process will stay in state 1 forever. Hence, in this case $\bar{P}_{12}(t) > \int_t^{+\infty} p_{12}(s)\mathrm{d}s$.

Moreover, in the definition of Z^X, we deliberately ignore the situation when $X_t = 0$. The reason is that

$$\int_0^t \mathbf{1}_{\{X_u = 0\}} \mathrm{d}u = 0.$$

We will now show how to get $p_{ij}(t)$. We use $\hat{P}_{ij}(\beta)$ to represent the Laplace transform of $p_{ij}(t)$, i.e.

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta t} p_{ij}(t) \mathrm{d}t = E\left(e^{-\beta U_{i,k}^X}\right). \tag{5.11}$$

Consider the equation

$$-\beta + cv_\beta + \lambda\left(\frac{\alpha}{v_\beta + \alpha} - 1\right) = 0, \tag{5.12}$$

which has two roots,

$$v_\beta^+ = \frac{\sqrt{(c\alpha + \beta + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - \beta - \lambda)}{2c}, \tag{5.13}$$

and

$$v_\beta^- = \frac{-\sqrt{(c\alpha + \beta + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - \beta - \lambda)}{2c}. \tag{5.14}$$

First of all, we want to look at the length of an excursion below 0, i.e. $U_{2,k}^X$, $k = 1, 2, 3, \ldots$. Define the stopping time

$$T_x = \inf\left\{t > 0, X_t = 0 \mid X_0 = x, x < 0\right\}.$$

It has been shown in [25] that

$$E\left(\exp\left(-\beta T_x\right)\right) = \exp\left(v_\beta^+ x\right).$$

According to the definitions of the process X and $U_{2,k}^X$ and the argument above, every excursion below 0 starts from an point below 0 whose absolute value is exponentially distributed with parameter α. We have therefore

$$\begin{aligned}\hat{P}_{21}(\beta) &= E\left(e^{-\beta U_{2,k}^X}\right) = \int_0^\infty E\left(e^{-\beta T_{-x}}\right)\alpha e^{-\alpha x}\mathrm{d}x \\ &= \int_0^\infty \exp\left(-v_\beta^+ x\right)\alpha e^{-\alpha x}\mathrm{d}x \\ &= \frac{2c\alpha}{\sqrt{(\beta+\lambda+c\alpha)^2-4c\lambda\alpha}+(\beta+\lambda+c\alpha)}.\end{aligned}$$

Inverting this Laplace transform with respect to β gives the transition density

$$p_{21}(t) = \sqrt{\frac{c\alpha}{\lambda}}e^{-(\lambda+c\alpha)t}t^{-1}I_1\left(2t\sqrt{c\lambda\alpha}\right). \tag{5.15}$$

The formulae for the inversion can be found in [7].

For the length of an excursion above 0, i.e. $U_{1,k}^X$, $k = 1,2,3,\ldots$, we define the stopping time

$$T_0 = \inf\{t > 0, X_t < 0 \mid X_0 = 0\}.$$

By results in [26], [27] and [28] and the independence of the time and the size of the overshoot, i.e. T_0 and X_{T_0} we have

$$E\left(e^{-\beta T_0}\right)E\left(\exp\left(v_\beta^- X_{T_0}\right)\right) = 1.$$

And we also know that X_{T_0} follows exponential distribution with parameter α.

Therefore

$$\begin{aligned}
\hat{P}_{12}(\beta) &= E\left(e^{-\beta U_{1,k}^X}\right) = E\left(e^{-\beta T_0}\right) = \frac{1}{E\left(\exp\left(v_\beta^- X_{T_0}\right)\right)} \\
&= \frac{1}{\int_0^\infty \exp\left(-v_\beta^- x\right) \alpha e^{-\alpha x} \mathrm{d}x} \\
&= \frac{2\lambda}{\sqrt{(\beta+\lambda+c\alpha)^2 - 4c\lambda\alpha} + (\beta+\lambda+c\alpha)}.
\end{aligned}$$

Inverting $\hat{P}_{12}(\beta)$ gives

$$p_{12}(t) = \sqrt{\frac{\lambda}{c\alpha}} e^{-(\lambda+c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \tag{5.16}$$

(see [7] for the formulae).

5.3 The Laplace Transform of τ_d^X

In this section we give the Laplace transform of τ_d^X for the cases when $x = 0$ and when $x > 0$ together with the proofs.

Theorem 5.3.1 *For X with $x = 0$, we have*

$$E\left(e^{-\beta\tau_d^X}\right) = \frac{e^{-\beta d}\bar{P}_{21}(d)\hat{P}_{12}(\beta)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)}, \tag{5.17}$$

where

$$\begin{aligned}
\bar{P}_{21}(d) &= 1-\int_0^d \sqrt{\frac{c\alpha}{\lambda}} e^{-(\lambda+c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \mathrm{d}t, && (5.18)\\
\tilde{P}_{21}(\beta) &= \int_0^d \sqrt{\frac{c\alpha}{\lambda}} e^{-(\beta+\lambda+c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \mathrm{d}t, && (5.19)\\
\hat{P}_{12}(\beta) &= \frac{2\lambda}{\sqrt{(\beta+\lambda+c\alpha)^2-4c\lambda\alpha}+(\beta+\lambda+c\alpha)}, && (5.20)
\end{aligned}$$

and $I_1(x)$ is the modified Bessel function of the first kind.

