Valuing Credit Spread Options under Stochastic Volatility/Interest Rates

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ABSTRACT
This thesis studies the pricing of credit spread options in a continuous time setting. Our main examples are credit spreads between US government bonds and highly risky emerging market bonds, such as Argentina, Brazil, Mexico, etc. Based on empirical findings we model the credit spread options as a geometric Brownian Motion with stochastic volatility. We implement and compare several one-factor stochastic volatility models, namely the Vasicek, Cox-Ingersoll-Ross and Ahn/Gao. As a stochastic model for the credit risk free interest rate, we use the Vasicek model.

As a further new ingredient we introduce dependence between the spread rate and interest rate in our pricing model (stochastic volatility is assumed to be independent of the other factors). The mean reverting property of the short rate models enables us to view the mean reverting stochastic volatility models as moment generating function of a time integral of positive diffusion. The moment generating function of the average variance of the credit spread price process is evaluated. The Numerical Laplace inversion method is used to invert the moment generating function to obtain the density of the average variance. This average variance density is then used in the analytic pricing formulae. We compare the credit spread option prices under the closed form and the numerical formula in the cases of no correlation and some correlation between the credit spreads and the short rate under the Vasicek, Cox/Ross and Ahn/Gao(Alternative) mean reverting stochastic volatility model.

We also look at the delta hedge parameters for the credit spread options under the various stochastic volatility models. Further analysis is carried out on the effects of correlation between the credit spread, the short rate and various mean reversion parameters on the pricing and hedging of the credit spread options.

We finally compare our credit spread option price/hedging stochastic volatility model with the Longstaff and Schwartz model on mean reverting credit spreads under constant volatility.

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Contents

Chapter 1

Introduction

Credit risk is the risk whereby counterparties fail to make payments to transactions they are obligated. It is sometimes referred to as counterparty risk. A great deal of attention has been placed in credit risk management by academics and practitioners over the years and further research is ongoing in this new area.

In the last decade, a new innovation in credit risk management appeared, the credit derivative instruments. Credit derivatives are designed to segregate market risk from credit risk and to allow separate trading of credit risk. They allow a more efficient allocation of credit risk, which greatly benefits those who borrow, lend and transact interest rate and credit derivatives. This ensures that premiums associated with default risk are appropriate for that level of risk. Credit derivatives may also be defined as contracts that pass credit risk from one counterparty to another. They allow credit risk to be stripped from loans, bonds and swaps and placed in a different market. Their performance is based on a credit spread, credit rating or default status.

Besides the obvious advantage of eliminating and trading credit risk, other benefits include the diversification of highly concentrated portfolios (reducing liquidity risk), balance sheet management and the reduction of regulatory capital for better return on capital.

Credit derivative instruments include total return swaps(TORS), defaults swaps, credit default swaps(CDS), credit spread options(CSO), credit spread forwards, credit linked notes(CLNs), credit debt obligations(CDOs), credit loan obligations(CLOs) and synthetic CDOs etc.

The instruments of interest in our research paper are credit spread options. These are options where the underlying is the spread on a 3rd party security. Suppose a trader feels that the rating agencies have given the 3rd party too low a credit rating and want to exploit this. Given the current credit spread as 100 basis points, the trader pays a premium to go long the credit spread option. If spreads narrow or fall below the current level of 100 basis points he or she profits. But if the spread widens he or she will abandon the option and the maximum loss is the premium paid. Also banks and insurance companies may sell credit spread option protection for a fee because they think that the asset is undervalued and that the credit spread widening is not indicating a default event.

A trader holding the bond issued by the 3rd party might purchase a credit spread option that pays if spreads widen as a result of bond issuer or reference asset being downgraded. CSOs can also be used to hedge index related basis risk by say an insurance company purchasing an option on the credit spread between a bond index and an index on a similar duration and maturity US Treasuries.

Overall the credit spread option offers the flexibility where it can be used to reduce credit exposure, hedge against potential default risk, speculate on credit spreads widening or narrowing and the selling of credit protection.

In this paper we propose the valuation of credit spread options under a continuous time model with the following assumptions:

(a) credit spread is a traded asset. This is a standard assumption in CSO pricing literature which can be justified by holding a portfolio of long credit risk bond and short a default free bond with same characteristics.

(b) credit spread exhibits stochastic volatility in contrast to Longstaff and

Schwartz spread option model. This is motivated by our prime example: the spreads between US government bonds and emerging market bonds.

(c) interest rates are stochastic (Vasicek short rate model), which is standard in this context.

(d) the credit spread and the short rate are correlated, is supported by empirical evidence, although the evidence on the sign of the correlation is mixed.

We use the following model.

Assume a stochastic basis $\left(\Omega, F, Q, \tilde{F}\right)$; $\tilde{F} = (F_t)$ filtration satisfying usual conditions.

Let

$$dX_t = \pi_0 \sigma_t^2 dt + \sigma_t dW_1(t) \tag{1.1}$$

where $X_t = \log S_t$ (log of credit spread).

The SDE in (1.1) is supported by the empirical study on derivatives of credit spreads. See working paper on the valuation of derivatives on credit spreads by Rudiger Kiesel.

For σ_t the volatility, we consider one-factor mean reverting volatility processes of the following forms:

Vasicek SDE:

$$d\sigma_t^2 = \theta \left(c - \sigma_t^2\right) dt + a_1 dW_3(t) \tag{1.2}$$

Cox, Ingersoll and Ross SDE:

$$d\sigma_t^2 = \theta \left(c - \sigma_t^2\right) dt + a_1 \sigma_t dW_3(t) \tag{1.3}$$

Ahn/Gao SDE:

$$d\sigma_t^2 = \theta \left(c - \sigma_t^2\right) \sigma_t^2 dt + a_1 \sigma_t^3 dW_3(t) \tag{1.4}$$

where W_1 and W_3 are independent Brownian motions.

Given the framework above, the credit spread option model is valued under a stochastic interest rate and stochastic volatility dynamics. Under stochastic interest rates we consider cases of zero, positive and negative correlation between the credit spreads and the Vasicek short rate process in our pricing and hedging models. The choice of interest rate process is purely from an analytic tractability point of view. Other one-factor short interest rate process could be applied to our model.

For the stochastic volatility dynamics we consider 2 well known one-factor mean reverting stochastic processes, Vasicek, Cox, Ingersoll and Ross as well as an alternative one-factor stochastic process Ahn Gao. The credit spread has no correlation with the stochastic volatility processes.

Chapter 2 discusses the one-factor mean reverting stochastic processes Vasicek, Cox, Ingersoll and Ross and Ahn Gao in a more general setting rather than focusing on its use in the interest rate world. In Chapters 3 and 4, we build a general closed form credit spread valuation model under a risk neutral price framework, from which we derive a pricing formula for European call credit spreads under the risk neutral measure as the expectation of the Black Scholes call credit spread conditional on the realised volatility. The realised volatility is obtained from the average variance over the life time of the credit spread call option.

In chapters 5, 6 and 7 we derive the credit spread call formula for closed form and numerical form under the three one-factor mean reverting stochastic volatility processes (Vasicek, Cox, Ingersoll and Ross and Ahn Gao). For each of the one-factor mean reverting stochastic volatility process, its stochastic differential equation is expressed as a random variable for the average variance. The moment generating function for the average variance is obtained by applying a concept well known from bond pricing, where the mean reverting short rate are viewed as moment generating functions of the time integral of positive diffusions.

The three mean reverting stochastic volatility processes in the form of

Vasicek, Cox, Ingersoll and Ross and Ahn Gao SDE's have closed form solutions for the moment generating functions. Obtaining the distribution for the average variance analytically by Inverse Laplace transform techniques is only possible for simple moment generating functions. The complex moment generating functions for the mean reverting volatility processes requires numerical inversion for the distribution.

We apply the Abate/Whitt method to numerically invert these moment generating functions to obtain the distribution of the average variance for the chosen one-factor stochastic volatility process. The density of the average variance is then used in evaluating the risk neutral expectation of the credit spread option price conditional on the distribution of the average variance process.

Credit spread options prices are evaluated for the closed form and numeric models (Monte Carlo) under zero and non zero correlation for a given stochastic volatility model of either Vasicek, Cox, Ingersoll and Ross or Ahn/Gao. Credit spread prices from the closed and numeric models are compared to establish how close they are. The numerical and closed form prices are listed in the appendix sections for chapters 5 to 7.

Further analysis is done on how the structural or mean reverting parameters and the correlation between the credit spreads and the short rate affect credit spread option prices. Calculations are done for in and out of the money credit spread call prices for our credit spread option model.

In chapter 8 we derive the hedge parameters for the credit spread option model. Chapter 9 briefly mentions the Longstaff/Schwartz credit spread option pricing and hedging model. We calculate in and out of the money credit spread call prices for the Longstaff/Schwartz model. Delta call hedge parameter values are calculated for both our credit spread option and Longstaff/Schwartz models.

Chapter 10 compares the spread option prices and delta hedge values for our credit spread option model with the Longstaff/Schwartz credit spread model by analysing the graphs of credit spread option price against underlying spread and Delta against underlying credit spread plots for maturities of 0.5, 1.0 and 1.5 years.

Finally we discuss our findings in the conclusion in chapter 11.

Chapter 2

One-factor Mean reverting models

In this chapter we discuss the three classes of one-factor mean reverting models mentioned in the introduction.

Two of the models are of the general form

$$dv = k(\theta - v)dt + \sigma v^{\beta} dz \tag{2.1}$$

The third model is of the general form

$$dv = k(\theta - v)vdt + \sigma v^{\beta} dz \tag{2.2}$$

These models have widely been used for the modelling of the short rate in the interest rate literature. We try to give a more general view of these models. The choice of β in (2.1) is usually dictated by a compromise between analytic tractability and reasonableness of the resulting distribution. The empirical issue of which exponent gives the best description of the process to be modelled has not been settled yet, although Chan et al(1991) indicate that a higher β might describe the observed underlying better whether it is volatility or short rates.

In our study we consider the following cases

(i) $\beta = 0$ under (2.1);

(ii) $\beta = 0.5$ under (2.1);

(iii) $\beta = 1.5$ under (2.2);

The case of $\beta = 0$ in (2.1) is the Vasicek process of the form

$$dv = k(\theta - v)dt + \sigma dz \tag{2.3}$$

The Vasicek process is related to the Ornstein-Uhlenbeck process.

As a Gaussian process it produces a symmetric distribution. It allows for negative values for the underlying asset process, which is an undesirable property. As a mean reverting model, when the underlying asset is above (below) a long term level it experiences a downward (upward) pull towards this level.

The case of $\beta = 0.5$ in (2.1) gives us the Cox, Ingersoll and Ross stochastic process of the form

$$dv = k(\theta - v)dt + \sigma\sqrt{v}dz \tag{2.4}$$

It is widely known as the squared gaussian process with a non central chi-square distribution i.e. a skewed distribution with a fatter right hand tail. For large t, i.e. t $\rightarrow \infty$, its distribution approaches a Gamma distribution. For k, $\theta > 0$ this corresponds to a continuous time first order autoregressive process whereby the random volatility is elastically pulled toward a long term value θ. The mean reversion parameter k which determines the speed of adjustment can reach 0 if $\sigma^2 > 2k\theta$. If $2k\theta \geq \sigma^2$, the upward shift is sufficient to make the origin inaccessible. This implies an initially non negative volatility can never become negative.

Other analytical properties imply the following:

(a) negative volatilities are precluded;

(b) if volatility reaches θ, it can subsequently become positive;

(c) the absolute variance of the volatility increases when volatility increases, as the volatility process is proportional to the square root of the volatility process;

The Vasicek and Cox, Ingersoll and Ross stochastic processes have the same drift volatility specifications. These processes can be extended further by making the speed of the mean reversion and the long term mean reversion of the volatility process a function of time rather than a constant. However the extended Vasicek process still exhibits the properties of negative values.

Empirical studies by Ahn Gao on the parametric nonlinear model of term structure dynamics propose a mean reverting SDE of the one factor form as in Chan et al, where $\beta = 1.5$ but the drift is non linear.

The Ahn Gao SDE is of the form

$$dv = k(\theta - v)vdt + \sigma v^{1.5}dz. \tag{2.5}$$

The long term parameter θ is the threshold of the volatility process, where the drift is 0. The assumption of $k > 0$ is necessary for the stationarity of the volatility process. The drift is positive if long term reversion is above the volatility process, implying the volatility process reverts to the normal range. When the volatility process exceeds long term mean reversion θ, the drift becomes negative. A negative drift pulls long term mean reversion to its normal range. The mean reversion parameter k also determines the curvature of the drift which measures the degree of nonlinearity in the drift. When the drift climbs until the volatility process reaches $\frac{\theta}{2}$, the drift hits its maximum $\frac{k\theta^2}{4}$. Once the volatility passes this point, the drift begins to decline, reaching 0 when the volatility process is θ. The further away the volatility process is from θ, the faster the mean reversion. In contrast to the linear drift models, the mean reversion speed remains the same. A negative K with a diffusion of $\sigma v^{1.5}$ causes the volatility process to explode. The density of Ahn/Gao stochastic volatility process is a form of the modified Bessel function of the first kind of order q.

The three single factor models discussed above exhibit the affine term structure property and are thus analytically tractable with closed form solutions for derivative prices.

Chapter 3

Credit Spreads Option Valuation

The continuous time model for the credit spreads in (1.1) are general Markovian models for all continuous time stochastic volatility models.

We assume we model under an equivalent martingale measure Q.

By Itô's Lemma, from (1.1)

$$dS(t) = S(t)\left(U\sigma_t^2 dt + \sigma_t dW_1(t)\right) \tag{3.1}$$

where $U = \pi_0 + \frac{1}{2}$.

We assume the Vasicek type short rate r has Q dynamics

$$dr(t) = (b - ar(t))dt + \sigma(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)) \tag{3.2}$$

where ρ is based on the assumption that there is a correlation between the short rate and the credit spread.

(W_1,W_2) are uncorrelated Brownian motions.

Our aim is to model the contingent claims with S as underlying.

The price process $\Pi(t;X)$ of any such integrable T contingent claim X is obtained with the risk neutral valuation technique by computing the Q expectation i.e. $\Pi(t;X) = E^Q[X \exp(-\int_t^T r(u)\,du) \mid F_t]$.

If we are interested in a contingent claim, $X = \Phi(r(T))$ with sufficiently smooth Φ then using the Feyman Kac formula, the arbitrage free price process is given by $\Pi(t;X) = F(t, r(t))$, where F is the solution of the partial differential equation, also known as the term structure equation.

$$F_t + (b - ar)\,F_r + \frac{\sigma^2}{2} F_{rr} - rF = 0 \tag{3.3}$$

with terminal condition $F(T, r) = \Phi(r)$ for all $r \in \mathbb{R}$.

For example zero bond prices with maturity T are given by $p(t,T) = F(t, r(t)); T)$, where F is a solution to (3.3) and terminal condition F(T,r;T) = 1.

The bond price process will be of the form

$p(t,T) = A(t,T)\exp(-B(t,T)r),\ 0 \leq t \leq T,$

where r follows the Vasicek short rate process in (3.2), with A(t,T) and B(t,T) deterministic functions.

We can find the process $p(t,T) = F(t, r(t)); T)$, by solving (3.3) with terminal condition F(T,r;T) = 1.

$\Rightarrow F(t,r;T) = A(t,T)\exp(-B(t,T)r)$

where

$$B(t,T) = \frac{1}{a}\left[1 - \exp\left(-a(T-t)\right)\right], \tag{3.4}$$

$$A(t,T) = \exp\left(\left[\left((T-t) - B(t,T) \right) \left[\frac{b}{a} - \frac{\sigma^2}{2a^2} \right] + \frac{\sigma^2}{4a} B^2(t,T) \right] \right). \tag{3.5}$$

See appendix A at end of chapter 3 on the derivation of $B(t,T)$ and $A(t,T)$.

Define $Z(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$, then $Z(t)$ is a Brownian Motion and the Vasicek short rate process in (3.2) can be specified as

$$dr(t) = (b - ar(t))\,dt + \sigma dZ(t). \tag{3.6}$$

Given that the bond price are of the form,

$p(t,T) = A(t,T)\exp\left(-B(t,T)r\right),\ 0 \le t \le T$, where r follows the Vasicek short rate process in (3.2), with A(t,T) and B(t,T) as deterministic functions, then Ito's formula implies

$dP(t,T) = P_t(t,T)dt + P_r(t,T)dr + \frac{1}{2}P_{rr}(t,T)d < r >.$

Thus

$$\begin{aligned}
dP(t,T) &= [A_t(t,T)\exp(-B(t,T)r) - rA(t,T)B_t(t,T)\exp(-B(t,T)r)]\,dt \\
&\quad - [A(t,T)B(t,T)\exp(-B(t,T)r)]\,dr \\
&\quad + \frac{1}{2}\left[A(t,T)B^2(t,T)\exp(-B(t,T)r)\right] d<r> \\
&= [A_t(t,T)\exp(-B(t,T)r) - rB_t(t,T)P(t,T)]\,dt - P(t,T)B(t,T)dr \\
&\quad + \frac{1}{2}B^2(t,T)P(t,T)d<r> \\
&= A_t(t,T)\exp(-B(t,T)r)\,dt - P(t,T) \\
&\quad \left[rB_t(t,T)dt + B(t,T)\left((b-ar(t))\,dt + \sigma dZ(t)\right) - \frac{1}{2}B^2(t,T)\sigma^2 dt\right] \\
&= A_t(t,T)\exp(-B(t,T)r)\,dt - P(t,T) \\
&\quad \left[\left(rB_t(t,T) - \frac{1}{2}B^2(t,T)\sigma^2 + B(t,T)(b-ar(t))\right)dt + \sigma B(t,T)dZ(t)\right].
\end{aligned}$$

From the derivation of $A(t,T)$ and $B(t,T)$ in appendix A of this chapter

$$\begin{aligned}
dP(t,T) &= A(t,T)\exp(-B(t,T)r)\left(bB(t,T) - \frac{\sigma^2}{2}B^2(t,T)\right)dt \\
&\quad - P(t,T)\left[(B_t(t,T)r - \frac{1}{2}B^2(t,T)\sigma^2 + bB(t,T) - aB(t,T)r)dt + \sigma B(t,T)dZ(t)\right. \\
&= P(t,T)\left(bB(t,T) - \frac{\sigma^2}{2}B^2(t,T) - rB_t(t,T) + \frac{1}{2}\sigma^2B^2(t,T) - bB(t,T)\right)dt \\
&\quad + P(t,T)\left(aB(t,T)rdt - \sigma B(t,T)dZ(t)\right) \\
&= P(t,T)\left((aB(t,T) - B_t(t,T))\,rdt - \sigma B(t,T)dZ(t)\right) \\
&= P(t,T)\left(rdt - \sigma B(t,T)dz(t)\right).
\end{aligned}$$

Hence the bond prices under the Vasicek short rate model are given by

$$dP(t,T) = P(t,T)\left(r(t)dt - \sigma B(t,T)\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right)\right). \tag{3.7}$$

3.1 Contingent Claim Pricing

We want to find the arbitrage price Π_G of $G(S(T))$ (G an appropriate function)

From chapter 8 of Bingham-Kiesel(Bingham and Kiesel 1998)

$$\begin{aligned}\Pi_g(t) &= B(t).E_Q\left[\frac{G}{B(T)}|F_t\right] \\ &= E_Q\left[G\exp(-\int_t^T r(s)\,ds)|F_t\right].\end{aligned}$$

Also $P(t,T) = E_Q\left[\exp(-\int_t^T r(s)\,ds)|F_t\right]$.

The risk neutral pricing formula is

$\Pi_g(t) = E_Q(\exp(-\int_t^T r(u)\,du)\ G(S(T))|F_t)$.

By the change of numéraire theorem we find

$\Pi_g(t) = P(t,T)E_{\tilde{Q}}\left[G(S(T))|F_t\right]$,

where the expectation is under $\tilde{Q}$ measure.

The change of measure is such that $P(t,T)$ is the new numeraire thus

$\tilde{S}(t) = \frac{S(t)}{P(t,T)}$ has to be a martingale.

By the product rule

$d\tilde{S}(t) = S(t).d(\frac{1}{P(t,T)}) + dS(t).\frac{1}{P(t,T)} + < S(t), \frac{1}{P(t,T)} >$.

Now

$dS(t) = S(t)\left(U\sigma_t^2 dt + \sigma_t.dW_1(t)\right)$ from (3.1) and

$dP(t,T) = P(t,T)\left(r(t)dt - \sigma B(t,T)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))\right)$ from (3.9).

We start with evaluating $d\left(\frac{1}{P(t,T)}\right)$.

By Ito's lemma, with $f = \frac{1}{x}$,

$(\Rightarrow\ f_t = 0,\ ,f_x = \frac{-1}{x^2}, f_{xx} = \frac{2}{x^3})$

we have

$df(t,x) = f_t.dt + f_x.dx + \frac{1}{2}f_{xx}.dx^2,$

$$\begin{aligned} d\left[\frac{1}{P(t,T)}\right] &= \frac{-1}{P(t,T)^2}.P(t,T)\left(r(t)dt - \sigma B(t,T)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))\right) \\ &\quad + \frac{2}{P(t,T)^3}.\frac{P(t,T)^2}{2}\left(\sigma^2 B^2(t,T)(\rho^2 + 1 - \rho^2)dt\right) \\ &= \frac{-1}{P(t,T)}\left(r(t)dt - \sigma B(t,T)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))\right) \\ &\quad + \frac{1}{P(t,T)}\left(\sigma^2 B^2(t,T)dt\right), \end{aligned}$$

$$\begin{aligned} < S(t), \frac{1}{P(t,T)} > &= S(t)\{\sigma_t.dW_1(t)\} \times \frac{\sigma B(t,T)}{P(t,T)}(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)) \\ &= \frac{S(t)}{P(t,T)}\sigma\sigma_t\rho B(t,T)dt, \end{aligned}$$

$$\begin{aligned} d\tilde{S}(t) &= \frac{S(t)}{P(t,T)}\left(U\sigma_t^2 dt + \sigma_t . dW_1(t)\right) \\ &-\frac{S(t)}{P(t,T)}\left(r(t)dt - \sigma B(t,T)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))\right) \\ &+\frac{S(t)}{P(t,T)}\left(\sigma^2 B^2(t,T)dt\right) + \frac{S(t)}{P(t,T)}\left(\sigma\sigma_t\rho B(t,T)dt\right), \end{aligned}$$

$$\begin{aligned} \Rightarrow d\tilde{S}(t) &= \tilde{S}(t)\left(\left(U\sigma_t^2 + \sigma^2 B^2(t,T) + \sigma\sigma_t\rho B(t,T) - r(t)\right)\right) dt \\ &+\tilde{S}(t)\left(\left(\sigma_t + \rho\sigma B(t,T)\right) dW_1(t) + \sigma B(t,T)\sqrt{1-\rho^2}dW_2(t)\right). \end{aligned}$$

Let $W = (W_1, W_2)$ be a 2-dimensional Brownian motion defined on a filtered probability space $(\Omega, F, Q, \tilde{F})$ if $\tilde{F} = (F_t)$ is the Brownian filtration, any pair of equivalent probability measures $\tilde{Q} \sim Q$ on $F = F_t$ is a Girsanov pair, i.e.

$$\left.\frac{d\tilde{Q}}{dQ}\right|_{F_t} = L(t)$$

with

$$L(t) = \exp\left(-\int_0^t(\lambda_1(u)\, dW_1(u) + \lambda_2(u)\, dW_2(u)) - \tfrac{1}{2}\int_0^t(\lambda_1^2(u) + \lambda_2^2(u))\, du\right)$$

where $(\lambda(t) : 0 \le t \le T)$ a measurable, adapted 2 dimensional process with $\int_0^T \lambda_i^2(t)\, dt < \infty$ a.s., $i = 1, 2$.

