Finding combinatorial structures

Peter Allen

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Department of Mathematics

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is my own work, with the following exceptions.

The contents of Chapter 1 are well-known results due to various authors not including myself.

The contents of Chapter 2 are joint work with Graham Brightwell and Jozef Skokan.

The contents of Chapter 5 are joint work with Vadim Lozin and Michaël Rao.

Abstract

In this thesis we answer questions in two related areas of combinatorics: Ramsey theory and asymptotic enumeration.

In Ramsey theory we introduce a new method for finding desired structures. We find a new upper bound on the Ramsey number of a path against a kth power of a path: $R(P_n, P_n^k) \leq \left(k + 1 + \frac{1}{k+1}\right) n + o(n)$.

Using our new method and this result we obtain a new upper bound on the Ramsey number of the kth power of a long cycle:

$$R(C_n^k) \leq \left(2k + 4 + \frac{2}{k+2}\right) n + o(n) \ .$$

As a corollary we show that, while graphs on n vertices with maximum degree k may in general have Ramsey numbers as large as $c^k n$, if the stronger restriction that the bandwidth should be at most k is given, then the Ramsey numbers are bounded by the much smaller value $\left(2k + 2 + \frac{2}{k+1}\right) n + o(n)$.

We go on to attack an old conjecture of Lehel: by using our new method we can improve on a result of Łuczak, Rödl and Szemerédi [60]. Our new method replaces their use of the Regularity Lemma, and allows us to prove that for any $n > 2^{18000}$, whenever the edges of the complete graph on n vertices are two-coloured there exist disjoint monochromatic cycles covering all n vertices.

In asymptotic enumeration we examine first the class of bipartite graphs with some forbidden induced subgraph H. We obtain some results for every H, with special focus on the cases where the growth speed of the class is factorial, and make some comments on a connection to clique-width. We then move on to a detailed discussion of 2-SAT functions. We find the correct asymptotic formula $\big(1 + o(1)\big)2^{\binom{n}{2}+n}$ for the number of 2-SAT functions on n variables (an improvement on a result of Bollobás, Brightwell and Leader [13], who found the dominant term in the exponent), the first error term for this formula, and some bounds on smaller error terms. Finally we obtain various expected values in the uniform model of random 2-SAT functions.

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Contents

1

Introduction and preliminaries

1.1 Introduction

Given an object G and a property P, one can ask 'Does G possess the property P?' This may be a trivially easy question, or an exceptionally hard question: it depends, of course, on the object and the property given.

For example, one could be asked to check if the graph G has chromatic number 2. In principle this is easy to check, as one can simply check every possible bipartition of the vertex set; in practice this is a very slow algorithm. But one can instead colour one vertex at a time, if possible choosing a neighbour of a previously coloured vertex and if possible maintaining a proper 2-colouring. This gives a 2-colouring if one exists, and is quick even for quite large graphs.

On the other hand, the question might be whether the graph G is Hamiltonian. Now this is again in principle easy to check: but the brute force method is still slow, and this time there is no known quick method.

Alternatively, one might be asked whether the graph G possesses the Erdős-Hajnal property [30] (that there is an upper bound, polynomial in n, on the order of a graph containing none of K_n, E_n and G as an induced subgraph).

In this case it is not clear that there is any algorithm, fast or slow, which will reveal an answer, and indeed the answer is not known for most graphs.

In this thesis we will examine several properties which amount to checking whether G contains some specific substructure. Of course, these properties are in principle easy to check, and in fact there will invariably be an algorithm whose running time is polynomial in the order of G. On the other hand, we will spend very little time checking individual graphs. We will instead answer questions such as 'How many graphs possess property P?', 'What do graphs with property P look like?' and 'How can we find this substructure?'.

The difficulty of answering these questions is closely related to the difficulty of checking whether individual graphs possess the property. It is usually quite easy to analyze a quick algorithm and determine whether it will answer 'Yes' or 'No' for most graphs, because such an algorithm gives a 'good reason' along with the answer, but it is usually hard to analyze a brute-force algorithm. For example, if a graph G does not have chromatic number 2 then the above algorithm will find either no edges or an odd cycle in G. Now we have an idea: we know that most graphs contain a triangle, so we know that most graphs do not have chromatic number 2. On the other hand, if one runs the brute-force Hamiltonicity algorithm on a graph and finds that it is not Hamiltonian, then the 'reason' will be the rather unhelpful one that all the vertex orders were tried and none worked.

We will answer some questions in two major areas of combinatorics dealing with these problems: Ramsey theory and asymptotic enumeration. Our notation will follow standard usage (see for example Bollobás [10]), and in particular logarithms will be to base 2 unless stated otherwise.

1.2 Thesis outline

The rest of **Chapter 1** is an introduction to the ideas and methods used in Ramsey theory and asymptotic enumeration. We discuss outstanding problems and state (sometimes with proof) well-known theorems which will be useful in the succeeding chapters. We pay particular attention to the Szemerédi Regularity Lemma and its associated results.

In **Chapter 2** we introduce a method for solving certain problems in Ramsey theory. Given a two-edge-coloured complete graph G we construct an auxiliary two-edge-coloured complete graph whose nodes are disjoint monochromatic cliques in G. We prove a variant on the Blow-up Lemma which allows us to find results on some relatively hard problems by applying simpler Ramsey results to the auxiliary graph.

The kth power of a graph G is defined to be the graph G^k on the same vertex set with xy an edge of G^k whenever x and y are at distance not more than k in G.

Our simple Ramsey result is that when the edges of the complete graph on $N(n)$ vertices are coloured red or blue in any way, there is either a red P_n or a blue P_n^k, where $N(n)$ is a certain function of the form $\left(k+1+\frac{1}{k+1}\right)n+o(n)$. Our new method then gives us bounds on the Ramsey number of the kth power of a long cycle:

$$\left(k+\frac{1}{k+1}\right)n+o(n) \leq R(C_n^k) \leq \left(2k+4+\frac{2}{k+2}\right)n+o(n) \ .$$

This immediately gives the same upper bound on the Ramsey number of any n-vertex graph with bandwidth k; the best previous bound was that of Graham, Rödl and Ruciński, who showed that there exists c such that any n-vertex graph with maximum degree $2k$ (so covering graphs with bandwidth k) has Ramsey number at most $c^{k\log k}n$.

This chapter is joint work with Graham Brightwell and Jozef Skokan.

In **Chapter 3** we prove that the vertices of every two-edge-coloured complete graph on at least 2^{18000} vertices can be covered by a red cycle and a blue cycle which are vertex-disjoint. This is a partial solution to Lehel's conjecture (which states that the result holds for every two-edge-coloured complete graph) and an improvement on a previous result of Łuczak, Rödl and Szemerédi [60] (which required a much larger number of vertices). Our improvement comes from avoiding in the proof any use of the Regularity Lemma; we use instead the method described in the preceding chapter.

A version of this chapter has appeared in Combinatorics, Probability and Computing [3].

Chapter 4 turns to asymptotic enumeration. Given a bipartite graph H we find the speed of the class of labelled H-free bipartite graphs: that is, we estimate the number of labelled bipartite graphs on n vertices with no induced copy of H.

We focus particularly on the cases where bounds of the form $n^{cn+o(n)}$ exist. Here we either find the correct value of c, or at least place it within a small interval (with the single exception of P_7 where we have no good upper bounds); we also give some details of the structure of a typical H-free graph in these cases. Our upper bounds come from finding fairly natural compact ways to record the information in an H-free graph, while our lower bounds are straightforward constructions.

A version of this chapter will appear in the Journal of Graph Theory [4].

Chapter 5 is a short digression on clique-width in hereditary graph classes.

We repeat the definition of clique-width of a graph given by Courcelle, Engelfriet and Rozenberg [24], and recall that a class of graphs $\mathcal{X}$ is said to have bounded clique-width if there is some constant c such that no graph in $\mathcal{X}$ has clique-width exceeding c. We observe that the hereditary classes considered in the previous chapter have bounded clique-width if and only if their speed is bounded by a function of the form $n^{n+o(n)}$.

We show that a weak version of this is true in general: if $\mathcal{X}$ is a hereditary graph class whose speed is eventually bounded above by the Bell number (which counts the number of ways to partition $[n]$ into disjoint subsets and is a function of the form $n^{n+o(n)}$) then its clique-width is bounded, while if for every c the speed of $\mathcal{X}$ eventually exceeds $n!c^n$ then its clique-width is unbounded.

However we also show that a strong version is not true: there is no function $f(n)$ such that the hereditary graph class $\mathcal{X}$ has bounded clique-width if and only if its speed is eventually bounded by $f(n)$.

This chapter is joint work with Vadim Lozin and Michaël Rao.

Finally in **Chapter 6** we examine 2-SAT functions in some detail.

A Boolean variable has a positive and a negative literal, which are True respectively with and against the Boolean variable.

A 2-SAT formula (on n variables) is a collection of sets (called clauses) each of two literals chosen from the $2n$ literals corresponding to the n variables. If there exists an assigment of truth values to the variables such that every clause of the formula contains at least one True literal, then the formula is called satisfiable, and the assignment is a satisfying assignment.

A 2-SAT function, then, is a Boolean function defined by the satisfying assignments of a 2-SAT formula. One obvious large class of 2-SAT functions is the class of unate 2-SAT functions. These are obtained by choosing for each of the n variables one of its two literals, then constructing a 2-SAT formula using only the chosen literals. Bollobás, Brightwell and Leader [13] showed that there are $2^{n^2/2+o(n^2)}$ 2-SAT functions on n variables, and conjectured that almost every 2-SAT function is unate.

We are able to prove their conjecture, and thus show that the number of 2-SAT functions is $\big(1+o(1)\big)2^{\binom{n}{2}+n}$. We proceed to find the next largest class of 2-SAT functions and so determine the first error term in the above estimate, and go on to bound the sizes of the next error terms. Our work allows us to approximate various expected values in the uniform model of random 2-SAT functions.

A version of this chapter has appeared in the Israel Journal of Mathematics [2].

1.3 Ramsey theory

Ramsey theory deals with properties that hold for every sufficiently large structure. We recall Ramsey's theorem [66], describing a graph version essentially due to Erdős and Szekeres [32] (we give a slightly weaker statement for simplicity).

Theorem 1.1. *Every graph G on at least 4^k vertices contains either a clique or an independent set of k vertices.*

Proof. Let $G(V,E)$ be a graph on 2^{2k} vertices. Pick a vertex x_1 of G. It has either at least 2^{2k-1} neighbours or at least that many non-neighbours among the other vertices of G. Let V_1 be the larger of $\Gamma(x_1)$ and $V - \Gamma(x_1)$. Now for each $2 \leq i \leq 2k-1$ in succession, pick $x_i \in V_{i-1}$, and let V_i be the larger of $V_{i-1} \cap \Gamma(x_i)$ and $V_{i-1} - \Gamma(x_i)$, choosing the former in the event of a tie. This is always possible since $|V_i| \geq 2^{2k-i}$.

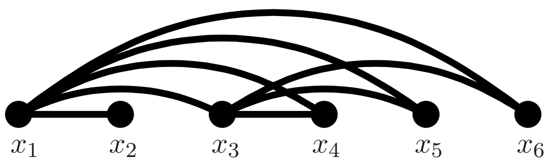


Figure 1.1 $G[x_2, x_4, x_5, x_6]$ forms an independent set.

The graph $G[\{x_1, \ldots, x_{2k-1}\}]$ has the property that each x_i sends edges forward to either all or none of the x_j, $j > i$. Now if there are k vertices among the x_i which do send edges forward then these k vertices form a clique: but if there are not k such vertices, then there are k vertices which do not send edges forward, and these form an independent set. $\square$

It is often easier to describe and prove Ramsey-style theorems in an equivalent setting: instead of examining simple graphs, we will discuss complete graphs in which the edges are coloured either red or blue. In this setting, Ramsey's theorem becomes: Every two-coloured complete graph on 4^k vertices contains a monochromatic complete subgraph on k vertices. For convenience we use normal graph terms to discuss these graphs, prefixing with 'red-' rather than the cumbersome 'In the subgraph given by taking only the red edges'.

The *Ramsey number* $R(H)$ of a graph H is defined to be the smallest n such that any n-vertex two-coloured complete graph contains a monochromatic H. The existence of $R(H)$ for any H is guaranteed by Ramsey's theorem (as H is a subgraph of the complete graph on $|H|$ vertices), but correct values are known only for a very few graphs H, and even good bounds are not known in most cases. Often it is clear that an existing upper bound is far from optimal, but if H is a complicated graph then finding implications of the statement 'This two-coloured complete graph does not contain a red copy of H' that help us find instead a blue copy of H is hard. Indeed, even in the seemingly simple case $H = K_k$ and after considerable work, the best upper bound on $R(K_k)$ is only a little better than the trivial argument above: Conlon [21] proved that there exists a constant $C > 0$ such that

$$R(K_{k+1}) \leq k^{-C\frac{\log k}{\log\log k}} \binom{2k}{k} .$$

On the other hand, Erdős [27] proved the first exponential lower bound: that $R(K_k) > \frac{k}{e\sqrt{2}} 2^{\frac{k}{2}}$, and the best lower bound, due to Spencer[68], is $R(K_k) > \frac{k\sqrt{2}}{e} 2^{\frac{k}{2}}$—leaving a huge gap between the lower and upper bounds.

If one is presented with a sparse graph H then it is clear that the Ramsey number should not be exponential in $V(H)$: the condition 'This two-coloured complete graph contains no red path on k vertices' is a much stronger condition than denying the existence of a red clique on k vertices; and at the same time we do not need especially strong conditions on the blue edges to construct a blue P_k. We should expect to find a monochromatic path covering a significant fraction of any two-coloured complete graph. On the other hand, even though P_k is as sparse as a connected graph can be, we cannot guarantee to cover all, or even almost all, of the vertices. We can construct a graph G on vertex set $A \sqcup B$, with A of size $k-1$ and B of size $\lfloor \frac{k}{2} \rfloor - 1$ forming red cliques, and the remaining edges between A and B all blue.

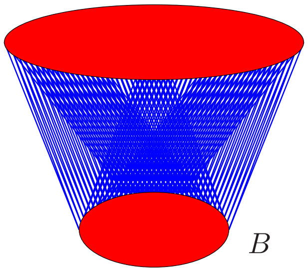


Figure 1.2 No monochromatic P_k exists

In this graph the longest monochromatic path covers only about $\frac{2}{3}$ of the vertices, since the longest red path covers A, while the vertices of any blue path must alternate between A and B, so again no blue path can contain more than $k-1$ vertices. In fact, the paths are one of the few graph classes where the Ramsey numbers are known exactly. Gerencsér and Gyárfás [37] proved an upper bound to match this construction: $R(P_k) = k + \lfloor \frac{k}{2} \rfloor - 1$ for $k \geq 2$.

We will be particularly interested in finding bounds for Ramsey problems involving sparse graphs; and we will seek linear sized bounds. Naturally, we need to be precise about what constitutes a 'sparse' graph. Consider the 'furry ball graph' G_k: this is a connected graph on k vertices, $2(\log k)^2$ of which form a clique, with the remaining vertices all of degree one. We know from Spencer's bound that $R(G_k) \geq k^{\log k}$, which is super-polynomial, even though G_k is overall only slightly denser than the path on k vertices.

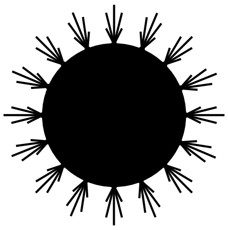


Figure 1.3 The furry ball

On the other hand, Chvátal, Rödl, Szemerédi and Trotter [20] showed that there is a constant $c(\Delta)$ such that every graph on k vertices with maximum degree Δ—so allowing graphs with almost $\frac{\Delta}{2}$ times more edges than the furry ball graphs—has Ramsey number bounded above by $c(\Delta)k$.

Recall that a graph G is called *d-degenerate* if it and all its subgraphs have minimum degree at most d. Burr and Erdős [17] conjectured that bounded degeneracy should be a good definition of 'sparse': that is, there should exist $c(d)$ such that the d-degenerate graphs on k vertices have Ramsey number no larger than $c(d)k$. This remains one of the major open problems in Ramsey theory.

A major problem with finding both upper and lower bounds on Ramsey numbers is that we do not in general know very much about what the right extremal structures should look like. For paths and cycles we do know exactly, and we can find the exact Ramsey numbers; but for most other graph classes we have only some ideas and guesses—and relatively poor bounds.

1.4 Asymptotic enumeration

When one is presented with a graph property, there are two questions one could naturally ask: does a typical graph possess this property, and what does a typical graph with this property look like?

For many interesting graph properties, the answer to the first question is 'no', and rather than considering the probability that a graph on n vertices chosen uniformly at random possesses the property (which usually approaches zero rapidly), it is convenient to replace the first question with: how many labelled graphs on n vertices possess this property? Balogh, Bollobás and Weinreich [6], defined the *speed* of a class of graphs $\mathcal{X}$ to be the function $n \to |\mathcal{X}_n|$, where $\mathcal{X}_n$ is the subclass of $\mathcal{X}$ consisting of graphs on n vertices.

Of course, for a general graph class nothing of interest can be said about the speed. Recall that a class of graphs $\mathcal{X}$ is called *hereditary* if whenever $G \in \mathcal{X}$, every induced subgraph of G is also in $\mathcal{X}$. Even the weak constraint that the class $\mathcal{X}$ is hereditary is enough to lead to interesting results. Again it is clear that little of interest can be said about $|\mathcal{X}_n|$ for small values of n: but the asymptotics of the speed of hereditary graph classes are sharply constrained. Scheinerman and Zito [67] originally showed that speeds must lie in one of several broad categories (eventually constant, polynomial, exponential, factorial or superfactorial) and Balogh, Bollobás and Weinreich [6], [7], [8] sharpened their results. In particular, they showed that, while hereditary classes of graphs have highly constrained and well-behaved speeds when those speeds are bounded above by $n^{(1-\varepsilon)n}$ or below by $2^{\varepsilon n^2}$ for any $\varepsilon > 0$, this is no longer true for hereditary classes whose speeds lie within that gap. For example, some such classes have speeds which oscillate between n^{cn} and $2^{n^{2-\varepsilon}}$.

It remains interesting to ask what the speed of a specific graph property is. Lerchs [54] defined a *cograph* as follows: G is a cograph if it is a single vertex, or if it is the disjoint union of cographs, or if it is the complement of a cograph. The property of being a cograph is a hereditary property, and one with a simple forbidden subgraph characterisation due to Corneil, Lerchs and Stewart Burlingham [23]. Note that by P_4 we mean the path on four vertices (not with four edges).

Lemma 1.2. *A graph G is a cograph if and only if it is P_4-free.*

Proof. First we show that the property of being a cograph is hereditary.

Suppose not: let G be a cograph of minimum order which has an induced subgraph that is not a cograph, and let H be the smallest induced subgraph of G which is not a cograph. The singleton graph K_1 is a cograph, so G has at least three vertices. Since the complements $\overline{G}$ and $\overline{H}$ also form another minimal pair, we can assume that G is the disjoint union of two smaller cographs G' and G''. By minimality of G, neither G' nor G'' contains H, and so by minimality of H both $G' \cap H$ and $G'' \cap H$ are cographs. But H is the disjoint union of $G' \cap H$ and $G'' \cap H$, which is a contradiction.

Now we prove the characterisation.

Since P_4 is connected and its complement is P_4, it is not a cograph. The property of being a cograph is hereditary, so every cograph is P_4-free.

Now suppose that G is a minimal P_4-free graph which is not a cograph. Let x be a vertex. Since $G - x$ is still P_4-free it is a cograph. As K_1, K_2 and E_2 are cographs $G - x$ has at least two vertices, so either it is disconnected or its complement is disconnected. Without loss of generality assume it is disconnected.

Since G is a minimal P_4-free non-cograph, it is connected and its complement is connected. In particular, there is a vertex $y \neq x$ not adjacent to x. Choose a vertex z in a component of $G - x$ not containing y.

Because G is connected, there is a minimum-length path from y to z. This path must go through x, as y and z are in different components of $G - x$, and it does not go directly from y to x. Therefore it is a path on at least four vertices, and by its minimality any four consecutive vertices induce a copy of P_4 in G. □

Of course, if G is any cograph we may write a formula for it in terms of the smaller cographs from which it is built: using the symbols $\oplus$ and `COMPLEMENT` for the graph operations of disjoint union and taking complement, together with brackets and the vertex labels for the single-vertex basic cographs.