Proof: A_k denotes the event that the first time the length of the excursion in state 2, i.e. below 0, reaches d happens during the kth excursion in this state, i.e.

$$\{A_k\} = \left\{\tau_d^X \text{ is achieved in the } k\text{th excursion in state } 2\right\}.$$

So we have

$$E\left(e^{-\beta\tau_d^X}\right) = \sum_{k=1}^{\infty} E\left(e^{-\beta\tau_d^X} \mid A_k\right) P\left(A_k\right). \tag{5.21}$$

Notice that given A_k, τ_d^X is comprised of k full excursions above 0, $k-1$ below 0 with the length less than d and last one with the length d, i.e.

$$\tau_d^X \mid A_k = \sum_{n=1}^{k-1}\left(U_{1,n}^X + U_{2,n}^X\right) + U_{1,k}^X + d \mid U_{2,1}^X < d, \cdots, U_{2,k-1}^X < d, U_{2,k}^X \geq d.$$

More importantly, $U_{1,n}^X$'s have distribution P_{12}; $U_{2,n}^X$'s have distribution P_{21} and all these variables are independent of each other. As a result,

$$
\begin{aligned}
& E\left(e^{-\beta\tau_d^X} \mid A_k\right) \\
= \; & E\left(e^{-\beta\left(\sum_{n=1}^{k-1}(U_{1,n}^X+U_{2,n}^X)+U_{1,k}^X+d\right)} \,\middle|\, U_{2,1}^X < d, \cdots, U_{2,k-1}^X < d, U_{2,k}^X \geq d\right) \\
= \; & \sum_{k=1}^{\infty} e^{-\beta d}\left\{\int_0^{+\infty} e^{-\beta u} p_{12}(u)\mathrm{d}u\right\}^k \left\{\int_0^d e^{-\beta u}\frac{p_{21}(u)}{P_{21}(d)}\mathrm{d}u\right\}^{k-1}.
\end{aligned}
$$

Also

$$
P(A_k) = P_{21}(d)^{k-1}\bar{P}_{21}(d).
$$

We have therefore

$$
\begin{aligned}
& E\left(e^{-\beta\tau_d^X}\right) \\
= \; & \sum_{k=1}^{\infty} E\left(e^{-\beta\tau_d^X} | A_k\right) P\left(A_k\right) \\
= \; & \sum_{k=1}^{\infty} e^{-\beta d}\left\{\int_0^{+\infty} e^{-\beta u} p_{12}(u)\mathrm{d}u\right\}^k \left\{\int_0^d e^{-\beta u}\frac{p_{21}(u)}{P_{21}(d)}\mathrm{d}u\right\}^{k-1} P_{21}(d)^{k-1}\bar{P}_{21}(d) \\
= \; & \frac{e^{-\beta d}\bar{P}_{21}(d)\int_0^{+\infty} e^{-\beta s}p_{12}(s)\mathrm{d}s}{1-\int_0^{+\infty} e^{-\beta s}p_{12}(s)\mathrm{d}s \int_0^d e^{-\beta s}p_{21}(s)\mathrm{d}s}.
\end{aligned}
$$

□

We should also consider the case when $x > 0$.

Theorem 5.3.2 *For X, with $X_0 = x > 0$ we have*

$$
E\left(e^{-\beta\tau_d^X}\right) = \frac{\alpha + v_\beta^-}{\alpha} e^{-\beta d + v_\beta^- x} \frac{\bar{P}_{21}(d)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)}. \tag{5.22}
$$

Proof: When $x > 0$, we need to find the Laplace transform for $U_{1,1}^X$, which has different distribution from $U_{1,k}^X$, $k = 2, 3, 4, \ldots$.

Applying the optional sampling theorem to the martingale $e^{-\beta t + v_\beta^- X_t}$ (it is easy to check that $e^{-\beta t + v_\beta^- X_t}$ is a martingale), we have

$$E\left(e^{-\beta U_{1,1}^X + v_\beta^- X_{U_{1,1}^X}} \,\Big|\, X_0 = x\right) = e^{v_\beta^- x}.$$

Since the distribution of the overshoot of this process, i.e. $-X_{U_{1,1}^X}$, is still an exponential distribution with parameter α and it is independent of the time of overshoot, i.e. $U_{1,1}^X$, we have

$$E\left(e^{-\beta U_{1,1}^X + v_\beta^- X_{U_{1,1}^X}} \,\Big|\, X_0 = x\right) = \frac{\alpha}{\alpha + v_\beta^-} E\left(e^{-\beta U_{1,1}^X} \,\Big|\, X_0 = x\right).$$

Therefore

$$E\left(e^{-\beta U_{1,1}^X} \,\Big|\, X_0 = x\right) = \frac{\alpha + v_\beta^-}{\alpha} e^{v_\beta^- x}. \tag{5.23}$$

As a result,

$$\begin{aligned}
& E\left(e^{-\beta \tau_d^X}\right) \\
= \; & E\left(e^{-\beta \tau_d^X} \mathbf{1}_{\{U_{2,1}^X \geq d\}}\right) + E\left(e^{-\beta \tau_d^X} \mathbf{1}_{\{U_{2,1}^X < d\}}\right) \\
= \; & e^{-\beta d} E\left(e^{-\beta U_{1,1}^X} \mathbf{1}_{\{U_{2,1}^X \geq d\}}\right) + E\left(e^{-\beta \left(U_{1,1}^X + U_{2,1}^X\right)} \mathbf{1}_{\{U_{2,1}^X < d\}}\right) E\left(e^{-\beta \tau_2^{\tilde{X}}}\right),
\end{aligned}$$

where $\tilde{X}$ is the same process with $X_0 = 0$. $E\left(e^{-\beta\tau_2^{\tilde{X}}}\right)$ has been calculated in Theorem 5.3.1. Since $U_{1,1}^X$ and $U_{2,1}^X$ are independent, we have