If L is a continuous local martingale which satisfies Novikov's condition:

$$E\left(\exp\{\tfrac{1}{2}\int_0^T(\|\lambda_1^2(u)\| + \|\lambda_2^2(u)\|)\, du\}\right) < \infty,$$

then L is a martingale with $E\left(L\left(T\right)\right)=1$, therefore we can change measure to $\tilde{Q}$. Then Girsanov's theorem implies

$\Rightarrow dW_1 = d\tilde{W_1} - \lambda_1(t)dt$

$\Rightarrow dW_2 = d\tilde{W_2} - \lambda_2(t)dt$

Substituting dW_1 and dW_2 above in $d\tilde{S}(t)$,

and grouping terms together

$$\begin{aligned}\Rightarrow d\tilde{S}(t) &= \tilde{S}(t)\left[\left(U\sigma_t^2+\sigma\sigma_t\rho B(t,T)+\sigma^2B^2(t,T)-r(t)\right)dt\right] \\ &\quad -\tilde{S}(t)\left[\left(\left(\sigma_t+\rho\sigma B(t,T)\right)\lambda_1(t)\right)dt\right] \\ &\quad -\tilde{S}(t)\left[\left(\sigma B(t,T)\sqrt{1-\rho^2}\lambda_2(t)\right)dt\right] \\ &\quad +\tilde{S}(t)\left[\left(\sigma_t+\rho\sigma B(t,T)\right)d\tilde{W_1}+\sigma B(t,T)\sqrt{1-\rho^2}d\tilde{W_2}\right].\end{aligned}$$

Under $\tilde{Q}$ risk neutral T forward measure, $\tilde{S}$ has to be a local $\tilde{Q}$ martingale, hence the drift coefficient has to be zero.

So

$$U\sigma_t^2+\sigma\sigma_t\rho B(t,T)+\sigma^2B^2(t,T)-r(t)-\sigma_t\lambda_1(t)-\sigma B(t,T)\left(\rho\lambda_1(t)+\sqrt{1-\rho^2}\lambda_2(t)\right)=0$$

so we choose $\lambda_1(t)$, $\lambda_2(t)$ for a unique equivalent martingale.

Let $\rho\lambda_1(t)+\sqrt{1-\rho^2}\lambda_2(t)=1$

$\Rightarrow \lambda_1(t) = \frac{\{U\sigma_t^2+\sigma\sigma_t\rho B(t,T)+\sigma^2B^2(t,T)-r(t)-\sigma B(t,T)\}}{\sigma_t}$ where $\sigma_t \neq 0$.

For the λ_2 solution, substitute λ_1 in

$\lambda_2 = \frac{(1-\rho\lambda_1(t))}{\sqrt{1-\rho^2}}$,where $-1 < \rho < 1$.

Hence under $\tilde{Q}$ measure

$$d\tilde{S}(t) = \tilde{S}(t)\left[(\sigma_t + \rho\sigma B(t,T))\,d\tilde{W}_1(t) + \sigma B(t,T)\sqrt{1-\rho^2}d\tilde{W}_2(t)\right] \quad (3.8)$$

without any effect on W_3 the Brownian motion for the stochastic volatility process.

3.2 Appendix A: Proof of Vasicek Zero bond price

The Vasicek bond price process is of the form

$F(t,r;T) = A(t,T)\exp(-B(t,T)r)$.

Taking partial derivatives of $F(t,r;T)$,

$$\begin{aligned} F_t &= A_t(t,T)\exp(-B(t,T)r) - A(t,T)B_t(t,T)r\exp(-B(t,T)r) \\ &= A_t(t,T)\exp(-B(t,T)r) - B_t(t,T)rF, \end{aligned}$$

$$\begin{aligned} F_r &= -A(t,T)B(t,T)\exp(-B(t,T)r) \\ &= -B(t,T)F, \end{aligned}$$

$$\begin{aligned} F_{rr} &= A(t,T)B^2(t,T)\exp(-B(t,T)r) \\ &= B^2(t,T)F. \end{aligned}$$

Substituting the partial derivatives above into the term-structure equation (3.3),

$A_t(t,T)\exp(-B(t,T)r) - B_t(t,T)rF - (b-ar)\,B(t,T)F + \frac{\sigma^2}{2}B^2(t,T)F -$

$rF = 0$

$\Rightarrow \frac{A_t(t,T)F}{A(t,T)} - B_t(t,T)rF - (b - ar)\,B(t,T)F + \frac{\sigma^2}{2}B^2(t,T)F - rF = 0.$

$\frac{A_t(t,T)}{A(t,T)} - B_t(t,T)r - (b - ar)\,B(t,T) + \frac{\sigma^2}{2}B^2(t,T) - r = 0$

$\Rightarrow \frac{A_t(t,T)}{A(t,T)} - bB(t,T) + \frac{\sigma^2}{2}B^2(t,T) - (B_t(t,T) + 1 - aB(t,T))\,r = 0.$

$A(T,T) = 1$ and $B(T,T) = 0$ satisfies terminal condition $F(t,r;T) = 1$

if $r = 0$ or $1 + B_t(t,T) - aB(t,T) = 0, B(T,T) = 0.$

$$\frac{A_t(t,T)}{A(t,T)} - bB(t,T) + \frac{\sigma^2}{2}B^2(t,T) = 0,\ A(T,T) = 1. \tag{3.9}$$

We solve $B_t(t,T) - aB(t,T) + 1 = 0,$ where $B(T,T) = 0$

$$B_t(t,T) - aB(t,T) = -1. \tag{3.10}$$

Multiplying through by the integrating factor

$$\exp\left(a(T-t)\right)B_t(t,T) - a\exp\left(a(T-t)\right)B(t,T) = -\exp\left(a(T-t)\right)$$

gives

$$d(B(t,T)\exp\left(a(T-t)\right) = -\exp\left(a(T-t)\right),$$

so

$$\Rightarrow B(t,T)\exp\left(a(T-t)\right) = -\int \exp\left(a(T-t)\right),$$

or

$$\begin{aligned} B(t,T)\exp(a(T-t)) &= K + \frac{1}{a}\left[\exp\left(a(U-t)\right)\right]_t^T \\ &= K + \frac{1}{a}\left[\exp\left(a(T-t)\right) - 1\right]. \end{aligned}$$

When $t = T$, $0 = k + 0$, so $k = 0$, and

$$B(t,T)\exp\left(a(T-t)\right) = \frac{1}{a}\left[\exp\left(a(T-t)\right) - 1\right],$$

$$B(t,T) = \frac{1}{a}\left[\exp\left(a(T-t)\right) - 1\right]\exp\left(-a(T-t)\right),$$

$$B(t,T) = \frac{1}{a}\left[1 - \exp\left(-a(T-t)\right)\right], \tag{3.11}$$

$$\exp\left(-a(T-t)\right) = 1 - aB(t,T). \tag{3.12}$$

from (3.4) $\frac{A_t(t,T)}{A(t,T)} = bB(t,T) - \frac{\sigma^2}{2}B^2(t,T)$

$\Rightarrow \int_t^T \frac{A_t(s,T)}{A(s,T)}\,ds = K + b\int_t^T B(s,T)\,ds - \frac{\sigma^2}{2}\int_t^T B^2(s,T), ds$

$\Rightarrow \ln A(t,T) = K + b\int_t^T B(s,T)\,ds - \frac{\sigma^2}{2}\int_t^T B^2(s,T), ds$

when $t = T$, $A(T,T) = 1$, $\Rightarrow K = 0$.

$$
\begin{aligned}
\ln A(t,T) &= b\int_t^T B(s,T)\,ds - \frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds \\
&= b\int_t^T \frac{1}{a}\left[1-\exp\left(-a(T-s)\right)\right]ds - \frac{\sigma^2}{2a^2}\int_t^T \left(1-\exp\left(-a(T-s)\right)\right)^2 ds \\
&= \frac{b}{a}\int_t^T \left[1-\exp\left(-a(T-s)\right)\right]ds - \frac{\sigma^2}{2a^2}\int_t^T \left(1-\exp\left(-a(T-s)\right)\right)^2 ds \\
&= \frac{b}{a}\left[s-\frac{1}{a}\exp\left(-a(T-s)\right)\right]_t^T \\
&\quad -\frac{\sigma^2}{2a^2}\left[s-\frac{2}{a}\exp\left(-a(T-s)\right)+\frac{1}{2a}\exp\left(-2a(T-s)\right)\right]_t^T \\
&= \frac{b}{a}\left[T-\frac{1}{a}-\left(t-\frac{1}{a}\exp\left(-a(T-t)\right)\right)\right] \\
&\quad -\frac{\sigma^2}{2a^2}\left[T-\frac{2}{a}+\frac{1}{2a}-\left(t-\frac{2}{a}\exp\left(-a(T-t)\right)+\frac{1}{2a}\exp\left(-2a(T-t)\right)\right)\right] \\
&= \frac{b}{a}\left[(T-t)-\frac{1}{a}\left(1-\exp\left(-a(T-t)\right)\right)\right] \\
&\quad -\frac{\sigma^2}{2a^2}\left[(T-t)-\frac{2}{a}\left(1-\exp\left(-a(T-t)\right)\right)+\frac{1}{2a}\left(1-\exp\left(-2a(T-t)\right)\right)\right].
\end{aligned}
$$

Also from (3.12) $\exp\left(-2a(T-t)\right) = 1 - 2aB(t,T) + a^2B^2(t,T)$

Substituting (3.11) and (3.12) for $\ln A(t,T)$ above

$$\begin{aligned}
\ln A(t,T) &= \frac{b}{a}\left[(T-t)-B(t,T)\right]-\frac{\sigma^2}{2a^2}\left[(T-t)-2B(t,T)+\frac{1}{2a}(2aB(t,T)-a^2B^2(t,T))\right]\\
&= \frac{b}{a}\left[(T-t)-B(t,T)\right]-\frac{\sigma^2}{2a^2}\left[(T-t)-2B(t,T)+B(t,T)-\frac{aB^2(t,T)}{2}\right]\\
&= ((T-t)-B(t,T))\left[\frac{b}{a}-\frac{\sigma^2}{2a^2}\right]+\frac{\sigma^2}{4a}B^2(t,T),
\end{aligned}$$

or

$$A(t,T)=\exp\left(\left[((T-t)-B(t,T))\left[\tfrac{b}{a}-\tfrac{\sigma^2}{2a^2}\right]+\tfrac{\sigma^2}{4a}B^2(t,T)\right]\right).$$

Chapter 4

Closed form Pricing of Credit Spread Call

We now focus on a European credit spread call option G on S with maturity

T and strike K

So $G(S(T)) = (S(T) - K)^+$.

The risk-neutral pricing formula gives

$\Pi_c(t) = P(t,T).E_Q\left((S(T) - K)^+ \,|F_t\right)$.

Under the $\tilde{Q}$ measure

$\Pi_c(t) = P(t,T).E_{\tilde{Q}}\left(\left(\tilde{S}(T) - K\right)^+ |F_t\right)$.

By the law of iterated conditioning,

$$E_{\tilde{Q}}\left(\left(\tilde{S}(T) - K\right)^+ |F_t\right) = E_{\tilde{Q}}\left(E_{\tilde{Q}}\left(\left(\tilde{S}(T) - K\right)^+ |(\sigma_u)_{t\le u\le T}\right) |F_t\right),$$

conditional on $(\sigma_u)_{t\le u\le T}$ realised volatility,

From (3.8)

$$\tilde{S}(t) = \tilde{S}(0) \exp\left(\xi(t) - \frac{1}{2}v^2(t)\right) \tag{4.1}$$

where

$\xi(t) = \int_0^t (\sigma_u + \sigma B(u,T)\rho)\, d\tilde{W}_1(u) + \int_0^t \sigma B(u,T)\sqrt{1-\rho^2}\, d\tilde{W}_2(u).$

So $\tilde{S}(t)$ is a Gaussian variable with mean $= 0$ and variance

$$v^2 = \int_0^t (\sigma_u + \sigma B(u,T)\rho)^2\, du + \int_0^t \sigma^2 B^2(u,T)(1-\rho^2)\, du. \tag{4.2}$$

4.1 General Pricing Formula

$$\begin{aligned}\Pi_c(t) &= P(t,T).E\left(\left(\tilde{S}(T) - K\right)^+ |F_t\right) \\ &= P(t,T).E_{\tilde{Q}}\left(E_{\tilde{Q}}\left(\left(\tilde{S}(T) - K\right)^+ |(\sigma_u)_{t\le u\le T}\right) |F_t\right),\end{aligned}$$

by iteration of conditional expectation

Substituting (4.1) in $\Pi_c(t)$ above

$\Pi_c(t) = E_{\tilde{Q}}\left[P(t,T).\tilde{S}(0)\exp\left(\xi(T) - \frac{1}{2}v^2(T)\right) - KP(t,T)|F_t\right].$

Here $\tilde{Q}$ denotes the risk neutral T forward measure, and P(t,T) the price process of a zero coupon bond.

The expectation is separated into two terms:

$\Pi_c(t) = E_1 - E_2$,

where $E_1 = E_{\tilde{Q}}\left[P(t,T)\tilde{S}(0)\exp\left(\xi(T) - \frac{1}{2}v^2(T)\right).1_{\tilde{S}(T)>K}\right]$

and

$$\begin{aligned}
E_2 &= KP(t,T)\tilde{Q}\left(\tilde{S}(T) > K\right) \\
&= KP(t,T)\tilde{Q}\left(\tilde{S}(0)\exp\left(\xi(T) - \frac{1}{2}v^2(T)\right) > K\right) \\
&= KP(t,T)\tilde{Q}\left(\xi(T) - \frac{1}{2}v^2(T) > \ln\left(\frac{K}{\tilde{S}(0)}\right)\right) \\
&= KP(t,T)\tilde{Q}\left[\xi(T) > \ln\left(\frac{K}{\tilde{S}(0)}\right) + \frac{1}{2}v^2(T)\right] \\
&= KP(t,T)\tilde{Q}\left[-\xi(T) < \ln\left(\frac{\tilde{S}(0)}{K}\right) - \frac{1}{2}v^2(T)\right].
\end{aligned}$$

Since $\tilde{S}(0) = \frac{S(0)}{P(0,T)}$ we have

$E_2 = KP(t,T)\tilde{Q}\left[-\xi(T) < \ln\left(\frac{S(0)}{KP(0,T)}\right) - \frac{1}{2}v^2(T)\right]$.

Divide $\xi(T)$ by $\sqrt{v^2(T)}$ to normalise to $N(0,1)$.

$$\begin{aligned}
E_2 &= KP(t,T)\tilde{Q}\left(\frac{-\xi(T)}{v(T)} < \frac{\ln(\frac{S(0)}{KP(0,T)}) - \frac{1}{2}v^2(T)}{v(T)}\right) \\
&= KP(t,T)N\left(-\left(\frac{\ln(\frac{S(0)}{KP(0,T)}) - \frac{1}{2}v^2(T)}{v(T)}\right)\right) \\
&= KP(t,T)N(d),
\end{aligned}$$

where

$$d = -\left(\frac{\ln(\frac{S(0)}{KP(0,T)}) - \frac{1}{2}v^2(T)}{v(T)}\right). \tag{4.3}$$

For $E_1 = E_{\tilde{Q}}\left[P(t,T)\tilde{S}(0)\exp(\xi(T) - \frac{1}{2}v^2(T)).1_{\tilde{S}(T)>K}\right]$

$\xi(t)$ is a normal random variable with law $N(0, {v_t}^2)$.

By normalising we obtain $\tilde{Z} = \frac{\xi(T)}{v(T)}$ which has law $N(0,1)$.

$$\begin{aligned} E_1 &= P(t,T)\int_{-\infty}^{d} \tilde{S}(0).\exp\left(-\frac{1}{2}v^2(T) + \tilde{Z}v(T)\right).\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\tilde{Z}^2)d\tilde{Z} \\ &= P(t,T)\int_{-\infty}^{d} \tilde{S}(0).\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\tilde{Z} - v(T)\right)^2\right)d\tilde{Z}. \end{aligned}$$

Compute E_1 by letting $U = \tilde{Z} - v(T)$

$\Rightarrow E_1 = P(t,T)\int_{-\infty}^{d-v(T)} \tilde{S}(0).\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}U^2\right)dU.$

d depends on the evaluation of the indicator function

$\Rightarrow E_1 = P(t,T)\tilde{S}(0).N\left(d - v(T)\right).$

Substituting the definition of d as in (4.3) in E_1

$$\begin{aligned} \Rightarrow E_1 &= P(t,T)\tilde{S}(0).N\left(-\frac{\ln(\frac{S(0)}{KP(0,T)}) + \frac{1}{2}v^2(T)}{v(T)} - v(T)\right) \\ &= P(t,T)\tilde{S}(0).N\left(-\frac{\ln(\frac{S(0)}{KP(0,T)}) - \frac{1}{2}v^2(T)}{v(T)}\right). \end{aligned}$$

Therefore

$\Pi_c(t) = \frac{P(t,T).S(0)}{P(0,T)}.N\left(-\left[\frac{\ln(\frac{S(0)}{KP(0,T)})+\frac{1}{2}v^2(T)}{v(T)}\right]\right)-KP(t,T)N\left(\left[-\frac{\ln(\frac{S(0)}{KP(0,T)})-\frac{1}{2}v^2(T)}{v(T)}\right]\right).$

So for a European Call on credit spread

$\Pi_c(0) = E\left(\Pi_{c,\sigma}(0)|(\sigma_t)\ 0 \leq t \leq T\right).$

let $\bar{v^2} = \frac{v^2(T)}{T}$ be average variance

$$\Rightarrow \Pi_c(0) = E\left(C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v^2}}, T\right)|\bar{v^2}\right), \tag{4.4}$$

where C_{BS} is the Black-Scholes formula for a European Call Option.

From (4.2) we need to find the distribution of Average variance $\bar{v^2}$

where $v^2(T) = \int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\ du + \int_0^T \sigma^2 B^2(u,T)\left(1-\rho^2\right) du.$

$v^2(T)$ depends on σ_u^2 a mean reverting volatility process of the SDE form in (1.2), (1.3) or (1.4).

The distribution of $\bar{v}^2$ can be obtained from the inversion of the moment generating function of the average variance.

One obtains that the moment generating function of the average variance, $\bar{v^2}$ i.e. $\frac{v^2}{T}$ is $I(\lambda)$,

where

$$\begin{aligned}
I(\lambda) &= E\left(\exp\left(-\frac{\lambda v^2(T)}{T}\right)\right) \\
&= E\left(\exp\left(-\lambda \bar{v^2}\right)\right) \\
&= E\left(\exp\left(\frac{-\lambda}{T}\int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\,du + \frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\,du\right)\right) \\
&= \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\,du\right)\; E\left(\exp\frac{-\lambda}{T}\int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\,du\right),
\end{aligned}$$

since $\sigma^2 B^2(t,T)\,(1-\rho^2)$ is deterministic, see $B(t,T)$ as in (3.4).

Chapter 5

European Call Vasicek Stochastic Volatility Model

5.1 Analytic Credit Spread Call under no correlation

In this section we derive the closed form formula of a European call on the credit spread under the assumption of zero correlation between the credit spread and the short rate.

We model the short rate and the stochastic volatility of the spread as a Vasicek process.

We know from chapter 4 that the moment generating function of the average variance is given by

$$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\,du\right)\ E\left(\exp\frac{-\lambda}{T}\int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\,du\right). \tag{5.1}$$

Let $\rho = 0$, then the moment generating function of the average variance from (5.1) is

$$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\,du\right) E\left(\exp\frac{-\lambda}{T}\int_0^T \sigma_u^2\,du\right), \qquad (5.2)$$

where $\sigma^2 B^2(u,T)$ is deterministic and the SDE for σ_t^2 is of the Vasicek form as in (1.2).

i.e. $d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)dt + a_1 dW_3(t)$.

We now find a closed form solution for σ_t^2 using L.C.G Rogers[1995] paper.

Let $K_t = \int_0^t \theta\,du$,

$\Rightarrow K_t = \theta t$.

Multiplying the SDE for σ_t^2 in (1.2) by $\exp(K_t)$, i.e. $\exp(\theta t)$, and using the product rule,

$$\begin{aligned} d\left(\exp(\theta t)\,\sigma_t^2\right) &= d\left(\exp(\theta t)\right).\sigma_t^2 + \exp(\theta t)\,.d\sigma_t^2 + <\sigma_t^2, \exp(\theta t)> \\ &= \theta\exp(\theta t)\,.\sigma_t^2 dt + \exp(\theta t)\left[\theta c dt - \theta\sigma_t^2 dt + a_1 dW_3(t)\right] \\ &= \exp(\theta t)\left[a_1 dW_3(t) + \theta c dt\right]. \end{aligned}$$

Integrating $d\left(\exp\left(\theta t\right)\sigma_t^2\right)$ above,

$$\sigma_t^2 = \exp\left(-\theta t\right)\left[\sigma_0^2 + \int_0^t \exp\left(\theta u\right)\left(a_1\, dW_3(u) + \theta c\, du\right)\right]. \qquad (5.3)$$

Differentiating equation (5.3),

$$\begin{aligned} d(\sigma_t^2) &= -\theta \exp\left(-\theta t\right)\left[\sigma_0^2 + \theta c \int_0^t \exp\left(\theta u\right)\, du + \int_0^t \exp\left(\theta u\right) a_1\, dW_3(u)\right] dt \\ &\quad + \exp\left(-\theta t\right)\left[\theta c \exp\left(\theta t\right) dt + \exp(\theta t) a_1\, dW_3(t)\right] \end{aligned}$$

$\Rightarrow d\sigma_t^2 = \theta\left(c - \sigma_t^2\right) dt + a_1 dW_3(t).$

Hence equation (5.3) is the solution to $d\sigma_t^2 = \theta\left(c - \sigma_t^2\right) dt + a_1 dW_3(t)$ from (1.2).