If H is any fixed graph, Prömel and Steger [64] considered the property of being *H-free*: that is, of not containing H as an induced subgraph. Every hereditary property is characterised by a set of forbidden induced subgraphs, so the classes they consider are the simplest hereditary classes.

Theorem 1.3. *If H is an induced subgraph of P_4 then the speed of the H-free graphs is bounded above by $n^{n+o(n)}$. Otherwise the speed is bounded below by $2^{\left(\frac{1}{4}+o(1)\right)n^2}$.*

Proof. If H is an induced subgraph of P_4 then an H-free graph is certainly P_4-free, therefore it is a cograph by Lemma 1.2.

The formula for a cograph consists of the symbols `COMPLEMENT`, $\oplus$, (,) and the vertex labels. Since taking the complement of the complement of a graph has no effect, we may assume that there are at most n of any of the symbols appearing in the formula for a cograph on n vertices, so that the formula consists of a string of length at most $4n$ of symbols chosen from an alphabet of five (we include a blank character to allow for the cases when less than n of any symbol is used) with n vertex labels in some order inserted at various positions. It follows that the number of cograph formulae for cographs on

n vertices is at most $\binom{5n}{n}n!5^n = n^{n+o(n)}$, and therefore this is also a bound on the speed of the H-free graphs.

If on the other hand H is not an induced subgraph of P_4 then H contains one of K_3, E_3, C_5 or C_4 as an induced subgraph: by Ramsey's theorem if H has six or more vertices one of the first two holds, and it is easy to check for graphs on five or less vertices.

Bipartite graphs do not contain either K_3 or C_5 as induced subgraphs, while the complements of bipartite graphs do not contain E_3 as an induced subgraph. The *split graphs*, defined as those graphs whose vertex set admits a partition into a clique and an empty set, do not contain C_4 as an induced subgraph. It is easy to check that each of these three graph classes has size $2^{\left(\frac{1}{4}+o(1)\right)n^2}$; since the H-free graphs must contain at least one of these three classes, their speed is at least this great. $\square$

This theorem relies on a good description of the P_4-free graphs, and provides a reasonably accurate count of the P_4-free graphs. It is also not hard to describe the H-free graph classes for H a proper induced subgraph of P_4:

The only K_1-free graph is the null graph: the speed is $n \to 0$

The K_2-free graphs are the empty graphs, and the E_2-free graphs are the complete graphs, each with speed $n \to 1$.

The P_3-free graphs are disjoint unions of complete graphs, and the $\overline{P_3}$-free graphs are their complements, both by definition having speed equal to the Bell function $n \to \mathcal{B}_n = n^{n+o(n)}$.

However for the larger graph classes the theorem does not say so much: for example most K_4-free graphs are not bipartite graphs, split graphs or complements of bipartite graphs, and there are significantly more than $2^{\left(\frac{1}{4}+o(1)\right)n^2}$ of them (every tripartite graph is K_4-free)[1]. This is an illustration of a general phenomenon. Answers to the two basic questions tend to come together: more accurate description of the typical structure within a graph class and more accurate bounds on the speed of the class are often

[1] Erdős, Kleitman and Rothschild [31] gave a good asymptotic estimate of the number of K_4-free graphs, by showing that they are almost surely tripartite.

produced by the same argument.

1.5 Quasirandomness and Regularity

1.5.1 Quasirandomness

If U and V are disjoint subsets of vertices of some graph, then we define the *density* of the pair (U, V) to be $d(U, V) = \frac{e(U,V)}{|U||V|}$.

A (large) graph on m vertices is called *quasirandom* (with density d and parameter ε) if it has the property that for any disjoint subsets U and V of the vertices, each of size at least εm, the density $d(U, V)$ is within ε of d. Clearly a random graph whose edges are chosen independently with probability d has this property with high probability. Chung, Graham and Wilson [18] showed that being quasirandom (for sufficiently large graphs with sufficiently small values of ε) implies several other properties which a random graph satisfies with high probability.

For any $\varepsilon > 0$ almost every graph is quasirandom with density $\frac{1}{2}$, so that a structure theorem should capture the idea of quasirandomness. However many graphs—for example complete balanced bipartite graphs—are far from being quasirandom.

If disjoint sets of vertices X_1, X_2 have the property that for any pair of subsets $X'_i \subset X_i$, $|X'_i| \geq \varepsilon|X_i|$, the density $d(X'_1, X'_2)$ is within ε of the density $d(X_1, X_2)$ then (X_1, X_2) is called an *ε-regular pair*; this is the bipartite equivalent to quasirandomness. Observe that if the vertices V of any quasirandom graph (with parameter ε^2) are partitioned into ε^{-1} parts of equal size, then pairs of parts are ε-regular pairs; while if any partition of the vertices of a complete bipartite graph that refines the bipartition is taken, again all pairs of parts are ε-regular.

One might hope that the vertices of any graph could be usefully partitioned such that all pairs are ε-regular: but this is not possible for small ε unless the parts all have size one (which is not a useful partition), as is demonstrated by the 'half-graph': the bipartite graph on vertex set $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$

with u_i adjacent to v_j whenever $i \le j$.

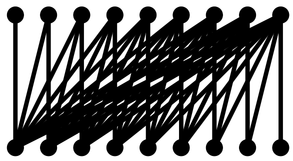


Figure 1.4 The half-graph

A partition into t parts is called ε-regular if all but at most εt^2 pairs of parts are ε-regular: the half-graph does have an ε-regular partition into ε^{-1} parts of equal size.

1.5.2 Szemerédi's Regularity Lemma

In 1975 Szemerédi [69] solved a 1936 question of Erdős and Turán, showing that for any $d > 0$ and k, any set of dN integers not larger than N contains a k-term arithmetic progression, provided that N is sufficiently large.

An important lemma in the proof was described in detail by Szemerédi [70] in 1976. Called the Szemerédi Regularity Lemma, this amounts to a coarse structure theorem for general graphs. There now exist several versions.

Theorem 1.4. *(Szemerédi Regularity Lemma) For any $\varepsilon > 0$ and k, there exists a constant $K = K(\varepsilon, k)$ such that every graph has a partition into parts $V_0, V_1, \ldots, V_t$, $k \le t \le K$, where $|V_0| \le \varepsilon |V(G)|$ and the sets $V_1, \ldots, V_t$ form an ε-regular partition into parts of equal size.*

Other conditions can be imposed: such as insisting that the regular partition refines some given partition, or that the partition should be ε-regular with respect to two graphs on the same vertex set. Proofs can be found in e.g. Bollobás [10]. However the growth of K as ε approaches zero is rapid (a standard proof gives K as an exponential tower of 2s of height ε^{-5}), and Gowers [42] has shown this is necessary (even under weakened regularity assumptions K must grow at least as fast as a tower of 2s of height $\varepsilon^{-\frac{1}{16}}$), so that results based on the Szemerédi Regularity Lemma are results about very large graphs.

In most applications of the Regularity Lemma one is not concerned with the precise density of a regular pair, but rather that it should not be too close (in terms of ε) to zero (or sometimes either to zero or one); we will simply say that a regular pair whose density is not too close to zero is *dense*, while a regular pair whose density is close to zero will be called *sparse*, and a non-ε-regular pair will be called *bad*.

Given a graph G with an ε-regular partition we define the *reduced graph* $R(G)$ to be the graph whose nodes are the parts $V_1, \ldots, V_t$ of the ε-regular partition $V_0, V_1, \ldots, V_t$ of G, with edges coloured either 'bad', 'sparse' or 'dense' to reflect the nature of the underlying pairs of parts.

1.5.3 Regularity and subgraphs

If $d > 0$ then with high probability a sufficiently large random graph with edge probability d contains any given fixed subgraph. This remains true for large quasirandom graphs with small enough values of ε, and the bipartite version holds for ε-regular pairs.

For example, if each pair of the three disjoint subsets U, V, W of $V(G)$ is ε-regular with density at least 2ε, $\varepsilon < \frac{1}{2}$, then there is a triangle uvw within G. To see this, let U_1 be the vertices in U which are adjacent to less than $\varepsilon|V|$ vertices of V. Since $d(U_1, V) < \varepsilon$, $|U_1| < \varepsilon|U|$ by ε-regularity of (U, V). Similarly the set U_2 of vertices of U adjacent to less than $\varepsilon|W|$ vertices of W has size less than $\varepsilon|U|$. Choose any $u \in U - (U_1 \cup U_2)$, which exists since $\varepsilon < \frac{1}{2}$. Let V' be the neighbours of u in V, and W' the neighbours of u in W. By ε-regularity of (V, W) the density $d(V', W')$ is at least ε; in particular there is an edge vw between the two sets as required.

If the density is bounded away from 1 as well, we can find induced subgraphs.

Theorem 1.5. *Fix a constant d, $0 < d < 1$, and a bipartite graph H with bipartition (U, V). Let G be a (necessarily large) bipartite graph whose parts (X, Y) form an ε-regular pair with density d. If ε is sufficiently small then H is an induced subgraph of G.*

Proof. Let $U = \{u_1, \ldots, u_s\}$. By the same argument as before, from at least

$(1-2\varepsilon)|X|$ possibilities we can choose $x_1 \in X$ with between $(d-\varepsilon)|Y|$ and $(d+\varepsilon)|Y|$ neighbours in Y.

Now the neighbours and non-neighbours of x_1 divide Y into two parts which are not too dissimilar in size. We wish now to find x_2 whose neighbours and non-neighbours divide these two parts into four parts all similar in size: i.e. we choose x_2 from the at least $(1-4\varepsilon)(|X|-1)$ possibilities which have both $(d-\varepsilon)\Gamma(x_1)$ to $(d+\varepsilon)\Gamma(x_1)$ neighbours in $\Gamma(x_1)$ and $(d-\varepsilon)(Y-\Gamma(x_1))$ to $(d+\varepsilon)(Y-\Gamma(x_1))$ neighbours in $Y-\Gamma(x_1)$.

Continuing this process, finally we choose x_s from the $(1-2^s\varepsilon)(|X|+1-s)$ possibilities that refine the partition of Y given by $\{x_1,\ldots,x_{s-1}\}$ into sets which are all of comparable size. The smallest of these should be of size $\min((d-\varepsilon)^s|Y|,(1-d-\varepsilon)^s|Y|)$, and for the argument to work this must be at least $\varepsilon|Y|$. However we can certainly choose $\varepsilon>0$ small enough that this condition will be satisfied, and also small enough that $(1-2^s\varepsilon)>0$.

Now if G is large enough that $\varepsilon|Y|\geq|V|$, then we can find H as an induced subgraph of G; the vertices $\{u_1,\ldots,u_s\}$ map to the vertices $\{x_1,\ldots,x_s\}$ and there are sufficient vertices in each of the 2^s parts of Y to accomodate any possible adjacencies between U and V. $\square$

Finally, the Blow-up Lemma [50] of Komlós, Sárközy and Szemerédi says that when embedding graphs of bounded degree we can treat dense regular pairs almost as if they were complete bipartite graphs.

An (ε,δ)-*super-regular pair* is a pair of parts X, Y such that whenever $X'\subset X$ has size at least $\varepsilon|X|$ and $Y'\subset Y$ at least $\varepsilon|Y|$, so $d(X',Y')>\delta|X'||Y'|$, and furthermore every $x\in X$ has degree at least $\delta|Y|$, and every $y\in Y$ has degree at least $\delta|X|$. It is clear that an ordinary $\frac{\varepsilon}{2}$-regular pair of density $\delta+\varepsilon$ already satisfies the first condition. Then removing at most $\frac{\varepsilon}{2}|X|$ vertices of low degree from X, and a similar number from Y, yields an (ε,δ)-super-regular pair.

Given a graph G whose vertices are partitioned into r parts $V_1,\ldots,V_r$, let $\mathbf{R}(G)$ be the graph on the same vertex set with xy an edge of $\mathbf{R}(G)$ if and only if xy lies between vertices of an (ε,δ)-super-regular pair V_i,V_j.

Theorem 1.6. *(Blow-up Lemma [50]) Fix Δ, δ, r and $c > 0$. There exist positive $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ and $\alpha = \alpha(\delta, \Delta, r, c)$ such that the following is true.*

Let G be a graph partitioned into r parts $V_1, \ldots, V_r$ each of size m, and let H be a graph of maximum degree Δ. Suppose that there exists an embedding $\phi : H \to \mathbf{R}(G)$: then there also exists an embedding $\psi : H \to G$, and for each $v \in H$, $\phi(v)$ and $\psi(v)$ are in the same part V_i.

Furthermore, we may choose vertices $x_1, \ldots$ of H and for each x_i a subset C_{x_i} of the part V_j containing $\phi(x_i)$, and insist that the image under ψ of each x_i is in C_{x_i}, provided that each C_{x_i} has size at least cm and that not more than αm of the x_i should be mapped into any one of the V_j.

This lemma is very useful in many situations: for example, as Łuczak [59] observed, it allows us to reduce the problem of finding a long path in a graph G to the much easier one of finding a large connected matching in the dense edges of the reduced graph $R(G)$. Although we will never actually use these results, we will make use of something closely resembling the Blow-up Lemma, and the following theorem is a good example of the way in which we will apply it.

Theorem 1.7. *Fix $0 < d \leq 1$. Choose $\varepsilon < \varepsilon(\frac{d}{2}, 2, 2, \frac{d}{2})$ and $\alpha < \alpha(\frac{d}{2}, 2, 2, \frac{d}{2})$ as in the Blow-up Lemma. Let G be a graph whose vertices are partitioned into sets $U_1, V_1, \ldots, U_t, V_t, W_1, W_2, \ldots$ each of size m. Let $R(G)$ be the graph whose nodes are these vertex sets, with edges between parts whenever they form ε-regular pairs of density at least d. If U_iV_i is an edge of $R(G)$ for each i and these edges are connected in $R(G)$, then if $(d - 2\varepsilon)m - t > \frac{d}{2}m$ and $\alpha m > 1$ there is a path covering $2mt - 2\varepsilon mt - 2t^2$ vertices of G.*

Proof. Since $R(G)$ is connected, for each $i = 1, \ldots, t-1$ we may choose $P_i = p_{i,1}p_{i,2} \ldots p_{i,|P_i|}$ such that $V_iP_iU_{i+1}$ is a path in $R(G)$: except that we insist that no P_i is empty, so if V_iU_{i+1} is an edge of $R(G)$ we choose $P_i = U_{i+1}V_i$. We will now construct sequentially $t-1$ disjoint paths in G, $Q_i = q_{i,1}q_{i,2} \ldots q_{i,|P_i|}$, where $q_{i,j} \in p_{i,j}$ for each i, j, where $q_{i,1}$ is adjacent to at least $(d - \varepsilon)m$ vertices of V_i and $q_{i,|P_i|}$ is adjacent to at least $(d - \varepsilon)m$ vertices of U_{i+1}.

Suppose that paths $Q_1, \dots, Q_{i-1}$ have been constructed. There are in total m vertices in $p_{i,1}$, of which less than εm are adjacent to less than $(d-\varepsilon)m$ vertices of V_i (otherwise the εm vertices together with V_i would contradict ε-regularity). Similarly at most εm vertices fail to be adjacent to $(d-\varepsilon)m$ vertices of $p_{i,2}$, and at most t vertices may be vertices of previously chosen paths $Q_1, \dots, Q_{i-1}$. It follows that we can choose a vertex $q_{i,1}$ of $p_{i,1}$ not in any previous path which is adjacent to at least $d - \varepsilon m$ vertices of each of U_i and $p_{i,2}$. Now for each $j = 2, \dots, |P_i|$ we may choose $q_{i,j} \in p_{i,j}$ a neighbour of $q_{i,j-1}$ and adjacent to at least $(d-\varepsilon)m$ vertices of $p_{i,j+1}$ (or of U_{i+1} if $j = |P_i|$). This gives the path Q_i.

Observe that if (X, Y) is an ε-regular pair of parts of G with density at least d, then there are less than εm vertices of X adjacent to less than $(d-\varepsilon)$ vertices of Y, and vice versa. It follows that we can remove $\varepsilon m + t$ vertices from each U_i and V_i to create U_i' and V_i' which are disjoint from $Q_1 \cup \ldots \cup Q_{t-1}$ (which meets any one of the parts of G in at most $t-1$ vertices) to leave $(2\varepsilon, \frac{d}{2})$-super-regular pairs of parts U_i', V_i' of size $(1-\varepsilon)m - t$.

Since $(d-2\varepsilon)m - t > \frac{d}{2}m$, for each $1 \leq i \leq t-1$, $q_{i,1}$ is adjacent to at least $\frac{d}{2}m$ vertices of V_i and $q_{i,|P_i|}$ is adjacent to at least $\frac{d}{2}m$ vertices of U_{i+1}.

Applying the Blow-up Lemma to the pair U_i', V_i' we can find a path R_i in $U_i' \cup V_i'$, starting in U_i' and ending in V_i', covering every vertex of $U_i' \cup V_i'$, constrained to start in one of the $\frac{d}{2}m$ neighbours of $q_{i-1,|P_{i-1}|}$ in U_i (if $i > 1$), and constrained to end in one of the $\frac{d}{2}m$ neighbours of $q_{i,1}$ in V_i (if $i < t$). Now $R_1 Q_1 R_2 Q_2 \dots Q_{t-1} R_t$ is a path in G of the required size. $\square$

1.5.4 Regularity and enumeration

One often encounters the problem: how many graphs are there which do not contain this given structure? Sometimes we can answer this by showing that a regular partition must have lots of regular pairs whose density is far from $\frac{1}{2}$.

Although there are 2^{m^2} labelled bipartite graphs with parts each of size m, almost all of them have density close to $\frac{1}{2}$. The number with density less than $d < \frac{1}{2}$ is at most $dm^2\binom{m^2}{dm^2}$, which for large m is much smaller than 2^{m^2}. This can be used to bound, for instance, the number of bipartite graphs not containing an induced copy of H (originally due to Prömel and Steger [64]).

Theorem 1.8. *Let H be any fixed bipartite graph. Then there are $2^{o(n^2)}$ bipartite graphs on n vertices which do not contain an induced copy of H.*

Proof. We will show that for any $\delta > 0$, for sufficiently large n there are at most $2^{\delta n^2}$ such bipartite graphs.

Choose $d > 0$ to be any number small enough that $\binom{n^2}{dn^2}2^{4dn^2} < 2^{\delta n^2}$ holds for all sufficiently large n. Now choose $0 < \varepsilon < d$ sufficiently small that we may apply Theorem 1.5 with the graph H to a large ε-regular pair.

If G is any large graph not containing H, then we can apply the Regularity Lemma to discover a partition respecting the bipartition into parts $V_0, V_1, \ldots, V_t$, where $\frac{1}{\varepsilon} < t < K = K(\varepsilon)$. Let $m = |V_1| = \cdots = |V_t|$. Each pair of parts is either bad or has density less than d, or by Theorem 1.5 there would be an induced copy of H.

For each t, there are less than $(t+1)^n$ ways to partition $V(G)$, and given the partition there are at most $\binom{t^2}{\varepsilon t^2}$ choices for which pairs should be bad. There are $2^{\varepsilon n^2}$ possible ways to arrange the edges within and from V_0, and $2^{t\binom{m}{2}}$ ways to arrange the edges within the parts $V_1, \ldots, V_t$. Finally, there are 2^{m^2} choices for edges within each of the εt^2 bad pairs, and $dm^2\binom{m^2}{dm^2}$ choices for the edges in the less than $\binom{t}{2}$ sparse pairs. The total number of possible graphs is thus at most

$$\sum_{t=\varepsilon^{-1}}^{K} (t+1)^n \binom{t^2}{\varepsilon t^2} 2^{\varepsilon n^2} 2^{t\binom{m}{2}} 2^{\varepsilon t^2 m^2} (dm^2)^{\binom{t}{2}} \binom{m^2}{dm^2}^{\binom{t}{2}}$$

and since $tm \leq n$, this is bounded above by

$$K(K+1)^n \binom{K^2}{\varepsilon K^2} m^{K^2} 2^{3\varepsilon n^2} \binom{n^2}{dn^2} < 2^{\delta n^2} K(K+1)^n \binom{K^2}{\varepsilon K^2} n^{K^2} 2^{-dn^2}$$

which is smaller than $2^{\delta n^2}$ for sufficiently large n. □

This method can be very useful: but because of the bad pairs of parts and the (usually many) sparse pairs it can only produce the correct coefficient for the n^2 term in the exponent, and not any linear or other smaller terms. If a more precise estimate is desired then another method must be used to deal with the typical cases.