$$\begin{aligned}
& E\left(e^{-\beta\tau_d^X}\right) \\
= \; & e^{-\beta d}E\left(e^{-\beta U_{1,1}^X}\right)P\left(U_{2,1}^X \geq d\right) + E\left(e^{-\beta U_{1,1}^X}\right)E\left(e^{-\beta U_{2,1}^X}1_{\{U_{2,1}^X<d\}}\right)E\left(e^{-\beta\tau_2^{\tilde{X}}}\right) \\
= \; & \frac{\alpha+v_\beta^-}{\alpha}e^{-\beta d+v_\beta^- x}\int_d^\infty p_{21}(t)\mathrm{d}t + \frac{\alpha+v_\beta^-}{\alpha}e^{v_\beta^- x}\int_0^d e^{-\beta t}p_{21}(t)\mathrm{d}tE\left(e^{-\beta\tau_d^{\tilde{X}}}\right) \\
= \; & \frac{\alpha+v_\beta^-}{\alpha}e^{-\beta d+v_\beta^- x}\frac{\bar{P}_{21}(d)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)}.
\end{aligned}$$

□

Remark: One can define a perpetual Parisian digital option as follows: the owner of a perpetual Parisian digital down-in option will get £1 when the length of excursion below a given level L reaches d for the first time. The underlying stop price is defined by (5.2) and set $l = \ln L$ as above. We have therefore

$$\begin{aligned}
P_{digital-down-in} &= E\left(e^{-r\tau_{L,d}^S}\right) = E\left(e^{-r\tau_{l,d}^X}\right) \\
&= \frac{\alpha+v_r^-}{\alpha}e^{-rd+v_r^-(x-l)}\frac{\bar{P}_{21}(d)}{1-\hat{P}_{12}(r)\tilde{P}_{21}(r)}.
\end{aligned}$$

By taking $\beta = 0$ in (5.22), we obtain the probability that τ_d^X will ever be achieved.

Corollary 5.3.2.1 *For X with $X_0 = x > 0$, we have*

$$P\left(\tau_d^X < \infty\right) = \frac{\lambda}{c\alpha}e^{\left(\frac{\lambda}{c}-\alpha\right)x}\frac{c\alpha\bar{P}_{21}(d)}{c\alpha-\lambda P_{21}(d)}. \tag{5.24}$$

Remark: From (5.24) we can see that the Parisian ruin probability actually equals to the ruin probability multiplied by a constant. In fact, the Parisian type ruin probability can also be calculated in the following way:

$$P\left(\tau_d^X < \infty\right) = P\left(T_x < \infty\right) \int_0^\infty P\left(\tau_d^{\bar{X}} < \infty \mid X_{T_x} = -y\right) \alpha e^{-\alpha y} \mathrm{d}y, \tag{5.25}$$

where T_x has the same definition as in (5.3) and $\bar{X}$ is the risk process with $\bar{X}_0 = X_{T_x}$. Therefore $P\left(T_x < \infty\right)$ is the ruin probability which has been well studied,

$$P\left(T_x < \infty\right) = \frac{\lambda}{c\alpha} e^{\left(\frac{\lambda}{c} - \alpha\right)x}. \tag{5.26}$$

By using the same method in Theorem 5.3.1, we can calculate that

$$\int_0^\infty E\left(e^{-\beta \tau_d^{\bar{X}}} \mid X_0 = -y\right) \alpha e^{-\alpha y} \mathrm{d}y = \frac{e^{-\beta d} \bar{P}_{21}(d)}{1 - \hat{P}_{12}(\beta) \bar{P}_{21}(d)}. \tag{5.27}$$

By taking $\beta = 0$ in (5.27) we have

$$\int_0^\infty P\left(\tau_d^{\bar{X}} < \infty \mid X_{T_x} = -y\right) \alpha e^{-\alpha y} \mathrm{d}y = \frac{c\alpha \bar{P}_{21}\left(d_2\right)}{c\alpha - \lambda P_{21}\left(d_2\right)}. \tag{5.28}$$

Substituting (5.26) and (5.28) in (5.25) gives the same result as in Corollary 5.3.2.1.

5.4 Parisian options pricing

In this section we focus on pricing the Parisian options we define in Section 5.1. We start with a Lemma.

Lemma 5.4.1 *For X defined by (5.1), we have*

$$\begin{aligned} & E\left(e^{\eta X_{\tilde{T}}} \mathbf{1}_{\{\tau_d^X < \tilde{T}\}} \mid X_0 = x, x > 0\right) && (5.29) \\ = & \frac{\gamma}{\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) - c\eta} E\left(\exp\left\{-\left(\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) + c\eta\right)\tau_d^{X^*}\right\}\right), \end{aligned}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ and is independent of X and

$$X_t^* = x + ct - \sum_{i=0}^{N_t^*} Y_i^*, \tag{5.30}$$

and $\{N_t^*\}_{t\geq 0}$ is a Poisson process with parameter $\lambda^* = \frac{\lambda\alpha}{\alpha+\eta}$. The sequence $\{Y_k\}$, $k = 1, 2, \ldots$, are independent and identically exponentially distributed with parameter $\alpha^* = \alpha + \eta$.

Proof: Let T be the final time. It is not difficult to show that

$$E\left(e^{-\beta\tau_d^{X^*}}\right) = E\left(\int_0^\infty \beta e^{-\beta T} \mathbf{1}_{\{\tau_d^{X^*} < T\}} \mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-\{\beta + \lambda(\frac{\alpha}{\alpha+\eta} - 1) + c\eta\}T - \eta x} E\left(e^{\eta X_T} \mathbf{1}_{\{\tau_d^X < T\}}\right) \mathrm{d}T,$$

where X has the same definition as (5.1) and X_t^* is defined by (5.30).