From L.C.G Rogers[1995],

$\sigma_t^2 = \exp\left(-\theta t\right)\left[\sigma_0^2 + \int_0^t \exp\left(\theta u\right)\left(a_1\, dW_3(u) + \theta c\, du\right)\right]$

is a Gaussian process with

mean

$\mu_t = E\left(\sigma_t^2\right) = \exp\left(-\theta t\right)\left[\sigma_0^2 + \int_0^t \exp\left(\theta u\right)\theta c, du\right]$

and covariance

$\rho(s,t) = cov(\sigma_s^2, \sigma_t^2).$

By Itô's Product rule, the covariance is

$\rho(s,t) = \exp(-\theta t - \theta s) \int_0^{s \wedge t} \exp(2\theta u)\, a_1^2 \, du.$

Thus $\int_0^t \sigma_u^2 \, du$ has Normal Distribution $N(m(t), \nu(t))$, where

$$\begin{aligned} m(t) &= \int_0^t \exp(-\theta u) \left(\sigma_0^2 + \int_0^u \exp(\theta s)\, \theta c, ds \right) du \\ &= \int_0^t \exp(-\theta u) \left(\sigma_0^2 + c\left(\exp(\theta u) - 1\right)\right) du \\ &= \int_0^t \exp(-\theta u) \left(\sigma_0^2 - c \right) du \\ &= ct - \left(\frac{1}{\theta} \left(\sigma_0^2 - c\right) \left(\exp(-\theta t) - 1\right) \right), \end{aligned}$$

and the variance $\nu(t)$ is obtained by setting $t = s$ in $cov(\sigma_s^2, \sigma_t^2)$ to obtain

$$
\begin{aligned}
\nu_t &= 2\int_0^t du \int_0^u ds \int_0^s a_1^2 \exp(2\theta y)\exp(-\theta s)\exp(-\theta u)\ dy \\
&= 2a_1^2 \int_0^t du \int_0^u ds \exp(-\theta s - \theta u)\left[\frac{1}{2\theta}\exp(2\theta y)\right]_0^s \\
&= \frac{a_1^2}{\theta}\int_0^t du \int_0^u \exp(-\theta s - \theta u)\left[\exp(2\theta s) - 1\right] ds \\
&= \frac{a_1^2}{\theta}\int_0^t du \int_0^u \exp(\theta s - \theta u) - \exp(-\theta s - \theta u)\ ds \\
&= \frac{a_1^2}{\theta}\int_0^t \exp(-\theta u)\left[\frac{1}{\theta}\exp(\theta s) + \frac{1}{\theta}\exp(-\theta s)\right]_0^u du \\
&= \frac{a_1^2}{\theta}\int_0^t \exp(-\theta u)\left[\frac{1}{\theta}\exp(\theta u) + \frac{1}{\theta}\exp(-\theta u) - \frac{2}{\theta}\right] du \\
&= \frac{a_1^2}{\theta}\int_0^t \frac{1}{\theta} + \frac{1}{\theta}\exp(-2\theta u) - \frac{2}{\theta}\exp(-\theta u)\ du \\
&= \frac{a_1^2}{\theta}\left[\frac{u}{\theta} - \frac{\exp(-2\theta u)}{2\theta^2} + \frac{2}{\theta^2}\exp(-\theta u)\right]_0^t \\
&= \frac{a_1^2}{\theta}\left[\frac{t}{\theta} - \frac{\exp(-2\theta t)}{2\theta^2} + \frac{2}{\theta^2}\exp(-\theta t) + \frac{1}{2\theta^2} - \frac{2}{\theta^2}\right] \\
&= \frac{a_1^2}{\theta}\left[\frac{2\theta t - \exp(-2\theta t) + 4\exp(-\theta t) + 1 - 4}{2\theta^2}\right] \\
&= \frac{a_1^2}{2\theta^3}\left[2\theta t - \exp(-2\theta t) + 4\exp(-\theta t) - 3\right].
\end{aligned}
$$

From (5.2) we evaluate

$E\left(\exp\left(\frac{-\lambda}{T}\int_0^T \sigma_u^2\, du\right)\right)$,

where σ_u^2 is the solution to the Vasicek SDE in (1.2).

Let $I^*(\lambda) = E\left(\exp\left(\frac{-\lambda}{T}\int_0^T \sigma_u^2\, du\right)\right)$.

We already know that

$\int_0^t \sigma_u^2\, du \sim N(m(t), \nu(t))$,

$\Rightarrow \frac{-\lambda}{T}\int_0^t \sigma_u^2\, du \sim N\left(\frac{-\lambda}{T}m(t), \frac{\lambda^2}{T^2}\nu(t)\right)$.

Hence

$$\begin{aligned} I^*(\lambda) &= E\left(\exp\left(\frac{-\lambda}{T}\int_0^T \sigma_u^2\, du\right)\right) \\ &= \exp\left(\frac{-\lambda}{T}m(T) + \frac{1}{2}\frac{\lambda^2}{T^2}\nu(t)\right), \end{aligned}$$

which is the moment generating function of $N\left(\frac{-\lambda}{T}m(T), \frac{\lambda^2}{T^2}\nu(t)\right)$.

Hence the moment generating function for $I(\lambda)$ from (5.2) is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\, du\right)\exp\left(\frac{-\lambda}{T}m(T) + \frac{1}{2}\frac{\lambda^2}{T^2}\nu(t)\right)$.

We can express this moment generating function as $I(\lambda) = E\left(\exp\left(-\lambda\bar{v}\right)\right)$

where $\bar{v}$ is the average variance.

Hence $I(\lambda) = \int_{-\infty}^{\infty} \exp\left(-\lambda\bar{v}\right) m(\bar{v})\, d\bar{v}$.

The distribution of the average variance is obtained by the classical inversion theorem, written as

$m(\bar{v}) = \frac{1}{2\Pi i}\int_{c-i\infty}^{c+i\infty} \exp(\lambda\bar{v})\, I(\lambda)\, d\lambda$ for $\bar{v} \in [0, \infty)$.

This integral is also known as the Bromwich Mellin contour integral, where c is a vertical contour in the complex plane chosen so that all singularities of $m(\bar{v})$ are to the left of it.

Make the change of variables $\lambda = c + iz$ in the above integral to reduce it to the Laplace inversion integral:

$$\begin{aligned} m(\bar{v}) &= \frac{\exp(c\bar{v})}{2\Pi}\int_{-\infty}^{\infty} \exp(-i\bar{v}z)\, I(c+iz)\, dz \\ &= \frac{\exp c\bar{v}}{2\Pi}\int_{-\infty}^{\infty} \left[\Re\left(I(c+iz)\right)\cos\bar{v}z - \Im\left(I(c+iz)\right)\sin\bar{v}z\right]\, dz \end{aligned}$$

where $\Re\left(I(c+iz)\right)$ and $\Im\left(I(c+iz)\right)$ are real and imaginary parts of $I(c+iz)$ respectively.

The Abate and Whitt numerical inversion method reduces the Laplace inversion integral to give us the density of the average variance as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi}\int_0^{\infty} \Re\left(I(c+iz)\right)\cos\bar{v}z\, dz.$

Hence the European call on credit spread, where the correlation between the credit spread and the short rate is zero under a Vasicek stochastic volatility process, is

$$\begin{aligned}\Pi_c(0) &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ m(\bar{v})\ d\bar{v} \\ &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ \left(2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos \bar{v}z\, dz\right)\ d\bar{v},\end{aligned}$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

The numerical inversion techniques can be performed using techniques from Mark Craddock, David Heath and Eckhard Platen[2000].

5.2 Numerical Credit Spread Call Price under no Correlation

We price numerically the European call on credit spreads with no correlation between the credit spread and the Vasicek short rate under Vasicek stochastic volatility by Monte Carlo simulation. The Vasicek variance process is simulated over a thousand random samples of the standard normal distribution. An average variance is calculated and substituted in the Black Scholes call price formula to obtain a simulated call price.

As the convergence rate of the Monte Carlo estimate is $\frac{1}{\sqrt{N}}$, the variance reduction by the antithetic method is used to increase precision and speed up the numerical computation. The method of antithetic variates is based on the observation that if sample ϵ is N(0,1) so is $-\epsilon$. An average variance is then calculated from the anithetic standard normal variables $(\epsilon, -\epsilon)$.

About 100 simulated call prices are obtained for antithetic pairs of average variances $(\bar{V_j}, \bar{V_k})$. Two sets of average call prices are obtained and the best estimate of the call price is the mean of the 2 average call prices.

From (1.2), σ_t the volatility process is of the Vasicek form

$d\sigma_t^2 = \theta \left(c - \sigma_t^2\right) dt + a_1 dW_3(t)$

Let $V_t^* = \sigma_t^2$

$\Rightarrow dV_t^* = \theta \left(c - V_t^*\right) dt + a_1 dW_3(t)$.

For much better accuracy we simulate the closed form solution of the above Vasicek SDE

$$V_{i+1,j}^* = V_{i,j}^* \exp\left(-\theta \triangle t\right) + \theta \left(c - V_{i,j}^*\right) \triangle t + a_1 \epsilon_i \sqrt{\frac{1 - exp\left(-2\theta \triangle t\right)}{2\theta}}, \quad (5.4)$$

where ϵ is random sample from N(0,1).

Given a 1000 steps of interval $\triangle t$ we end up with total time T, where

$T = \triangle t L$ and $\triangle t$ is the time interval and L is the number of steps i.e. 1000 in this case.

We calculate a 1000 $V^*_{i,j}$ by simulating from the random sample $\epsilon_i \sim N(0,1)$ in equation (5.4).

For each ϵ_i, we generate the $-\epsilon_i$ sample values, which is used to calculate another 1000 $V^*_{i,k}$ using

$$V^*_{i+1,k} = V^*_{i,k} \exp\left(-\theta \triangle t\right) + \theta \left(c - V^*_{i,k}\right) \triangle t - a_1 \epsilon_i \sqrt{\frac{1 - exp\left(-2\theta \triangle t\right)}{2\theta}}. \quad (5.5)$$

We obtain two sets of average variances:

$\bar{V_j} = \frac{1}{L} \sum_{j=1}^{L} V^*_{i,j}$ from ϵ_i random samples of N(0,1),

$\bar{V_k} = \frac{1}{L} \sum_{k=1}^{L} V^*_{i,k}$ from $-\epsilon_i$ random samples of N(0,1),

where $L = 1000$ and for each SDE we use the Riemann sum approximation of the integral. We calculate 100 $\bar{V_j}$ and calculate the average call price as $y_1 = \frac{1}{n} \sum_{i=1}^{n} C_{BS}(\sqrt{\bar{V_i}})$

where $n = 100$

Also calculate 100 $\bar{V_k}$ and calculate the average call price as $y_2 = \frac{1}{n} \sum_{k=1}^{n} C_{BS}(\sqrt{\bar{V_k}})$

where $n = 100$

N.B in equations (5.4) and (5.5) all instances of negative $\bar{V_j}$ or $\bar{V_k}$ are discarded.

Hence the best estimate of the European call price is $\frac{(y_1+y_2)}{2}$.

5.3 Analysis of Closed/Numerical Call Prices under no Correlation

In this section we carry out a sensitivity analysis of the closed form and the Monte Carlo simulation of European call credit spread prices for zero correlation between the credit spread and the Vasicek short rate under the Vasicek stochastic volatility process

$d\sigma_t^2 = \theta\,(c - \sigma_t^2)\,dt + a_1 dW_3(t)$

where θ is the speed of mean reversion, c is the long term mean, a_1 is the volatility of variance and σ_t^2 at $t = 0$, the initial variance.

See Appendix A at the end of this chapter for a table of closed form and Monte Carlo credit spread call option prices.

For the closed form solution, we compute the credit spread option prices for contour values $c = 0.05$ with inner upper bound of average variance density integral set to 25 and outer bound integral set to 60. See Ted Huddleston(1999) on numerical inversion of Laplace transforms on the choice of contour values and the inner upper boundary of the average variance density integral. The average computation time is 20 seconds. The use of larger contour values or average density integral upper bounds increases computational time over 5 minutes which we find unacceptable for a closed form solution. This increase in computation time is due to the large terms obtained in the expansion of the call spread option price integral. The optimum choice of the upper bound of the outer integral is non trivial as our Maple computation package cannot compute option price when it is set to infinity. We choose upper bound of 60 for which the double integral credit spread price formula converges to the Monte Carlo simulation prices.

For Monte Carlo simulation, the variance reduction method of antithetic variates is employed which increases our computation time by 2 minutes to 27 minutes on average, but does not improve the price accuracy by much. For convenience we set the initial variance in the Monte Carlo Vasicek equation to the long term mean.

We investigate the effects of stochastic volatility mean reversion parameters and no correlation between the credit spread and the short rate on the credit spread option prices obtained from either analytic form or numerical form.

For the numerical form increasing the mean reversion reduces the credit spread price as the variance decreases. Increasing the long term mean, initial variance or volatility of variance increases the credit spread option price. For the closed form, increasing the long term mean, the initial variance or volatility of variance increases the credit spread option price. An increase in the mean reversion reduces the credit spread option price due to the variance increasing.

The difference in prices between closed form and numerical form prices is within the convergence rate of the Monte Carlo estimate $\frac{1}{\sqrt{N}}$.

5.4 Analytic Credit Spread Call with correlation.

In this section we derive the closed form formula of the European call credit spread price for the case where there is correlation between the credit spread and the short rate. Again we model the volatility and the short rate with a Vasicek process.

From (4.2) we need to compute the dynamics of the process

$$\tilde{X}_t = (\sigma_t + \sigma\rho B(t,T))^2$$

where the dynamics of the volatility process is given by

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)dt + a_1 dW_3(t).$

Although the dynamics of the process $\tilde{X}_t$ can be obtained using Itô's lemma there is little hope of obtaining an explicit expression for its distribution. We therefore use as our analytical proxy the standard Vasicek type SDE

$$dX_t \quad = \quad \theta\left(c - X_t\right)dt + a_1 dW_t$$

with initial condition $X_0 = \sigma_0 + \sigma\rho B(0,T)$.

So as in the no-correlation setting, one can compute an analytical solution, which we compare with the numerical results.

Using L.C.G Rogers[1995] approach as in previous section, we now find a closed form solution for X_t as

$X_t = \exp\left(-\theta t\right)\left[X_0 + \int_0^t \exp\left(\theta u\right)\left(a_1\, dW_3(u) + \theta c\, du\right)\right]$

X_t is a Gaussian process with

mean

$\mu_t = E\left(X_t\right) = \exp\left(-\theta t\right)\left[X_0 + \int_0^t \exp\left(\theta u\right)\theta c, du\right]$

and covariance

$\rho(s,t) = cov(X_s, X_t)$

By Itô's Product rule, the covariance is

$\rho(s,t) = \exp(-\theta t - \theta s) \int_0^{s\wedge t} \exp(2\theta u)\, a_1^2\, du.$

Thus $\int_0^t X_u^2\, du$ has Normal Distribution $N(m(t), \nu(t))$ where

$$\begin{aligned} m(t) &= \int_0^t \mu_u\, du \\ &= \int_0^t \exp(-\theta u) \left[X_0 + \int_0^u \exp(\theta s)\, \theta c, ds \right] du \\ &= \int_0^t \exp(-\theta u) \left(X_0 + c(\exp(\theta u) - 1) \right) du \\ &= \int_0^t \exp(-\theta u) \left((\sigma_0 + \sigma\rho B(0,T))^2 + c(\exp(\theta u) - 1) \right) du \\ &= ct - \left(\frac{1}{\theta} \left((\sigma_0 + \sigma\rho B(0,T))^2 - c \right) (\exp(-\theta t) - 1) \right) \end{aligned}$$

and the variance $\nu(t)$ is obtained by setting $t = s$ as in the previous

section to obtain

$\nu_t = \frac{a_1^2}{2\theta^3} \left[2\theta t - \exp(-2\theta t) + 4\exp(-\theta t) - 3 \right].$

Hence the moment generating function $I(\lambda)$ for the correlation case from

(5.1) is

$I(\lambda) = \exp\left(\frac{-\lambda}{T} \int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\, du \right) \ \exp\left(\frac{-\lambda}{T} m(T) + \frac{1}{2}\frac{\lambda^2}{T^2}\nu(t) \right).$

We apply the Abate and Whitt numerical inversion method as in the previous section to give us the density of the average variance as $m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz$,
where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.
Hence the European call on credit spread, for correlation between the credit spread and the short rate under a Vasicek stochastic volatility process, is

$$\begin{aligned}
\Pi_c(0) &= \int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ \left(2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz\right)\ d\bar{v},
\end{aligned}$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

5.5 Numerical Credit Spread Call Price with Correlation

We price numerically the European call credit spreads with non zero correlation between the credit spread and the short rate by Monte Carlo simulation.

We model the short rate with the SDE as in (3.2)

$dr(t) = (b - ar(t))\, dt + \sigma \left(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)\right),$

where ρ is the correlation between the short rate and the credit spread.

We also assume (1.2), so σ_t the volatility process is of the Vasicek form.

$d\sigma_t^2 = \theta\,(c - \sigma_t^2)\, dt + a_1 dW_3(t).$

Let $V_t^* = \sigma_t^2$.

Then $dV_t^* = \theta\,(c - V_t^*)\, dt + a_1 dW_3(t)$.

We also evaluate the Vasicek SDE using an Euler scheme,

$$V_{i+1,j}^* = V_{i,j}^* \exp\left(-\theta \triangle t\right) + \theta \left(c - V_{i,j}^*\right) \triangle t + a_1 \epsilon_{1i} \sqrt{\frac{1 - exp\left(-2\theta \triangle t\right)}{2\theta}}, \quad (5.6)$$

where ϵ_{1i} is a random sample from N(0,1).

From (3.1) the credit spread SDE is of the form

$dS(t) = S(t)\left(U\sigma_t^2 dt + \sigma_t dW_1(t)\right),$

where $U = \pi_0 + \frac{1}{2}$.

We simulate the credit spread SDE by

$$S_{i+1,j} = S_{i,j} + S_{i,j}\left(U\sigma_t^2 \triangle t + \sigma_t \sqrt{\triangle t}\epsilon_{2i}\right), \quad (5.7)$$

where ϵ_{2i} is the random sample from N(0,1) independent of ϵ_{1i}.

We simulate the short rate SDE by

$$\begin{aligned} r_{i+1,j} &= r_{i,j}\exp\left(-a\triangle t\right)+\frac{b}{a}\left(1-\exp\left(-a\triangle t\right)\right) \\ &+\sigma\left(\rho\epsilon_{2i}\sqrt{\frac{1-exp\left(-2a\triangle t\right)}{2a}}+\sqrt{1-\rho^2}\epsilon_{3i}\sqrt{\frac{1-exp\left(-2a\triangle t\right)}{2a}}\right), \end{aligned}$$

where ϵ_{3i} is the random sample from N(0,1) independent of ϵ_{1i} and ϵ_{2i}.

In our simulation, we take a 1000 steps of interval $\triangle t$.

We obtain the average variance by

$\bar{V_j} = \frac{1}{L}\sum_{j=1}^{L} V^*_{i,j}$.

We obtain the average spreads :

$\bar{S_j} = \frac{1}{L}\sum_{j=1}^{L} S_{i,j}$,

and the average short rates :

$\bar{r_j} = \frac{1}{L}\sum_{j=1}^{L} r_{i,j}$,

where $L = 1000$. For each SDE we use the Reimann sum approximation of the integral.

We then calculate 100 of $\bar{V_j}$, $\bar{S_j}$, $\bar{r_j}$ and then calculate the call credit spread price for $n = 100$ as

$y_1 = \frac{1}{n}\sum_{j=1}^{n} C_{BS}(\bar{S_j}, K, \bar{r_j}, \sqrt{\bar{V_j}})$.

5.6 Analysis of Closed/Numerical Call Prices with Correlation

In this section we carry out a sensitivity analysis of the closed form and the numerical form of European call credit spread prices with correlation between the credit spread and the short rate under a stochastic volatility process.
The volatility process is of the form $d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)dt + a_1 dW_3(t)$, where θ is the speed of mean reversion, c is the long term mean, a_1 is the volatility of volatility and σ_0^2 the initial variance.
Appendix B at the end of this chapter contains the table of closed form and Monte Carlo credit spread call option prices for various parameter values.
We compute credit spread option prices using choice of contour values and upper boundary of inner integral as mentioned previously in the no correlation sensitivity analysis on closed and numerical form. The average computation time is 20 seconds.
Once again we observe that large contour values or upper bounds for inner integral, increases computation time with no convergence. We try different upper bounds for the inner integral and realise that values beyond 200 result in an indefinite integral. We choose 60 as it converges much closer

to credit spread call prices produced by the Monte Carlo simulation.

The Monte Carlo simulation is done without the variance reduction method of antithetic variates. Our initial trial with variance reduction increases our computational time to several hours with no convergence. Hence all simulation results are obtained without variance reduction.

We analyse the effects of the mean reversion parameters and correlation on either the analytic or numerical credit spread option pricing model. For the Monte Carlo approach, increasing the long term mean, the initial variance and the volatility of variance drives up the credit spread call price. Increasing the speed of mean reversion decreases the credit spread call price. In general correlation regardless of whether its positive or negative results in higher credit spread call prices than the no correlation case. Average computation time is about 1 hour. We also realise that the correlation effect increases the computation time of the credit spread option price.

The analytical form produces similar effects as observed for the numerical form where increasing the long term mean, the initial variance and the volatility of variance increases the credit spread call price. Increasing the speed of mean reversion decreases the credit spread price as the volatility decreases. Average computation time is about 10 seconds.

The correlation effect is the same for both the analytic and numerical models as increase in positive correlation drives up the credit spread price but increase in negative correlation reduces the credit spread price.
We observe that at high long term mean variances or initial variances, the Monte Carlo credit spread call prices deviate from the closed form call prices.
In the case of low long term mean variances or initial variances the price difference between models for numerical and closed form is within the convergence rate of the Monte Carlo estimate $\frac{1}{\sqrt{N}}$.

5.7 Appendix A: Table of Call Prices under no correlation

Table of closed form and Monte Carlo simulation of European call credit spread prices for no correlation between the credit spread and the Vasicek short rate under Vasicek stochastic volatility process.

θ is speed of mean reversion,

σ_0^2 is the initial variance,

c is the long term mean variance,

a_1 is the volatility of volatility,

Given underlying spread price = 0.3, strike = 0.1, risk free rate= 0.06.