2

Ramsey numbers for C_n^k

2.1 Introduction

Recall that the Ramsey number $R(H)$ of a graph H is defined to be the minimum N such that however the edges of K_N are two-coloured, there exists a monochromatic copy of H.

Although it would be interesting to find the Ramsey number $R(H)$ exactly for any graph H, this has only been possible for very simple, very sparse families of graphs such as paths and cycles, or for small graphs where a brute force search may be effective. Indeed, for graphs which are not very sparse even the right order of magnitude is not known. For sparse graphs, however, there are a great many results. Chvátal, Rödl, Szemerédi and Trotter [20] proved that for each d there is a constant c_d such that, when H is a graph on n vertices with maximum degree d, $R(H) \leq c_d n$. Their original constant c_d was very large—a tower of 2s of height approximately d^5—but it was subsequently improved by Eaton [26] to $2^{2^{O(d)}}$ and again by Graham, Rödl and Ruciński [43] to $2^{O(d \log d)}$. Alon [5] showed that if a graph H on n vertices has no edge between vertices of degree greater than two, then $R(H) \leq 12n$. Recently Kostochka and Sudakov [51] proved a bound almost linear in n on the Ramsey number of any d-degenerate n-vertex graph, and Fox and Sudakov [35] gave improvements on most of these bounds for bipartite graphs.

All of these proofs, of course, at some point must contain a phrase like 'Either this structure in the red edges, or that structure in the blue edges, exists', just as the proof of Theorem 1.1 reaches a step where it insists that either k of the selected vertices send edges forward, or k of them do not send edges forward. However it is notable that in most cases at least one of these conditions amounts to (half of) a quasirandomness condition 'Provided $X' \subset X$ and $Y' \subset Y$ are not too small, the density $d(X', Y')$ is not too small', and the method of embedding the graph H is precisely that which one would use to show H is a subgraph of a quasirandom graph. For example the proof of Chvátal et al. makes use of the Szemerédi Regularity Lemma followed by an argument similar to that of Erdős and Szekeres (Lemma 1.1) to eventually find $d + 1$ nodes in the reduced graph which form either a red-dense or blue-dense clique, while the proof of Graham et al. shows that either there is a large set of vertices with very high red-density or a large quasirandom subgraph with positive blue-density.

The argument of Chvátal et al., having established the existence of $d + 1$ parts, each pair of which is ε-regular and has (without loss of generality) red-density at least $\frac{1}{2}$, goes on to make use of only this $(d+1)$-partite structure to embed a red copy of H. It makes no use of any red edges within parts: because of this the strong quasirandomness condition on pairs of parts is required. However one can easily imagine that, if there were guaranteed to be many red edges within parts, then some much weaker condition on pairs of parts might suffice. In this chapter we describe a method based on this idea.

2.2 First ideas

The *kth power* of a graph G is the graph G^k on the same vertex set, with xy an edge of G^k if the distance from x to y in G is at most k.

The *bandwidth* of a graph G on n vertices is the smallest k such that G is a subgraph of P_n^k.

There is an easy lower bound $k(n-1)+\left\lfloor \frac{n}{k+1} \right\rfloor \leq R(P_n^k)$. We wish eventually to find an upper bound of $\left(2k+2+\frac{2}{k+1}\right)n+o(n)$. We note that this automatically gives an upper bound on the Ramsey number of all the subgraphs of P_n^k—all other n-vertex graphs of bandwidth at most k—which is significantly smaller than the previous best bound (since $\Delta(P_n^k)=2k$, it follows from Graham, Rödl and Ruciński's result [43] that $R(P_n^k) \leq 2^{O(k\log k)}n$). Our method strongly suggests that the lower bound is in fact correct: it is not hard to see, for example, where the factor of two is 'lost'.

We will mainly focus on a slightly different result: we will prove a bound $R(C_n^k) \leq \left(2k+4+\frac{2}{k+2}\right)n+o(n)$ on the kth power of a cycle.

It is worth considering how one could approach this problem. One could attempt to argue directly—suppose there is no red copy of C_n^k: what can we immediately deduce about G that might let us find a blue copy? But without some tools this is hard; C_n^k is not, for example, a subgraph of C_m^k for other values of m (excluding the trivial cases where n is small).

One obvious tool springs to mind: the combination of the Regularity and Blow-up Lemmas. With these it is an easy argument that, if there is a linear sized upper bound on $R(P_n^{k+1})$, then $R(C_n^k) \leq \big(1+o(1)\big)R(P_n^{k+1})$. But this only reduces the problem to that of bounding $R(P_n^k)$; and it would also mean that, to obtain any non-trivial bound, n would have to grow very rapidly indeed with k, which is not ideal.

Bounding $R(P_n^k)$ is still not a trivial problem, but there is another tool we can consider now: we can assume G does not contain a red copy of P_n^k, and since P_n^k is a supergraph of P_m^k for all $m \leq n$, we can consider a largest red kth power of a path in G. This has fewer than n vertices: and every vertex outside it is blue-adjacent to at least one of every $2k$ consecutive vertices.

By choosing successively disjoint maximum-sized red kth powers of paths in G, we can argue that G contains a large $(k+1)$-partite blue subgraph with equal parts and a blue density of at least $\frac{1}{2k}$ between pairs of parts. But it is still not clear how we might use this to find a blue P_n^k.

A method which will work is to find a reasonably large quasi-random subgraph of G, or even a collection of $k+1$ disjoint vertex sets such that every pair of vertex sets is ε-regular with sufficiently large density in one colour, and use this structure to find the monochromatic copy of C_n^k. This is certainly possible: one needs only to borrow the method (or even just apply the result) from the paper of Graham, Rödl and Ruciński. However this sort of method also has a major limitation: any upper bound on $R(C_n^k)$ obtained this way would be an upper bound on $R(H)$ for any graph H on n vertices with maximum degree at most k. Graham, Rödl and Ruciński showed that there are such graphs H with Ramsey number at least $c^k n$, for some $c > 1$: and as we would like to prove an upper bound for $R(C_n^k)$ which is linear in both n and k, we cannot use this method.

For our proof, we will need two tools: a partitioning method, and an embedding lemma. We recall that Chvátal et al. used the Regularity Lemma and (in effect) the Blow-up Lemma to 'blow up' the easy result that a two-edge-colouring of a large, very dense graph contains a monochromatic clique on $d+1$ vertices into a proof of linear-sized Ramsey numbers for graphs with maximum degree d. In the same spirit, we use our tools to 'blow up' an easier Ramsey result into our desired upper bound on $R(C_n^k)$.

2.3 The embedding lemma

We say that two disjoint sets of vertices in a two-coloured complete graph are *red-adjacent* if there is a red $K_{s,s}$ between them, and *blue-adjacent* if there is not. By the Kövari-Sós-Turán theorem [52], being blue-adjacent strongly limits the number and distribution of red edges.

Lemma 2.1. *(Kövari, Sós and Turán [52]) Let G be a bipartite graph with parts X and Y of sizes a and b which does not contain any $K_{s,s}$. If $2\left(\frac{s}{b}\right)^{\frac{1}{s}} \le p \le 1$, then at most $\frac{2s}{p}$ vertices in X have degree greater than pb.*

Proof. Suppose that there were a set Z of $\frac{2s}{p}$ vertices in X with degree greater than pb.

Choose a set C of s vertices in Z uniformly at random. The probability that a given vertex $y \in Y$ is adjacent to every vertex in C is

$$\frac{\binom{d_Z(y)}{s}}{\binom{\frac{2s}{p}}{s}} \; .$$

and so by linearity, Jensen's inequality and the fact that

$$2sb \le \sum_{z\in Z} d_Y(z) = \sum_{y\in Y} d_Z(y) \; ,$$

the expected number of such vertices is

$$\sum_{y\in Y} \frac{\binom{d_Z(y)}{s}}{\binom{\frac{2s}{p}}{s}} \ge b\frac{\binom{2s}{s}}{\binom{\frac{2s}{p}}{s}} \ge b\left(\frac{2s-s}{\frac{2s}{p}-s}\right)^s = b\left(\frac{p}{2-p}\right)^s \; .$$

which is greater than s. It follows that there is some $C \subset Z$ of size s which does have s common neighbours, giving a copy of $K_{s,s}$. □

Finally, we can give our embedding lemma. This lemma is essentially what the Blow-up Lemma would be with pairs of parts which have density approaching 1 rather than being ε, δ-super-regular.

Lemma 2.2. *Let $V_1, \ldots, V_t$ be disjoint sets of vertices of the graph G, each of size at most m. Define a graph G' on disjoint vertex sets $V'_1, \ldots, V'_t$, where $|V'_i| = \max(|V_i| - \left\lfloor 4s^2 m^{\frac{2s-1}{2s}}(d+1) \right\rfloor, 0)$ for each i, by putting edges between all vertices in V'_i and V'_j whenever there is no red $K_{s,s}$ between V_i and V_j (i.e. V_i and V_j are blue-adjacent). If H is any subgraph of G' with maximum degree d, and $m \geq d^2$, then G contains a blue copy of H.*

Proof. If $s = 1$ then G' is a subgraph of the graph of blue edges of G, and the result is trivially true. We will assume $s \geq 2$.

Let $p = 4s^2 m^{-1/2s}$: then for each i, $|V_i| - |V'_i| \leq p(d+1)m$. Note that if $p \geq \frac{1}{d+1}$ then each set V'_i is empty and there is nothing to prove: so we can assume $p < \frac{1}{d+1}$. By Lemma 2.1, if X and Y are vertex sets within a blue-adjacent pair (V_i, V_j) and $|Y| \geq \frac{\sqrt{m}}{2^s s^{2s-1}}$ then at most $2s/p$ vertices in X have red-degree greater than $p|Y|$.

Choose an embedding $\psi : V(H) \to V(G')$. Let $V(H) = \{x_1, \ldots\}$. We will choose successively vertices $\phi(x_1), \ldots \in V(G)$ which give an embedding ϕ of H into the blue edges of G. For each $x_i \in H$ set $A_{x_i,1} = V_j$, where V'_j is the part of G' containing $\psi(x_i)$.

The set $A_{x_i,t}$ is called the *allowed set* of x_i at time t; we invariably choose $\phi(x_t)$ to be within its allowed set at time t. We maintain two properties. First, that if $x_i x_j \in E(H)$ and x_i has been embedded, then the allowed set of x_j is entirely within the blue-neighbourhood of x_i, and second, that if, at time t, x_i has not yet been embedded, then its allowed set has size larger than $pm/2 = 2s^2 m^{\frac{2s-1}{2s}}$. This is definitely bigger than the $\frac{\sqrt{m}}{2^s s^{2s-1}}$ required to apply Lemma 2.1. At time 1 the first condition is trivially satisfied, and the second is true by the choice of the sizes of the V'_i.

At time t we choose a vertex $\phi(x_t) \in A_{x_t,t}$ which is blue-adjacent to at least $(1-p)|A_{x_l,t}|$ of the vertices of $A_{x_l,t}$ for each $l > t$ with x_l adjacent to x_t. This is possible since by Lemma 2.1, for each of the at most d neighbours of x_t not yet embedded, at most $\frac{2s}{p}$ vertices in $A_{x_t,t}$ fail to be blue-adjacent to $(1-p)|A_{x_l,t}|$ of the vertices of $A_{x_l,t}$, and $|A_{x_t,t}| \geq pm/2 > d\frac{2s}{p}$ by choice of m.

Having chosen $\phi(x_t)$, for each $l > t$ we set $A_{x_l,t+1}$ equal to $A_{x_l,t} - \{\phi(x_t)\}$

if $x_t x_l \notin E(H)$, and equal to $A_{x_l,t} \cap \Gamma_{blue}(\phi(x_t))$ if x_l is adjacent to x_t. It is clear that the allowed sets maintain the first property. If x_i is a vertex not yet embedded, with $\psi(x_i) \in V_j'$, then there are two reasons why a vertex $v \in V_j$ should not be in $A_{x_i,t+1}$: first, it might not be blue-adjacent to one of the at most d embedded neighbours of x_i, and second, it might be the image under ϕ of some preceding vertex (in V_j') of H. Thus we have

$$|A_{x_i,t+1}| \geq (1-p)^d|V_j| - |V_j'| > (1-pd)|V_j| - (|V_j| - p(d+1)m) \geq \frac{pm}{2}$$

so that the allowed sets maintain both the required conditions. It follows that this algorithm successfully embeds H into the blue edges of G. □

2.4 The easier problem

Just as Chvátal et al. [20] needed the Ramsey result that there exists a monochromatic $(d+1)$-clique within a two-edge-coloured large dense graph, we need a simple Ramsey result to blow up. Unfortunately our result is not quite so easy to prove: on the other hand it may have some independent interest. We seek an upper bound on the Ramsey number $R(P_n, P_n^k)$. Observe that in light of results such as Theorem 1.7 we would expect to find that it is relatively easy to bound this number: long paths are not hard to construct.

In fact we will not need to use the Regularity or Blow-up lemmas on the way to proving our bound $R(P_n, P_n^k) \leq \left(k+1+\frac{1}{k+1}\right) n + o(n)$; though we will need the embedding lemma from the previous section (which we use in preference to the Blow-up Lemma because it gives more easily explicit bounds on the error $o(n)$ term).

For contrast, we provide a lower bound.

Theorem 2.3. *The Ramsey number $R(P_n, P_n^k)$ is bounded below by*

$$k(n-1) + \left\lfloor \frac{n}{k+1} \right\rfloor = \left(k + \frac{1}{k+1}\right) n + o(n) \ .$$

Proof. Let G be a two-edge-coloured complete graph on $k(n-1) + \left\lfloor \frac{n}{k+1} \right\rfloor - 1$

vertices, whose graph of red edges is the disjoint union of k cliques each on $n-1$ vertices and one clique on $\left\lfloor \frac{n}{k+1} \right\rfloor - 1$ vertices.

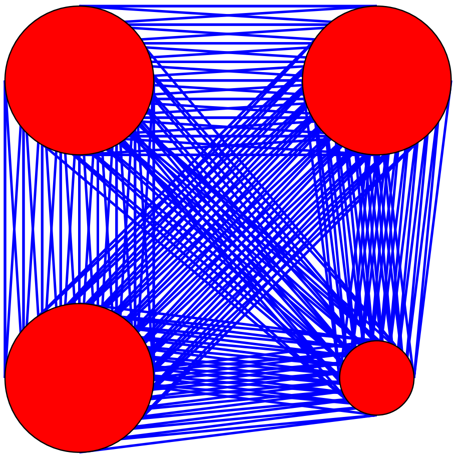


Figure 2.1 G contains neither a red P_n nor a blue P_n^k.

There is up to isomorphism only one proper $(k+1)$-vertex-colouring of the graph of blue edges of G; and one of its parts has size $\left\lfloor \frac{n}{k+1} \right\rfloor - 1$. Now there is also only one proper $(k+1)$-vertex-colouring of P_n^k, and all of its parts have size at least $\left\lfloor \frac{n}{k+1} \right\rfloor$. It follows that G contains no blue P_n^k, and since its red components have size at most $n-1$ it contains no red P_n either, as required. $\square$

Before giving our upper bound, we need three preliminary results.

First, the Erdős-Gallai extremal theorem for cycles [28]:

Theorem 2.4. *Let G be a graph on n vertices and c an integer, $3 \le c \le n$. Then either G contains a cycle of length at least c or*

$$e(G) < (c-1)(n-1)/2 + 1 \ .$$

Second, we adapt a result of Kohayakawa, Simonovits and Skokan [49] on maximum cycles in graphs, giving a much weaker but (for us) more convenient form:

Lemma 2.5. *Given a graph G containing vertex disjoint cycles C_t and $C_{t'}$, if G contains no cycle of length greater than t, then the bipartite graph $G[V(C_t), V(C_{t'})]$ contains no copy of $K_{s,s}$, where $s = \left\lceil \frac{t}{t'} \right\rceil + 2$.*

Proof. Suppose not; and let G, C_t, $C_{t'}$ be a counterexample. Now G contains a copy of the bipartite graph $K_{s,s}$ whose parts are in $V(C_t)$ and $V(C_{t'})$, so in particular there are two vertices of this complete bipartite graph in C_t which are joined in C_t by a path P of length at least $\frac{s-1}{s}t$, and two more in $C_{t'}$ joined by a path P' in $C_{t'}$ of length at least $\frac{s-1}{s}t'$. The vertices $V(P) \cup V(P')$ form a cycle of length at least $\frac{s-1}{s}(t+t') > t$, which is a contradiction. □

Third, a standard greedy method allows us to find a copy of P_n^k in a very dense graph on only slightly more than n vertices:

Lemma 2.6. *Given $0 < \varepsilon \leq (k+3)^{-1}$ and $n > 3\varepsilon^{-2}$, if H is any graph on at least $n + (k+2)\varepsilon n$ vertices such that $\overline{H}$ contains no cycle of length $\varepsilon^2 n$ or greater, then H contains a copy of P_n^k.*

Proof. By Theorem 2.4, $\overline{H}$ has at most $(v(H)-1)(\varepsilon^2 n-1)/2+1 < \varepsilon^2 v(H)n/2$ edges. If $\overline{H}$ had less than $n + k\varepsilon n$ vertices of degree smaller than εn, then it would have at least $(v(H) - n - k\varepsilon n)\frac{\varepsilon n}{2}$ edges, which is a contradiction. So at least $n + k\varepsilon n$ vertices of $\overline{H}$ have degree less than εn. Let H' be the graph with maximum co-degree εn obtained by removing these vertices from H. The neighbourhood of any set of k vertices of $\overline{H'}$ contains at most $k\varepsilon n$ vertices: so in H' every set of k vertices has at least n common neighbours. We can embed P_n^k into H' by a simple greedy procedure: we choose any vertex to be the first vertex of the path, any neighbour to be the second vertex of the path, and so on. At each embedding step we only need to find a vertex which is adjacent to all of the last k vertices embedded, and which has not yet been used in the embedding. Such a vertex is guaranteed to exist since any k vertices of H' have at least n common neighbours, and we only need to embed a total of n vertices. □

Now we can prove our result.

Lemma 2.7. *The Ramsey number $R(P_n, P_n^k)$ is bounded above by*

$$\left(k+1+\frac{1}{k+1}\right)n + o(n) \ .$$

Proof. We show that, for any $0 < \varepsilon \leq (k+3)^{-1}$, the Ramsey number $R(P_n, P_n^k)$ is bounded above by

$$\left(k+1+\frac{1}{k+1}+(k+3)\varepsilon\right)n \text{ for } n > \left(16(2k+1)\varepsilon^{-8}\right)^{4\varepsilon^{-2}} .$$

Let G be a two-edge-coloured complete graph on $\left(k+1+\frac{1}{k+1}+(k+3)\varepsilon\right)n$ vertices which contains no red P_n. We choose successively vertex-disjoint maximum-length red cycles in G. Let V_1 be the vertex set of the longest red cycle of G, V_2 the vertex set of the longest red cycle of $G - V_1$, and so on.

Since $P_n \subset C_n$, we have $n-1 \geq |V_1| \geq |V_2| \geq \ldots$. Let r be the greatest index such that $|V_r| \geq \varepsilon^2 n$, and let $W = V(G) - \bigcup_{i=1}^r V_i$. Since the sets V_i are disjoint, $r \leq (k+\frac{1}{k+1}+(k+3)\varepsilon)\varepsilon^{-2} < \varepsilon^{-3}$ is bounded independently of n.

If $|W| \geq n + (k+2)\varepsilon n$ then the graph of blue edges in W satisfies the conditions of Lemma 2.6, so G contains a copy of P_n^k. Therefore we will assume $|W| < n + (k+2)\varepsilon n$.

Let $s = \left\lceil \frac{n}{\varepsilon^2 n} \right\rceil + 2 < 2\varepsilon^{-2}$. By Lemma 2.5 for any $1 \leq i < j \leq r$, there is no red copy of $K_{s,s}$ in G whose parts are in V_i and V_j respectively. We wish to use this together with Lemma 2.2 to find a blue copy of P_n^k (which has maximum degree $2k$). We will use the fact that P_n^k is a subgraph of the complete $(k+1)$-partite graph with parts of size $\left\lceil \frac{n}{k+1} \right\rceil$. Observe that no part V_i has size greater than n, and the union of all the parts has size at least $\left(k+\frac{1}{k+1}+\varepsilon\right)n$.