Setting $\gamma = \beta + \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) + c\eta$ gives

$$\begin{aligned}
E\left(e^{-\beta\tau_d^{X^*}}\right) &= \int_0^\infty \left(\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) - c\eta\right) e^{-\gamma T} E\left(e^{\eta X_T}\mathbf{1}_{\{\tau_d^X < T\}}\right) \mathrm{d}T \\
&= \frac{\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) - c\eta}{\gamma} E\left(e^{\eta X_{\tilde{T}}}\mathbf{1}_{\{\tau_d^X < \tilde{T}\}}\right),
\end{aligned}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of X. Therefore we have

$$\begin{aligned}
& E\left(e^{\eta X_{\tilde{T}}}\mathbf{1}_{\{\tau_d^X < \tilde{T}\}}\right) \\
&= \frac{\gamma}{\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) - c\eta} E\left(e^{-\beta\tau_d^{X^*}}\right) \\
&= \frac{\gamma}{\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) - c\eta} E\left(\exp\left\{-\left(\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) + c\eta\right)\tau_d^{X^*}\right\}\right).
\end{aligned}$$

□

We can see that $E\left(\exp\left\{-\left(\gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) + c\eta\right)\tau_d^{X^*}\right\}\right)$ has been obtained in Theorem 5.3.2 (5.22) with $\beta = \gamma - \lambda\left(\frac{\alpha}{\alpha+\eta} - 1\right) + c\eta$, $\alpha = \alpha^*$ and $\lambda = \lambda^*$. We have therefore obtained the moment generating function of the joint probability of $X_{\tilde{T}}$ and $\tau_d^X < \tilde{T}$. Define the inversion of (5.29) with the respect to η to be

$$p(y, x, d) = \mathscr{L}^{-1}\left\{E\left(e^{\eta X_{\tilde{T}}}\mathbf{1}_{\{\tau_d^X < \tilde{T}\}} \mid X_0 = x, x > 0\right)\right\}. \tag{5.31}$$

We have therefore

$$P\left(X_{\tilde{T}} \in \mathrm{d}y, \tau_d^X < \tilde{T} \mid X_0 = x, x > 0\right) = p(y, x, d)\mathrm{d}y.$$

For a Parisian down-in-call option, its price formula is given by

$$P_{down-in-call} = e^{-rT} E\left((S_T - K)^+ \mathbf{1}_{\{\tau_{L,d}^S < T\}}\right),$$

where r is the risk-free rate, K is the strike price and S is the underlying stock price satisfying

$$S_t = \exp X_t, \quad S_0 = e^x > L.$$

Set

$$l = \ln L, \quad b = \ln K.$$

We have

$$P_{down-in-call} = e^{-rT} E\left(\left(e^{X_T} - K\right)^+ \mathbf{1}_{\{\tau_{l,d}^X < T\}}\right).$$

Define

$$P^*_{down-in-call} = e^{rT} P_{down-in-call}.$$

We are going to show that we can obtain the Laplace transform of $P^*_{down-in-call}$ w.r.t T, denoted by $\mathscr{L}_T$.

$$
\begin{aligned}
& E\left(\left(e^{X_{\tilde{T}}}-K\right)^+ \mathbf{1}_{\{\tau^X_{l,d}<\tilde{T}\}}\right) \\
= & \int_b^\infty \left(e^y-K\right) P\left(X_{\tilde{T}} \in \mathrm{d}y, \tau^X_{l,d} < \tilde{T} \mid X_0 = x, x > 0\right) \\
= & \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty \left(e^y-K\right) P\left(X_T \in \mathrm{d}y, \tau^X_{l,d} < T \mid X_0 = x, x > 0\right) \mathrm{d}T \\
= & \gamma \int_0^\infty e^{-\gamma T} E\left(\left(e^{X_T}-K\right)^+ \mathbf{1}_{\{\tau^X_{l,d}<T\}}\right) \mathrm{d}T \\
= & \gamma \mathscr{L}_T
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathscr{L}_T &= \frac{1}{\gamma}\int_b^\infty \left(e^y-K\right) P\left(X_{\tilde{T}} \in \mathrm{d}y, \tau^X_{l,d} < \tilde{T} \mid X_0 = x, x > 0\right) \\
&= \frac{1}{\gamma}\int_b^\infty \left(e^y-K\right) P\left(X_{\tilde{T}} \in \mathrm{d}y - l, \tau^X_{d} < \tilde{T} \mid X_0 = x - l\right) \\
&= \frac{1}{\gamma}\int_b^\infty \left(e^y-K\right) p(y-l, x-l, d)\mathrm{d}y.
\end{aligned}
$$