Time to maturity is 6 months.

θ	σ_0^2	c	a_1	Closed form Price	Monte Carlo Price
0.05	0.05	0.05	0.10	0.2032125907	0.2029554558
	0.06	0.06		0.2033142221	0.2029554626
	0.25	0.25		0.2052549110	0.2029716572
0.2	0.2	0.2	0.40	0.2048549003	0.2030120233
	0.3	0.3		0.2058817398	0.2030846868
	0.6	0.6		0.2089932438	0.2037329933
0.3	0.3	0.3	0.60	0.2060320831	0.2031974429
	0.4	0.4		0.2070648231	0.2033495644
	0.8	0.8		0.2072768672	0.2046975370
0.5	0.09	0.09	0.20	0.2036416106	0.2029564758
	0.10	0.10		0.2037434571	0.2029567698
	0.60	0.60		0.2089012470	0.2036378389
0.5	0.09	0.09	0.40	0.2037306403	0.2029725462
	0.10	0.10		0.2038325310	0.2029745372
	0.60	0.60		0.2089925762	0.2037203970
4	0.09	0.09	0.40	0.2037269257	0.2029572654
	0.10	0.10		0.2038288148	0.2029576826
	0.30	0.30		0.2058773283	0.2030178641

5.8 Appendix B: Table of Call Prices with correlation

Table of closed form and Monte Carlo simulation of European call credit spread prices for correlation between the credit spread and the Vasicek short rate under Vasicek stochastic volatility process.

θ is speed of mean reversion,

V_0 is the initial variance,

c is the long term mean variance,

a_1 is the volatility of volatility,

ρ is the correlation between credit spread and short rate,

S_i is the initial credit spread,

r_i is the initial short rate,

The constants a and b in the Vasicek short rate SDE are set to 0.5

for this Monte Carlo simulation. Time to maturity is 6 months.

θ	V_0	c	a_1	ρ	S_i	r_i	Closed form	Monte Carlo
0.5	0.09	0.09	0.20	0.2	0.3	0.06	0.2032569813	0.2131646733
				-0.2			0.2031297228	0.2131182330
				0.4			0.2033257050	0.2132029562
				-0.4			0.2030711819	0.2131114595
0.5	0.09	0.09	0.40	0.2	0.3	0.06	0.2046254670	0.2171814597
				-0.2			0.2041134871	0.2175215639
				0.4			0.2049023533	0.2169406820
				-0.4			0.2038782906	0.2176155111
4	0.09	0.09	0.40	0.2	0.3	0.06	0.2041613430	0.2063988377
				-0.2			0.2039115854	0.2070990111
				0.4			0.2042893246	0.2059812749
				-0.4			0.2037898015	0.2073760953
0.5	0.10	0.10	0.20	0.2	0.3	0.06	0.2032686893	0.2141766567
				-0.2			0.2031414237	0.2141107070
				0.4			0.2033374172	0.2142394880
				-0.4			0.2030828793	0.2141090925
0.5	0.10	0.10	0.40	0.2	0.3	0.06	0.2046372540	0.2161956656
				-0.2			0.2041252448	0.2153616905
				0.4			0.2049141562	0.2160777968
				-0.4			0.2038900345	0.2154665130
0.5	0.60	0.60	0.20	0.2	0.3	0.06	0.2038549632	0.2169431960
				-0.2			0.2037273303	0.2170657790
				0.4			0.2039238889	0.2169064108
				-0.4			0.2036686170	0.2171518108
0.005	2.0	2.0	0.20	0.2	0.3	0.06	0.2032333601	0.2702459460
				-0.2			0.2030886349	0.2701967694
				0.4			0.2033102632	0.2702917256
				-0.4			0.2030230042	0.2701934584
0.005	2.2	2.2	0.20	0.2	0.3	0.06	0.2032347983	0.2790800248
				-0.2			0.2030911715	0.2790181936
				0.4			0.2033128024	0.2791316490
				-0.4			0.2030255402	0.2790080588
0.005	3.0	3.0	0.20	0.2	0.3	0.06	0.2032449518	0.3166480382
				-0.2			0.2031013177	0.3165493624
				0.4			0.2033229598	0.3167163370
				-0.4			0.2030356832	0.3165190206

Chapter 6

European Call Cox/Ross Stochastic Volatility Model

6.1 Analytic Credit Spread Call under no correlation

In this section we derive the closed form formula of a European call on credit spread for zero correlation between the credit spreads and the short rate. We assume a Cox, Ingersoll and Ross stochastic volatility process and model the short rate via Vasicek's model.

From (5.1) with $\rho = 0$, the moment generating function of the average variance is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\,du\right)\; E\left(\exp\frac{-\lambda}{T}\int_0^T \sigma_u^2\,du\right),$

where $\sigma^2 B^2(t,T)$ is deterministic and the SDE for σ_t^2 is of the Cox, Ingersoll and Ross form as in (1.3)

i.e. $d\sigma_t^2 = \theta\,(c - \sigma_t^2)\,dt + a_1 \sigma_t dW_3(t)$.

Let $I^*(\lambda) = E\left(\exp\left(\frac{-\lambda}{T}\int_0^T \sigma_u^2\,du\right)\right)$

and $V_t^* = \frac{\lambda}{T}\sigma_t^2$.

Then $I^*(\lambda) = E\left[\exp\left(-\int_0^T V^*(s)\,ds\right)\right]$,

where $V_t^* = \frac{\lambda}{T}\sigma_t^2$ follows the process

$dV_t^* = \left(\frac{\lambda}{T}\theta c - \theta V_t^*\right)dt + a_1\sqrt{\frac{\lambda}{T}}\sqrt{V_t^*}dW_3.$

Using the Cox, Ingersoll and Ross result in Ball and Roma (1994),

a closed form for $I^*(\lambda)$ is

$I^*(\lambda) = \exp\left(N^*(T) + M^*(T)V_t^*\right),$

where

$N^*(T) = \frac{2\theta c}{a_1^2}\ln\left(\frac{2\gamma\exp\left((\theta-\gamma)\frac{T}{2}\right)}{g(T)}\right),$

$M^*(T) = \frac{-2(1-\exp(-\gamma T))}{g(T)},$

$\gamma = \sqrt{\theta^2 + 2\left(\frac{\lambda}{T}\right)a_1^2},$

$g(T) = 2\gamma + ((\theta-\gamma)(1-\exp(-\gamma T))).$

Hence the moment generating function under the Cox, Ingersoll and Ross

stochastic volatility model is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\,du\right)\ \exp\left(N^*(T) + M^*(T)V_t^*\right),$

where $N^*(T)$ and $V^*(T)$ are as above.

Let $\bar{v}$ be the average variance $= \frac{1}{T}\int_0^T \left(\sigma_u^2 + \sigma^2 B^2(u,T)\right)du$

for zero correlation from (4.2).

We already know that the moment generating function can be expressed as

$I(\lambda) = E(\exp(-\lambda\bar{v}))$.

Therefore $I(\lambda) = \int_0^\infty \exp(-\lambda\bar{v})\, m(\bar{v})\, d\bar{v}$.

We apply the Abate and Whitt numerical inversion method for inverse Laplace transforms to invert the moment generating function to obtain the density of average variance under the Cox, Ingersoll and Ross stochastic volatility model as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi} \int_0^\infty \Re(I(c+iz)) \cos\bar{v}z\, dz$,

where $\Re(I(c+iz))$ is the real part of $I(c+iz)$.

Hence the European call on credit spread, where correlation between credit spread and short rate is zero under the Cox, Ingersoll and Ross stochastic volatility process, is

$$\begin{aligned} \Pi_c(0) &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\; m(\bar{v})\; d\bar{v} \\ &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\; \left(2\frac{\exp c\bar{v}}{\Pi} \int_0^\infty \Re(I(c+iz)) \cos\bar{v}z\, dz \right)\; d\bar{v}, \end{aligned}$$

where $\Re(I(c+iz))$ is the real part of $I(c+iz)$.

6.2 Numerical Credit Spread Call Price under no Correlation

We apply the numerical approach of Monte Carlo simulation to price European call credit spreads with no correlation between the credit spread and the short rate under Cox, Ingersoll and Ross stochastic volatility. The short rate is modelled by a Vasicek process.

From (1.3), σ_t the volatility process is of the Cox, Ingersoll and Ross form

$d\sigma_t^2 = \theta\,(c - \sigma_t^2)\,dt + a_1\sigma_t dW_3(t)$.

Let $V_t^* = \sigma_t^2$, then

$\Rightarrow dV_t^* = \theta\,(c - V_t^*)\,dt + a_1\sqrt{V_t^*}dW_3(t)$.

Discretizing the above SDE,

$$V_{i+1,j}^* = V_{i,j}^* + \theta\left(c - V_{i,j}^*\right)\Delta t + a_1\sqrt{V_{i,j}^*}\sqrt{\Delta t} \times \epsilon_i, \qquad (6.1)$$

where ϵ_i is random sample from N(0,1).

We simulate over a 1000 steps from the random sample $\epsilon_i \sim N(0,1)$.

The variance reduction method of antithetic variates is employed, where for each ϵ_i, we also calculate the $-\epsilon_i$ sample values, which is used to calculate a 1000 $V_{i,k}^*$ using

$$V_{i+1,k}^* = V_{i,k}^* + \theta\left(c - V_{i,k}^*\right)\Delta t + a_1\sqrt{V_{i,k}^*}\sqrt{\Delta t} \times -\epsilon_i. \qquad (6.2)$$

Two sets of average variances are calculated as follows:

$\bar{V_j} = \frac{1}{L}\sum_{j=1}^{L} V_{i,j}^{*}$ from ϵ_i random samples of N(0,1),

$\bar{V_k} = \frac{1}{L}\sum_{k=1}^{L} V_{i,k}^{*}$ from $-\epsilon_i$ random samples of N(0,1),

where $L = 1000$.

We calculate 100 $\bar{V_j}$ and calculate the average call price as $y_1 = \frac{1}{n}\sum_{j=1}^{n} C_{BS}(\sqrt{\bar{V_j}})$,

where $n = 100$.

We also calculate 100 $\bar{V_k}$ and calculate the average call price as $y_2 = \frac{1}{n}\sum_{k=1}^{n} C_{BS}(\sqrt{\bar{V_k}})$

where $n = 100$.

Hence the best estimate of European call price is $\frac{(y_1+y_2)}{2}$.

6.3 Analysis of Closed/Numerical Call Prices under no Correlation

We analyse the sensitivity of the closed form and the Monte Carlo simulation of the European call credit spread prices for zero correlation between the credit spread and the short rate under Cox, Ingersoll and Ross stochastic volatility process,

$d\sigma_t^2 = \theta\,(c - \sigma_t^2)\,dt + a_1\sigma_t dW_3(t)$,

where θ is the speed of mean reversion, c is the long term mean, a_1 is the volatility of variance and σ_0^2 the initial variance.

See Appendix A at the end of this chapter for the table of closed form and Monte Carlo credit spread call option prices.

In the analytic case, the credit spread prices are computed using contour values $c = 0.05$ and inner upper bound of average variance density integral of 25 as mentioned in previous chapters. The choice of the outer bound integral is set to 20.

Large contour values or upper bounds for average density integral results in more expansion terms for call spread price integral, which increases computation time to over 5 minutes. We choose a value of 20 for the upper bound of the outer integral at which the analytic form converges to the Monte Carlo estimate. Higher upper bound values result in an undefined integral due to singularity. The average computation time is 30 seconds.

Monte Carlo simulation is used for the numerical form. A variance reduction method of antithetic variates is employed to improve the simulation. We observe close credit spread prices obtained for variance reduction and without variance reduction techniques, but at the expense of increased computation time. As in previous chapters we set the initial variance in the Monte Carlo Cox, Ingersoll and Ross equation to the long term mean.

An additional table of credit spread option prices for the Monte Carlo simulation without variance reduction is listed in appendix A to show that including variance reduction does not improve the credit spread price considerably for the increased computation time. The average computation time is about 27 minutes.

We investigate the effects of mean reversion parameters for stochastic volatility and no correlation between the credit spread and the short rate under the analytic or numerical pricing models.

For both numerical and analytic form, increasing (decreasing) the long term mean, the initial variance of credit spread or the volatility of variance increases (reduces) the credit spread option price. Increasing (decreasing) the speed of mean reversion reduces (increases) the credit spread option price as the volatility of variance is reduced (increased).

The difference in European call prices between the analytic and the Monte Carlo pricing models is within the convergence rate of the Monte Carlo estimate $\frac{1}{\sqrt{N}}$.

6.4 Numerical Credit Spread Call Price with Correlation

In this section we price numerically European call credit spreads with correlation between the credit spread and the short rate under Cox, Ingersoll and Ross stochastic volatility by Monte Carlo simulation.

We model the short rate in (3.2) by a Vasicek process,

$dr(t) = (b - ar(t))\,dt + \sigma\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right),$

where ρ is the correlation between the short rate and the credit spread.

From (1.3), σ_t the volatility process is of the Cox, Ingersoll and Ross form

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)dt + a_1\sigma_t dW_3(t).$

Let $V_t^* = \sigma_t^2$

$\Rightarrow dV_t^* = \theta\left(c - V_t^*\right)dt + a_1\sqrt{V_t^*}dW_3(t).$

Discretizing the above SDE,

$$V_{i+1,j}^* = V_{i,j}^* + \theta\left(c - V_{i,j}^*\right)\Delta t + a_1\sqrt{V_{i,j}^*}\sqrt{\Delta t}\epsilon_{1i}, \qquad (6.3)$$

where ϵ_{1i} is a random sample from N(0,1).

From (3.1) the credit spread SDE is of the form

$dS(t) = S(t)\left(U\sigma_t^2 dt + \sigma_t dW_1(t)\right),$

where $U = \pi_0 + \frac{1}{2}$.

We simulate the credit spread SDE by,

$$S_{i+1,j} = S_{i,j} + S_{i,j}\left(U\sigma_t^2\Delta t + \sigma_t\sqrt{\Delta t}\epsilon_{2i}\right), \tag{6.4}$$

where ϵ_{2i} is random sample from N(0,1), $V_t^* = \sigma_t^2$ and ϵ_{2i} is independent of ϵ_{1i}.

We simulate the short rate SDE using the Vasicek closed form,

$$\begin{aligned} r_{i+1,j} &= r_{i,j}\exp(-a\Delta t) + \frac{b}{a}(1-\exp(-a\Delta t)) \\ &+\sigma\left(\rho\epsilon_{2i}\sqrt{\frac{1-exp(-2a\Delta t)}{2a}} + \sqrt{1-\rho^2}\epsilon_{3i}\sqrt{\frac{1-exp(-2a\Delta t)}{2a}}\right), \end{aligned}$$

where ϵ_{3i} is random sample from N(0,1), independent of ϵ_{2i}, ϵ_{1i}.

We simulate a 1000 $V_{i,j}^*$ from the random sample $\epsilon_{1i} \sim N(0,1)$ in the above Cox, Ingersoll and Ross SDE.

We calculate a 1000 $S_{i,j}$ by simulating from the random sample $\epsilon_{2i} \sim N(0,1)$ in above credit spread SDE.

For each ϵ_{2i} random samples we calculate $r_{i,j}$ by simulating from random sample $\epsilon_{3i} \sim N(0,1)$ for short rate SDE values.

We obtain average variance,

$\bar{V}_j = \frac{1}{L}\sum_{j=1}^{L} V_{i,j}^*$ from ϵ_{1i} random samples of N(0,1).

We obtain average spreads,

$\bar{S_j} = \frac{1}{L}\sum_{j=1}^{L} S_j$ from ϵ_{2i} random samples of N(0,1).

We obtain average short rates,

$\bar{r_j} = \frac{1}{L}\sum_{j=1}^{L} r_j$ from ϵ_{2i} and ϵ_{3i} random samples of N(0,1)

where $L = 1000$.

We calculate 100 $\bar{V_j}$, $\bar{S_j}$, $\bar{r_j}$

We then calculate the call credit spread price for $n = 100$ as

$y_1 = \frac{1}{n}\sum_{j=1}^{n} C_{BS}\left(\bar{S_j}, K, \bar{r_j}, \sqrt{\bar{V_j}}\right)$.

6.5 Analytic Credit Spread Call with correlation.

In this section we derive the closed form formula of European call credit spread price for non zero correlation between the credit spread and the short rate under a Cox, Ingersoll and Ross stochastic volatility process. The short rate is modelled by a Vasicek process. In the case of correlation between the credit spread and the Vasicek short rate, the expectation of credit spread prices remains conditional on the volatility process.

However from (5.1), we have the correlation term ρ in the expression for the moment generating function of the average variance

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\,du\right)\ E\left(\exp\frac{-\lambda}{T}\int_0^T(\sigma_u + \sigma B(u,T)\rho)^2\,du\right)$

where $\sigma^2 B^2(u,T)\,(1-\rho^2)$ is deterministic.

In order to evaluate

$E\left(\exp\left(\frac{-\lambda}{T}\int_0^T(\sigma_u + \sigma B(u,T)\rho)^2\,du\right)\right)$

we use the moment generating approach for square root mean reverting volatility processes, as in Ball and Roma (1994).

As in previous section for the Vasicek case we assume that the SDE process for $(\sigma_t + \sigma B(t,T)\rho)^2$ is of the Cox, Ingersoll and Ross form.

We thus compare again the numerical solution with an analytical solution, where we change the initial value of the process only (compare the discussion on pages 50,51).

So we assume that now

$I^*(\lambda) = E\left[\exp\left(-\int_0^T V^*(s)\,ds\right)\right]$, where V_t^* follows the SDE

$dV_t^* = (\frac{\lambda}{T}\theta c - \theta V_t^*)dt + a_1\sqrt{\frac{\lambda}{T}}\sqrt{V_t^*}dW.$

with initial condition $V_0^* = (\sigma_0 + \sigma B(0,T)\rho)$

Using the Cox, Ingersoll and Ross result in the Ball and Roma paper (1994), the closed form for $I^*(\lambda)$ is

$I^*(\lambda) = \exp\left(N^*(T) + M^*(T)V_t^*\right),$

where

$N^*(T) = \frac{2\theta c}{a_1^2} \ln \left(\frac{2\gamma \exp\left((\theta-\gamma)\frac{T}{2}\right)}{g(T)} \right)$,

$M^*(T) = \frac{-2(1-\exp(-\gamma T))}{g(T)}$,

$\gamma = \sqrt{\theta^2 + 2\left(\frac{\lambda}{T}\right) a_1^2}$,

$g(T) = 2\gamma + ((\theta - \gamma)(1 - \exp(-\gamma T)))$.

Hence the moment generating function under the Cox, Ingersoll and Ross stochastic volatility model with correlation term is

$I(\lambda) = \exp\left(\frac{-\lambda}{T} \int_0^T \sigma^2 B^2(t,T)(1-\rho^2)\, du \right) \ \exp\left(N^*(T) + M^*(T)V_t^*\right)$,

where $N^*(T)$, $M^*(T)$ and V_t^* is as above.

Let $\bar{v}$ average variance $= \frac{1}{T}\int_0^T \left((\sigma_u + \sigma B(u,T)\rho)^2 + \sigma^2 B^2(u,T)(1-\rho^2)\right) du$.

From section (4.1), we already know that the moment generating function can be expressed as

$I(\lambda) = E\left(\exp\left(-\lambda \bar{v}\right)\right)$.

Therefore $I(\lambda) = \int_{-\infty}^{\infty} \exp\left(-\lambda \bar{v}\right) m(\bar{v})\, d\bar{v}$.

We apply the Abate and Whitte numerical inversion method for the inverse Laplace transform to invert the moment generating function to obtain the density of average variance with correlation term under Cox, Ingersoll and Ross stochastic volatility model as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi} \int_0^\infty \Re\left(I(c+iz)\right) \cos \bar{v}z\, dz$, where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

Hence the European call on credit spread, for correlation between the credit spread and the short rate under the Cox, Ingersoll and Ross stochastic volatility process, is

$$\begin{aligned}\Pi_c(0) &= \int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\; m(\bar{v})\; d\bar{v} \\ &= \int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\; \left(2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz\right)\; d\bar{v}\end{aligned}$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

6.6 Analysis of Closed/Numerical Call Prices with Correlation

In this section we carry out a sensitivity analysis of closed form and Monte Carlo simulation of European call credit spread prices with correlation between the credit spread and the short rate under the Cox, Ingersoll and Ross stochastic volatility process

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)dt + a_1\sigma_t dW_3(t)$,

where θ is the speed of mean reversion, c is the long term mean,

a_1 the volatility of variance and σ_0^2 the initial variance.

See Appendix B at the end of this chapter for table of closed form and Monte Carlo credit spread call option prices.

Once again we use contour values $c = 0.05$, inner upper bound of average variance density integral set to 25 and outer bound integral of 40 to compute the credit spread option prices. The choice of contour values is based on our computations in the previous sections. The average computation time is 20 seconds.

We select an upper bound of 40 for outer integral as option prices obtained converge to numerical form. Upper bound values above 40 result in singularity for the double integral credit spread option price formula.

For Monte Carlo simulation, no variance reduction is used as we already know that it increases computation time with no improvement. Our average computation time is 1 hour. The introduction of correlation increases the computation time. For convenience we set the initial variance in the Monte Carlo Cox, Ingersoll and Ross equation to the long term mean.

The Monte Carlo and closed form pricing models produce similar mean reversion effects as increasing the long term mean, the initial variance and the volatility of variance drives up the credit spread option prices. Increasing the speed of mean reversion increases the credit spread option price. We note that this effect is opposite to the no correlation case as in the previous section. The introduction of correlation does increase the credit spread option prices.

We observe closer credit spread call prices between the analytic and numerical models for the no correlation case than correlation case. The difference between prices for analytic and numerical pricing models is within the Monte Carlo convergence rate of $\frac{1}{\sqrt{N}}$.

6.7 Appendix A: Table of Call Prices under no correlation

The table of closed form and Monte Carlo simulation of European call credit spread prices for no correlation between the credit spread and the Vasicek short rate under Cox, Ingersoll and Ross Stochastic Volatility process.

θ is the speed of mean reversion,

σ_0^2 is the initial variance,

c is the long term mean variance,

a_1 is the volatility of variance,

Underlying spread price = 0.3, strike = 0.1, Risk free rate= 0.06.

Time to maturity is 6 months.