Now choose ℓ_1 to be the smallest index such that

$$\sum_{i=1}^{\ell_1}\left(|V_i| - 4s^2 n^{\frac{2s-1}{2s}}(2k+1)\right) \geq \left\lceil \frac{n}{k+1} \right\rceil .$$

Since $4s^2 n^{\frac{2s-1}{2s}}(2k+1)r < \varepsilon n$, this is possible and furthermore $\sum_{i=1}^{\ell_1}|V_i| < n$ (in fact, this sum can only exceed $2\left\lceil \frac{n}{k+1} \right\rceil + \varepsilon n$ when $\ell_1 = 1$).

For each $2 \leq j \leq k$ in succession, let ℓ_j be the smallest index such that

$$\sum_{i=\ell_{j-1}+1}^{\ell_j}\left(|V_i| - 4s^2 n^{\frac{2s-1}{2s}}(2k+1)\right) \geq \left\lceil \frac{n}{k+1} \right\rceil .$$

Again this is possible, with $\sum_{i=\ell_{j-1}+1}^{\ell_j} |V_i| < n$.

We apply Lemma 2.2 to the parts $V_1, \ldots, V_r$ of G. Let $V_1', \ldots, V_r'$ be the parts of G' as in the lemma; since for each $1 \leq i < j \leq r$ the sets V_i and V_j are blue-adjacent the parts V_i' and V_j' span a complete bipartite graph. Let $W_1 = \bigcup_{i=1}^{\ell_1} V_i'$, $W_j = \bigcup_{i=\ell_{j-1}+1}^{\ell_j} V_i'$ for each $2 \leq j \leq k$, and $W_{k+1} = \bigcup_{i=\ell_k+1}^{r} V_i'$. By choice of n we are guaranteed to find that $|W_{k+1}| \geq \left\lceil \frac{n}{k+1} \right\rceil$. The W_j form the parts of a complete $(k+1)$-partite subgraph of G', so that P_n^k can be embedded into G'. By Lemma 2.2 G contains a blue copy of P_n^k. □

Observe that while the $o(n)$ error term does not decay especially quickly, it is vastly smaller than any error term arising from a Regularity Lemma proof would be.

We note that while there are few exact results on Ramsey numbers, there are some results on the Ramsey number of a path against some other graph: in addition to Gerencsér and Gyárfás' result [37] on $R(P_m, P_n)$, Parsons [62] showed that $R(P_m, K_n) = (m-1)(n-1) + 1$, and various other authors (see Radziszowski [65] for details) found some results for a few other simple graph classes. It seems likely that the method above can be extended to give an exact value for $R(P_m, P_n^k)$ (for large m and n) with some additional work.

2.5 Bounding $R(C_n^k)$

Theorem 2.8. *The Ramsey number $R(C_n^k)$ is bounded above by*

$$\left(2k + 4 + \frac{2}{k+2}\right) n + o(n) \ .$$

Proof. We show that for every sufficiently small $\varepsilon > 0$,

$$R(C_n^k) \leq \left(2k + 4 + \frac{2}{k+2} + (4k+15)\varepsilon\right) n$$

for all sufficiently large n.

Let G be a two-coloured complete graph on $\left(2k+4+\frac{2}{k+2}+(4k+15)\varepsilon\right)n$ vertices, and assume $\varepsilon n > \sqrt{n}$. By Ramsey's theorem we can find successively disjoint monochromatic cliques of size $\frac{1}{4}\log n$ in G covering all but $\sqrt{n}$ vertices of G. Suppose without loss of generality that at least half of these cliques, $V_1, \ldots, V_{\left(k+2+\frac{1}{k+2}\right)t+(2k+7)\varepsilon t}$, are red cliques, where $t = \frac{4n}{\log n}$. Consider the two-coloured complete graph whose nodes are these cliques, with two cliques red-adjacent if there is a red $K_{4k,4k}$ between them and blue-adjacent otherwise.

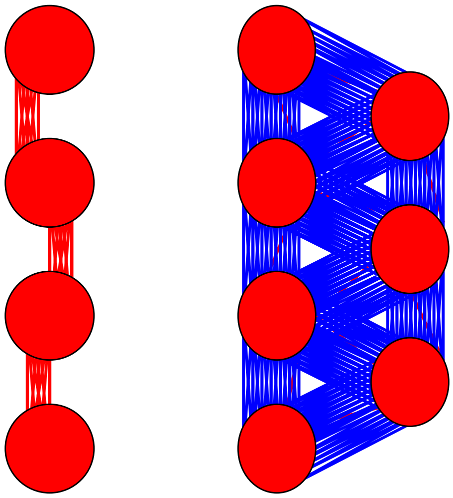


Figure 2.2 A red-adjacent P_4 and a blue-adjacent P_7^2

By Lemma 2.7, if t is sufficiently large, then either there is a red-adjacent $P_{t+\varepsilon t}$ or a blue-adjacent $P_{t+\varepsilon t}^{k+1}$. In the first case we can trivially embed a red C_n^k into G using the red-adjacent path of red cliques, while in the second case for sufficiently large n Lemma 2.2 can be applied to find a blue copy of C_n^k. □

Again the decay of the $o(n)$ error term in this theorem is not especially pleasant: $t = \frac{4n}{\log n}$ must be large enough to make the $o(t)$ error term in $R(P_t, P_t^{k+1}) \leq \left(k+2+\frac{1}{k+2}\right)t + o(t)$ small, and $\log n$ must be large enough to make the fraction of each red clique covered by Lemma 2.2 close to 1, but the error term is certainly (eventually) smaller than $\frac{n}{\log\log n}$. By contrast, a Regularity Lemma proof, if one exists, would give an error term resembling $\frac{n}{\log_* n}$ (where $\log_* n$ is the smallest h such that n is less than a tower of 2s of height h).

Note that in the above proof P_t^{k+1} is required rather than P_t^k because the chromatic number of C_n^k is $k+2$ when k does not divide $n > 4k^2$; when k does divide $n > 4k^2$ the chromatic number is $k+1$, only P_t^k is required and the bound may be improved to $\left(2k+2+\frac{2}{k+1}\right)n + o(n)$.

Corollary 2.9. *Whenever G is an n-vertex graph with bandwidth k,*

$$R(G) \le \left(2k+2+\frac{2}{k+1}\right)n + o(n) \ .$$

Proof. By the same logic as in the previous proof,

$$R(P_n^k) \le \left(2k+2+\frac{2}{k+1}\right)n + o(n) \ :$$

and every n-vertex graph with bandwidth k is a subgraph of P_n^k. □

3

Lehel's conjecture

3.1 Introduction

In the previous chapter we were concerned with showing that some Ramsey numbers were linear-sized. Another way of looking at this is to say that we can guarantee that any two-edge-coloured K_n contains a monochromatic C_r^k, where

$$r = \big(1 - o(1)\big)\frac{n}{2k + 4 + \frac{2}{k+2}}\ .$$

This is a structure covering a positive fraction of the n vertices. Now we will push this to an extreme, and ask for a structure covering all of the n vertices.

We could, for example, ask for a monochromatic spanning tree. It is easy to see that this structure always exists: given a two-edge-coloured complete graph G, either the red edges form a connected graph, in which case they contain a spanning tree, or there is a red component on vertex set C. In this case every edge between C and $V(G) - C$ is blue, so the blue edges form a connected graph and there is a blue spanning tree. But spanning trees are not especially pleasant structures—and although, for example, a spanning path would be a nicer structure, G need not contain a monochromatic spanning path. Since we wish to find a nice structure covering $V(G)$ we need to make use of both colours.

If the vertices of a two-edge-coloured complete graph G can be partitioned into two sets C_r and C_b, with C_r possessing a red Hamilton cycle and C_b a blue Hamilton cycle, we say that G has a *two-cycle partition.* We use slightly non-standard notation here: we allow the existence of cycles on zero, one and two vertices, being respectively the null graph, the graph on one vertex, and the complete graph on two vertices.

In 1972 Lehel [53] made the following conjecture.

Conjecture 3.1. *Every two-edge-coloured complete graph has a two-cycle partition.*

The conjecture is known to be true for sufficiently large graphs: in 1998 Łuczak, Rödl and Szemerédi [60] used the Regularity Lemma to show that there exists n_0 such that, for any $n \geq n_0$, there is a two-cycle partition of G. Their argument went through the following steps often found in Regularity Lemma proofs.

They identified an extremal structure (a large monochromatic complete bipartite subgraph) whose absence guarantees the existence of a useful substructure (short and well-behaved monochromatic paths exist joining any pair of large vertex sets). They gave a proof that if the extremal structure is present in G, then G has a two-cycle partition. They did not need to use the Regularity Lemma to establish this result.

They then embarked on the proof of the general result: they took an ε-regular partition (with $\varepsilon = 10^{-60}$) of G, and examined the reduced graph, finding sets covering a large fraction of G in which (by applying ideas similar to Theorem 1.7) it is easy to construct paths and cycles. They applied the extremal result to generate many short monochromatic paths joining these sets, and finally they found ways to connect the left-over vertices one at a time to the sets (which is the most difficult part of the proof), giving the result.

Of course, their use of the Regularity Lemma makes their n_0 extremely large. In this chapter we describe a different proof method, not using the Regularity Lemma, giving the following theorem.

Theorem 3.2. *For all $n \geq 2^{18000}$ and all two-edge-coloured graphs G on n vertices, there exists a two-cycle partition of G.*

We will borrow the first part of their proof, but instead of taking a regular partition we will apply a similar method to that in the previous chapter of 'blowing up' a preliminary Ramsey result by applying it to a graph whose nodes are monochromatic cliques in G with suitably defined adjacencies. The preliminary result is a theorem of Gyárfás which 'almost' gives a two-cycle partition of G; most of the difficulty in our proof is, as with the argument of Łuczak, Rödl and Szemerédi, in finding ways to ensure the left-over vertices (that are not in cliques, and so on) are incorporated into the final cycles.

3.2 The preliminary result

It is certainly the case that for every G, Lehel's conjecture is 'almost' true: Gyárfás [44] proved the following theorem.

Theorem 3.3. *For any two-coloured complete graph G we can find within G a red cycle and a blue cycle which together cover the vertices of G and have at most one vertex in common.*

Although the proof of this theorem is quite short, it involves creating an intermediate structure on the way to the final result, and the choice of this intermediate structure is not obvious.

Proof. Given a path or cycle in G, the number of colour changes on the path or cycle is the number of vertices where a red and a blue edge of the path or cycle meet.

We first construct inductively a sequence of paths with at most one colour change, starting with P_1 a single vertex of G. Suppose that we have constructed a path $P_t = (x_1, \ldots, x_t)$ covering t vertices of G with at most one colour change. We now want to give a path P_{t+1} incorporating another vertex y of G.

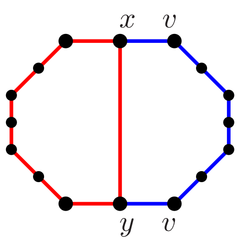
If the path P_t has no colour changes or the colour of the edge $x_{t-1}x_t$ is equal to the colour of x_ty then we set $P_{t+1} = (x_1, \ldots, x_t, y)$, which is a path with at most one colour change.

If this is not the case, but the colours of x_1x_2 and yx_1 are equal, then we set $P_{t+1} = (y, x_1, \ldots, x_t)$.

If neither of the above is acceptable, then the edges x_1y and x_ty are of different colours, and without loss of generality we may assume x_1x_t and x_1y are the same colour, which must also be the colour of $x_{t-1}x_t$. Then we set $P_{t+1} = (x_2, x_3, \ldots, x_t, x_1, y)$.

Now P_n is a path covering all of G with at most one colour change. If P_n is monochrome and the edge joining its end vertices into a cycle is of the same colour, then this cycle together with the zero-vertex cycle of the opposite colour satisfy the theorem.

Otherwise, the cycle C consisting of P_n together with the edge between its end vertices has two colour changes, occurring at vertices x and y. Incorporating the edge xy we obtain a monochromatic cycle and a path of the opposite colour joining consecutive vertices x and y on the cycle whose interior vertices are not in C_1. Without loss of generality assume the monochromatic cycle C_1 is red and the path Q_1 is blue.



Now let u and v be the second and penultimate vertices of Q_1. If $u = v$ or u and v are consecutive vertices of Q_1 then they form a blue cycle, disjoint from C_1, and the theorem is satisfied. If uy or vx is blue then C_1 together with the blue cycle through y or x respectively are as desired, while if uv is blue we have our disjoint cycles of opposite colours covering G. But if all three edges are red, then we obtain a red cycle C_2 by replacing xy with $xvuy$ which contains two more vertices than C_1, and a path $Q_2 = Q_1 - \{x, y\}$

joining consecutive vertices of C_2 whose interior vertices are disjoint from C_2. Repeating this step to form ever longer red cycles and shorter blue paths C_i and Q_i is only possible at most $\frac{|Q_1|}{2}$ times before at some step we obtain the desired red and blue cycles intersecting in at most one vertex. $\square$

It is clear that this proof depends substantially on the extra freedom given by allowing the two cycles to intersect in one vertex. Some attempts to modify it (dividing into cases depending on the colours of edges near the colour changes) to yield a proof of Lehel's conjecture ran into rapidly expanding numbers of cases that do not quite solve the problem, with no obvious way out.

3.3 The extremal case

We make use of two theorems from the paper of Łuczak, Rödl and Szemerédi [60]. The first gives our result in the case that G contains a large monochromatic complete bipartite graph.

Theorem 3.4. *If there exists a partition $V(G) = V_1 \sqcup V_2 \sqcup V_3$, such that $\min(|V_1|, |V_2|) \geq 5 + 2|V_3|$ and V_1, V_2 form the parts of a blue complete bipartite graph, then there is a two-cycle partition of G.*

The second theorem is a variant on 'Fact 4.3' from the same paper, adapted to give us greater control over the relatively small number of paths we will need to claim exist.

Theorem 3.5. *For every $k \geq 2$ and $n \geq 63k$, the following holds. Either $V(G)$ may be partitioned into three sets satisfying the conditions of Theorem 3.4, or given disjoint subsets A, B and C of $V(G)$, where $|A|, |B| \geq \frac{n}{2k}$ and $|C| \leq \frac{n}{5k}$, there exists a red path of length at most $100k$ whose initial vertex is in A, whose final vertex is in B, and whose interior vertices are in $V(G) - (A \cup B \cup C)$.*

Proof. Let R be the graph whose edges are the red edges of G on the vertex set $V(G) - C$. Let N_r be the set of vertices at distance exactly r from the set A, and N'_r be the set of vertices at distance exactly r from the set B.

If both $\sum_{r=1}^{50k} |N_r| > \frac{n}{2}$ and $\sum_{r=1}^{50k} |N'_r| > \frac{n}{2}$, then there must be a path of length at most $100k$ from A to B within R as desired.

If there does not exist any such path, then we may assume without loss of generality that $\sum_{r=1}^{50k} |N_r| \leq \frac{n}{2}$, and so there must be r_0, $1 \leq r_0 \leq 50k$, such that $|N_{r_0}| \leq \frac{n}{100k}$.

Now let $V_1 = A \cup \bigcup_{r=1}^{r_0-1} N_r$, $V_3 = N_{r_0} \cup C$ and $V_2 = V(G) - (V_1 \cup V_3)$.

We have

$$|V_1| \geq |A| \geq \frac{n}{2k} \geq 5 + 2\left(\frac{n}{5k} + \frac{n}{100k}\right) \geq 5 + 2|V_3| \ ,$$

and similarly $|V_2| \geq 5 + 2|V_3|$. By definition of the N_i and N'_i, all the edges between V_1 and V_2 must be blue, satisfying the conditions of Theorem 3.4.

□

3.4 The non-extremal case

Throughout this section we assume that G does not possess any large complete bipartite graph of either colour, so that we can apply Theorem 3.5 with either colour. When $\mathcal{U} = (U_1, \ldots, U_u)$ is a list and we refer to an element U_i, $i > u$ we mean the element $U_{i \mod u}$.

We first give a basic partitioning result which gives us a structure that would allow us to cover a large fraction of G with two disjoint cycles, one red and one blue (if we wanted to). We then describe the desired 'correction' of this structure which allows us to find a two-cycle partition of G. Finally we show how to get the 'corrected' structure.

3.4.1 Clique-cycles

Suppose that $\mathcal{U} = (U_1, \ldots, U_u)$, $u \geq 3$, is a list of disjoint red cliques within $V(G)$. Suppose further that there are specified disjoint red *linking paths* $u_{i,i+1 \mod u}$ between each pair U_i and U_{i+1} whose interior vertices are not in any U_j. We call this structure an *on-colour* red *clique-cycle*. In general the linking paths will be paths on only two vertices (i.e. single red edges), and never on more than four.

Suppose that $\mathcal{V} = (V_1, \ldots, V_v)$, $v \geq 3$, is a list of disjoint blue cliques within $V(G)$, with each pair V_i, V_{i+1} spanning a red complete bipartite subgraph of G. We will call this an *off-colour* red clique-cycle.

If the U_i and the V_j are disjoint, and furthermore there exist disjoint red paths P_1 and P_2 between U_1 and V_1, each of length at most $18000 + u + v$, neither of which meet either $u_{1,2}$ or $u_{u,1}$, and whose interior vertices are not in any of the U_i or V_j, we call $(\mathcal{U}, P_1, P_2, \mathcal{V})$ a red *clique-cycle pair* (see Figure 3.1(iii)). We do permit one or both of the sets $\mathcal{U}$ and $\mathcal{V}$ to be empty, in which case we require that the paths P_1 and P_2 are empty.

Given a red clique-cycle pair, it is trivial to see that there exists a red cycle which passes through every vertex of the on-colour red clique-cycle, both paths P_1 and P_2, and $\min_i |V_i|$ vertices of each of the V_i.

We define similarly blue on-colour and off-colour clique-cycles and a blue clique-cycle pair.

The purpose of this subsection is to establish the following lemma.

Lemma 3.6. *When $n \geq 2^{18000}$ there exists a partition of the vertices of G into the following three parts:*

a red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$,

a blue clique-cycle pair $(\mathcal{X}, Q_1, Q_2, \mathcal{Y})$, and

a 'leftover set' L_1.

The leftover set has size at most $2^{17990} + \frac{n}{80} + 6(v + y)$ (where $v = |\mathcal{V}|$ and $y = |\mathcal{Y}|$), and all of the cliques in the off-colour clique-cycles have size between 8981 and 8989. Furthermore, when two of the clique-cycles are not empty we have $|L_1| \leq 2^{17990} + \frac{n}{120} + 6(v + y)$.

Proof. By Ramsey's Theorem, we can guarantee that any set of 4^{8995} vertices of G contains either a red or a blue clique of size 8995.

Thus we can find a partition of $V(G)$ into a collection $\mathcal{R} = R_1, \ldots$ of red cliques each of size 8995, a collection $\mathcal{B} = B_1, \ldots$ of blue cliques each of size 8995, and a set L_0 of size at most $2^{17990} < \frac{n}{1000}$.

We say that two red cliques are *red-adjacent* if there exists a red matching of size at least four between them, and *blue-adjacent* otherwise. This defines a two-coloured complete graph with vertex set $\mathcal{R}$.

By Theorem 3.3 (Gyárfás' result), there exist red and blue cycles C_r and C_b within this graph which cover $\mathcal{R}$ and which intersect in at most one member of $\mathcal{R}$ (Figure 3.1(i)).

We let $\mathcal{U}' = C_r - C_b$. Note that C_r and C_b may intersect in at most one clique. If they do intersect in a clique $(C_b)_j$, so that $C_r = (\ldots, U'_s, (C_b)_j, U'_{s+1}, \ldots)$, then there is a red path $u_{s,s+1}$ on either three or four vertices from U'_s to U'_{s+1} through $(C_b)_j$. Since every other pair of sets U'_i, U'_{i+1} has a red matching of size four between them, we can construct all the desired disjoint red paths $u_{i,i+1 \mod u}$ as single red edges from the matchings. With these paths, the list $\mathcal{U}'$ becomes an on-colour red clique-cycle.

We let $\mathcal{Y}'$ be the cliques in C_b, with the exception that if $C_r \cap C_b = \{(C_b)_j\}$ we replace $(C_b)_j$ with $(C_b)_j - u_{s,s+1}$.