5.5 A diffusion approximation

Set

$$
c = \mu + \frac{\sigma^2\alpha}{2}, \quad \lambda = \frac{\sigma^2\alpha^2}{2,}
$$

with $\mu > 0$ and let $\alpha \to +\infty$. The process $X_t - \mu t - x$ converges weakly in $D\left[0,\infty\right)$ to a standard Brownian motion W with $W_0 = 0$ and hence X converges to a Brownian motion with drift

$$
W^\mu_t = x + \mu t + \sigma W_t.
$$

See for example [4], pp 117-118 and also [32], pp 159-160. Moreover, the events

$$\{\tau_d^X \leq t\}$$

and

$$\left\{\sup_{0\leq s\leq t}\left\{\mathbf{1}_{\{X_s<0\}}\left(s-g_s^X\right)\right\}\geq d\right\}$$

are identical and since

$$\sup_{0\leq s\leq t}\left\{\mathbf{1}_{\{X_s<0\}}\left(s-g_s^X\right)\right\}$$

is a continuous functional of X_t on $D\,[0,\infty)$ a.e., we can conclude that

$$\lim_{\alpha\to\infty} P\left(\tau_d^X \leq t\right) = P\left(\tau_d^{W^\mu}\leq t\right) \tag{5.32}$$

for all t; and therefore

$$\lim_{\alpha\to\infty} E\left(e^{-\beta\tau_d^X}\mid X_0=x, x>0\right) = E\left(e^{-\beta\tau_d^{W^\mu}}\mid W_0^\mu = x, x>0\right). \tag{5.33}$$

As a result, by taking the limit $\alpha\to+\infty$ in (5.22) and applying the approximation for the modified Bessel function of the first kind (see [29])

$$I_1(z)\sim\frac{e^z}{\sqrt{2\pi z}}, \tag{5.34}$$

we have

$$E\left(e^{-\beta\tau_d^{W^\mu}}\mid W_0^\mu = x, x>0\right) = \frac{e^{-\beta d}e^{-\frac{\sqrt{\mu^2+2\beta\sigma^2}+\mu}{\sigma^2}x}\int_d^\infty\frac{1}{2\sqrt{2\pi t^3}}e^{-\frac{\mu^2}{2\sigma^2}t}dt}{\sqrt{\frac{\mu^2}{\sigma^2}+2\beta}+\int_d^\infty\frac{1}{2\sqrt{2\pi t^3}}e^{-\beta t}e^{-\frac{\mu^2}{2\sigma^2}t}dt}. \tag{5.35}$$

Calculating the integrals in (5.35) gives

$$E\left(e^{-\beta\tau_d^{W^\mu}} \mid W_0^\mu = x, x > 0\right) \tag{5.36}$$
$$= \frac{e^{-\beta d}e^{-\frac{\sqrt{\mu^2+2\beta\sigma^2}+\mu}{\sigma^2}x}\left\{\frac{1}{\sqrt{2\pi d}}e^{-\frac{\mu^2}{2\sigma^2}d} - \frac{\mu}{\sigma}\mathcal{N}\left(-\frac{\mu}{\sigma}\sqrt{d}\right)\right\}}{\sqrt{\frac{\mu^2}{\sigma^2}+2\beta} + \frac{1}{\sqrt{2\pi d}}e^{-\left(\beta+\frac{\mu^2}{2\sigma^2}\right)d} - \sqrt{2\beta+\frac{\mu^2}{\sigma^2}}\mathcal{N}\left(-\sqrt{\left(2\beta+\frac{\mu^2}{\sigma^2}\right)d}\right)}.$$

The same result with $x = 0$ and $\sigma = 1$ has been obtained in [13], [38] and Chapter 2 using different approaches. It is an important result for pricing the Parisian options. And actually, but taking the limit as $\alpha \to \infty$ in (5.29) and set $\mu = 0$, we can get the moment generating function

$$E\left(e^{\eta W_{\tilde{T}}}\mathbf{1}_{\{\tau_d^W < \tilde{T}\}} \mid W_0 = x, x > 0\right)$$

which is invertible with respect to η, where W is a standard Brownian motion; therefore we have the explicit expression for $p(y, x, d)$ and then the explicit expression for the Laplace transform for the option price. For details see Chapter 2.

Letting $\beta = 0$ in (5.35) and (5.36), we have the Parisian type ruin probability for a Brownian motion with positive drift,

$$P\left(\tau_d^{W^\mu} < \infty \mid W_0^\mu = x, x > 0\right)$$
$$= \frac{e^{-\frac{2\mu}{\sigma^2}x}\int_d^\infty \frac{1}{2\sqrt{2\pi t^3}}e^{-\frac{\mu^2}{2\sigma^2}t}\mathrm{d}t}{\frac{\mu}{\sigma} + \int_d^\infty \frac{1}{2\sqrt{2\pi t^3}}e^{-\frac{\mu^2}{2\sigma^2}t}\mathrm{d}t} \tag{5.37}$$
$$= \frac{e^{-\frac{2\mu}{\sigma^2}x}\left\{\frac{1}{\sqrt{2\pi d}}e^{-\frac{\mu^2}{2\sigma^2}d} - \frac{\mu}{\sigma}\mathcal{N}\left(-\frac{\mu}{\sigma}\sqrt{d}\right)\right\}}{\frac{1}{\sqrt{2\pi d}}e^{-\frac{\mu^2}{2\sigma^2}d} + \frac{\mu}{\sigma}\mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)}. \tag{5.38}$$

Remark: It is tempting to derive the Parisian ruin probability by taking the limit as $\alpha \to \infty$ in (5.24). However, the argument used to get (5.32) does not generalise in the case of an infinite horizon so we can not argue directly from (5.24). See [3] pp 196,199, [4] pp 119, [30], [31] and [32] pp 165-166 for more details. A simple way to proceed is via (5.35) or (5.36) as we did.

Chapter 6

Parisian Type Ruin Probabilities in Infinite Time Horizon

Abstract

In this paper, we extend the concept of ruin in risk theory to the Parisian type of ruin. For this to occur, the surplus process must fall below zero and stay negative for a continuous time interval of specified length. We obtain the probability of ruin in the infinite horizon for the case when the process starts from zero and the asymptotic form of the probability of ruin in the infinite horizon for the case when the process starts from the point far above zero. We show that in the small claim case an asymptotic formula similar to Cramér's formula is true.