θ	σ_0^2	c	a_1	Closed form Price	Monte Carlo Price
4	0.09	0.09	0.40	0.2036205802	0.2029554999
	0.10	0.10		0.2036783741	0.2029554999
	0.30	0.30		0.2048376933	0.2029983349
0.05	0.05	0.05	0.10	0.2032786272	0.2029554484
	0.06	0.06		0.2032798872	0.2029554484
	0.25	0.25		0.2033038277	0.2029680026
0.005	2	2	0.20	0.2033251871	0.2131450556
	2.2	2.2		0.2033277650	0.2147307951
	3.0	3.0		0.2033378830	0.2209322592
0.2	0.2	0.2	0.40	0.2034558183	0.2029676970
	0.3	0.3		0.2035050185	0.2030207558
	0.6	0.6		0.2036526909	0.2036852839
0.3	0.3	0.3	0.60	0.2037032447	0.2030591318
	0.4	0.4		0.2037759151	0.2032084570
	0.8	0.8		0.2040668565	0.2046544886
0.5	0.09	0.09	0.40	0.2034261171	0.2029557487
	0.10	0.10		0.2034378316	0.2029559249
	0.45	0.45		0.2038482654	0.2032404457
0.5	0.09	0.09	0.20	0.2033416869	0.2029554596
	0.10	0.10		0.2033533992	0.2029554675
	0.60	0.60		0.2037637532	0.2036267784

See table of credit spread prices for Monte Carlo case without variance reduction.

Underlying spread price = 0.3, strike = 0.1, Risk free rate= 0.06

Time to maturity is 6 months.

θ	Initial Variance	c	a_1	Closed form Price	Monte Carlo Price
4	0.09	0.09	0.40	0.2036205802	0.2029554589
	0.10	0.10		0.2032798872	0.2029554739
	0.30	0.30		0.2048376933	0.2029911729
0.05	0.05	0.05	0.10	0.2032786272	0.2029554485
	0.06	0.06		0.2032798872	0.2029554485
	0.25	0.25		0.2033038277	0.2029687860
0.005	2	2	0.20	0.2033251871	0.2129228815
	2.2	2.2		0.2033272650	0.2144982606
	3.0	3.0		0.2033378830	0.2206740173
0.2	0.2	0.2	0.40	0.2034558183	0.2029635161
	0.3	0.3		0.2035050185	0.2030246685
	0.6	0.6		0.2036526909	0.2035171963
0.3	0.3	0.3	0.60	0.2037032447	0.2030275652
	0.4	0.4		0.2037759151	0.2031426631
	0.8	0.8		0.2040668565	0.2043979407
0.5	0.09	0.09	0.40	0.2034261171	0.2029555881
	0.10	0.10		0.2034378316	0.2029556830
	0.45	0.45		0.2038482654	0.2031881831
0.5	0.09	0.09	0.20	0.2033416869	0.2029554543
	0.10	0.10		0.2033533992	0.2029554633
	0.60	0.60		0.2037637532	0.2035773025

6.8 Appendix B: Table of Call Prices with correlation

The table of closed form and Monte Carlo simulation of European call credit spread prices for correlation between the credit spread and the short rate under Cox, Ingersoll and Ross stochastic volatility process. The short rate is modelled under a Vasicek process.

θ is Speed of mean reversion,

V_0 is the initial variance,

c is the long term mean variance,

a_1 is the volatility of variance,

ρ is the correlation between the credit spread and the short rate,

S_i is the initial credit spread,

r_i is the initial short rate,

The constants a and b in the Vasicek short rate model are set to 0.5 for this Monte Carlo simulation. Time to maturity is 6 months.

θ	V_0	c	a_1	ρ	S_i	r_i	Monte Carlo	Closed form
0.5	0.09	0.09	0.20	0.2	0.3	0.06	0.2061986814	0.2030125600
				-0.2			0.2063770752	0.2030198592
				0.4			0.2061332100	0.2031774913
				-0.4			0.2064902354	0.2029740847
0.5	0.09	0.09	0.40	0.2	0.3	0.06	0.2070851661	0.2032619016
				-0.2			0.2023134732	0.2030585006
				0.4			0.2097252736	0.2032857932
				-0.4			0.2027138958	0.2028865841
0.5	0.10	0.10	0.20	0.2	0.3	0.06	0.2062651706	0.2030380251
				-0.2			0.2064435616	0.2029382278
				0.4			0.2061997014	0.2030929879
				-0.4			0.2065567204	0.2028933885
0.5	0.10	0.10	0.40	0.2	0.3	0.06	0.2071309395	0.2031774401
				-0.2			0.2019929080	0.2029778432
				0.4			0.2099311454	0.2032974997
				-0.4			0.2023933225	0.2028982675
0.5	0.60	0.60	0.20	0.2	0.3	0.06	0.2168929332	0.2036235932
				-0.2			0.2170146360	0.2035235081
				0.4			0.2168564896	0.2036787145
				-0.4			0.2171001378	0.2034785395
0.005	2.0	2.0	0.20	0.2	0.3	0.06	0.2702087756	0.2029609287
				-0.2			0.2701602636	0.2028487989
				0.4			0.2702543336	0.2030231401
				-0.4			0.2701573966	0.2027988740
				0.8			0.2704091502	0.2031598740
0.005	2.2	2.2	0.20	0.2	0.3	0.06	0.2790113878	0.2029634635
				-0.2			0.2789502388	0.2028513323
				0.4			0.2790627872	0.2030256757
				-0.4			0.2789405624	0.2028014067
0.005	3.0	3.0	0.20	0.2	0.3	0.06	0.3164031988	0.2029736018
				-0.2			0.3163052028	0.2028614650
				0.4			0.3164771283	0.2030358171
				-0.4			0.3162753278	0.2028115370
4	0.09	0.09	0.40	0.2	0.3	0.06	0.2082775851	0.2034563684
				-0.2			0.2032751341	0.2033576921
				0.4			0.2111244460	0.2035086922
				-0.4			0.2023339154	0.2033113367

Chapter 7

European Call Ahn/Gao Stochastic Volatility Model

7.1 Numerical Credit Spread Call Price under no Correlation

In this section we price numerically European Call credit spreads with no correlation between the credit spread and the short rate under the Ahn/Gao stochastic volatility process by Monte Carlo simulation. The short rate is modelled by a Vasicek process.

From (1.4), σ_t the volatility process is of the Ahn Gao form

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)\sigma_t^2 dt + a_1 \sigma_t^3 dW_3(t).$

The Ahn/Gao mean reverting one-factor SDE is classified as an alternative mean reverting stochastic process to normal ones such as Vasicek or Cox, Ingersoll and Ross. It has a non linear drift and diffusion which is used to model the explosiveness of a one-factor mean reverting stochastic process.

See the Jesper Andreasen (2000) working paper on credit explosions about the explosive stochastic process.

Let $V_t^* = \sigma_t^2$

$\Rightarrow dV_t^* = \theta\left(c - V_t^*\right) V_t^* dt + a_1 V_t^{*\frac{3}{2}} dW_3(t)$.

Discretizing the above SDE,

$$V_{i+1,j}^* = V_{i,j}^* + \theta\left(c - V_{i,j}^*\right) V_{i,j}^* \triangle t + a_1 V_{i,j}^{*\frac{3}{2}} \sqrt{\triangle t}\epsilon_i, \qquad (7.1)$$

where ϵ_i is random sample from N(0,1).

We take a 1000 steps of interval $\triangle t$ which results in total time T, where $T = \triangle tL$ and $\triangle t$ is the time interval and L is the number of steps i.e. 1000 in this case.

We calculate a 1000 $V_{i,j}^*$ by simulating from the random sample $\epsilon_i \sim N(0,1)$.

For each ϵ_i, we also calculate the $-\epsilon_i$ sample values, which is used to calculate a 1000 $V_{k,j}^*$ using

$$V_{k+1,j}^* = V_{k,j}^* + \theta\left(c - V_{k,j}^*\right) V_{k,j}^* \triangle t + a_1 V_{k,j}^{*\frac{3}{2}} \sqrt{\triangle t} \times -\epsilon_i. \qquad (7.2)$$

We calculate two sets of average variances as follows:

$\bar{V_j} = \frac{1}{L}\sum_{i=1}^{L} V^*_{i,j}$ from ϵ_i random samples of N(0,1),

$\bar{V_k} = \frac{1}{L}\sum_{i=1}^{L} V^*_{i,k}$ from $-\epsilon_i$ random samples of N(0,1),

where $L = 1000$.

We calculate 100 $\bar{V_j}$ and calculate the average call price as $y_1 = \frac{1}{n}\sum_{j=1}^{n} C_{BS}\left(\sqrt{\bar{V_j}}\right)$

where $n = 100$.

Also calculate 100 $\bar{V_k}$ and calculate the average call price as $y_2 = \frac{1}{n}\sum_{k=1}^{n} C_{BS}\left(\sqrt{\bar{V_k}}\right)$

where $n = 100$.

Hence best estimate of European Call price is $\frac{(y_1+y_2)}{2}$.

7.2 Analytic Credit Spread Call under no correlation

In this section we derive the closed form formula of European call on credit spread for zero correlation between the credit spread and the short rate under the Ahn/Gao stochastic volatility process. The short rate is modelled by a Vasicek process.

From (5.1) where $\rho = 0$ and moment generating function of average variance is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\,du\right)\; E\left(\exp\frac{-\lambda}{T}\int_0^T \sigma_u^2\,du\right)$,

where $\sigma^2 B^2(t,T)$ is deterministic and SDE for σ_t^2 is of the Ahn/Gao form as in (1.4), i.e. $d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)\sigma_t^2 dt + a_1\sigma_t^3 dW_3(t)$.

Let $I^*(\lambda) = E\left(\exp\left(\frac{-\lambda}{T}\int_0^T \sigma_u^2\, du\right)\right)$ and

$V_t^* = \frac{\lambda}{T}\sigma_t^2$.

Then $I^*(\lambda) = E\left[\exp\left(-\int_0^T V^*(s)\, ds\right)\right]$,

where $V_t^* = \frac{\lambda}{T}\sigma_t^2$ follows the process

$dV_t^* = \theta\left(\frac{\lambda}{T}c - V_t^*\right)V_t^* dt + a_1\sqrt{\frac{\lambda}{T}}V_t^{*\frac{3}{2}} dW_3$.

Using the Cox and Ross result in the Ball and Roma paper (1994), $I^*(\lambda)$ is analogous to time 0 price of bond with maturity at time T, whose short rate is of the $\frac{3}{2}$ Ahn Gao one-factor non-linear drift model.

Hence closed form for $I^*(\lambda)$ using Ahn/Gao paper (1994) is

$I^*(\lambda) = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}.M\left(\gamma, \beta, -\chi\left(V_0^*, 0, T\right)\right).\chi\left(V_0^*, 0, T\right)^{\gamma}$

where $M\left(.,.,.\right)$ is the confluent hypergeometric function (or Kummer function), represented as

$M\left(\gamma, \beta, -\chi\left(V_0^*, 0, T\right)\right) = \frac{\Gamma(\beta)}{\Gamma(\gamma).\Gamma(\beta-\gamma)}\int_0^1 \exp\left(-\chi\left(V_0^*, 0, T\right)Z\right)Z^{\gamma-1}\left(1-Z\right)^{\beta-\gamma-1}dz$.

So $I^*(\lambda) = \frac{1}{\Gamma(\gamma)}\int_0^1 \exp\left(-\chi\left(V_0^*, 0, T\right)Z\right)Z^{\gamma-1}\left(1-Z\right)^{\beta-\gamma-1}dz$,

where

$\chi\left(V_0^*, 0, T\right) = \frac{2\theta\frac{\lambda}{T}c}{\sigma^2\left(\exp\left(\theta\frac{\lambda}{T}c\right)T-1\right)V_0^*}$,

$V_0^* = \frac{\lambda}{T}\left(\sigma_o^2\right)$,

$\gamma = \frac{1}{\sigma^2}\left[\sqrt{\phi^2 + 2\sigma^2} - \phi\right]$,

$\beta = \frac{2}{\sigma^2}\left[\theta + (1+\gamma)\,\sigma^2\right]$,

$\phi = \theta + \frac{1}{2}\sigma^2$,

$\sigma^2 = a_1^2 \frac{\lambda}{T}$.

Therefore the moment generating function under the Ahn and Gao stochastic volatility model is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)\,du\right) \; * I^*(\lambda)$.

Let $\bar{v}$ average variance $= \frac{1}{T}\int_0^T \left(\sigma_u^2 + \sigma^2 B^2(u,T)\right)\,du$

for zero correlation from (5.1).

We already know that the moment generating function can be expressed

$I(\lambda) = E\left(\exp\left(-\lambda\bar{v}\right)\right)$.

Therefore $I(\lambda) = \int_0^\infty \exp\left(-\lambda\bar{v}\right) m(\bar{v})\,d\bar{v}$.

We apply the Abate and Whitt numerical inversion method for inverse Laplace transform to invert the moment generating function to obtain density of the average variance with correlation term under Ahn and Gao stochastic volatility model as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz$,

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

Hence the European call on credit spread, where correlation between the credit spread and the short rate is zero under the Ahn and Gao stochastic volatility process, is

$$\begin{aligned}\Pi_c(0) &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ m(\bar{v})\ d\bar{v} \\ &= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ \left(2\frac{\exp c\bar{v}}{\Pi}\int_0^\infty \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz\right)\ d\bar{v},\end{aligned}$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

7.3 Analysis of Closed/Numerical Call Prices under no Correlation

This section investigates the sensitivity analysis of the closed form and Monte Carlo simulation of European call credit spread prices for zero correlation between the credit spread and the Vasicek short rate under the Ahn Gao stochastic volatility process,

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)\sigma_t^2 dt + a_1\sigma_t^3 dW_3(t),$

where θ is the speed of mean reversion, c is the long term mean, a_1 is the volatility of variance and σ_0^2 the initial variance.

See Appendix A at the end of this chapter for table of closed form and Monte Carlo credit spread call option prices.

In the analytic case, the credit spread options prices are calculated for contour values of $c = 0.05$, inner upper bound of average variance density integral of 25 and outer upper bound integral of 8.88. We select the same contour values and inner upper bounds integral as in previous section. The outer upper bound of 8.88 produces stable spread option prices which converge to the Monte Carlo prices. Values higher than 8.88 result in singularity causing the double integral credit spread option formula to be undefined. The average computation time is 60 seconds.

For Monte Carlo simulation, the variance reduction method is not applied as we already observe that the increase in computation time does not improve the credit spread option prices. In our simulation we set the initial variance to the long term mean. The average computation time is 1 hour.

We analyse the effects of the stochastic volatility mean reversion parameters and no correlation on the credit spread option price obtained from either analytic form or numerical form.

For closed form, an increase in initial variance or long term mean, increases the credit spread option price. Increasing the volatility of variance gives us an unexpected result of reducing the credit spread option price. Increasing the speed of mean reversion, reduces the credit spread option price as expected.

For numerical form, increasing the initial variance and the long term mean increases the credit spread option price. Increasing the volatility of variance, increases the credit spread option price by a negligible amount. Increasing the mean reversion, reduces the credit spread option price. The difference in numerical form and closed form prices are within the convergence rate of the Monte Carlo estimate $\frac{1}{\sqrt{N}}$.

7.4 Numerical Credit Spread Call Price with Correlation

In this section we price numerically European call credit spreads with correlation between the credit spread and the short rate under Ahn/Gao stochastic volatility by Monte Carlo simulation.

From (3.2) the short rate is modelled by a Vasicek process whose SDE is

$dr(t) = (b - ar(t))\,dt + \sigma\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right),$

where ρ is the correlation between the short rate and the credit spread.

From (1.4), σ_t the volatility process is of the Ahn and Gao form

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)\sigma_t^2 dt + a_1\sigma_t^3 dW_3(t)$

Let $V_t^* = \sigma_t^2$, then

$dV_t^* = \theta\left(c - V_t^*\right)V_t^* dt + a_1 V_t^{*\frac{3}{2}} dW_3(t)$

As in previous sections, we simulate the variance process by discretizing the SDE

$$V_{i+1,j}^* = V_{i,j}^* + \theta\left(c - V_{i,j}^*\right)V_{i,j}^*\Delta t + a_1 V_{i,j}^{*\frac{3}{2}}\sqrt{\Delta t}\epsilon_{1i} \qquad (7.3)$$

where ϵ_{1i} is random sample from N(0,1).

From (3.1), we simulate the credit spread SDE as ,

$$S_{i+1,j} = S_{i,j} + S_{i,j}\left(U\sigma_t^2\Delta t + \sigma_t\sqrt{\Delta t}\epsilon_{2i}\right) \qquad (7.4)$$

where ϵ_{2i} is a random sample from N(0,1) independent of ϵ_{1i}.

The short rate is simulated using the closed form of Vasicek SDE ,

$$\begin{aligned} r_{i+1,j} &= r_{i,j}\exp\left(-a\Delta t\right) + \frac{b}{a}\left(1 - \exp\left(-a\Delta t\right)\right) \\ &+\sigma\left(\rho\epsilon_{2i}\sqrt{\frac{1 - exp\left(-2a\Delta t\right)}{2a}} + \sqrt{1-\rho^2}\epsilon_{3i}\sqrt{\frac{1 - exp\left(-2a\Delta t\right)}{2a}}\right), \end{aligned}$$

where ϵ_{3i} is a random sample from N(0,1) and independent of ϵ_{2i}.

Applying the simulation approach as in previous sections for correlation case, we obtain the average variance as:

$\bar{V_j} = \frac{1}{L}\sum_{i=1}^{L} V_{i,j}^*$ from ϵ_{1i} random samples of N(0,1), the average credit spread as:

$\bar{S}_j = \frac{1}{L}\sum_{i=1}^{L} S_{i,j}$ from ϵ_{2i} random samples of N(0,1),

the average short rate :

$\bar{r}_j = \frac{1}{L}\sum_{i=1}^{L} r_{i,j}$ from ϵ_{2i} and ϵ_{3i} random samples of N(0,1),

where L number of steps is 1000.

We then calculate the credit spread call price for $n = 100$ as

$y_1 = \frac{1}{n}\sum_{j=1}^{n} C_{BS}(\bar{S}_j, K, \bar{r}_j, \sqrt{V_j})$.

7.5 Analytic Credit Spread Call with correlation

In this section we derive the closed form formula of European call credit spread price for correlation between the credit spread and the short rate under Ahn/Gao stochastic volatility process. The short rate is modelled by a Vasicek process.

For correlation between the credit spread and the short rate, the expectation of credit spread price remains conditional on the volatility process.

However from (5.1), we have the correlation term ρ in the expression for the moment generating function of average variance,

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)(1-\rho^2)\,du\right)\ E\left(\exp\frac{-\lambda}{T}\int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\,du\right)$

where $\sigma^2 B^2(u,T)\,(1-\rho^2)$ is deterministic.

In order to evaluate

$E\left(\exp\left(\frac{-\lambda}{T}\int_0^T (\sigma_u + \sigma B(u,T)\rho)^2\, du\right)\right)$

we use the moment generating approach for the square root mean

reverting volatility process as in Ball and Roma (1994).

We assume that $(\sigma_t + \sigma B(t,T)\rho)^2$ has Ahn Gao SDE

Here, we approximate the volatility process with a volatility process of

Ahn Gao type where again (as in the corresponding section before) the

correlation parameter is only used to find an initial value for the process.

Let $I^*(\lambda) = E\left[\exp\left(-\int_0^T V^*(s)\,ds\right)\right]$,

where V_t^* follows the SDE

$dV_t^* = \theta\left(\frac{\lambda}{T}c - V_t^*\right) V_t^* dt + a_1\sqrt{\frac{\lambda}{T}} V^{*\frac{3}{2}} dW_3$

with initial condition $V_0^* = (\sigma_0 + \sigma B(0,T)\rho)$.

The closed form $I^*(\lambda)$ is obtained by applying the Cox and Ross result in the

Ball and Roma paper (1994) for the Ahn/Gao SDE as in the previous section.

Hence the closed form for $I^*(\lambda)$ is

$I^*(\lambda) = \frac{1}{\Gamma(\gamma)}\int_0^1 \exp\left(-\chi\left(V^*, 0, T\right) Z\right) Z^{\gamma-1}\left(1-Z\right)^{\beta-\gamma-1}\, dz,$

where

$\chi(V_0^*, 0, T) = \frac{2\theta\frac{\lambda}{T}c}{\sigma^2\left(\exp\left(\theta\frac{\lambda}{T}c\right)T-1\right)V_0^*}$,

$\gamma = \frac{1}{\sigma^2}\left[\sqrt{\phi^2 + 2\sigma^2} - \phi\right]$,

$\beta = \frac{2}{\sigma^2}\left[\theta + (1+\gamma)\,\sigma^2\right]$,

$\phi = \theta + \frac{1}{2}\sigma^2$,

$\sigma^2 = a_1^2\frac{\lambda}{T}$.

Hence the moment generating function under the Ahn and Gao stochastic volatility model is

$I(\lambda) = \exp\left(\frac{-\lambda}{T}\int_0^T \sigma^2 B^2(u,T)1 - \rho^2)\,du\right) \;* I^*(\lambda)$.

Let $\bar{v}$ be the average variance $= \frac{1}{T}\int_0^T \left(\sigma_t + \sigma B(t,T)\rho\right)^2 \, du$.

We already know that the moment generating function can be expressed

$I(\lambda) = E\left(\exp\left(-\lambda\bar{v}\right)\right)$.

Therefore $I(\lambda) = \int_{-\infty}^{\infty} \exp\left(-\lambda\bar{v}\right) m(\bar{v})\, d\bar{v}$.

We apply the Abate and Whitt numerical inversion method for the Laplace transform to obtain the density of average variance under Ahn and Gao stochastic volatility model as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi}\int_0^{\infty} \Re\left(I(c+iz)\right)\cos\bar{v}z\,dz$,

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

Hence the European call on credit spread, for correlation between the credit spread and the short rate under the Ahn and Gao stochastic volatility process is

$$\Pi_c(0) = \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ m(\bar{v})\ d\bar{v}$$
$$= \int_0^\infty C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T)\ \left(2\frac{\exp c\bar{v}}{\Pi} \int_0^\infty \Re\left(I(c+iz)\right) \cos \bar{v}z\, dz \right)\ d\bar{v},$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

7.6 Analysis of Closed/Numerical Call Prices with Correlation

In this section we analyse the sensitivity of closed form and Monte Carlo simulation of European Call credit spread prices based on correlation between the credit spread and the short rate under the Ahn/Gao stochastic volatility process

$d\sigma_t^2 = \theta\left(c - \sigma_t^2\right)\sigma_t^2 dt + a_1 \sigma_t^3 dW_3(t),$

where θ is the speed of mean reversion, c is the long term mean, a_1 the volatility of variance and σ_t^2 at $t = 0$, the initial variance.

The short rate is modelled under a Vasicek process as in previous sections. See Appendix B at the end of this chapter for the table of numerical and analytic credit spread call option prices.