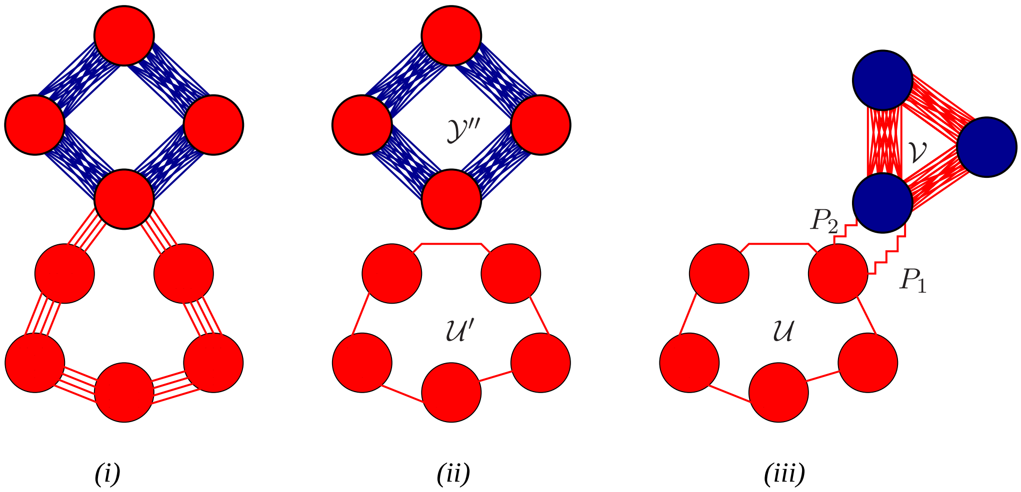


Figure 3.1 (i) The red cliques covered by a red and a blue cycle, (ii) The clique-cycles $\mathcal{U}'$ and $\mathcal{Y}''$ obtained, and (iii) A red clique-cycle pair.

Now since there is no red matching of size four between any pair Y'_j, Y'_{j+1} we can remove six vertices from each Y'_j to obtain Y''_j such that each pair Y''_j, Y''_{j+1} spans a blue complete bipartite graph. The list $\mathcal{Y}''$ is an off-colour blue clique-cycle; each clique in it has size between 8987 and 8989. The two clique-cycles $\mathcal{U}'$ and $\mathcal{Y}''$ are disjoint, as in Figure 3.1(ii).

Similarly we say that two blue cliques are blue-adjacent if there exists a blue matching of size at least four between them, and red-adjacent otherwise. By applying the theorem of Gyárfás in the same way to $\mathcal{B}$ we obtain the disjoint on-colour blue clique-cycle $\mathcal{X}'$ and off-colour red clique-cycle $\mathcal{V}''$.

We will now construct $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$.

First, if $|\bigcup\mathcal{U}'| \leq \frac{n}{240}$ then we set $\mathcal{U} = \emptyset$ and $u = 0$, and similarly for $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$.

For each $1 \leq i \leq u$ either $u_{i,i+1}$ is a path on three or four vertices or we can identify a red matching of size four between U_i' and U_{i+1}' including the edge $u_{i,i+1}$. We can similarly identify blue matchings of size four between pairs in $\mathcal{X}'$. Let C_1 be the union of all the vertices in these identified matchings and the linking paths.

If both $\mathcal{U}$ and $\mathcal{V}$ are non-empty then let $A_1 = \bigcup\mathcal{U}' - C_1$, and $B_1 = \bigcup\mathcal{V}''$. Now $|C_1| \leq 8\frac{n}{8995} < \frac{17n}{18000}$, $|A_1|, |B_1| \geq \frac{n}{360}$ and the sets A_1, B_1 and C_1 are disjoint by construction. Thus we can apply Theorem 3.5 with $k = 180$ to obtain a minimal red path P_1 of length at most 18000 from A_1 to B_1 which does not pass through any vertices of C_1.

Note that $18000 < \frac{n}{18000}$. We let $C_2 = C_1 \cup P_1$, $A_2 = \bigcup\mathcal{U}' - C_2$ and $B_2 = \bigcup\mathcal{V}'' - C_2$. These three sets still satisfy the conditions of Theorem 3.5, so applying it we obtain a second minimal red path P_2' of length at most 18000 between A_2 and B_2 which avoids the vertices of C_2.

Continuing this, if both $\mathcal{X}$ and $\mathcal{Y}$ are non-empty we obtain blue paths Q_1 and Q_2' between $\bigcup\mathcal{X}' - C_1$ and $\bigcup\mathcal{Y}''$ which are of length at most 18000 and such that the paths P_1, P_2', Q_1, Q_2' are pairwise disjoint.

We renumber the lists $\mathcal{U}'$ and $\mathcal{V}''$ if necessary such that the path P_1 goes from U_1' to V_1''. The path P_2' does not necessarily go from U_1' to V_1''. But there is a chain of sets $U_1', \ldots, U_p'$ such that P_2' terminates in U_p' and such that each pair of sets U_i', U_{i+1}', $1 \leq i < p$, spans a red matching of size four contained in C_1 (we may assume that if there is a path $u_{i,i+1}$ of length greater than one then it comes after p). In each matching one of the four red edges must be disjoint from the linking paths; thus we can find a red path P_2 extending P_2' into U_1' and into V_1'' (the latter since consecutive cliques in $\mathcal{V}''$ span red complete bipartite graphs) such that P_2 does not intersect any of P_1, Q_1, Q_2' (since these avoid C_1) or the linking paths. The path P_1 is of length at most 18000, while $|P_2| \leq 18000 + u + v$.

Similarly we can assume Q_1 goes from X_1' to Y_1'' and extend Q_2' to obtain Q_2 which also starts and ends in those sets. Again $|Q_1| \leq 18000$ and $|Q_2| \leq 18000 + x + y$.

Finally we obtain $\mathcal{U} = (U_1, \ldots, U_u)$ by letting U_i contain all the vertices in U_i' that are not interior vertices of any of the paths P_1, P_2, Q_1, Q_2, and $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$ similarly. We let L_1 contain all the vertices which are not in either clique-cycle pair.

Observe that since the paths P_1 and P_2' are of minimal length, neither path intersects any one of the red cliques $\mathcal{Y}''$ in more than two places, and by construction the paths Q_1 and Q_2 intersect each clique in at most one place. Thus for each i, $|Y_i''| - |Y_i| \leq 6$, so that each clique in $\mathcal{Y}$ has size between 8981 and 8989. The same holds for the cliques $\mathcal{V}$.

Since a vertex can only be in L_1 if it was either in L_0, or was removed from either $\mathcal{V}'$ or $\mathcal{Y}'$, or was in a clique-cycle of size at most $\frac{n}{240}$, we obtain the desired bounds on $|L_1|$.

This partition fulfills the requirements of the lemma. $\square$

3.4.2 Corrected cycle pairs

Given a partition of $V(G)$ into a red clique-cycle pair, a blue clique-cycle pair and a leftover set, as provided by Lemma 3.6, we would like to say that there is a red cycle which covers the red clique-cycle pair and some of the leftover set and a blue cycle which covers everything else. Unfortunately this is not quite true. We will need to use a small number of vertices in the blue clique-cycle pair in constructing our red cycle, and vice versa. In this subsection we will define a similar concept to a red clique-cycle pair: a red *corrected cycle pair*. We will see that it can be covered by a red cycle.

First we must define some terms, in each case with respect to a given partition of G into red and blue clique-cycle pairs and a leftover set (as is provided by Lemma 3.6). A red *pickup path* is a red path whose start and end vertices are in the same clique in one clique-cycle, and whose interior vertices are alternately vertices within the leftover set and within other clique-cycles. We will see that disjoint pickup paths can be constructed covering every vertex of the leftover set.

A red *balance path* is a red path whose initial and final vertices are in the same clique in a clique-cycle; its purpose is to cover some excess vertices within off-colour clique-cycles.

We say that a *free* vertex is any vertex which is not contained in any pickup or balance path, any of the linking paths in the on-colour clique-cycles, or the paths P_1, P_2, Q_1, Q_2.

When S is a subset of $V(G)$, we let $Pick(S)$ be the number of pickup paths which start and end in S, $Bal(S)$ be the number of balance paths which start and end in S, and $Free(S)$ be the number of free vertices in the set S.

Finally, when V_i is a clique in an off-colour clique-cycle $\mathcal{V}$, we define $Spin(V_i)$ by

$$\begin{aligned} Spin(V_i) &= Free(V_i) + Pick(V_i) + Bal(V_i) && (i \geq 2 \text{ or } i = 1, P_1 = \emptyset) \\ Spin(V_1) &= Free(V_1) + Pick(V_1) + Bal(V_1) + 1 && (i = 1, P_1 \neq \emptyset). \end{aligned}$$

We say that the off-colour clique-cycle $\mathcal{V}$ is *balanced* if all its cliques have the same spin.

We define a red *corrected cycle pair* to be a collection $(\mathcal{U}, P_1, P_2, \mathcal{V}, J_r)$ consisting of a red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$ together with a set J_r of red pickup and balance paths, such that the pickup and balance paths are disjoint from each other, from the linking paths in the on-colour clique-cycle, and from the paths P_1, P_2, and such that the off-colour clique-cycle is balanced.

Lemma 3.7. *If G possesses a red corrected cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V}, J_r)$ then we can find a red cycle C_r in G covering exactly the vertices of the corrected cycle pair.*

Proof. We construct C_r as follows.

If neither $\mathcal{U}$ nor $\mathcal{V}$ are empty, we start at the start vertex of P_2 in U_1.

If this is the start vertex of a pickup or balance path we follow the path to its end vertex, by definition also in U_1. Now if there are any pickup or balance paths remaining in U_1 we move directly to the start vertex and then along each in turn. We then move to each free vertex in U_1 that we have not yet visited in succession, and eventually to the start vertex of the path $u_{1,2}$ and along it.

We now repeat the above procedure for each U_i, $2 \leq i \leq u$. On returning to U_1 along $u_{u,1}$ we move to the start vertex of P_1 and along it to V_1.

We now apply the following process. If the vertex in V_i we are currently at is one end of a pickup or balance path, we follow the path until we return to V_i. We now select if possible a vertex in $V_{i+1 \mod v}$ which is the start vertex of a pickup or balance path which we have not yet visited and move to it; if this is not possible we move to any free vertex in $V_{i+1 \mod v}$ which we have not yet visited.

We repeat this process until we are forced to stop. When this occurs, we are at a vertex in some clique V_i, having travelled every pickup and balance path and visited every free vertex in $V_{i+1 \mod v}$. Thus we have been around the clique-cycle $Spin(V_{i+1 \mod v})$ times. Since the off-colour clique-cycle is balanced, we are at V_v and have been along every pickup and balance path and through every free vertex in $\bigcup \mathcal{V}$. We move directly to the end vertex of P_2 in V_1 and along P_2 to U_1, completing the cycle C_r.

If both $\mathcal{U}$ and $\mathcal{V}$ are empty, we set $C_r = \emptyset$. If $\mathcal{U}$ is empty but $\mathcal{V}$ is not we start at the start vertex of a pickup or balance path in V_1 if this is possible, or any free vertex in V_1 if not, and follow the above procedure until we return to the start vertex and complete the cycle C_r. If $\mathcal{V}$ is empty but $\mathcal{U}$ is not we start at the end vertex of $u_{u,1}$ in U_1 and follow the clique-cycle $\mathcal{U}$ as above until we return to that vertex, completing the cycle C_r.

By the definition of a corrected cycle pair, the red cycle C_r covers exactly the vertices of the corrected cycle pair. $\square$

3.4.3 Correcting clique-cycle pairs

In this subsection we describe algorithms which take the partition of $V(G)$ into a leftover set, a red clique-cycle pair and a blue clique-cycle pair given by Lemma 3.6 and return the desired two-cycle partition, by way of Lemma 3.7.

We will need to use different algorithms depending on the sizes of the various parts $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$. In each case we will construct sequentially a set of pickup and balance paths on the way to giving corrected cycle pairs. We will again use the functions *Pick*, *Bal*, *Free* and *Spin* defined in the previous subsection; in each case with reference to the current set of pickup and balance paths at that point in the algorithm. The main obstacle which we must overcome is the requirement that the off-colour clique-cycles must be balanced: we start by giving a case where there is an easy 'quick fix', and the only difficulty is the (relatively easy) incorporation of the vertices in the leftover set into the corrected cycle pairs. We will then go on to the more involved cases where the 'quick fix' cannot be used; these are not really hard, but simply technical.

We will use the following lemma to obtain a set of pickup paths through all vertices of the leftover set.

Lemma 3.8. *Let $A_1, \dots, A_a, B_1, \dots, B_b, C$ be disjoint subsets of $V(G)$, where the A_i are subsets of cliques in one clique-cycle, the B_j are subsets of cliques in another, and C is a leftover set. Suppose that*

$$2|C| \leq \min(|A_1| + \cdots + |A_a| - 4a, |B_1| + \cdots + |B_b| - 4b) \ .$$

Then there exist collections J_r and J_b of disjoint red and blue pickup paths within $A_1 \cup \cdots \cup B_b \cup C$ such that each red path starts and ends in an A_i while each blue path starts and ends in a B_j and such that every vertex in C is in one of the paths. Furthermore in any A_i or B_j the number of vertices which are in none of the paths J_r or J_b (free vertices) is greater than the number of vertices which are interior vertices of the paths J_r or J_b.

Proof. We apply the following algorithm. First we mark all vertices as *active*. Now for each member c of C in succession, we proceed as follows.

If there are red edges between c and two active members x, y of some A_i then we record into J_r the red pickup path x, c, y and mark these vertices as *inactive*.

If there is no such red path through c, but there are blue edges between c and two active members x, y of some B_j then we record into J_b the blue pickup path x, c, y and mark these vertices as inactive.

If there are neither red nor blue pickup paths, we mark c as *remaining*.

We let the eventual set of remaining vertices be R. If it is empty, we are done. If not, then each $r \in R$ is red-adjacent to at most one active vertex in each A_i, and blue-adjacent to at most one active vertex in each B_j.

Observe that there must exist at least one pair of sets A_α and B_β which each contain at least five active vertices. Since any two-edge-colouring of $K_{5,5}$ has either a red or a blue matching of size three, we are guaranteed such between the active vertices of A_α and B_β. We assume without loss of generality that the former holds.

If $R = \{r_1\}$, then r_1 is blue-adjacent to at most one of the vertices of the red matching in B_β, so there is a red path on five vertices from A_α through r_1 and returning to A_α. We record this pickup path into J_r and are done.

If $|R| \geq 2$, then let $R = \{r_1, \ldots, r_r\}$. Since we have a red matching of size three between A_α and B_β we can choose active vertices a_1, b_1, b_{r+1}, a_2 such that a_1, b_1, r_1 and r_r, b_{r+1}, a_2 are both red paths from A_α to r_1 and r_r respectively. We mark these vertices as *interior-inactive*. By the original condition on $|C|$ there remain at least $2|R| + 4b - 2$ active vertices in $B_1 \cup \cdots \cup B_b$.

Now for each $1 \leq i \leq r - 1$ in succession, since there must be at least $2|R| + 4b - 1 - i$ active vertices in $B_1 \cup \cdots \cup B_b$ (one is made interior-inactive at each step) we can find an active vertex b_{i+1} in a set with at least four more active vertices than interior-inactive vertices which is red-adjacent to both r_i and r_{i+1}, and mark it as interior-inactive.

Finally we record the red pickup path $a_1, b_1, r_1, b_2, \ldots, b_r, r_r, b_{r+1}, a_2$ which passes through all of R into J_r. This path is the only path which has interior vertices in any of the A_i or B_j, and by its construction each of the A_i and B_j contains more active vertices (in none of the J_r or J_b) than interior-inactive vertices. $\square$

We now give the various algorithms for constructing two-cycle partitions.

Lemma 3.9. *If $n \geq 2^{18000}$ and G has a partition as in Lemma 3.6 in which both $|\bigcup\mathcal{U}|, |\bigcup\mathcal{X}| \geq \frac{n}{20}$, then G has a two-cycle partition.*

Proof. First we modify the off-colour clique-cycles $\mathcal{V}$, $\mathcal{Y}$ (if these are not empty) by removing vertices from each clique in these clique-cycles until $|V_1| = |Y_1| = 8981$ and all the other cliques have size 8980, to obtain $\mathcal{V}'$ and $\mathcal{Y}'$. If either clique-cycle is empty we do nothing to it. We create a new leftover set L_2 as the union of L_1 and the at most $9(v+y)$ vertices removed. The modified off-colour clique-cycles are balanced. Observe that $|L_2| \leq 2^{17790} + \frac{n}{120} + 15(v+y)$.

Now we let A_i be the set of free vertices in U_i for each i, B_j be the set of free vertices in X_j for each j, and $C = L_2$.

Observe that $|L_2| \leq 2^{17790} + \frac{n}{120} + 15\frac{n}{8995} < \frac{23n}{2000}$. Furthermore the number of free vertices in $\mathcal{U}$ is at least $|\bigcup\mathcal{U}| - 2u - 2 > \frac{90n}{2000} + 4u$, and similarly the number of free vertices in $\mathcal{X}$ is at least $\frac{90n}{2000} + 4x$. Thus the sets A_i, B_j and C satisfy the conditions of Lemma 3.8, and we can apply this lemma to obtain disjoint sets J_r and J_b consisting of pickup paths which are disjoint from each other, from the linking edges and paths in $\mathcal{U}$ and $\mathcal{X}$, and from the paths P_1, P_2, Q_1, Q_2. Every vertex in L_2 is in one of these paths. We modify $\mathcal{U}$ by removing every vertex in J_b to obtain $\mathcal{U}'$, and we modify $\mathcal{X}$ similarly to obtain $\mathcal{X}'$.

Now $(\mathcal{U}', P_1, P_2, \mathcal{V}', J_r)$ forms a red corrected cycle pair, which is disjoint from the blue corrected cycle pair $(\mathcal{X}', Q_1, Q_2, \mathcal{Y}', J_b)$. The two corrected cycle pairs cover $V(G)$. By Lemma 3.7 their vertices form the desired two-cycle partition. $\square$

This construction was made easier by the 'quick fix' of simply removing a small number of vertices from the off-colour clique-cycles to force them to be balanced. Unfortunately this is not possible in the remaining cases, and more technical work (rather than new ideas) will be required. We require the following trivial lemma.

Lemma 3.10. *If A and B are disjoint subsets of $V(G)$ each of size at least three, then either there exists a vertex in A red-adjacent to two vertices in B, or there exists a vertex in B blue-adjacent to two vertices in A.*

This lemma allows us to construct a balance path and so reduce the spin of the cliques containing A and B by one.

Lemma 3.11. *If $n \geq 2^{18000}$ and both $|\bigcup\mathcal{U}|, |\bigcup\mathcal{Y}| \geq \frac{n}{20}$ then we can find a two-cycle partition of G.*

Proof. We modify the off-colour clique-cycle $\mathcal{V}$ (if it is not empty) by removing vertices until each clique has size 8980, except for V_1 which has size 8981, to obtain $\mathcal{V}'$. We create a new leftover set L_2 consisting of L_1 together with the removed vertices. We observe that $|L_2| \leq \frac{n}{1024} + \frac{n}{120} + 15v + 6y < \frac{23n}{2000}$. Since $|\bigcup\mathcal{U}| \geq \frac{n}{20}$ we see that there are certainly at least $4|L_2| + 130u$ free vertices in $\bigcup\mathcal{U}$.

Now choose the largest $m \leq 8980$ such that $|\bigcup\mathcal{Y}| - my \geq 2|L_2| + 4y$. Observe that $m \geq 4480$. We apply a similar algorithm to that in the previous lemma. From each clique Y_j we choose a subset B_j consisting of $|Y_j| - m$ of the free vertices. Observe that

$$|B_1 \cup \cdots \cup B_y| = |\bigcup\mathcal{Y}| - my \in [2|L_2| + 4y, 2|L_2| + 5y)\ .$$

We let A_i be the set of free vertices in U_i for each i. We let $C = L_2$, and apply Lemma 3.8 to obtain sets J_r, J_b of pickup paths covering L_2.

By the definition of the spin of a clique Y_j, when a pickup path is constructed which starts and ends in Y_j (using two vertices of Y_j) it decreases the spin of the clique by one, while from Lemma 3.8 the number of vertices of Y_j which are interior vertices of any pickup path is exceeded by the number of vertices which are in no pickup path. It follows that the use of Lemma 3.8 to create a set of pickup paths causes the spin of the clique Y_j to decrease by at most $\frac{|B_j|}{2} \leq \frac{8981-m}{2}$. Thus at this point each clique Y_j has spin at least $b = \lfloor \frac{8980+m}{2} \rfloor \geq 6730$.

We now say that a clique Y_j is *balanced* if $Spin(Y_j) = b$, and *unbalanced* otherwise. We note that an unbalanced clique must have spin greater than b. We call the difference $Spin(Y_j) - b$ the excess spin of the clique Y_j.