Keywords: ruin, Parisian type of ruin, surplus process, ruin probability, adjustment coefficient.

6.1 Introduction

We consider a classical surplus process in continuous time $\{X_t\}_{t\geq 0}$

$$X_t = u + ct - \sum_{k=0}^{N_t} Y_k, \tag{6.1}$$

where $u \geq 0$ is the initial reserve, c is a constant rate of premium payment per time unit, and $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ representing the numbers of claims up to time t. The sequence $\{Y_k\}$, $k = 1, 2, ...$, are claim sizes which are independent and identically distributed non-negative random variables that are also independent of the number of claims. We also assume $c > \lambda E(Y_1)$ (the *net profit condition*). Define the stopping time

$$T = \inf\{t > 0 \mid X_t < 0\}. \tag{6.2}$$

The event of ruin in infinite time horizon can be expressed as $\{T < \infty\}$. The density of T and the probability of ruin have been widely studied. See for example [15], [16], [19], [20], [21], [22], [24], [27], [28], [26], [32], [33], [34], [36], [40], [44] and [45].

In this paper, we extend the concept of ruin to the Parisian type of ruin. The idea comes from Parisian options, the prices of which depend on the excursions of the underlying asset prices above or below a barrier. An example is a *Parisian down-and-out option*, the owner of which loses the option if the underlying asset price S reaches the level l and remains constantly below this level for a time interval longer than d. For details and extensions, see [13], [37], [38], Chapters 2, 3, 4 and 5.

Parisian type ruin will occur if the surplus falls below zero and stays below zero for a continuous time interval of length d. In some respects, this is a more appropriate measure of risk than classical ruin as it gives the office some time to put its finances back in order. In practice, the bankruptcy procedures in many countries allow for this "grace" period, such as the Chapter 11 bankruptcy of the United States' Bankruptcy Code. Similar bankruptcy regulations are also applied to Japan and France (see [10]).

In order to introduce the concept of Parisian type of ruin mathematically, we will first define the excursion. Set

$$g_t = \sup\{s < t \mid \operatorname{sign}(X_s)\operatorname{sign}(X_t) \leq 0\}, \tag{6.3}$$

$$d_t = \inf\{s > t \mid \operatorname{sign}(X_s)\operatorname{sign}(X_t) \leq 0\}, \tag{6.4}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$, where

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

The trajectory between g_t and d_t is the excursion of process X below or above zero which straddles time t. Assuming $d > 0$, we now define

$$\tau_d = \inf\{t > 0 \mid \mathbf{1}_{\{X_t<0\}}(t - g_t) \geq d\}. \tag{6.5}$$

We can see that τ_d is therefore the first time that the length of the excursion of process X below zero reaches given level d. We then define the events $\{\tau_d < \infty\}$

to be the Parisian type of ruin in the infinite horizon. We are interested in the corresponding probabilities

$$P(\tau_d < \infty).$$

In Section 6.2 we calculate the Parisian type ruin probability for the case when the initial reserve is zero. In Section 6.3 we study the case when the initial research is greater than zero. The asymptotic form of the Parisian type ruin probability will be given for the small claim case. We conclude our results in Section 6.4 and point out some directions for the future research.

6.2 The ruin probability for the case when the initial reserve is zero

In this section, we are going to consider a simplified case with no initial reserve, i.e.

$$X_t = ct - \sum_{k=0}^{N_t} Y_k. \tag{6.6}$$

Set

$$G(y) = P(Y_i < y), \quad \bar{G}(y) = P(Y_i > y);$$

$$m = E(Y_i), \quad \hat{g}(v) = \int_0^\infty e^{-vy} \mathrm{d}G(y).$$

Denote the ruin probabilities to be

$$\psi(u) = P(T < \infty \mid X_0 = u), \quad \psi_d(u) = P(\tau_d < \infty \mid X_0 = u).$$

Since $T < \tau_d$, it is clear that $\psi(u) > \psi_d(u)$.

Theorem 6.2.1 *For the process X defined by (6.6), we have that*

$$\psi_d(0) = \frac{\lambda m \bar{H}(d)}{c - \lambda m H(d)}, \tag{6.7}$$

where

$$\begin{aligned} H(d) &= \mathscr{L}_\beta^{-1}\left(\frac{cv_\beta^+ - \beta}{\lambda m \beta v_\beta^+}\right), & (6.8) \\ \bar{H}(d) &= 1 - H(d), & (6.9) \end{aligned}$$

and v_β^+ is the unique positive solution of

$$-\beta + cv_\beta + \lambda\left(\hat{g}\left(v_\beta\right) - 1\right) = 0. \tag{6.10}$$

Proof: It is well-known that

$$\psi(0) = \frac{\lambda m}{c}, \tag{6.11}$$

and that the overshoot $-X_T$ is a non-negative continuous random variable with density

$$\frac{\bar{G}(x)}{m}. \tag{6.12}$$

See for example [19], [20], [21], [22], [27], [28], [30], [31], [32] and [40]. Furthermore, define

$$T^* = \inf\{t > 0, X_t = 0 \mid X_0 = x, x < 0\}. \tag{6.13}$$

It has been shown in [25] that

$$E(\exp(-\beta T^*)) = \exp\left(v_\beta^+ x\right). \tag{6.14}$$

We use $h(t)$ to denote the density of the first (and actually any, due to the strong Markov property of the process X) excursion below zero. Its Laplace transform can be obtained as follows:

$$\begin{aligned}
\hat{h}(\beta) &= \int_0^\infty e^{-\beta t} h(t)\mathrm{d}t \\
&= \int_0^\infty E\left(\exp\left(-\beta T^* \mid X_0 = -y\right)\right) \frac{\bar{G}(y)}{m}\mathrm{d}y \\
&= \int_0^\infty \exp\left(-v_\beta^+ y\right) \frac{\bar{G}(y)}{m}\mathrm{d}y \\
&= \frac{1 - \hat{g}\left(v_\beta^+\right)}{m v_\beta^+} = \frac{c v_\beta^+ - \beta}{\lambda m v_\beta^+}.
\end{aligned}$$