In the analytic form, we compute the credit spread option prices for contour values $c = 0.05$ with inner upper bound of average variance density integral set to 25 and outer bound integral set to 8.88. Our choice of these contour value parameters is based on our analysis in the previous sections. The average computation time is 2 minutes. We observe that larger contour values increases computational time to over 5 minutes which we abort. This increase in computation time is due to the large terms obtained in the expansion of the call spread price integral. The optimum upper bound of the outer integral is set to 8.88 as the analytic pricing formula converges to the numerical form prices. Higher values of upper bound for outer integral result in singularity or the double integral analytic formula being undefined.

In the numerical form, we employ the no variance reduction method as variance reduction increases computational time to several hours, with no convergence. We limited the number of iterations for average call prices to 50 to reduce our computational time. Our average computation time is 1 hour. The initial variance is set to the long term mean for the Ahn/Gao simulation. We observe the effects of mean reverting stochastic volatility and correlation on the credit spread option prices for the analytic form and the numerical form.

For Monte Carlo form, increasing the long term mean variance, the initial variance and the volatility of variance drives up the credit spread prices. Increasing the speed of mean reversion reduces the credit spread.

For closed form, increasing either the long term mean or the initial variance increases the credit spread option price. Increasing volatility of variance reduces the credit spread price which is unexpected. Increasing the speed of mean reversion reduces the credit spread price. The difference in numerical and analytic credit spread prices is within the convergence rate of the Monte Carlo estimate $\frac{1}{\sqrt{N}}$. Overall we realise that the correlation effect produces higher option prices than the no correlation case with increased computation time.

7.7 Appendix A: Table of Call Prices under no correlation

Table of closed form and Monte Carlo simulation of European call credit spread prices for zero correlation between the credit spread and the Vasicek short rate under Ahn/Gao stochastic volatility process.

Underlying spread price = 0.3, strike = 0.1, Risk free rate= 0.06

Time to maturity is 6 months.

θ	Initial Variance	c	a_1	Monte Carlo Price	Closed form
0.1	0.09	0.09	0.20	0.2029554484	0.2038431019
	0.10	0.10		0.2029554492	0.2039599954
	0.30	0.30		0.2029897727	0.2093391089
0.1	0.09	0.09	0.40	0.2029554484	0.2027111190
	0.10	0.10		0.2029554484	0.2028304519
	0.30	0.30		0.2029921654	0.2081914763
0.5	0.09	0.09	0.20	0.2029554485	0.2040516137
	0.10	0.10		0.2029554491	0.2041334371
	0.45	0.45		0.2029897262	0.2093699787
0.05	0.09	0.09	0.60	0.2029554486	0.2008404950
	0.10	0.10		0.2029554501	0.2009692528
	0.60	0.60		0.2036664988	0.2781594754
4	0.09	0.09	0.20	0.2029554485	0.2038028366
	0.10	0.10		0.2029554492	0.2039544774
	0.30	0.30		0.2029894249	0.2093210829

7.8 Appendix B: Table of Call Prices with correlation

Table of closed form and Monte Carlo simulation of European call credit spread prices for correlation between the credit spread and the Vasicek short rate under Ahn/Gao stochastic volatility process.

θ is speed of mean reversion,

V_0 is the initial variance,

c is the long term mean variance,

a_1 is the volatility of variance,

ρ is the correlation between the credit spread and the short rate,

S_i is the initial credit spread,

r_i is the initial short rate,

The constants a and b in the Vasicek short rate model are set to 0.5 for this Monte Carlo simulation. Time to maturity is 6 months.

θ	V_0	c	a_1	ρ	S_i	r_i	Monte Carlo	Closed Form
0.1	0.10	0.10	0.20	0.2	0.3	0.06	0.2058258360	0.2038897745
				-0.2			0.2060042270	0.2040435593
				0.4			0.2057603670	0.2038064181
				-0.4			0.2061173858	0.2040933363
0.1	0.30	0.30	0.20	0.2	0.3	0.06	0.2090214858	0.2092698997
				-0.2			0.2091912138	0.2094370610
				0.4			0.2089606386	0.2092101873
				-0.4			0.2093003408	0.2094816760
0.1	0.10	0.10	0.40	0.2	0.3	0.06	0.2079323452	0.2025276654
				-0.2			0.2020692478	0.2031035458
				0.4			0.2110868368	0.2021860289
				-0.4			0.2004067506	0.2033344215
0.1	0.30	0.30	0.40	0.2	0.3	0.06	0.2123923894	0.2078824325
				-0.2			0.2017193167	0.2078824327
				0.4			0.2174766442	0.2084714860
				-0.4			0.1981831329	0.2075495378
0.5	0.09	0.09	0.20	0.2	0.3	0.06	0.2057479910	0.2037740748
				-0.2			0.2059263852	0.2037549999
				0.4			0.2056825196	0.2037828231
				-0.4			0.2060395454	0.2037549999
0.5	0.09	0.09	0.60	0.2	0.3	0.06	0.2098480375	0.2044173418
				-0.2			0.2021738862	0.2043982068
				0.4			0.2108736245	0.2043647672
				-0.4			0.1996584778	0.2043236550
4	0.09	0.09	0.4	0.20	0.3	0.06	0.2057478896	0.2010533491
				-0.2			0.2059262838	0.2010153479
				0.4			0.2056824184	0.2016394586
				-0.4			0.2060394440	0.2010153508

Chapter 8

Credit Spread Call Hedge Parameters

From the previous chapters, the closed form credit spread call prices is given as

$$\Pi_c(0) = \int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right) \; m(\bar{v}) \; d\bar{v},$$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

$m(\bar{v})$ is the density of the average variance process.

The distribution of average variance is obtained by the classical inversion theorem, written as

$m(\bar{v}) = \frac{1}{2\Pi i}\int_{c-i\infty}^{c+i\infty} \exp\left(\lambda\bar{v}\right) I(\lambda)\, d\lambda$ for $\bar{v} \in [0, \infty)$.

This integral is also known as the Bromwich-Mellin contour integral, where c is a vertical contour in the complex plane chosen so that all singularities of $m(\bar{v})$ are to the left of it.

Make change of variables $\lambda = c + iz$ in the above integral reducing it to the Laplace inversion integral

$$m(\bar{v}) = \frac{\exp(c\bar{v})}{2\Pi} \int_{-\infty}^{\infty} \exp(i\lambda\bar{v}z)\, I(c+iz)\, dz$$
$$= \frac{\exp(c\bar{v})}{2\Pi} \int_{-\infty}^{\infty} \left[\Re\left(I(c+iz)\right)\cos\bar{v}z - \Im\left(I(c+iz)\right)\sin\bar{v}z\right] dz,$$

where $\Re\left(I(c+iz)\right)$ and $\Im\left(I(c+iz)\right)$ are respectively real and imaginary parts of $I(c+iz)$.

The Abate and Whitt numerical inversion method reduces the Laplace inversion integral to give us the density of the average variance as

$m(\bar{v}) = 2\frac{\exp c\bar{v}}{\Pi} \int_0^{\infty} \Re\left(I(c+iz)\right)\cos\bar{v}z\, dz,$

where $\Re\left(I(c+iz)\right)$ is the real part of $I(c+iz)$.

$$\Pi_c(0) = \int_0^{\infty} C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right) \left(2\frac{\exp c\bar{v}}{\Pi} \int_0^{\infty} \Re\left(I(c+iz)\right)\cos\bar{v}z\, dz\right) d\bar{v}.$$

We already have closed form solution for $I(\lambda) = \int_0^{\infty} \exp(-\lambda\bar{v})m(\bar{v})\, d\bar{v}$, where $I(\lambda)$ is the Laplace transform of $m(\bar{v})$ and $m(\bar{v})$ is the density of the average variance process of the one-factor mean reverting stochastic volatility SDE as in (1.2) to (1.4).

The Black Scholes credit spread call price can be written as

$C_{BS}(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T) = S(0).N(d_1) - KP(0,T)N(d_2),$

where

$d_1 = \left(-\left(\frac{\ln(\frac{S(0)}{KP(0,T)}) + \frac{1}{2}\bar{v^2}(T)}{\bar{v}(T)}\right)\right),$

$d_2 = \left(-\left(\frac{\ln(\frac{S(0)}{KP(0,T)}) - \frac{1}{2}\bar{v^2}(T)}{\bar{v}(T)}\right)\right),$

$d_1 = d_2 - \bar{v}\sqrt{T},$

$\bar{v^2} = \frac{v^2}{T}.$

The discount bond price process is

$P(0,T) = A(0,T)\exp(-B(0,T)r),$

where r follows the Vasicek short rate process in (3.2)

with $A(0,T) = \exp\left([(T - B(0,T))\,[\frac{b}{a} - \frac{\sigma^2}{2a^2}] + \frac{\sigma^2}{4a}B^2(0,T)]\right)$

and $B(0,T) = \frac{1}{a}\,[1 - \exp\left(-a(T)\right)].$

8.1 Delta Call on Credit Spread

The delta of call on credit spread is obtained by differentiating the credit spread call price $\Pi_c(0)$ above with respect to underlying credit spread:

$$\begin{aligned}
\frac{d\Pi_c(0)}{d\tilde{S}} &= \frac{d}{d\tilde{S}}\left[\int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v}\right] \\
&= \int_0^\infty \frac{d}{d\tilde{S}} C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty N(d_1)\, m(\bar{v})\ d\bar{v},
\end{aligned}$$

where $N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{1}{2}x^2\right) dx$ is

the cumulative normal distribution function and

$d_1 = \left(-\left(\frac{\ln(\frac{S(0)}{KP(0,T)}) + \frac{1}{2}\bar{v}^2(T)}{\bar{v}(T)}\right)\right),$

$\tilde{S}(0) = \frac{S(0)}{P(0,T)}.$

8.2 Gamma Call on Credit Spread

The Gamma of call on credit spread is obtained by differentiating the delta call on credit spread with respect to underlying credit spread:

$$\begin{aligned}
\frac{d^2\Pi_c(0)}{d\tilde{S}^2} &= \frac{d}{d\tilde{S}}\left[\int_0^\infty N(d_1)\, m(\bar{v})\ d\bar{v}\right] \\
&= \int_0^\infty \frac{d}{d\tilde{S}}\left(N(d_1)\right)\ m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty \frac{d}{dd_1}\left(N(d_1)\right)\frac{dd_1}{d\tilde{S}} m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty \frac{-n(d_1)}{S\bar{v}} m(\bar{v})\, d\bar{v},
\end{aligned}$$

where $n(d_1)$ is the normal density function.

8.3 Vega Call on Credit Spread

The Vega of call on credit spread is obtained by differentiating the credit spread call price $\Pi_c(0)$ with respect to average credit spread variance $\bar{v}$.

Vega on stochastic volatility is critical.

$$
\begin{aligned}
\frac{d\Pi_c(0)}{d\bar{v}} &= \frac{d}{d\bar{v}}\left[\int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v}\right] \\
&= \int_0^\infty \frac{d}{d\bar{v}} C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty \left[S(0)n\left(d_1\right)\frac{dd_1}{d\bar{v}} - KP(0,T)\frac{d}{d\bar{v}}N\left(d_2\right) - n\left(d_2\right)KP(0,T)\frac{dd_2}{d\bar{v}}\right] m(\bar{v})\ d\bar{v} \\
&\quad + \left[C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\frac{d}{d\bar{v}}m(\bar{v})\right]\ d\bar{v} \\
&= \int_0^\infty -KP(0,T)\frac{d}{d\bar{v}}N\left(d_2\right)m(\bar{v}) + \left[\frac{d}{d\bar{v}}m(\bar{v})C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\right]\ d\bar{v},
\end{aligned}
$$

provided $\frac{S(0)}{K} = \frac{P(0,T)n(d_2)}{n(d_1)}$,

$S(0)$ is the underlying credit spread,

K is the strike price,

$P(0,T)$ is the price of the zero coupon bond of maturity T.

Under the Black/Scholes model,

$S(0)n(d_1) = KP(0,T)n(d_2)$, see Pg 265, Question 11.17(b) of John Hull(Hull 2000).

Proof

$$
\begin{aligned}
\frac{S(0)}{K} &= \frac{P(0,T)n\left(d_2\right)}{n\left(d_1\right)} \\
&= P(0,T)\frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_2^2}{2}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right)} \\
&= P(0,T)\frac{\exp\left(-\frac{d_2^2}{2}\right)}{\exp\left(-\frac{d_1^2}{2}\right)} \\
&= P(0,T)\exp\left(-\frac{1}{2}\left(d_2^2-d_1^2\right)\right) \\
&= P(0,T)\exp\left(-\frac{1}{2}\left(d_2+d_1\right)\left(d_2-d_1\right)\right) \\
&= P(0,T)\exp\left(-\frac{1}{2}\left(\frac{-2\ln\left(\frac{S(0)}{kP(0,T)}\right)}{\bar{v}}\right)\left(\frac{\bar{v}^2}{\bar{v}}\right)\right) \\
&= P(0,T)\exp\left(\ln\left(\frac{S(0)}{kP(0,T)}\right)\right).
\end{aligned}
$$

8.4 Theta Call on Credit Spread

The Theta of call on credit spread(Time decay) is obtained by differentiating the credit spread call price $\Pi_c(0)$ with respect to time T.

$$\begin{aligned}
\frac{d\Pi_c(0)}{dT} &= \frac{d}{dT}\left[\int_0^\infty C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v}\right] \\
&= \int_0^\infty \frac{d}{dT} C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\ m(\bar{v})\ d\bar{v} \\
&= \int_0^\infty \left[S(0)n\left(d_1\right)\frac{dd_1}{dT} - KP(0,T)\frac{d}{dT}N\left(d_2\right) - n\left(d_2\right)KP(0,T)\frac{dd_2}{dT}\right] m(\bar{v})\ d\bar{v} \\
&\quad + \left[C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\frac{d}{dT}m(\bar{v})\right]\ d\bar{v} \\
&= \int_0^\infty -KP(0,T)\frac{d}{dT}N\left(d_2\right)m(\bar{v}) + \left[\frac{d}{dT}m(\bar{v})C_{BS}\left(\tilde{S}(0), K, 0, \sqrt{\bar{v}}, T\right)\right]\ d\bar{v},
\end{aligned}$$

provided $\frac{S(0)}{K} = \frac{P(0,T)n(d_2)}{n(d_1)}$.

Chapter 9

Longstaff/Schwartz Credit Spread Option Model

In the Longstaff/Schwartz 1995 paper on valuing credit derivatives, the dynamics of the logarithm of the credit spread X is given by the SDE

$$dX = (a - bX)\,dt + sdZ_1, \tag{9.1}$$

where a, b and c are parameters and Z_1 is a standard Wiener process. This implies that the dynamics of the credit spreads are positive and conditionally log-normally distributed.

They introduce stochastic interest rates of the one-factor Vasicek model in which the dynamics of the short term interest rate r is given by

$$dr = (\alpha - Br)\,dt + \sigma dZ_2, \tag{9.2}$$

where α, B and σ are parameters and Z_2 is a standard Wiener process.

The correlation coefficient between dZ_1 and dZ_2 is ρ.

Since X denotes the logarithm of credit spread, the payoff function for this

option is

$$H(X) = max\,(0, \exp X - K)\,. \tag{9.3}$$

The value of the Longstaff/Schwartz European Call option on credit spread

is given by

$$C(X, r, T) = D(r, T) \exp\left(U + \frac{\eta^2}{2}\right) N(d_1) - KD(r, T)N(d_2), \tag{9.4}$$

where:

$D(r, T)$ is the price of the riskless discount bond,

$N\,(.)$ is the cumulative standard normal distribution,

$d_1 = \frac{-\ln K + U + \eta^2}{\eta}$,

$d_2 = d_1 - \eta$,

$\eta^2 = \frac{S^2[1-\exp(-2bT)]}{2b}$,

$U = \exp\,(-bT)\,X + \frac{1}{b}\left(a - \frac{\rho\sigma S}{B}\right)[1 - \exp\,(-bT)] + \frac{\rho\sigma S}{B(b+B)}\,(1 - \exp\,(-\,(b + B)\,T))$.

The expression of delta call on the credit spread is obtained

by differentiating equation (9.4) with respect to credit spread i.e.

$$D(r,T)\exp\left(U+\frac{\eta^2}{2}\right)N(d_1)\exp\left(-bT\right)\exp\left(-X\right). \tag{9.5}$$

Chapter 10

Sensitivity Analysis of Credit Spread/Delta Calls

In this chapter we discuss the sensitivity analysis of the graphs of credit spread call price against the credit spread and Call Deltas against credit spread. These graphs are listed after the bibliography section.

10.1 Cox/Ross

For non zero or zero correlation of out of the money credit spread options, the call price is an increasing function of the credit spread. Long dated credit spread options have a higher curvature than short to medium term credit spread options. This higher curvature effect is due to increase in term to maturity.

We also observe that the out of moneyness for credit spread option results in the convex shapes at different maturities. In the money call options gives a linear shape for all maturities. Also the credit spread option prices increases with maturity.

10.2 Vasicek

We observe convex shapes for plots of out of the money credit spread option prices against credit spread. The curvature of these plots increases with maturity, with curves for long dated credit spread options being steeper than short to medium term spread options. In the money call spread options have a linear shape.

For positive correlation under Vasicek stochastic volatility, the plots for short to medium term credit spread options are not entirely convex shaped as we observe kinks, followed by increase in steepness of convex shape. The spread option prices increases with maturity.

We observe that Vasicek in/out of money credit spread call prices are higher than Cox/Ross spread prices for zero/non zero correlation. Cox/Ross spread prices are higher than Ahn/Gao credit spread option prices.

10.3 Ahn/Gao

We realise linear shape for in the money calls. Out of the money call spread prices show convex shape plots. Long dated spread options have higher curvature than short to medium term spread options. The steepness of the curvature for out of the money options is more pronounced in case of positive and negative correlation. The spread option prices increases with maturity.

10.4 Longstaff/Schwartz

We observe linear shape at different maturities for in the money credit spread call prices. The out of the money credit spread call prices have a non linear shape, which is an increasing function of the credit spread.

In general the Longstaff/Schwartz credit spread option prices are much lower than option prices obtained from our stochastic volatility model for Cox/Ross, Vasicek and Ahn/Gao. This could be attributed to the assumption of constant volatility in the Longstaff/Schwartz model.

The spread option prices decrease with maturity for in/out of the money options, because of the mean reverting behaviour of the credit spread in the Longstaff/Schwartz model.

10.5 Delta Hedging

In general we observe positive deltas for at and out of the money options and negative deltas for in the money credit spread options. Under Black/Scholes option pricing delta calls are always positive but we realise negative deltas in our stochastic volatility model.

The delta calls for out of the money credit spread options decrease with maturity and credit spread. For low spreads delta is close to or above 1, whilst for high spreads it is a decreasing function of the credit spread. We observe this behaviour for the 3 stochastic volatility modes of the Vasicek, Cox, Ingersoll and Ross and Ahn/Gao forms. The plots are concave shaped. For in the money call spread options under Vasicek, Ahn/Gao and Cox, Ingersoll and Ross stochastic volatility, the delta decreases with increasing credit spread. At high spreads the delta is close to 0.

In the case of Longstaff/Schawartz out of the money calls, delta is an increasing function of the credit spread. For both in and out of the money call options delta decreases with maturity. The delta is a decreasing function of the credit spread for in the money call options.

A credit spread option under our stochastic volatility model can be hedged as follows; A negative delta would imply that if one is long the option then you can hedge shifts in credit spread by going long the credit spread. For positive delta, when long the credit spread option, you can hedge shifts in credit spread by going short credit spread. Likewise delta hedging a short position on the European credit spread call involves maintaining a long position on the credit spread.

Chapter 11

Conclusion

In our thesis we present both closed form and numerical form pricing models for credit spread options under stochastic volatility. The credit spread option process is independent of the stochastic volatility, but we consider cases of no correlation and some correlation between credit spread and short term interest rate. The stochastic volatility models under consideration are one-factor mean reverting stochastic processes of the forms, Vasicek, Cox/Ross and Ahn/Gao. We then evaluate credit spread options with no correlation or correlation under a chosen stochastic volatility process of the 3 forms above.

The numerical formulation is the simulation of various paths of the variance under stochastic volatility. An average Black/Scholes credit spread call price is obtained using several average variance values from the simulation. The closed form formulae is derived as the expectation of Black/Scholes credit spread option price conditional on the density of the average variance as the underlying credit spread is independent of the stochastic volatility process. Deriving the density of the average variance is not trivial. However the choice of the stochastic volatility models Vasicek, Cox, Ingersoll and Ross and Ahn/Gao belong to the general affine class models. These affine class models as seen in interest rate literature provide closed form expressions for transition and marginal densities of the interest rate as well as bond prices. We employ the bond pricing concept to obtain the moment generating function for the mean reverting average variance process. The moment generating function is inverted via the Abate and Whitt numerical Laplace inversion method to obtain the density of the average variance, which is then used in the analytical pricing formula. The density of the average variance involves an integral, hence our credit spread option model results in the evaluation of a double integral, which is non trivial. We select upper boundaries for which we obtain convergence of the closed form double integral spread option prices to the Monte Carlo spread option prices.

We observe closeness in option prices between closed form and numerical form for low variances of the underlying credit spread, regardless of correlation. At high variances of the underlying credit spread with no correlation between the credit spread and the short rate, the closed and numerical form spread option prices deviate slightly from each other. In the case of correlation combined with high variance of the credit spread, the credit spread option prices are higher than the intrinsic value of the credit spread option. This effect is due to a combination of high variances and simulated high credit spread option prices from correlation. The correlation between the credit spread and the short rate definitely produces higher spread option prices than the no correlation case. Given that credit spreads are correlated with short term interest rates, ignoring correlation entirely could lead to mispricing of credit spread options.

The Longstaff/Schwartz credit spread option model under constant volatility produces lower spread option prices than our stochastic volatility pricing model. This could be attributed to the difference in volatility between the models. We realise that our credit spread call option model under Vasicek stochastic volatility produces higher prices than Cox, Ingersoll and Ross stochastic volatility model. Once again the difference in volatility is a contributing factor. The Vasicek model produces a symmetric distribution, whilst Cox, Ingersoll and Ross produces a skewed distribution as its constrained to positive real numbers and has a fatter right hand tail. This constraint means that the Cox, Ingersoll and Ross model is likely to produce smaller volatility than Vasicek model. The Ahn/Gao model has the least credit spread option price than Cox, Ingersoll and Ross or Vasicek models, due to effects of stronger mean reverting drift and higher diffusion coefficient. The credit spread call option prices from any of the 3 chosen stochastic volatility model increases with maturity. This would imply that in/out of the money call options are likely to remain in/out of the money over time. In comparision with the Longstaff/Schwartz model, the mean reversion of the credit spread could result in credit spread reducing over time when above its long run mean or could increase over time when below its long run mean. This means that in the money call options are less likely to remain in the money over time, resulting in the likelihood of the call option on credit spread being less than its intrinsic value when only slightly in the money.