Since $|\bigcup \mathcal{U}| \geq \frac{n}{20}$ and not more than $2|L_2|$ vertices in $\mathcal{U}$ can be in any of the paths $J_r \cup J_b$ we observe that the number of free vertices in $\bigcup \mathcal{U}$ is still at least $2|L_2| + 130u$.

From the definitions of m and b the sum of the excess spins of all the cliques in $\mathcal{Y}$ cannot exceed

$$\frac{|\bigcup \mathcal{Y}| - my}{2} \leq \frac{2|L_2| + 5y}{2} \leq |L_2| + 3y < |L_2| + 60u \; .$$

We construct J'_r and J'_b by adding new balance paths sequentially to J_r and J_b as follows.

If Y_j is an unbalanced clique, then $Spin(Y_j) > 6730$, so

$$Free(Y_j) + Pick(Y_j) + Bal(Y_j) \geq 6730 \; .$$

But each pickup or balance path contributing to $Pick(Y_j)$ or $Bal(Y_j)$ uses two vertices from Y_j, and $|Y_j| \leq 8981$. Thus certainly $Free(Y_j) \geq 3$. Since $|\bigcup \mathcal{U}| \geq 4|L_2| + 130u$ there must be a clique U_i with $Free(U_i) \geq 3$. We can apply Lemma 3.10: there exists either a red balance path on three vertices whose ends are free vertices in U_i passing through a free vertex of V_j, or a blue balance path on three vertices whose ends are free vertices of V_j passing through a free vertex in U_i. We record the red balance path into J'_r if it exists, otherwise the blue balance path into J'_b. This procedure causes $Spin(Y_j)$ to decrease by one. We repeat this until every clique Y_i is balanced.

Finally we modify $\mathcal{U}$ by removing all vertices in J'_b to obtain $\mathcal{U}'$ and $\mathcal{Y}$ by removing all vertices in J'_r to obtain the balanced off-colour clique-cycle $\mathcal{Y}'$. Now $(\mathcal{U}', P_1, P_2, \mathcal{V}', J'_r)$ and $(\mathcal{X}, Q_1, Q_2, \mathcal{Y}', J'_b)$ are disjoint corrected cycle pairs covering $V(G)$, and the result follows by Lemma 3.7. $\square$

In the next case we have to balance simultaneously two off-colour clique-cycles; this case requires the most care.

Lemma 3.12. *If $n \geq 2^{18000}$ and both $|\bigcup \mathcal{V}|, |\bigcup \mathcal{Y}| \geq \frac{n}{20}$ then there exists a two-cycle partition of G.*

Proof. We begin similarly to the previous lemma. Choose the largest $m_r, m_b \leq 8980$ such that

$$|\bigcup \mathcal{V}| - m_r v \geq 2|L_1| + 4v \text{ and}$$

$$|\bigcup \mathcal{Y}| - m_b y \geq 2|L_1| + 4y \ .$$

Note that $m_r, m_b \geq 5300$.

For each i, choose a set A_i consisting of $|V_i| - m_r$ of the free vertices of V_i; for each j choose a set B_j of free vertices of Y_j of size $|Y_j| - m_b$. Let $C = L_1$, and apply Lemma 3.8 to obtain sets J_r, J_b of pickup paths covering every vertex in L_1. By an identical argument to that in the previous lemma, the spin of any clique V_i has decreased by at most $\frac{8989 - m_r}{2}$ so is at least $\frac{8980 + m_r}{2} \geq 7100$. Similarly each clique Y_j now has spin at least $\frac{8980 + m_b}{2} \geq 7100$. There remain at least $2|L_1| + 45y$ free vertices in $\mathcal{V}$, and at least $2|L_1| + 45v$ free vertices in $\mathcal{Y}$.

Now we must balance both off-colour clique-cycles. We must choose the parameters b_r and b_b which will be the spins of cliques in the red and blue off-colour clique-cycles in our eventual corrected cycle pairs.

For a b_r we define the excess spin of the clique-cycle $\mathcal{V}$ by

$$\mathit{Excess}(\mathcal{V}, b_r) = \sum_{k=1}^{v} (\mathit{Spin}(V_k) - b_r) \ .$$

If we apply Lemma 3.10 with A consisting of three free vertices in some V_i and B being three free vertices of Y_j then we obtain a three vertex balance path. Incorporating this into our set of balance paths decreases the spin of both V_i and Y_j by one: so if there exist $b_r, b_b < 7097$ which are not too small and such that $\mathit{Excess}(\mathcal{V}, b_r) = \mathit{Excess}(\mathcal{Y}, b_b)$, we can repeatedly apply Lemma 3.10 until both clique-cycles are simultaneously balanced. Since

$$\mathit{Excess}(\mathcal{V}, b_r - 1) = \mathit{Excess}(\mathcal{V}, b_r) + v \text{ and}$$

$$\mathit{Excess}(\mathcal{Y}, b_b - 1) = \mathit{Excess}(\mathcal{Y}, b_b) + y \ ,$$

choosing b_r and b_b amounts to solving a linear congruence. Unfortunately this congruence may not be soluble at all (if v and y have a common factor) or the solution may require b_r and b_b to be too big or small to be useful.

Since the cliques V_1 and Y_1 each have at least three free vertices, we can identify either a red or a blue matching between them of size two. Assume without loss of generality that it is a red matching (α, β), $(\gamma, \delta) \in V_1 \times Y_1$.

We can construct one more (long) balance path using this matching that allows us to make use of a solution to

$$\mathit{Excess}(\mathcal{Y}, b_b) + 10 \leq \mathit{Excess}(\mathcal{V}, b_r) \leq \mathit{Excess}(\mathcal{Y}, b_b) + v + 20\ .$$

This is soluble: we choose b_b to be the largest number such that

$$\mathit{Excess}(\mathcal{Y}, b_b) + 10 \leq \mathit{Excess}(\mathcal{V}, 7097)\ ,$$

and then choosing $b_r \leq 7097$ to be the largest number such that

$$\mathit{Excess}(\mathcal{V}, b_r) \leq \mathit{Excess}(\mathcal{Y}, b_b) + v + 20$$

gives a solution to both inequalities.

Since $20v \geq y \geq \frac{v}{20}$ we are guaranteed to find that one of b_r and b_b is between 7077 and 7097.

Since $\mathit{Excess}(\mathcal{Y}, 7097)$ cannot exceed $|L_1|$ we are guaranteed to find also that $b_r \geq 5000$, and similarly for b_b. We say that a clique in $\mathcal{V}$ is balanced if its spin is b_r, and similarly for $\mathcal{Y}$. Observe that an unbalanced clique must have at least three free vertices; at this point every clique in $\mathcal{V}$ has spin at least $b_r + 3$, and similarly for $\mathcal{Y}$.

If $\mathit{Excess}(\mathcal{V}, b_r) - \mathit{Excess}(\mathcal{Y}, b_b) = s$ is even, choose a free vertex ε in Y_1 not in the red matching of size two. Note that $v + 20 < 2v - 4$, so that $s < 2v - 4$ and $2 + \frac{s}{2} < v$.

Now choose from each clique $V_2, \ldots, V_{2+\lfloor \frac{s}{2} \rfloor}$ two free vertices, and let B be a red balance path which starts and ends in the chosen vertices in $V_{2+\lfloor \frac{s}{2} \rfloor}$ and whose interior vertices are the other chosen vertices, α, β, γ, δ and if s is even ε. We record this red balance path along with the paths J_r to create J'_r.

Now the spin of any clique (with respect to the new sets J_r', J_b) has decreased by at most three, so each clique in $\mathcal{V}$ has spin at least b_r and each clique in $\mathcal{Y}$ has spin at least b_b. The creation of B has decreased $Excess(\mathcal{Y}, b_b)$ by either two or three, depending on whether s is odd or even (the vertices β, δ and, if s is even, ε in Y_1 are no longer free). The creation of B has also decreased $Excess(\mathcal{V}, b_r)$ by either $s+2$ or $s+3$, again depending on whether s is odd or even (two vertices in each clique $V_1, \ldots, V_{2+\lfloor \frac{s}{2} \rfloor}$ are no longer free and one balance path has been created in $V_{2+\lfloor \frac{s}{2} \rfloor}$). Thus the creation of B gives us $Excess(\mathcal{Y}, b_b) = Excess(\mathcal{V}, b_r)$.

We apply Lemma 3.10 repeatedly to construct balance paths on three vertices between the free vertices of unbalanced pairs of cliques V_i and Y_j, each decreasing the spin of both V_i and Y_j by one. Eventually every clique in both off-colour clique-cycles is balanced. We let J_r'' be the union of J_r' and the red balance paths just constructed, and J_b' be the union of J_b and the blue balance paths just constructed. We modify $\mathcal{V}$ and $\mathcal{Y}$ to obtain the balanced clique-cycles $\mathcal{V}'$ and $\mathcal{Y}'$ by removing all vertices in J_b' and J_r'' respectively. Now $(\mathcal{U}, P_1, P_2, \mathcal{V}', J_r'')$ and $(\mathcal{X}, Q_1, Q_2, \mathcal{Y}', J_b')$ are disjoint corrected cycle pairs covering $V(G)$, and the result follows. $\square$

Finally we consider the possibility that one of the two clique-cycle pairs is small.

Lemma 3.13. *If $n \geq 2^{18000}$ and $|\bigcup \mathcal{X}|, |\bigcup \mathcal{Y}| \leq \frac{n}{20}$ then we have a two-cycle partition of G.*

Proof. We let $L_2 = L_1 \cup \bigcup \mathcal{X} \cup \bigcup \mathcal{Y} \cup Q_1 \cup Q_2$. Observe that

$$|L_2| \leq 2^{17990} + \frac{n}{80} + 6(v+y) + \frac{2n}{20} + 2(18000 + x + y) \leq \frac{12n}{100} .$$

Now either $|\bigcup \mathcal{U}| \geq \frac{42n}{100}$ or $|\bigcup \mathcal{V}| \geq \frac{42n}{100}$.

In the former case, we create L_3 by removing at most $8v$ vertices from $\mathcal{V}$ to obtain a balanced clique-cycle $\mathcal{V}'$. Then, for each $\ell \in L_3$ sequentially, we apply the following process to obtain a set J of pickup paths.

If ℓ is red-adjacent to two free vertices j_1, j_2 in any clique U_i then record into J the pickup path j_1, ℓ, j_2. Otherwise mark ℓ as remaining.

Let the set of remaining vertices be $R = \{r_1, \dots, r_r\}$. Each vertex is red-adjacent to at most one free vertex in any clique U_i. Since $|\bigcup \mathcal{U}| \geq \frac{42n}{100}$ and each vertex in $L_3 - R$ has given a path in J which uses up two vertices from $\bigcup \mathcal{U}$, the number of free vertices remaining in $\bigcup \mathcal{U}$ exceeds $|R| + 3u$. We can follow the same logic as in Lemma 3.8 to greedily construct a blue cycle C_b whose vertices are alternately the members of R and free vertices from $\bigcup \mathcal{U}$.

We modify $\mathcal{U}$ by removing all the vertices in C_b to obtain $\mathcal{U}'$. Then $(\mathcal{U}', P_1, P_2, \mathcal{V}', J)$ is a red corrected cycle pair which covers exactly the vertices of $V(G)$ not in C_b, so by Lemma 3.7 it is covered by a red cycle C_r.

In the latter case, let $m \leq 8981$ be the greatest number such that

$$\sum_{i=1}^{v} (\mathit{Spin}(V_i) - m - 5) \geq |L_2| \ .$$

Since $|L_2| \leq \frac{12n}{100}$ and $|\bigcup \mathcal{V}| \geq \frac{42n}{100}$ we certainly have that $m > 5000$. Thus any clique with spin greater than m must have at least 100 free vertices. For each $\ell \in L_2$ we apply the following process.

If ℓ is red-adjacent to two free members j_1, j_2 of a clique V_i which has $\mathit{Spin}(V_i) \geq m + 5$ then we record the red pickup path j_1, ℓ, j_2. If not, we mark ℓ as remaining.

Let the set of remaining vertices be R. Let

$$\mathit{Excess}(\mathcal{V}) = \sum_{i=1}^{v} (\mathit{Spin}(V_i) - m) \ .$$

Now every clique V_i has spin at least $m + 4$, and $\mathit{Excess}(\mathcal{V}) \geq |R|$. We say that a clique is balanced if it has spin m, and unbalanced otherwise.

We construct red balance paths on three vertices between the free vertices of pairs of unbalanced cliques V_i, V_j $(i, j \neq 1)$ until either

$$|R| \leq \mathit{Excess}(\mathcal{V}) \leq |R| + 1$$

or there remain no red balance paths on three vertices between free vertices of pairs of unbalanced cliques. Observe that each balance path constructed reduces $\mathit{Excess}(\mathcal{V})$ by two.

In the first case, since the spin of V_1 is at least $m+4$ we have $|R| \geq 3$ and we can greedily construct a blue cycle C_b passing through all members of R and either $|R|$ (by choosing vertices alternately from R and $\mathcal{V}$) or $|R|+1$ (by having an extra edge in V_1 in the cycle) free vertices in the unbalanced cliques of $\mathcal{V}$, as appropriate. Then we construct $\mathcal{V}'$ by removing all vertices of C_b from $\mathcal{V}$; this is a balanced clique-cycle.

In the second case, we have a collection of unbalanced cliques $V_1, V_{r_1}, \ldots$ such that any pair V_{r_i}, V_{r_j} do not have any red balance path between their free vertices. Since each clique has at least 100 free vertices, certainly there are blue edges between the free vertices of any such pair.

If $|R| \leq 1$ then we can find further red balance paths between the free vertices of V_1 and of the V_{r_i} until either all the cliques are balanced or any pair of our remaining unbalanced cliques have blue edges between their free vertices. In either case we can find a blue cycle C_b which passes through $Spin(V_i) - m$ of the free vertices of each such V_i; if $|R| = 0$ it passes through no other vertices, while if $|R| = 1$ it passes through the vertex in R also.

If $|R| \geq 2$ then we can find a blue cycle C_b covering exactly $Spin(V_i) - m$ free vertices of each unbalanced clique; between one free vertex in an unbalanced clique and the next along the cycle we may either have a blue edge or a blue path of length two passing through a member of R, as appropriate to cover all the members of R and to guarantee being able to pass from V_1 to the V_{r_i}.

In either case, we let J be the set of red balance paths we constructed and modify $\mathcal{V}$ by removing all vertices in C_b to obtain the balanced clique-cycle $\mathcal{V}'$. Then $(\mathcal{U}, P_1, P_2, \mathcal{V}', J)$ is a corrected clique-cycle which must be covered by a red cycle C_r, and C_b covers exactly the vertices of G not in it. $\square$

3.5 The final result

Let $n \geq 2^{18000}$. Suppose that G is a two-edge-coloured complete graph on n vertices.

If G possesses a large monochromatic complete bipartite subgraph satisfying the conditions of Theorem 3.4 then it possesses a two-cycle partition.

If G does not possess such a large bipartite subgraph then we may apply Lemma 3.6 to obtain a partition of $V(G)$ into disjoint red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$, blue clique-cycle pair $(\mathcal{X}, Q_1, Q_2, \mathcal{Y})$ and a leftover set.

At least one of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$ must cover at least $\frac{n}{20}$ vertices, since the leftover set is not larger than $2^{17790} + \frac{n}{120} + 6(v + y) < \frac{n}{100}$ by choice of n. Without loss of generality, assume that one of $\mathcal{U}$ or $\mathcal{V}$ covers at least $\frac{n}{20}$ vertices. If also either $\mathcal{X}$ or $\mathcal{Y}$ covers at least $\frac{n}{20}$ vertices then we may apply one of Lemmas 3.9, 3.11 (which of course also gives the result when $\mathcal{X}$ and $\mathcal{V}$ are large) or 3.12 to find that there exists a two-cycle partition of G. If on the other hand neither $\mathcal{X}$ nor $\mathcal{Y}$ covers $\frac{n}{20}$ vertices then we may apply Lemma 3.13 to discover a two-cycle partition of $V(G)$. $\square$

3.6 Further thoughts

It is not hard to find minor improvements to the proof above, which we do not give in the interests of a shorter and more readable proof. In particular, we can define red-adjacency in Lemma 3.6 with a matching of size only three; we can argue that the leftover set should always be much smaller, and so on. However even making the most optimistic assumptions—that there is some way to pick up vertices from the leftover set in long paths rather than one at a time, that the correct exponent in Ramsey's Theorem should be 2, and so on—it seems utterly impossible that this method could be made work with cliques of size smaller than 10 (and so with graphs on less than 1000 vertices). On the other hand, it is already out of the question to check by brute force computation all graphs on even 100 vertices, so while Lehel's conjecture certainly seems reasonable this method will not prove it in full.

We can read the proof of Theorem 3.5 as an algorithm which either produces the desired path on at most 18000 vertices (in quadratic time) or returns the large complete bipartite graph required for Theorem 3.4. The proof by Łuczak, Rödl and Szemerédi of that theorem is again a polynomial time algorithm finding the red and blue cycles explicitly, and it is easy to check that all our proofs amount to polynomial time algorithms, so that we have a polynomial time algorithm which returns the two-cycle partition of G (if it exists).

It seems reasonable that there should exist an extension of this result for larger numbers of colours: if the edges of K_n are k-coloured then we can find a partition of its vertices into k monochromatic cycles. This is a conjecture of Erdős, Gyárfás and Pyber [29]. They proved that a partition into $ck^2 \log k$ monochromatic cycles suffices, for some constant c; the best known bound is that a partition into at most $100k \log k$ monochromatic cycles exists for sufficiently large n, due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [45]. However the methods in this paper do not seem to be easily extended to dealing with even three colours. We can certainly apply Ramsey's theorem in a similar way to obtain a partition into small monochromatic cliques and a leftover set, and then describe two red cliques as red-adjacent if joined by a small red matching. But we would then have to define blue- and green-adjacency between two red cliques; and the obvious way to do this (colouring by the majority colour of edges) does not even allow us to construct blue paths along blue-adjacent paths of red cliques.

4

Forbidden induced bipartite graphs

4.1 Introduction

In 1992 Prömel and Steger considered the problem of finding the speeds of hereditary classes defined by one induced subgraph. They proved Theorem 1.3, and went on to find for every graph which is not an induced subgraph of P_4 the correct coefficient of n^2 in the exponent.

Theorem 4.1. *For any fixed $G \neq K_1$, the number of graphs on n vertices which do not contain G as an induced subgraph is*

$$2^{\left(1-\frac{1}{\tau(G)-1}\right)\frac{n^2}{2}+o(n^2)},$$

where $\tau(G)$ is defined to be the smallest number k such that for every $0 \leq i \leq k$ there is a partition of $V(G)$ into i cliques and $k-i$ independent sets.

The only graph with $\tau(G) = 1$ is K_1, while the graphs with $\tau(G) = 2$ must be bipartite graphs whose complement is bipartite: so they cannot have more than four vertices, and it is easy to check that in fact they correspond to P_4 and its induced subgraphs. Note that the lower bound in their theorem is easy: by the definition of $\tau(G)$, there is an i such that G is not an induced subgraph of any graph whose vertex set can be partitioned into i cliques and $\tau(G) - i - 1$ independent sets. The lower bound comes from counting graphs of this form.

One possibility would be to attempt to sharpen their results: for example Erdős, Kleitman and Rothschild showed that almost every triangle-free graph is bipartite (which of course gives a very accurate formula) and gave a similar result for K_l-free graphs, $l \geq 4$; and Prömel, Schickinger and Steger gave a related argument sharpening even that result (they showed that non-bipartite triangle-free graphs can almost surely be made bipartite by removing one vertex, and if that fails almost surely removing two vertices suffices, and so on) and similarly for K_l-free graphs, $l \geq 4$. However in this chapter we will pursue a different question.

Naturally, one can define speeds for structures other than graphs. Brightwell, Grable and Prömel [14] studied the problem of counting the partial orders on n elements not containing Q as an induced sub-order, finding for many partial orders Q the speed of the Q-free partial orders. Their results were much more complicated than Prömel and Steger's: they found sub-orders giving speeds in both ranges given by Prömel and Steger, but they also gave two further speed categories lying in the gap between those ranges: speeds bounded between n^{cn} and n^{Cn} for some $1 < c < C$, and speeds lying above that category but not faster than $2^{o(n^2)}$. They also left some sub-orders which could not be classified.