Define then the cumulative distribution function of T^* to be

$$H(d) = P(T^* < d). \tag{6.15}$$

We have actually

$$H(d) = \int_0^d h(t)\mathrm{d}t = \mathscr{L}_\beta^{-1}\left(\frac{\hat{h}(\beta)}{\beta}\right) = \mathscr{L}_\beta^{-1}\left(\frac{c v_\beta^+ - \beta}{\lambda m \beta v_\beta^+}\right). \tag{6.16}$$

Moreover, the number of excursions N below zero in infinite time horizon has a geometric distribution such that

$$P(N=n)=\left(1-\frac{\lambda m}{c}\right)\left(\frac{\lambda m}{c}\right)^n, \quad n=0,1,2,\ldots \tag{6.17}$$

As a result, the largest ever excursion below zero, denoted by L, is such that

$$P(L\leq d)=\sum_{i=0}^{\infty}(H(d))^i\left(1-\frac{\lambda m}{c}\right)\left(\frac{\lambda m}{c}\right)^i=\frac{1-\frac{\lambda m}{c}}{1-\frac{\lambda m}{c}H(d)}. \tag{6.18}$$

Hence we have

$$\psi_d(0)=1-P(L\leq d)=\frac{\lambda m\bar{H}(d)}{c-\lambda mH(d)}. \tag{6.19}$$

□

Remark: It is clear that $\psi_d(0)<\psi(0)$ by simply comparing (6.7) and (6.11). Also, we can obtain $\psi(0)$ by taking $d\to 0$ in (6.7).

6.3 An asymptotic formula for the Parisian ruin probability

In this section we focus on the asymptotic form for the Parisian ruin probability as $u\to\infty$. We assume that we have small claims.

Assumption: The Laplace transform $\hat{g}(v)$ is defined for all $v\in(\alpha,\infty)$ for some $\alpha<0$.

Theorem 6.3.1 *For the process X, $X_0 = u$, when $u \longrightarrow \infty$ we have that*

$$\psi_d(u) \sim C_d e^{-Ru}, \tag{6.20}$$

where

$$C_d = C\left\{1 - \frac{(c-\lambda m)R}{(c-\lambda m)(c-\lambda m H(d))} Q(d)\right\}, \tag{6.21}$$

$$C = \frac{c-m\lambda}{R\lambda}\left[\int_0^\infty y e^{Ry}\bar{G}(y)\mathrm{y}\right]^{-1}, \tag{6.22}$$

$$Q(d) = \mathscr{L}_\beta^{-1}\left(\frac{1}{v_\beta^+\left(v_\beta^+ + R\right)}\right), \tag{6.23}$$

and R is the adjustment coefficient which is the unique positive root of

$$-cR + \lambda\left(\hat{g}(-R) - 1\right) = 0. \tag{6.24}$$

Proof: First of all, the Parisian ruin probability can be written as follows:

$$\begin{aligned} &\psi_d(u) \\ =\ & P\left(\tau_d < \infty \mid X_0 = u\right) \\ =\ & P\left(\tau_d < \infty, T < \infty, T^* < d \mid X_0 = u\right) + P\left(\tau_d < \infty, T < \infty, T^* \geq d \mid X_0 = u\right) \\ =\ & P\left(T < \infty, T^* < d \mid X_0 = u\right) P\left(\tau_d < \infty \mid X_0 = 0\right) \\ & + P\left(T < \infty, T^* \geq d \mid X_0 = u\right). \end{aligned}$$

That last equality is due to the strong Markov property of X. We have obtained

$P(\tau_d < \infty \mid X_0 = 0)$ in (6.7). Furthermore, we have

$$
\begin{aligned}
& \int_0^\infty e^{-\beta d} \lim_{u\to\infty} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u)\,\mathrm{d}d \\
= & \lim_{u\to\infty} \int_0^\infty e^{-\beta d} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u)\,\mathrm{d}d \\
= & \lim_{u\to\infty} e^{Ru} E\left(\frac{e^{-\beta T^*}}{\beta} \mathbf{1}_{\{T<\infty\}} \middle| X_0 = u \right) \\
= & \lim_{u\to\infty} \int_0^\infty E\left(\frac{e^{-\beta T^*}}{\beta} \middle| -X_T = z \right) e^{Ru} P(T < \infty, -X_T \in \mathrm{d}z \mid X_0 = u) \\
= & \lim_{u\to\infty} \int_0^\infty E\left(\frac{e^{-\beta T^*}}{\beta} \middle| -X_T = z \right) P(-X_T \in \mathrm{d}z \mid T < \infty, X_0 = u)\, e^{Ru}\psi(u) \\
= & \int_0^\infty E\left(\frac{e^{-\beta T^*}}{\beta} \middle| -X_T = z \right) \lim_{u\to\infty} P(-X_T \in \mathrm{d}z \mid T < \infty, X_0 = u)\, e^{Ru}\psi(u).
\end{aligned}
$$

By (6.14) we have that

$$E\left(\frac{e^{-\beta T^*}}{\beta} \middle| -X_T = z \right) = \frac{e^{-v_\beta^+ z}}{\beta}.$$