From a hedging perspective, our spread option model does not reduce the delta of long term calls to zero as in the case of the Longstaff/Schwartz model where mean reversion reduces the delta of long term puts and calls to zero. The delta of calls in our spread option model ranges from positive to negative values. Hence our spread option model can hedge the risk of changes in the current credit spread for both short and long term maturities.

This thesis has concentrated on the pricing and hedging of European call option on a credit spread with some or no correlation with the short term interest rate under an independent stochastic volatility process. Closed form pricing will always be obtainable if the stochastic volatility process under the consideration is of the mean reverting form with analytical tractability. Numerical pricing is obtainable via Monte Carlo simulation. Our pricing model can be extended to European credit spread put options via the put-call parity relationship. However there are areas that can be considered for further research as follows:

(a) Optimum choice of upper bound of closed form spread price double integral.

(b) Correlation between credit spread and volatility.

(c) Empirical analysis of mean reversion parameters of stochastic volatility processes.

(d) American credit spread option under a tree approach.

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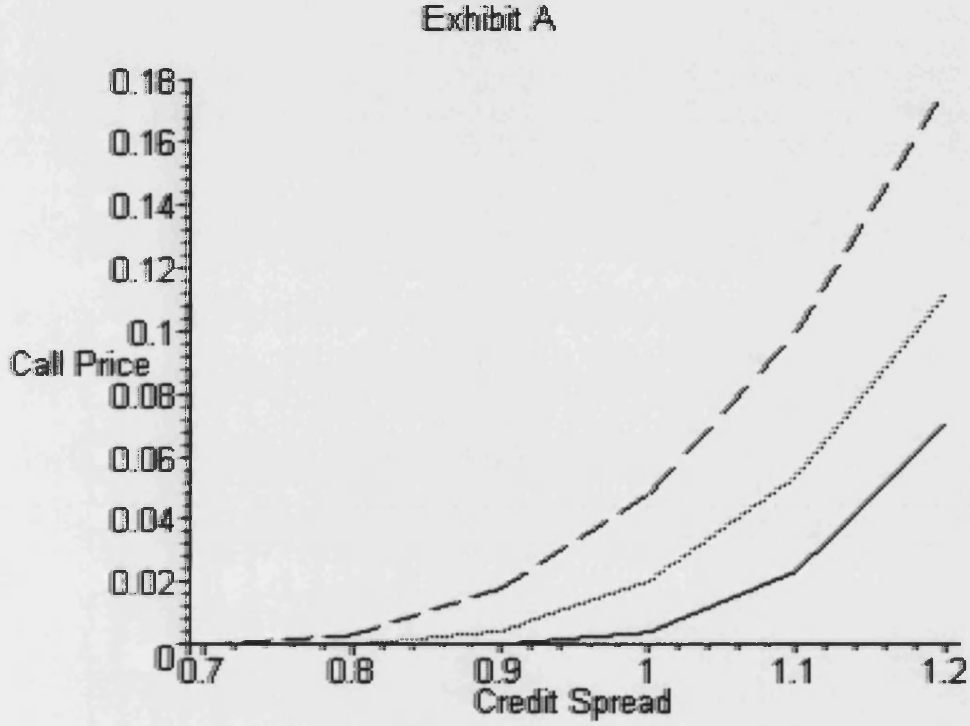
Plot of out of the money credit spread call prices obtained for Credit Spread model under the Cox Ross Stochastic volatility with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



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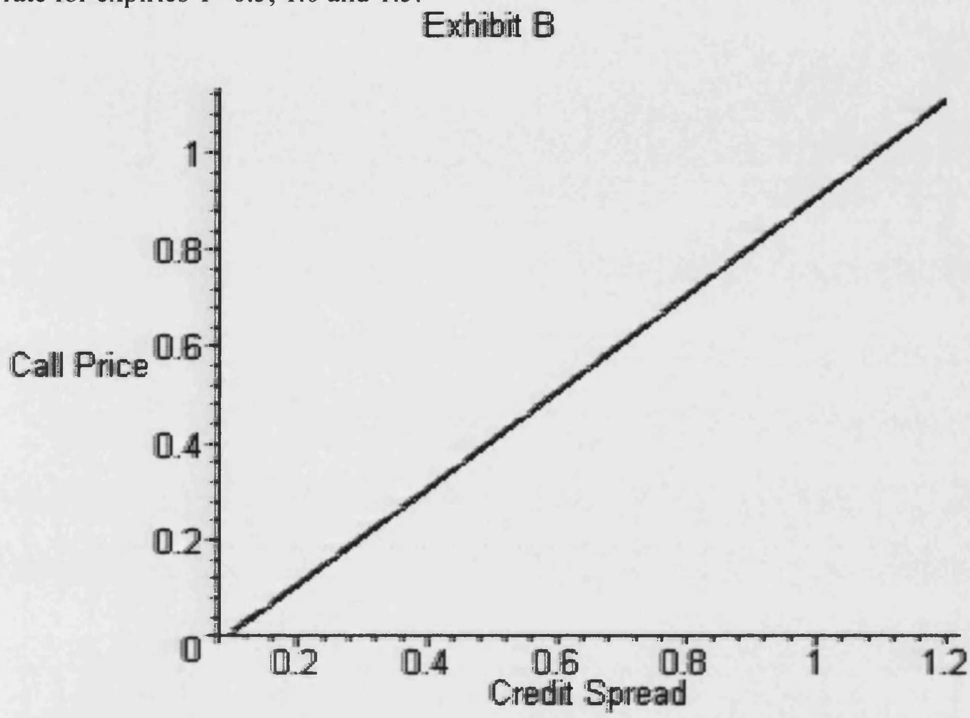
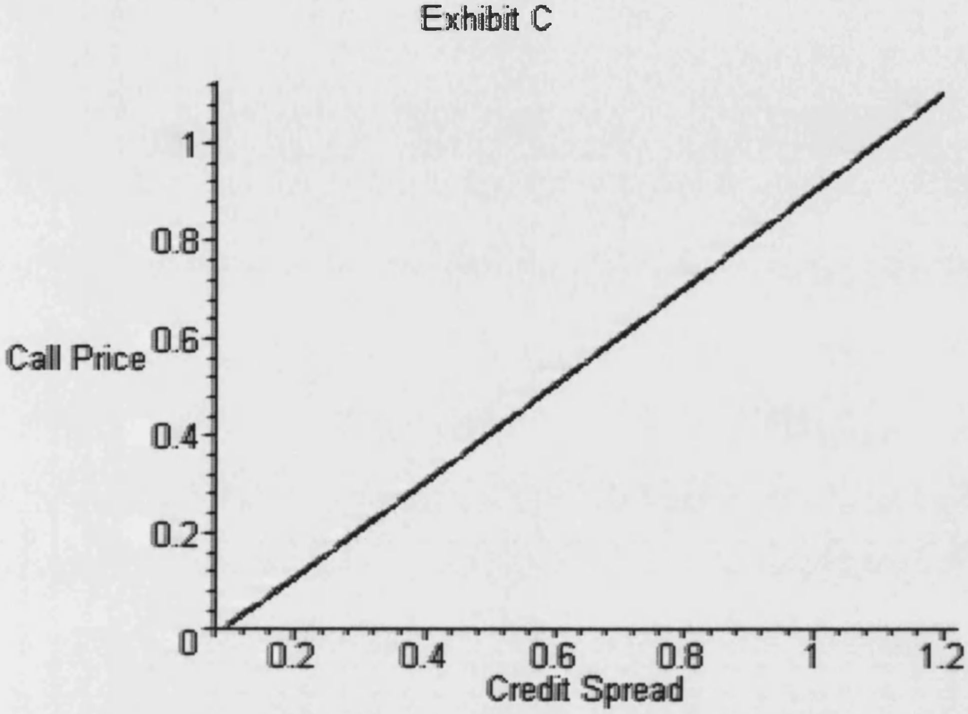


Plot of in the money credit spread call prices obtained for Credit Spread model under the Vasicek volatility with correlation of 0 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



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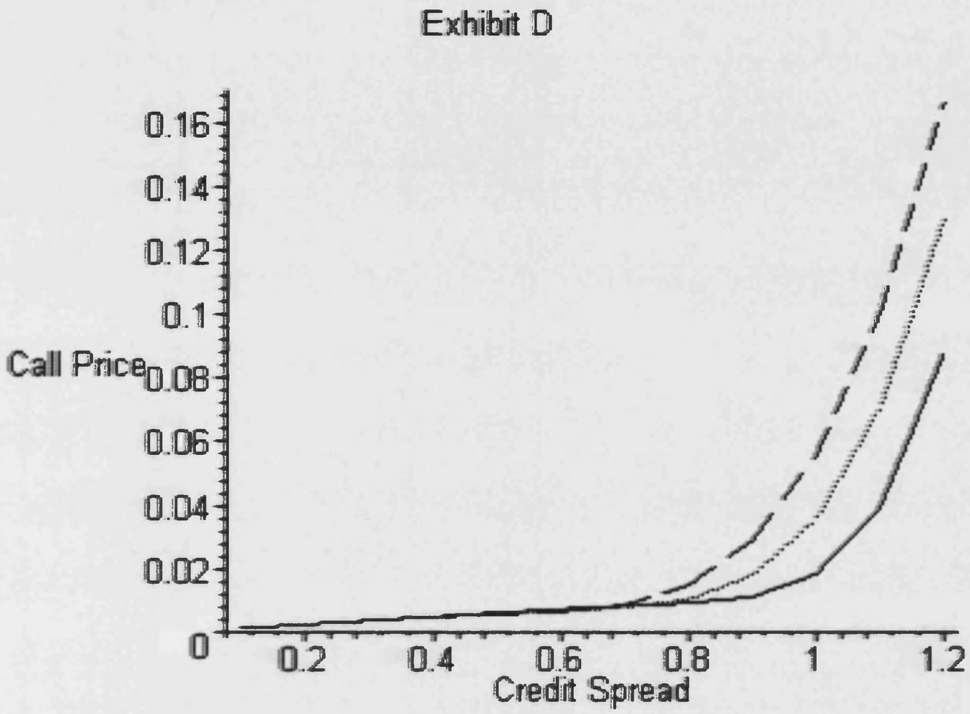
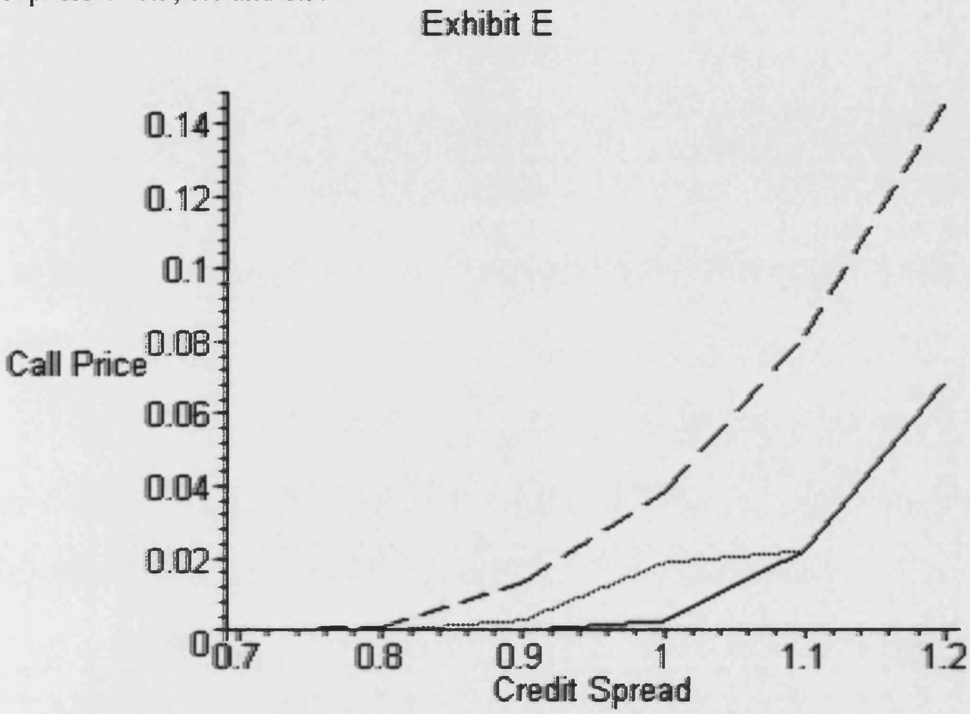
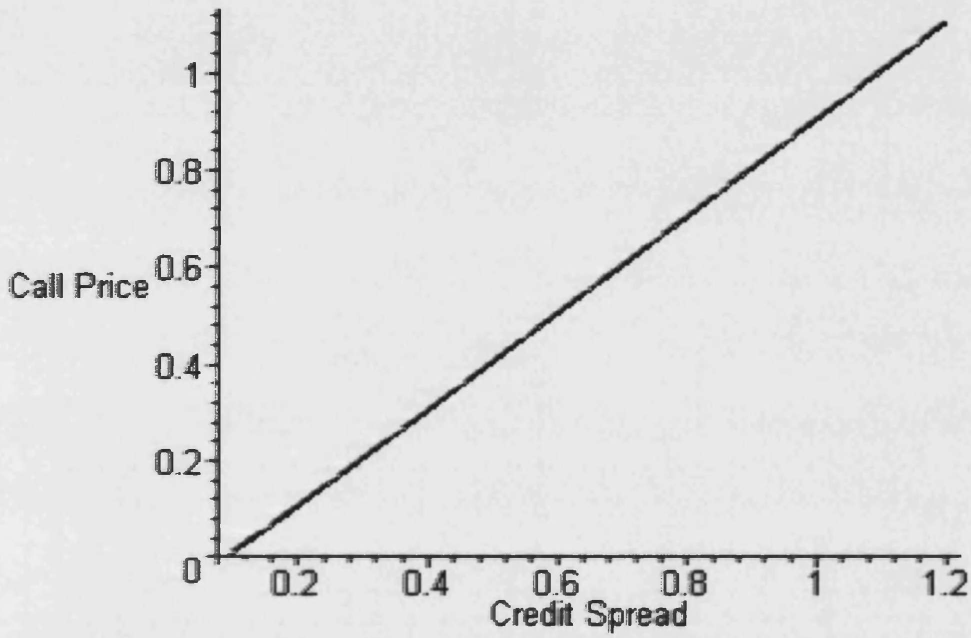


Plot of out of the money credit spread call prices obtained for Credit Spread model under the Ahn Gao volatility with correlation of 0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.

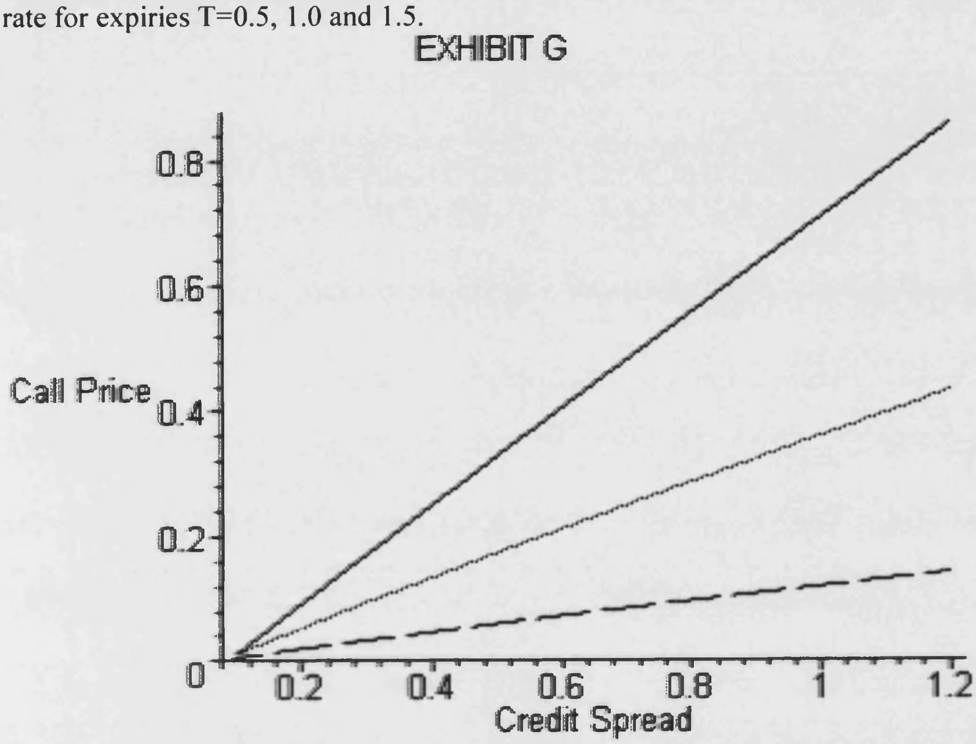


Plot of in the money credit spread call prices obtained for Credit Spread model under the Ahn Gao volatility with correlation of 0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.

Exhibit F



Plot of in the money credit spread call prices obtained for Credit Spread model under the Longstaff and Shwartz volatility with correlation of 0 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



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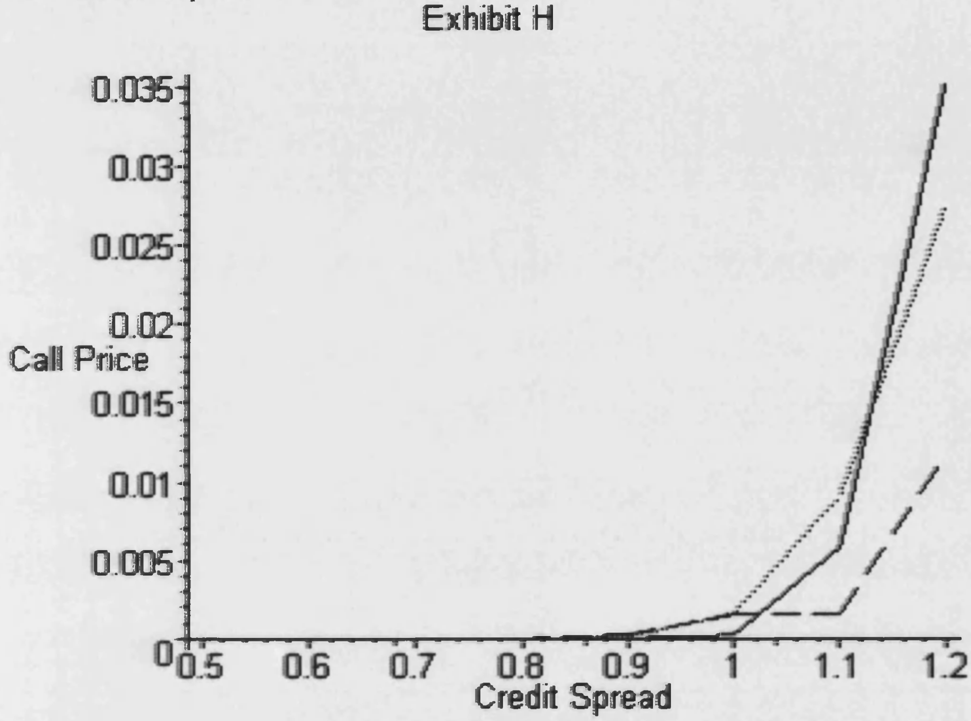

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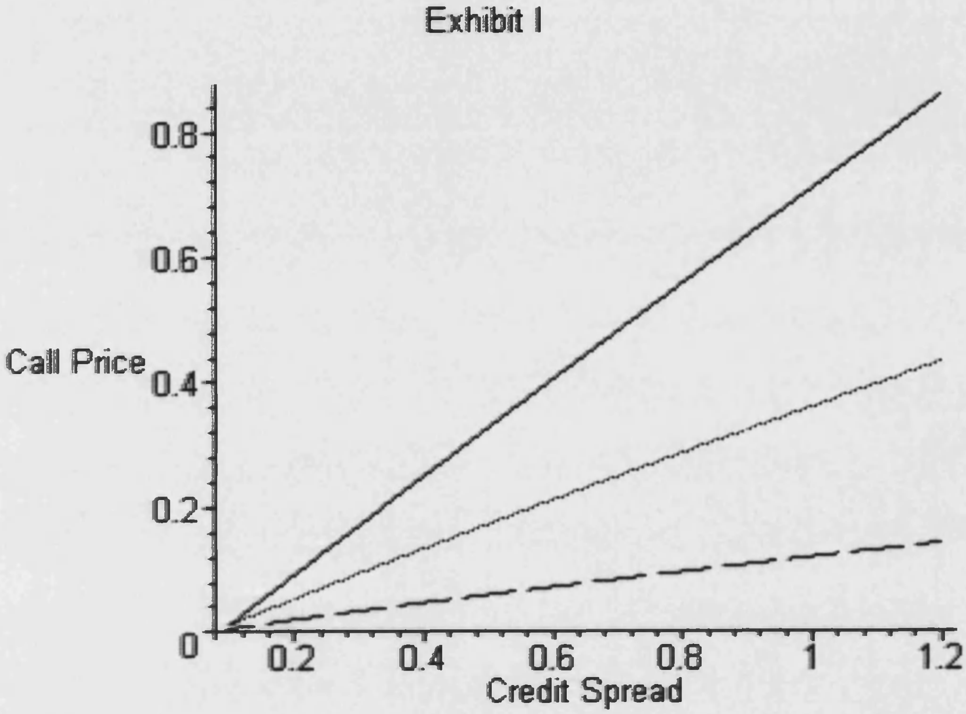