There is, of course, a relationship between graphs and partial orders. Every partial order has an underlying comparability graph, with edges between comparable pairs of points in the partial order. Every graph can be drawn on a sheet of paper, and a partial order obtained by directing edges up the paper and inserting further directed edges to satisfy transitivity. However this relationship is not one-to-one: most graphs are not comparability graphs and one cannot draw them in any way to avoid inserting edges. In particular, while there is a straightforward correspondence between bipartite graphs and partial orders of height two (one can direct edges from one part to the other without violating transitivity) there is no such correspondence between split graphs and partial orders. When considering the number of graphs without—for example—an induced copy of $K_{2,2}$ one immediately sees the large class of split graphs, obtained by replacing one of the independent set parts of a bipartite graph with a clique. These are $K_{2,2}$-free and their number is vastly greater than the number of $K_{2,2}$-free bipartite graphs, so it dominates the enumeration. But when one enumerates partial orders which do not contain the height two order Q obtained from $K_{2,2}$ by setting one part below the other, there is no large class of size $2^{\Theta(n^2)}$ dominating the count. The smaller classes—for example the height two partial orders with no induced Q—must be counted, and this is the cause of the extra complexity in the results of Brightwell, Grable and Prömel.

We wish to uncover the complexity in the graph case that has been hidden in the results of Prömel and Steger by the large classes like the split graphs: so we will enumerate the bipartite H-free graphs. Of course, we could obtain some bounds from the results of Brightwell, Grable and Prömel by translating back from partial orders of height two to graphs: but we will find that bipartite graphs are relatively easy to work with. We will obtain results on more graphs, and sharper results, by working with bipartite graphs directly.

Let $G = G[X, Y]$ be a bipartite graph with bipartition (X, Y). We say that X is the *lower part*, and Y the *upper part*, of G. We will draw diagrams accordingly. We say that the *bipartite complement* of G is the bipartite graph which has edges between X and Y exactly where G does not, together with the bipartition (X, Y). If z is a vertex in $G[X, Y]$, then as usual we say that the degree of z, $d(z)$, is the number of vertices (in the part not containing z) adjacent to z. We say that the *co-degree* of z is the number of vertices in the part not containing z which are not adjacent to z.

Let $G = G[X, Y]$ and $H = H[W, Z]$. We say that G *contains a copy* of H if there exist $W' \subset X$, $Z' \subset Y$, such that the induced subgraph of G on the vertices $W' \cup Z'$, with bipartition (W', Z'), is isomorphic to $H[W, Z]$.

We consider three closely related problems.

First, let $H = H[W, Z]$. We wish to estimate the number $Forb_{m,n}(H)$ of graphs with bipartitions $G[X, Y]$ which do not contain a copy of H, in terms of the sizes m, n of the parts X, Y of G. We will restrict our attention to the case $n = \Theta(m)$.

Second, let $H = H[W, Z]$. We wish to estimate the number $Forb_n(H)$ of bipartite graphs G on n vertices such that no bipartition of G contains a copy of H.

Third, let H be a fixed bipartite graph. We wish to estimate the number $Free_n(H)$ of bipartite graphs G on n vertices such that no bipartition of G contains a copy of any $H[W, Z]$, (W, Z) a bipartition of H (G is called *H-free*).

These three problems are obviously related. If an n-vertex graph G is H-free, then it certainly does not contain a copy of $H[W, Z]$ for any specific bipartition of H; if no bipartition of G contains a copy of $H[W, Z]$ then certainly $G[U, V]$ does not contain a copy of $H[W, Z]$. This gives us the inequalities

$$Free_n(H) \leq Forb_n(H[W, Z]) \leq 2^n Forb_{n-r,r}(H[W, Z])$$

where (W, Z) is any bipartition of H, and r maximises $Forb_{n-r,r}(H[W, Z])$; the first inequality becomes an equality when H has only one bipartition (for example, when it is connected). As we will consider many graphs which do have several very different bipartitions, this distinction will be important.

As an illustration of the differences between these three problems, consider the bipartite graph on four vertices $SI(2, 1)$, as shown in Figure 4.1, with the bipartition as shown there.

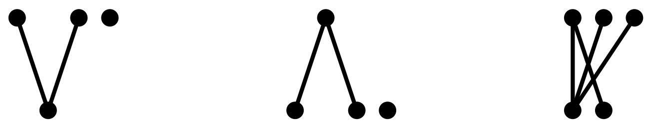


Figure 4.1 $SI(2, 1)$ and allowed graphs for the first and second problems

A bipartite graph $G[X, Y]$ containing no copy of $SI(2, 1)$ with the given bipartition has the property that for each $x \in X$, either X is adjacent to no vertex in Y, to exactly one vertex in Y, or to every vertex in Y, for a total of $n + 2$ possibilities for each of the m vertices in X. Since every graph with this property contains no copy of $SI(2, 1)$ with the given bipartition, $Forb_{m,n}(SI(2, 1)) = (n + 2)^m$. The second graph in Figure 4.1 contains no copy of $SI(2,1)$ with the given bipartitions – even though it is simply $SI(2,1)$ the other way up.

By contrast, suppose that G is a bipartite graph on n vertices such that no bipartition of G contains $SI(2,1)$ with the given bipartition. If G contains a vertex x of degree two or greater, then G must be connected and every vertex in the part not containing x must be adjacent to x. Thus G has three possible structures. First, G has only vertices of degree less than two. Second, G is a complete bipartite graph. Third, G is not a complete bipartite graph, but there are two adjacent vertices x and y in G such that every vertex in G is adjacent to either x or y, and every edge of G meets either x or y. The third graph in Figure 4.1 is an example of this third structure. It is clear that this condition is more restrictive than the condition for the first problem.

Finally, suppose that G is a bipartite graph on n vertices such that no bipartition of G contains a copy of $SI(2,1)$ with any bipartition. Then certainly G does not contain $SI(2,1)$ with the bipartition shown in Figure 4.1, so that G must be one of the three structures mentioned in the previous paragraph. But G also does not contain $SI(2,1)$ with the bipartition having two vertices in each part. If n is at least five, the third structure in the previous paragraph must contain a copy of $SI(2,1)$ with this alternative bipartition, so that (for $n \geq 5$) G is either a complete bipartite graph or contains only vertices of degree less than two.

We recall that a simple consequence of the Szemerédi Regularity Lemma was Theorem 1.8, which states that for any H these three functions are bounded above by $2^{o(n^2)}$. We will be interested in finding lower bounds and better upper bounds; we will be particularly interested in finding bounds of the form $n^{cn+o(n)}$ for constant c.

We will see that the bipartite graphs fall into the following classes: graphs containing cycles or the bipartite complements of cycles, five infinite families of graphs, and six exceptional graphs on six and seven vertices. The graphs in the five infinite families are the following:

- A star with k rays together with l isolated vertices, $SI(k,l)$.
- Two disjoint stars with respectively k and l rays, $DS(k,l)$.

- $DS(k,l)$ together with an isolated vertex, $DS^*(k,l)$.
- Two joined stars, obtained by taking $DS(k,l)$ and inserting a vertex adjacent to the centres of each star, $JS(k,l)$.
- Two joined stars together with an isolated vertex, $JS^*(k,l)$.

The six exceptional graphs are simply those bipartite graphs which are at once forests, complements of forests, and not in the five infinite families.

Given the example of $SI(2,1)$ where the three problems are clearly different, one might expect that each problem will need to be solved individually; that there will be graphs where knowing $Forb_n, m(H[W,Z])$ does not help us find good bounds on $Free_n(H)$. Somewhat surprisingly, this is not true. We are able to obtain good bounds on $Forb_n(H[W,Z])$ and $Free_n(H)$ simply by choosing the right (W,Z) and r and examining $Forb_{n-r,r}(H[W,Z])$.

This chapter is organised as follows.

In Section 2 we show that, since there are many (more than n^{cn} for any c) graphs on n vertices with large girth, the speed of $Forb_{m,n}(H)$ is large for all H which contain either a cycle or the bipartite complement of a cycle. This leaves only the five infinite families of graphs and six exceptional graphs.

It is obvious that any graph G with maximum degree (or co-degree) less than the maximum degree (or co-degree) of H cannot contain a copy of H. There are n^{km} bipartite graphs $G[U,V]$ with parts of sizes m and n respectively in which every vertex in U has degree at most k. One might perhaps guess that, when H does not contain a cycle or the complement of a cycle, the speed of $Forb_{m,n}(H)$ should depend principally upon the maximum degree or co-degree of H; and it is not too hard to show that for each of the infinite families this is true.

This would lead us to expect that the lower bounds on $Forb_{m,n}(H)$ should be given by families of graphs with small maximum degree or co-degree. Interestingly, this is not always the case. We find large families of graphs giving substantially better lower bounds than the obvious ones for four of the five infinite families: $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and $JS^*(k,l)$. We are able to show that these large families of graphs actually give the correct speed for the first three infinite families when $k = l$.

In Section 3 we describe ways to modify the lower bound examples from the previous section to obtain lower bounds for $Forb_n(H)$ and $Free_n(H)$. By making the right choices for (W, Z) and r, and examining $Forb_{n-r,r}(H[W, Z])$, we obtain good bounds for the two quantities.

Finally in Section 4 we use a structural result of Lozin [55] to obtain good upper bounds on $Forb_n(H)$ for all of the exceptional graphs except the path on seven vertices, P_7. We observe that this structural result does not suffice to bound $Forb_{m,n}(H[U, V])$ above for three more of the exceptional graphs (see Table 4.2).

Our results for each of the three problems are summarised in the Tables 4.2 and 4.3. We observe that the results for the second and third problems differ only in that forbidding certain graphs ($SI(0, l)$, $DS(k, 0)$ and $DS^*(k, 0)$) makes sense in the context of the second problem where their bipartition is fixed, but in the context of the third problem they are examples of simpler graphs (the empty graph on $l+1$ vertices, $SI(k, 1)$ and $SI(k, 2)$ respectively). Note that in a few cases we can find better bounds than those given in the tables; in particular we can show that the upper bound is correct for $Forb_n(JS(1, 0))$ and that the lower bounds are correct for $Forb_n(DS(k, 0))$.

A special case that might be of interest is that of the bipartite graphs on n vertices which do not contain the path on k vertices as an induced subgraph. Trivially when $k = 1, 2$ we have respectively zero and one bipartite graphs which are P_k-free. The P_3-free bipartite graphs are the sub-matchings (disjoint unions of copies of K_1 and K_2), of which there are $n^{\frac{n}{2}+o(n)}$. The P_4-free bipartite graphs are easily seen to be disjoint unions of complete bipartite graphs, and there are $n^{n+o(n)}$ such (we note that $P_4 = JS(1, 0)$; in this case the general lower bound in Tables 4.2 and 4.3 can be improved). The P_5-free bipartite graphs are disjoint unions of difference graphs ($2K_2$-free bipartite graphs), and the P_6-free bipartite graphs are a subclass of the bicographs introduced by Giakoumakis and Vanherpe [38]; in both cases there are $n^{n+o(n)}$ such bipartite graphs. The P_7-free bipartite graphs contain the $DS(2, 2)$-free graphs, so that there are at least $n^{\frac{3n}{2}+o(n)}$ P_7-free bipartite graphs on n vertices, but we have no good upper bounds. For $k \geq 8$, P_k contains the bipartite complement of C_4; and there are $2^{\Omega(n^{\frac{6}{5}})}$ graphs whose bipartite complements have girth at least six and so do not contain P_k.

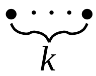
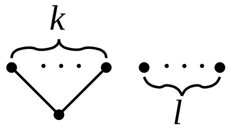
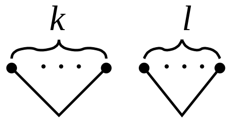
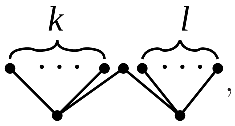
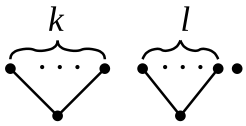
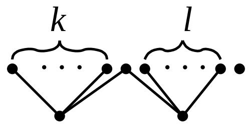
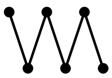
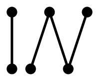
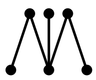
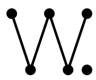
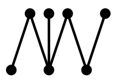
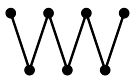
H	$Forb_{m,n}(H)$ Lower	$Forb_{m,n}(H)$ Upper
 $k \geq 1$	0 (for sufficiently large m, n)	
$SI(k,l)$  $k+l \geq 1$	$m^{\max(k-1,l-1)m+o(m)}$	
$DS(k,l)$  $k \geq l \geq 1$ or $k \geq 2$, $l=0$	$m^{\max((k-1)m,lm+n)+o(m)}$	$m^{km+n+o(m)}$
$JS(k,l)$ , $DS^*(k,l)$  $k \geq l \geq 1$ or $k \geq 1$, $l=0$	$m^{\max(km,lm+n)+o(m)}$	$m^{km+n+o(m)}$
$JS^*(k,l)$  $k \geq l \geq 1$ or $k \geq 1$, $l=0$	$m^{\max(km,lm+n)+o(m)}$	$m^{km+2n+o(m)}$
, 	$m^{m+n+o(m)}$	
, , 	$m^{m+n+o(m)}$	$2^{o(m^2)}$
	$m^{2m+n+o(m)}$	$2^{o(m^2)}$
All other bipartite graphs	$2^{\Omega(m^{\frac{6}{5}})}$	$2^{o(m^2)}$

Figure 4.2 Bounds obtained for the first problem

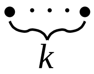
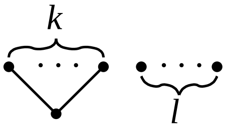
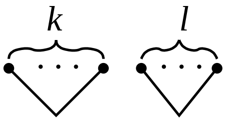
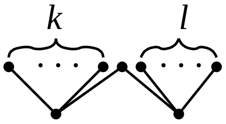
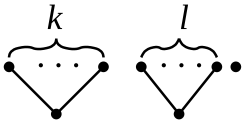
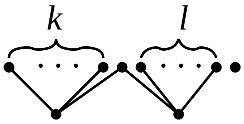
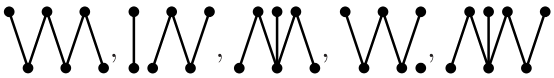
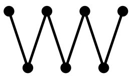
<table>
<tr><th rowspan="2">H</th><th colspan="2">$Forb_n(H)$, $Free_n(H)$</th></tr>
<tr><th>Lower</th><th>Upper</th></tr>
<tr><td>
$k \geq 1$</td><td colspan="2">0
(for sufficiently large n)</td></tr>
<tr><td>$SI(k,l)$

$k+l \geq 1^{(*)}$</td><td colspan="2">$n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)}$</td></tr>
<tr><td>$DS(k,l)$

$k \geq l \geq 1^{(*)}$</td><td>$n^{\max(\frac{(k-1)n}{2},\frac{(l+1)n}{2})+o(n)}$</td><td>$n^{\frac{(k+1)n}{2}+o(n)}$</td></tr>
<tr><td>$JS(k,l)$ , $DS^*(k,l)$ 
$k \geq l \geq 1$ or $k \geq 1$, $l = 0^{(*)}$,</td><td>$n^{\max(\frac{kn}{2},\frac{(l+1)n}{2})+o(n)}$</td><td>$n^{\frac{(k+1)n}{2}+o(n)}$</td></tr>
<tr><td>$JS^*(k,l)$

$k \geq l \geq 1$ or $k \geq 1$, $l = 0$</td><td>$n^{\max(\frac{kn}{2},\frac{(l+1)n}{2})+o(n)}$</td><td>$n^{\frac{(k+2)n}{2}+o(n)}$</td></tr>
<tr><td></td><td colspan="2">$n^{n+o(n)}$</td></tr>
<tr><td></td><td>$n^{\frac{3n}{2}+o(n)}$</td><td>$2^{o(n^2)}$</td></tr>
<tr><td>All other bipartite graphs</td><td>$2^{\Omega(n^{\frac{6}{5}})}$</td><td>$2^{o(n^2)}$</td></tr>
</table>

Figure 4.3 Bounds obtained for the second and third problems

$^{(*)}$ $SI(0,l)$, $DS(k,0)$ and $DS^*(k,0)$ apply only to the second problem.

4.2 Fixed bipartitions

In this section we obtain bounds on $Forb_{m,n}(H[W,Z])$ for all but a few exceptional graphs H. This section contains most of the work: we will deduce results on $Forb_n(H[W,Z])$ and $Free_n(H)$ from these bounds.

4.2.1 Short cycles

Most bipartite graphs have either a short cycle in the graph or a short cycle in the bipartite complement: we deal with these bipartite graphs now. We do not need to work hard for these results: but this is mainly because we accept quite crude bounds.

First we show that there are many graphs which do not contain short cycles. We make use of a geometric construction of Benson [9] showing that there exists a bipartite graph with large girth and many edges.

Theorem 4.2. *For q an odd power of 3, there exists a bipartite graph B_q with $q^5 + q^4 + q^3 + q^2 + q + 1$ vertices in each part, regular of degree $q + 1$, which has girth 12.*

We can now easily deduce the following corollary.

Corollary 4.3. *There are $2^{\Omega(m^{\frac{6}{5}})}$ bipartite graphs with bipartitions whose parts are of sizes m, $n = Theta(m)$, which are connected, whose bipartite complements are connected, and which have girth at least 12.*

Proof. Let q be the greatest odd power of 3 such that $q^5+q^4+q^3+q^2+q+1$ is not larger than either m or n. Then let $G[X,Y]$ be a graph obtained by adding sufficient vertices to the graph B_q given by Theorem 4.2 to ensure that the parts are of sizes m and n respectively, and sufficient edges to ensure that $G[X,Y]$ is connected, while creating no new cycles. This graph has at least $q^6 = \Omega(m^{\frac{6}{5}})$ edges, and girth 12. It is trivial to check that $G[X,Y]$ must have connected bipartite complement. Let T be a spanning tree of $G[X,Y]$. Then every spanning subgraph of $G[X,Y]$ which preserves the edges of T has girth at least 12, is connected, and has connected bipartite complement. There are at least $q^6 - m - n + 1 = \Omega(m^{\frac{6}{5}})$ edges of $G[X,Y]$ which are not edges of T, and hence there are $2^{\Omega(m^{\frac{6}{5}})}$ such graphs. $\square$

Although we do not need the connectedness part of the above corollary at this stage, it will be useful in a later section.

Corollary 4.3 provides a lower bound on $Forb_{m,n}(H)$ for all H which contain a cycle of length less than 12, or whose bipartite complement contains such a cycle. The following corollary allows us to list all the H which do not fall into that category.

Corollary 4.4. *If $H = H[U,V]$ is a bipartite graph on at least eight vertices, both of whose parts contain at least three vertices, and $n = \Theta(m)$, then*

$$Forb_{m,n}(H) = 2^{\Omega(m^{\frac{6}{5}})} .$$

Proof. If H contains a cycle, then either the shortest cycle in H is of length at most 8, or the bipartite complement of H contains a 4-cycle.

But if H is acyclic, then it has at most $|H| - 1$ edges, so its bipartite complement has at least $3(|H| - 3) - |H| + 1 = 2|H| - 8 > |H| - 1$ edges and must have a smallest subgraph which is a cycle; since H is acyclic this cycle is of length at most 8.

Therefore either H or its bipartite complement contains a cycle of length at most 8, and either the graphs given by Theorem 4.3 all do not contain a copy of H, or their bipartite complements all do not contain a copy of H. In either case, we obtain the given bound. $\square$

We now have to deal only with those H whose smaller part has zero, one or two vertices, together with a small number of exceptional cases on six and seven vertices. The various possibilities are set out in Table 4.2.

4.2.2 Acyclic families

Trivially if one part of H is empty, then for sufficiently large m, n, $Forb_{m,n}(H) = 0$.

If one part of H contains exactly one vertex, then $H = SI(k,l)$ for some k,l.

Theorem 4.5. *For $n = \Theta(m)$, $Forb_{m,n}(SI(k,l)) = m^{\max(k-1,l-1)m+o(m)}$.*

Proof. A graph G with bipartition (X,Y) which does not contain a copy of $SI(k,l)$ is precisely one in which every vertex in X is either adjacent to at most $k-1$ vertices in Y, or to all but at most $l-1$ vertices in Y. There are

$$\left(\binom{n}{0} + \ldots + \binom{n}{k-1} + \binom{n}{n-l+1} + \ldots + \binom{n}{n}\right)^m = m^{\max(k-1,l-1)m+o(m)}$$

such graphs (note that $n = \Theta(m)$, so that $n^m = m^{m+o(m)}$). □

We now consider bipartite graphs $H = H[W,Z]$ with two vertices in the lower part W. These graphs require the most work.