It is well-known that

$$\lim_{u\to\infty} P(-X_T \in \mathrm{d}z \mid T < \infty, X_0 = u) = \frac{\lambda R}{c - \lambda m} \int_0^\infty e^{Rx} \bar{G}(x+z)\mathrm{d}x\mathrm{d}z,$$

$$\lim_{u\to\infty} e^{Ru}\psi(u) = C = \frac{c - m\lambda}{R\lambda} \left[\int_0^\infty y e^{Ry} \bar{G}(y)\mathrm{y} \right]^{-1}.$$

For more details see [15], [24] and [40]. We have therefore that

$$\begin{aligned}
&\int_0^\infty e^{-\beta d} \lim_{u\to\infty} e^{Ru} P\left(T<\infty, T^* < d \mid X_0 = u\right) \mathrm{d}d \\
= \; & C\int_0^\infty \frac{e^{-v_\beta^+ z}}{\beta} \frac{\lambda R}{c-\lambda m} \int_0^\infty e^{Rx} \bar{G}(x+z) \mathrm{d}x \mathrm{d}z \\
= \; & \frac{C\lambda}{c-\lambda m} \frac{1}{\beta} \left(\frac{\hat{g}(-R) - \hat{g}(v_\beta^+)}{v_\beta^+ + R} - \frac{1-\hat{g}(v_\beta^+)}{v_\beta^+} \right) \\
= \; & \frac{CR}{c-\lambda m} \left(\frac{1}{v_\beta^+ \left(v_\beta^+ + R\right)} \right).
\end{aligned}$$

As a result,

$$\lim_{u\to\infty} e^{Ru} P\left(T<\infty, T^* < d \mid X_0 = u\right) = \frac{CR}{c-\lambda m} Q(d), \tag{6.25}$$

where

$$Q(d) = \mathscr{L}_\beta^{-1} \left(\frac{1}{v_\beta^+ \left(v_\beta^+ + R\right)} \right);$$

and hence

$$P\left(T<\infty, T^* < d \mid X_0 = u\right) \sim e^{-Ru} \frac{CR}{c-\lambda m} Q(d). \tag{6.26}$$

Also, we have

$$\begin{aligned}
&P\left(T<\infty, T^* \geq d \mid X_0 = u\right) \\
= \; & \psi(u) - P\left(T<\infty, T^* < d \mid X_0 = u\right) \\
\sim \; & Ce^{-Ru} \left(1 - \frac{R}{c-\lambda m} Q(d) \right).
\end{aligned} \tag{6.27}$$

We have therefore proved (6.20).

□

Remark 1: The constant C given by (6.22) is the well-know Cramér constant. This theorem gives the modified version of the Cramér constant, C_d for the Parisian ruin case, which is given by (6.21).

Remark 2: It is easy to see that $C_d < C$, and hence $\psi_d(u) < \psi(u)$.

6.4 Conclusion

In Section 6.3 we obtain the asymptotic result for the small claim case (see the *Assumption*). Note that it is not the case for the result in Section 6.2, which is true for all claim distributions. When $u > 0$, the difficulty with the large claim case is that we do not have a nice form for the distribution of overshoot on which the length of excursions below zero depend. The investigation of the large claim case can be a topic of future research.

For the small claim case, instead of asymptotic form we obtained here, it would also be nice to get a formula for $\psi_d(u)$ for a general $u > 0$. One of the difficulties is that the length of the excursions below zero depends on the length of the preceding excursion above zero since the overshoots depend on the length of the excursion above zero. However, for exponential distributed claims, we do not have such problem in which case the overshoot is independent of the excursion and the explicit form for $\psi_d(u)$ can be obtained (see Chapter 5).

Furthermore, as another direction of future research, one should try to study the Parisian ruin probability in finite time horizon, i.e. $P(\tau_d < t)$.

Chapter 7

Conclusion

This thesis mainly focus on two subjects, pricing Parisian options and calculating Parisian type ruin probabilities.

As for Parisian options, under the Black-Scholes assumptions, four types of Parisian options are priced, single barrier one-sided Parisian options, single barrier two-sided Parisian options, double barrier Parisian options and Parisian corridor options. The results are given in the explicit forms of the Laplace transforms of the option prices with respect to the maturity time T. The inversion of these Laplace transforms has not been attempted in this thesis. There are some works concerning the inversion for the the single barrier one-sided Parisian options (see for example [38]). The inversion for the other three types of Parisian options can be a topic of future research.

Additionally, pricing Parisian options under jump processes has been first attempted in this thesis. The case studied here is restricted to the single barrier one-sided Parisian options whose underlying asset prices follow a classical surplus process with negative exponential jumps. It is a relatively new area and more research can be done in the future to price a broader types of Parisian options with more general jump processes, for example the process with both negative and positive jumps. In this case, the excursion above and below an barrier does not preserve the independent structure anymore. The length of every excursion depends on its preceding excursion and therefore the methods used in this thesis cannot be applied. Different mathematical tools are needed in this case.

Regarding to the Parisian type ruin probabilities, the probability of ruin in the infinite horizon for zero initial reserve is calculated. When the initial reserve is larger than zero, only the asymptotic form can be calculated for very large initial reserve and the small claim distributions. The exact formula for any initial reserve larger than zero can only be obtained for the exponential claim. As a result, to obtain the exact formula for any initial reserve larger than zero and any small claim distributions and to study the large claim case can be a direction of future research. Furthermore, one can look at the ruin probability in finite time horizon, the exact form of which has only been obtained for the exponential claim case in this thesis.

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