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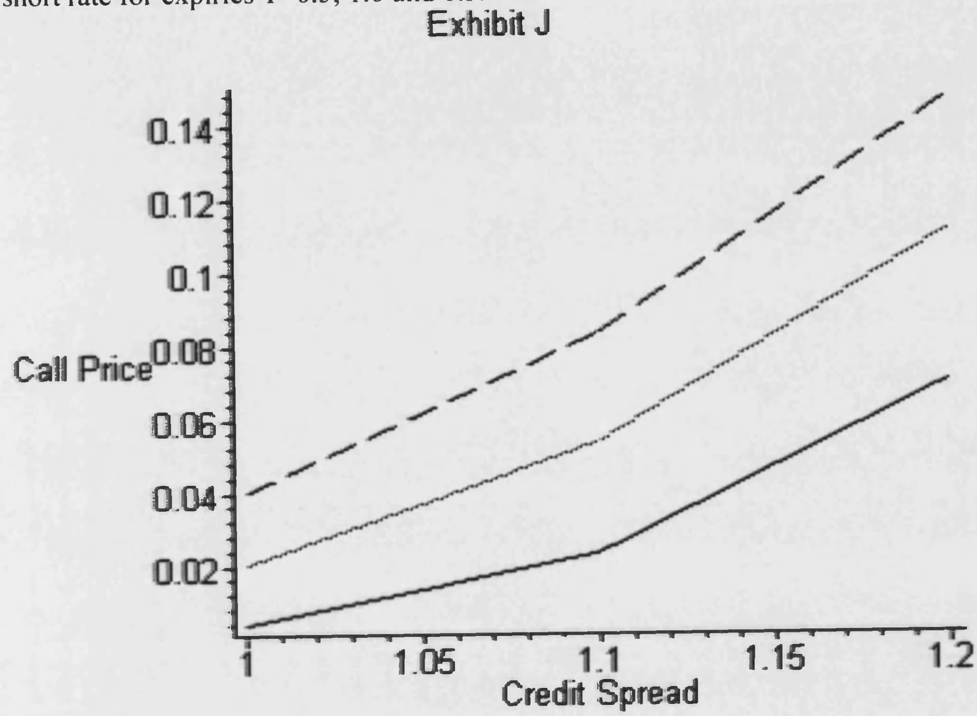
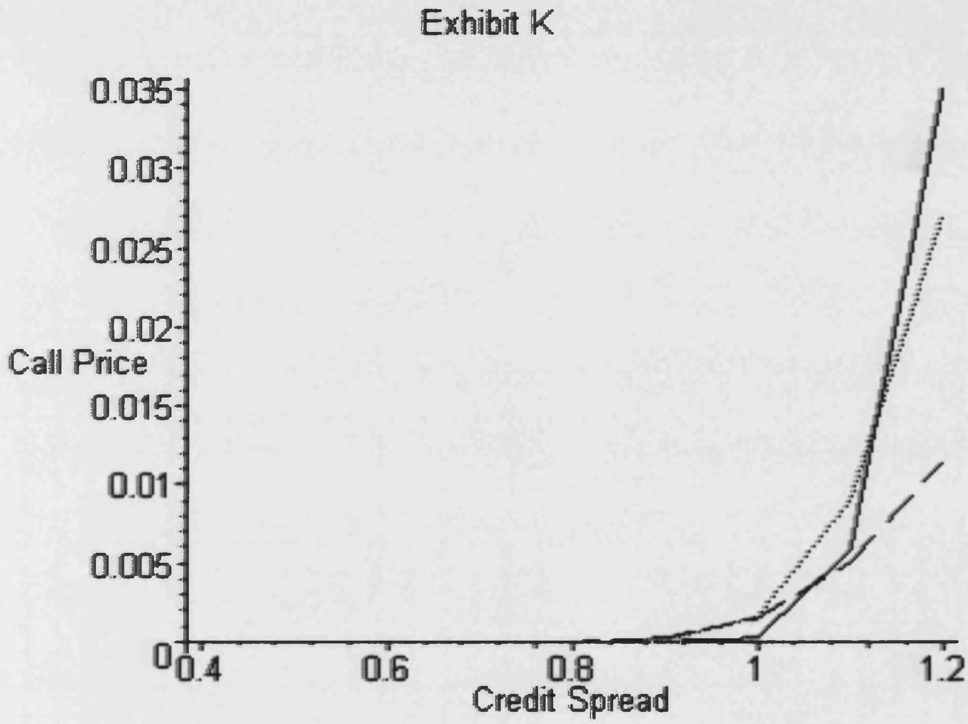
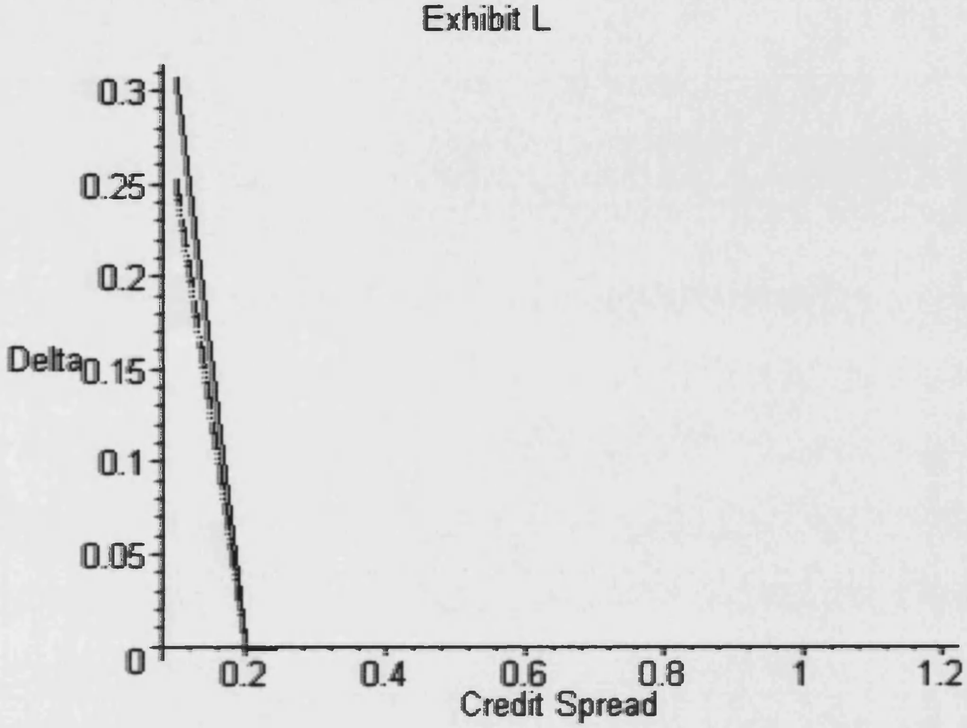


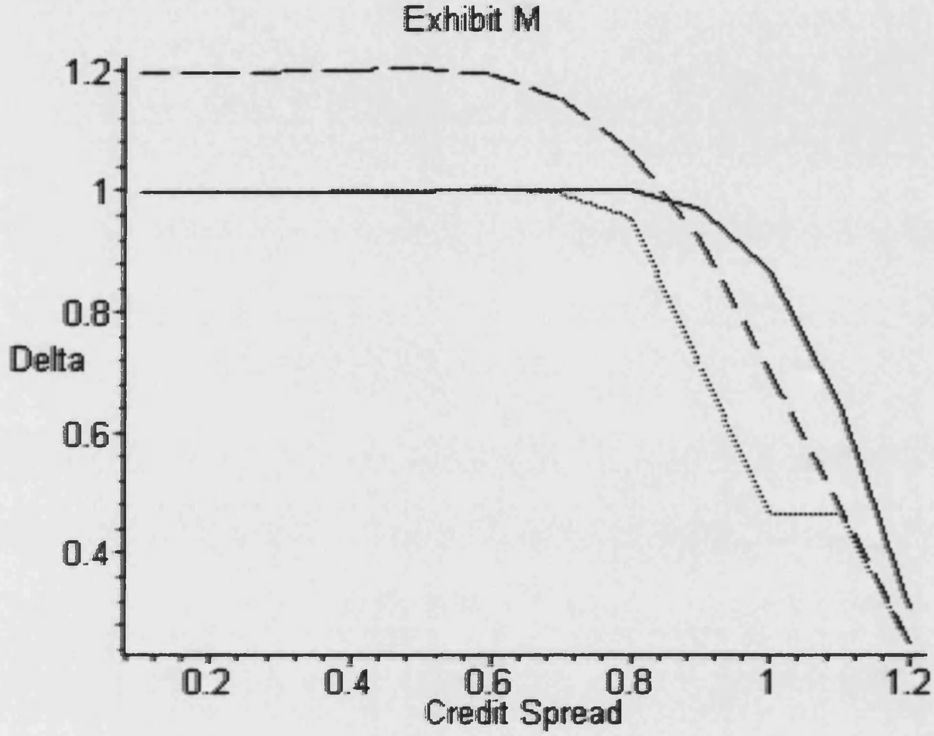
Plot of out of the money credit spread call prices obtained for Credit Spread model under the Longstaff and Shwartz volatility with correlation of 0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



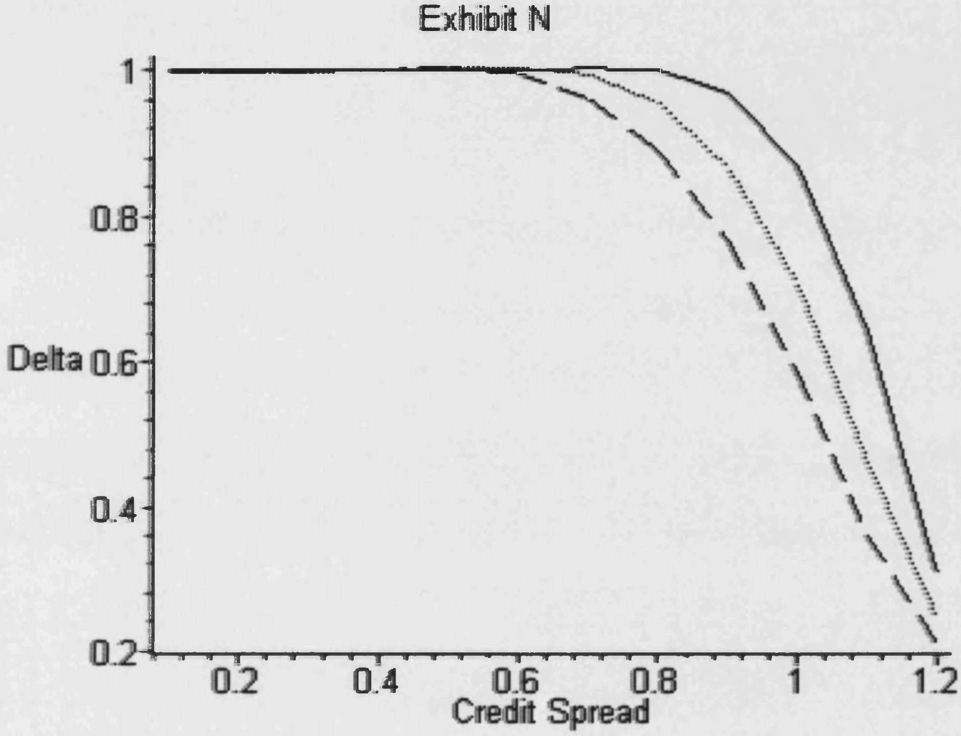
Plot of in the money Call deltas obtained for Credit Spread model under the Cox and Ross volatility with correlation of 0 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



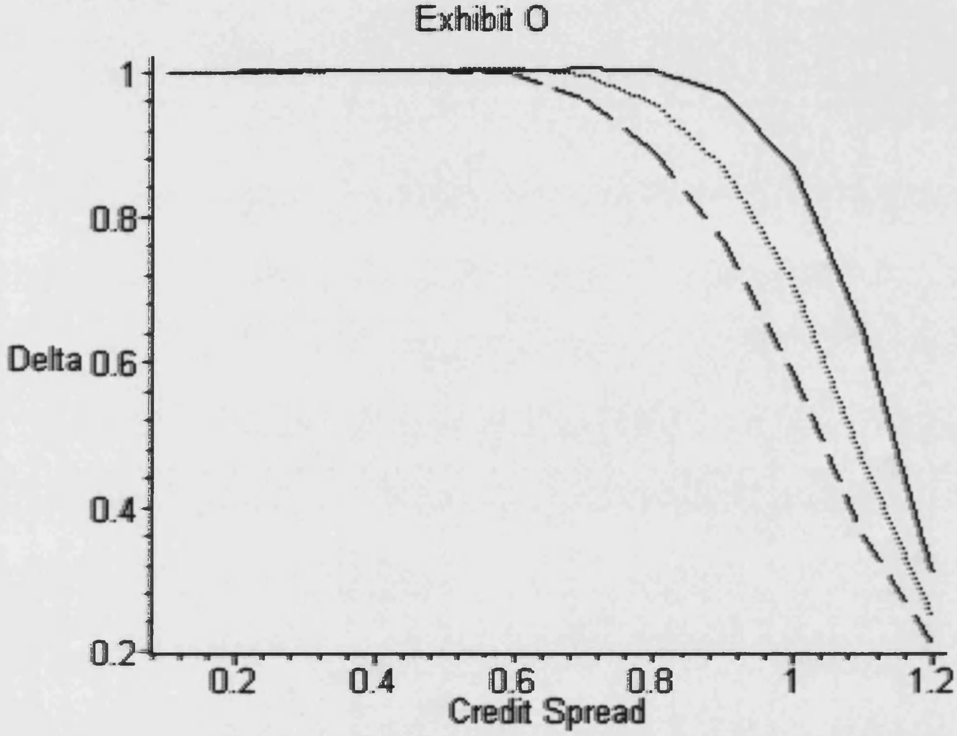
Plot of out of the money Delta call prices obtained for Credit Spread model under the Cox and Ross volatility with correlation of 0 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



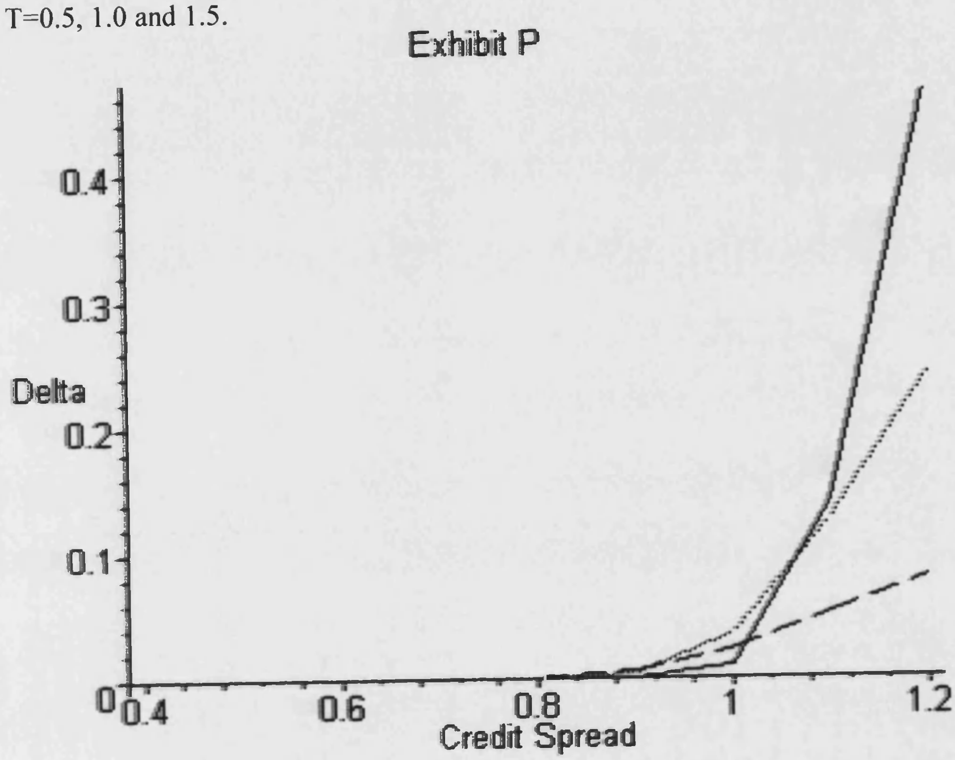
Plot of out of the money Call deltas obtained for Credit Spread model under the Cox and Ross volatility with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



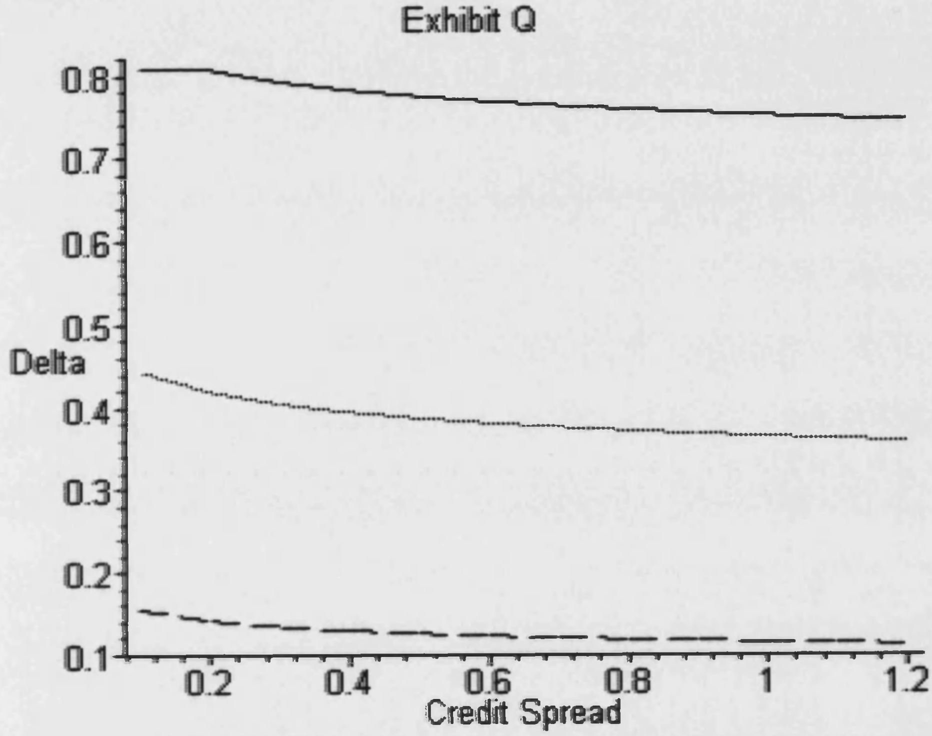
Plot out of the money Delta call prices obtained for Credit Spread model under the Vasicek volatility with correlation of 0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



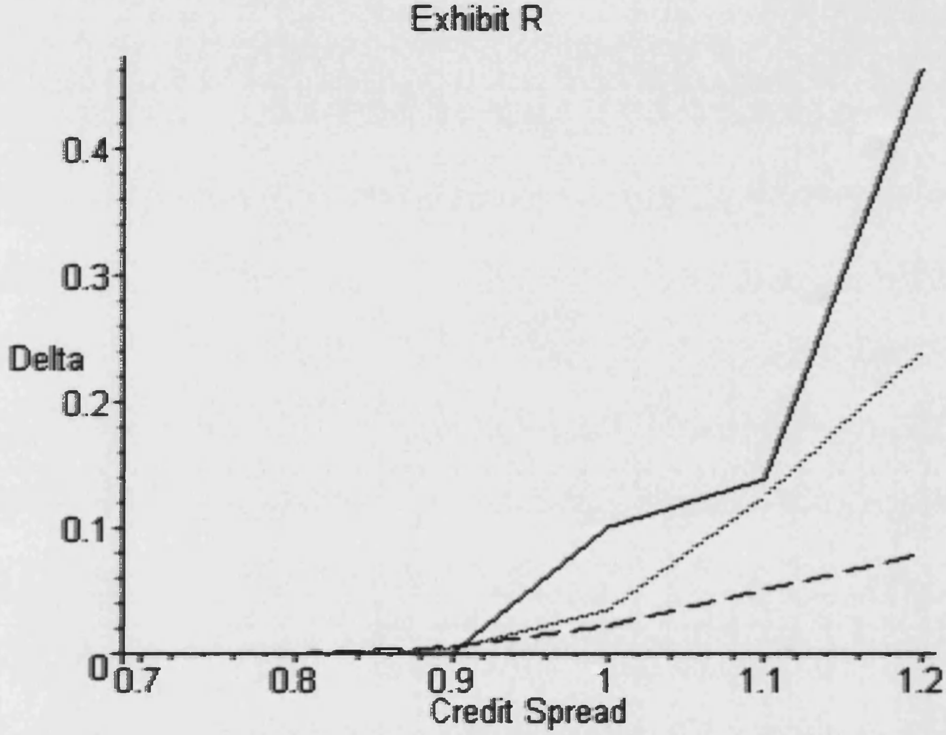
Plot of out of the money Delta call prices obtained for Credit Spread model under Longstaff Schwartz with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



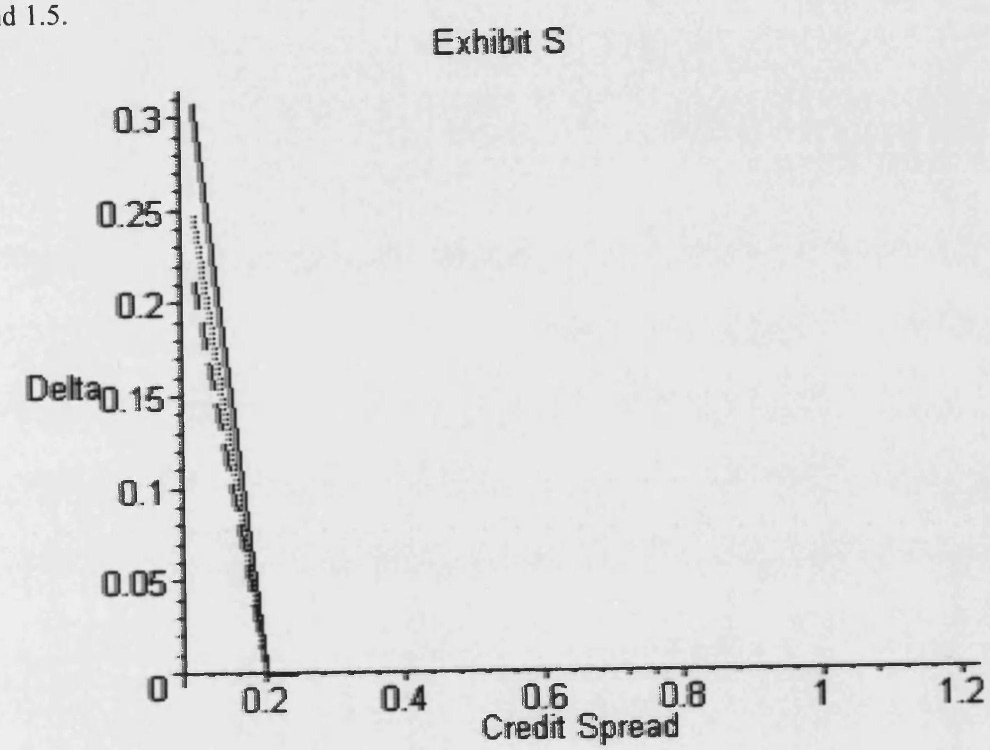
Plot of in the money Delta call prices obtained for Credit Spread model under Longstaff Schwartz with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



Plot of out of the money Delta call prices obtained for Credit Spread model under Longstaff Schwartz with correlation of 0 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



Plot of in the money Delta call prices obtained for Credit Spread model under Ahn Gao volatility with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.



Plot of out of the money Delta call prices obtained for Credit Spread model under Ahn Gao volatility with correlation of -0.2 between credit spread and short rate for expiries T=0.5, 1.0 and 1.5.