Observe that if the two vertices in the lower part have more than one common neighbour, or there are two isolated vertices in the upper part, then either H or its bipartite complement contains a cycle and so Theorem 4.3 gives us a lower bound on $Forb_{m,n}(H)$.

Therefore we need to find bounds for the four infinite families of bipartite graphs $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and $JS^*(k,l)$ (see Table 4.2). Note that the bipartite complement of $JS(k,l)$ is $DS^*(k,l)$, so that the bounds which we find for the former give immediately bounds for the latter.

Observe that if $G[X,Y]$ does not contain a copy of $DS(k,l)$, $l < k$, then it certainly contains no copy of $DS(k,k)$, so that it suffices to bound above $Forb_{m,n}(DS(k,k))$.

Theorem 4.6. *For $n = \Theta(m)$, $Forb_{m,n}(DS(k,k)) \leq m^{km+n+o(m)}$.*

Proof. We describe a process for recording information sufficient to reconstruct a bipartite graph $G[X,Y]$ containing no copy of $DS(k,k)$.

Choose any order $x_1, x_2, \ldots, x_m$ on X such that $d(x_i) \leq d(x_j)$ for every $1 \leq i < j \leq m$.

It is obvious that G contains no copy of $DS(k,k)$ if and only if $|\Gamma(x_i) - \Gamma(x_j)| \leq k-1$ for each $i \leq j$.

For each $2 \leq i \leq m$, let $U_{x_i} = \Gamma(x_{i-1}) - \Gamma(x_i)$, and let $V_{x_i} = \Gamma(x_i) - \Gamma(x_{i-1})$. Let $U_{x_1} = \emptyset$, and $V_{x_1} = \Gamma(x_1)$.

We call the sets U_{x_i} and V_{x_i} the *removed set* and *added set* at x_i.

It is clear that the following information, the *basic recording* of G, is sufficient to reconstruct G:

(X, Y)
$[V_{x_1}, x_1, V_{x_2}, x_2, \ldots, V_{x_m}, x_m]$
$[U_{x_1}, U_{x_2}, \ldots, U_{x_m}]$
where we write out the elements of each of the sets in the standard order. We call the first list $[V_{x_1}, x_1, \ldots]$ the *list of vertices*, and the second list $[U_{x_1}, \ldots]$ the *list of removals*.

Observe that the list of vertices is of length at most $m + n + (k-1)m$, since $n \geq |\Gamma(x_m)| = \sum_i (|V_{x_i}| - |U_{x_i}|) \geq \sum_i |V_{x_i}| - (k-1)m$.

This is already sufficient to give

$$Forb_{m,n}(DS(k,k)) \leq 2^{m+n}(m+n)^{km+n} \binom{n}{k-1}^{m-1} = m^{(2k-1)m+n+o(m)} ,$$

despite only using the fact that consecutive members x_i, x_{i+1} of X may not be the lower part of a copy of $DS(k,l)$.

In fact, no two members of X are the lower part of a copy of $DS(k,k)$. We can use this to show that, given the list of vertices, there are not $m^{(k-1)m+o(m)}$ choices for the list of removals, but only $m^{o(m)}$. Suppose that y appears in a removed set at some vertex between x_{i+1} and x_j, $i < j$, in the degree sequence order, but not in any added set at those vertices. Then y is adjacent to x_i but not to x_j. Since x_i and x_j are not the lower part of a copy of $DS(k,k)$, $|\Gamma(x_i) - \Gamma(x_j)| \leq k-1$. So we expect to find that most members of removed sets must also be members of added sets at nearby vertices in the degree sequence order.

We *compress* the information given in the removed sets U_{x_i}. Suppose that y is the jth member of the removed set at the vertex x_i. We define a *reference tag* $R_{x_i,j}$ as follows.

If there is a p, $-\log m \leq p \leq \log m$, such that the entry p after x_i in the list of vertices is y, then let $R_{x_i,j} = \mathtt{V}:p$. We say that the reference tag is a *good reference tag.*

If there is no such p, then let $R_{x_i,j} = \texttt{P} : y$. We say that this is a *bad reference tag.*

We now write out the *compressed recording* of G:

(X, Y)
$[V_{x_1}, x_1, V_{x_2}, x_2, \ldots, V_{x_m}, x_m]$
$[(R_{x_1,1}, R_{x_1,2}, \ldots), (R_{x_2,1}, R_{x_2,2}, \ldots), \ldots]$

It is clear that this recording gives enough information to reconstruct the basic recording, and hence G.

We will now show that for any $G[X, Y]$ with no copy of $DS(k, k)$, there are few bad reference tags.

We divide X into *blocks* $A_1, \ldots$ as follows. Let $A_1 = \{x_1, x_2, \ldots, x_a\}$ where x_a is within distance $\log m$ of x_1 in the list of vertices, but x_{a+1} is not. Let $A_2 = \{x_{a+1}, \ldots, x_b\}$, where x_b is within distance $\log m$ of x_{a+1} in the list of vertices, but x_{b+1} is not, and so on. Since the list of vertices is of length at most $km + n$, there are at most $\lceil \frac{km+n}{\log m} \rceil$ blocks.

Suppose that $R_{x_i,j} = \texttt{P} : y$ is a bad reference tag: so y is in the removed set at x_i, but it does not appear in the list of vertices within $\log m$ of x_i. If $x_i \in A_r = \{x_c, \ldots, x_d\}$, then y does not appear in an added set at any of $x_{c+1}, \ldots, x_d$. If $x_i \neq x_c$, then y is adjacent to x_c, but not to x_d. If there were k bad reference tags among those at vertices $x_{c+1}, \ldots, x_d$ then there would be k vertices in Y adjacent to x_c and not to x_d. This would mean that $|\Gamma(x_c) - \Gamma(x_d)| \geq k$, so $\{x_c, x_d\}$ would be the lower part of a copy of $DS(k, k)$. Therefore there can be at most $2(k-1)$ bad reference tags per block (at most $k-1$ at the first vertex in the block, and at most $k-1$ among those at the remaining vertices). Therefore there are at most $\frac{2k(km+n)}{\log m}$ bad reference tags.

There are $(1 + 2\log m)$ possible good reference tags, and n possible bad ones. Therefore we can bound above the number of possible compressed recordings by

$$2^{m+n}(m+n)^{km+n}2^{(k-1)m}(1+2\log m)^{(k-1)m}n^{\frac{2k(km+n)}{\log m}}$$

so $Forb_{m,n}(DS(k,k)) \leq m^{km+n+o(m)}$ as required. □

The upper bound in Theorem 4.6 gives the correct speed.

Theorem 4.7. *For $n = \Theta(m)$, $Forb_{m,n}(DS(k,k)) = m^{km+n+o(m)}$.*

Proof. We have the upper bound already; we construct a family of graphs which is of sufficient size.

Let $X = \{1, \ldots, m\}$, $Y = \{m+1, \ldots, m+n\}$. Let $X_0 = \{1, \ldots, \lfloor \frac{n}{\log m} \rfloor\}$. Let $Y_0 = \{m+1, \ldots, m + \lfloor \frac{m}{\log m} \rfloor\}$.

Partition $X - X_0$ into sets $X_1, X_2, \ldots$, each (except possibly the last) of size $\lfloor \log m \rfloor$. We can obtain such a partition by taking any order on $X - X_0$, which has size $m - \lfloor \frac{m}{\log m} \rfloor$, and letting X_1 be the first $\lfloor \log m \rfloor$ vertices in that order, X_2 the next $\lfloor \log m \rfloor$, and so on. There are $\left(m - \lfloor \frac{m}{\log m} \rfloor\right)! = m^{m-o(m)}$ ways to order $X - X_0$. The number of distinct orders which generate each partition is $|X_1|!|X_2|! \ldots \leq \lfloor \log m \rfloor !^{\lfloor \frac{m}{\log m} \rfloor + 1} = m^{o(m)}$. Therefore there are $m^{m-o(m)}$ such distinct partitions.

Partition $Y - Y_0$ into sets $Y_1, Y_2, \ldots$, each (except possibly the last) of size $\lfloor \log m \rfloor$. Similarly, there are $n^{n-o(n)}$ ways to do this.

Choose, for each vertex x_i in $X - X_0$, a set N_i of $k-1$ vertices in $Y - Y_0$. There are $n^{(k-1)\left((m - \lfloor \frac{m}{\log m} \rfloor) - o(m)\right)} = m^{(k-1)m+o(m)}$ ways to do this.

Construct a bipartite graph $G[X, Y]$ as follows. Put an edge from each $i \in X_0$ to each vertex in $Y_0 \cup Y_1 \cup \ldots \cup Y_{i-1}$. Put an edge from each $m + i \in Y_0$ to each vertex in $X_0 \cup X_1 \cup \ldots \cup X_{i-1}$. Put an edge from each $i \in X - X_0$ to each vertex in N_i.

Observe that whatever choices were made, G does not contain a copy of $DS(k,k)$. Furthermore, different choices imply different G. Therefore $Forb_{m,n}(DS(k,k)) = m^{km+n+o(m)}$ as required. □

Observe that if the recording method described in Theorem 4.6 were applied to a typical graph $G[X, Y]$ constructed as in Theorem 4.7, then given any $\epsilon > 0$ we would find the following.

There are no sets V_x of size greater than ϵn.

The list of vertices is of length at least $(k-\epsilon)m+n$.

There are at most ϵm vertices in X with any given degree.

There are at least $m^{1-\epsilon}$ different vertex degrees in X.

It is easy to check, by considering the recording method, that given $\epsilon > 0$, the speed of graphs $G[X,Y]$ which do not contain a copy of $DS(k,k)$ and which fail to satisfy any of the above conditions is at most $m^{km+n-\epsilon' m+o(m)}$, slower than the speed of $Forb_{m,n}(DS(k,k))$. In the first two cases, this is because there are not enough possibilities for the list of vertices, and in the last two, because there are $m^{\epsilon m}$ distinct orderings of X by increasing degree, so that each graph can be recorded in $m^{\epsilon m}$ different ways. So the graphs constructed in Theorem 4.7 are in some sense typical.

Since $K_{1,k} = SI(k,0)$ is an induced subgraph of $DS(k,l)$, any $G[X,Y]$ which does not contain $SI(k,0)$ does not contain $DS(k,l)$, so we have the lower bound $Forb_{m,n}(DS(k,l)) \geq m^{(k-1)m+o(m)}$. It is trivial to check that in the case $k \geq 2$, $l = 0$, this lower bound gives the correct speed.

Note that, if $1 \leq l \leq k-1$, since $DS(l,l)$ is an induced subgraph of $DS(k,l)$, the construction in Theorem 4.7 gives a lower bound

$$Forb_{m,n}(DS(k,l)) \geq m^{\max(lm+n,(k-1)m)+o(m)} \; .$$

When $l = k-1$ this bound is certainly better than the above, and it seems reasonable to conjecture that it is correct.

We now examine $JS(k,l)$. We will obtain an upper bound by modifying the argument used in Theorem 4.6; again we will find an upper bound on $Forb_{m,n}(JS(k,k))$ and observe that as $JS(k,l)$ is an induced subgraph of $JS(k,k)$ when $k \geq l$, this gives an upper bound for $Forb_{m,n}(JS(k,l))$.

Theorem 4.8. *For $n = \Theta(m)$, $Forb_{m,n}(JS(k,k)) \leq m^{km+n+o(m)}$.*

Proof. Again we will describe a process for recording bipartite graphs $G[X,Y]$ which contain no copy of $JS(k,k)$. Observe that if we have some guarantee that some vertices in X share a common neighbour in Y, then we can apply the same recording procedure as in Theorem 4.6 to these vertices.

Observe that $G[X, Y]$ contains no copy of $JS(k, k)$ if and only if whenever a pair of vertices $x, x' \in X$ share a common neighbour, with $d(x) \leq d(x')$, so $|\Gamma(x) - \Gamma(x')| \leq k - 1$.

It is convenient to record the graph G in several steps. First we find a way to record the neighbours of the set Q of vertices in X which have at most $\log \log m$ neighbours.

We do this as follows. First we construct a set $P \subset Q$ by reading through the vertices in Q in order of decreasing degree, and choosing for P every vertex whose neighbourhood is disjoint from all those previously chosen. Now any two vertices in P have disjoint neighbourhoods, and if $q \in Q - P$, then there is a $p \in P$ whose neighbourhood intersects that of q and which has $d(p) \geq d(q)$.

Let $\Gamma(P)$ be the set of vertices in Y which are neighbours of at least one vertex in P. Then we can record the neighbours of each vertex in P by writing down $\Gamma(P)$ and the partition of $\Gamma(P)$ into the sets $\Gamma(p)$ for $p \in P$.

Now let q be in $Q - P$. There is $p \in P$ with $d(p) \geq d(q)$ and such that p and q share at least one neighbour. Then $|\Gamma(q) - \Gamma(p)| \leq k - 1$, since $\{p, q\}$ is not the lower part of a copy of $JS(k, l)$. So we can record the neighbours of q by writing down the vertex p, the neighbours of p which are also neighbours of q, and the at most $k - 1$ vertices in $\Gamma(q) - \Gamma(p)$. This does not require us to have the vertices in $Q - P$ in any particular order, so we can record the set $Q - P$ by simply choosing them from X.

So we can record the neighbours of all the vertices in Q in at most

$$2^n m^{|\Gamma(P)|} 2^m \left(m 2^{\log \log m} n^{k-1}\right)^{|Q-P|} = m^{k|Q| + |\Gamma(P)| - k|P| + o(m)}$$

ways.

Now we record the neighbours of the remaining vertices $X' = X - Q$, each of which has degree at least $\log \log m > 2k - 1$.

We choose a set of vertices $S_1 \subset X'$ by reading through X' in order of increasing degree, and choosing for S_1 every vertex whose neighbours are disjoint from all those previously chosen. Let $X_1 = X' - S_1$. Now S_1 satisfies three properties. First, no two vertices in S_1 share a common neighbour.

Second, every vertex in X_1 shares at least one common neighbour with some vertex in S_1. Third, for every $x \in X_1$, there is an $s \in S_1$ which shares a common neighbour with x and satisfies $d(s) \leq d(x)$.

Observe that since $G[X, Y]$ contains no copy of $JS(k, k)$ and all vertices in X' have degree at least $\log \log m > 2k - 1$, these three properties imply that for every $x \in X_1$, every $s \in S_1$ which shares a common neighbour with x satisfies $d(s) \leq d(x)$. For if not, then let $s \in S_1$ be a vertex sharing a common neighbour with x and with $d(x) < d(s)$. Since x shares a neighbour with, and has degree not smaller than, some $s' \in S_1$, we must have $|\Gamma(s') - \Gamma(x)| \leq k - 1$ or $\{x, s'\}$ would be the bottom part of a copy of $JS(k, k)$. Then x has at least k neighbours in common with s', none of which are neighbours of s. So $|\Gamma(x) - \Gamma(s)| \geq k$, but then $\{x, s\}$ are the bottom part of a copy of $JS(k, k)$.

We assign to the vertices in X' *removed sets* and *added sets* U_x, and V_x by following the process below.

For each $s \in S_1$, let $U_s = \emptyset$ and let $V_s = \Gamma(s)$.

Let x_1 be a vertex in X_1 with minimum degree. We distinguish two possibilities.

If x_1 shares a common neighbour with only one $s_1 \in S_1$, then $d(x_1) \geq d(s_1)$ and we can write $\Gamma(x_1) = (\Gamma(s_1) - U_{x_1}) \cup V_{x_1}$, where as before we have $|V_{x_1}| \geq |U_{x_1}| \leq k - 1$. We let $S_2 = (S_1 - \{s_1\}) \cup \{x_1\}$, and $X_2 = X_1 - \{x_1\}$. We say that x_1 is part of the *degree sequence process* starting at s_1.

If x_1 shares a common neighbour with more than one member of S, then let these members be $s_1, \ldots, s_a$. Let $U_{x_1} = (\Gamma(s_1) \cup \Gamma(s_2) \cup \ldots \cup \Gamma(s_a)) - \Gamma(x_1)$, and let $V_{x_1} = \Gamma(x_1) - (\Gamma(s_1) \cup \ldots \cup \Gamma(s_a))$. Observe that $|U_{x_1}| \leq a(k - 1)$, since none of the sets $\Gamma(s_i) - \Gamma(x)$ have more than $k - 1$ members. We let $S_2 = (S_1 - \{s_1, \ldots, s_a\}) \cup \{x_1\}$, and $X_2 = X_1 - \{x_1\}$. We say that the vertex x_1 *joins the neighbourhoods* of the vertices $s_1, \ldots, s_a$.

By construction, no two vertices in S_2 share a common neighbour. If $x \in X_2$ shares a common neighbour with $s \in S_2$, then either $s \in S_1$, in which case $d(x) \geq d(s)$, or $s = x_1$, in which case $d(x) \geq d(s)$ by choice of x_1. If $x \in X_2$, then x shares a common neighbour with $s \in S_1$. Either $s \in S_2$, or s shares a common neighbour with x_1. In the latter case, both x and x_1 have degree

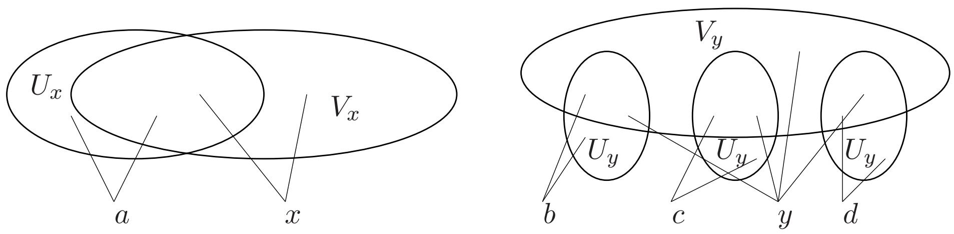


Figure 4.4 x follows a in a degree sequence process; y joins the neighbourhoods of b, c and d.

at least $d(s) > \log\log m > 2k-1$, so that x and x_1 are each adjacent to all but at most $k-1$ neighbours of s, and so must share a common neighbour. Therefore S_2 and X_2 satisfy the same conditions as S_1 and X_1, so we can continue this process with x_2, a vertex in X_2 with minimum degree, and the set S_2, and so on.

If we know that x follows a in a degree sequence process, then we can recover the neighbours of x given $\Gamma(a)$, U_x and V_x.

If we know that y joins the neighbourhoods of $b, \ldots, d$, then we can recover the neighbours of y given $\Gamma(b), \ldots, \Gamma(d)$, U_y and V_y.

Then we can write down a recording of $G[X, Y]$ as in the following example.

(X, Y)
Recording of the low degree vertices and their neighbours
$[V_{s_1}, s_1, V_{x_1}, x_1, \ldots]$
$[U_{s_1}, U_{x_1}, \ldots]$
$\ldots$
$[V_{s_{|S_1|}}, s_{|S_1|}, \ldots]$
$[U_{s_{|S_1|}}, \ldots]$
`JOIN` $:b, \ldots, d$
$[V_y, y, \ldots]$
$[U_y, \ldots]$
`JOIN` $:\ldots$
$\ldots$

Each of the pairs of lines $[V_{s_1}, s_1, V_{x_1}, x_1, \ldots]$, $[U_{s_1}, U_{x_1}, \ldots]$ et cetera represents a degree sequence process as in Theorem 4.6; so the neighbourhood of s_1 is V_{s_1}, the neighbourhood of x_1 is $\Gamma(s_1) \cup V_{x_1} - U_{x_1}$, and so on. The ordering of the degree sequence processes is immaterial.

Each triple of lines `JOIN` :$b, \ldots, d$, $[V_y, y, \ldots]$, $[U_y, \ldots]$ et cetera represents a new degree sequence process; in the example, the first vertex in this degree sequence process is y, whose neighbourhood is $(\Gamma(b) \cup \ldots \cup \Gamma(d)) - U_y) \cup V_y$. Again the ordering of these triples is immaterial.

As in Theorem 4.6 we call the lists $[V_{s_1}, s_1, V_{x_1}, x_1, \ldots]$ et cetera the *lists of vertices* and the lists $[U_{s_1}, U_{x_1}, \ldots]$ et cetera the *lists of removals*.

It is clear that we can reconstruct G from such a recording; we call this the *basic recording* of G.

Observe that $|S_1| \leq \frac{n}{\log\log m}$, since every member of S_1 has at least $\log\log m$ neighbours. If S_{i+1} is obtained from S_i by joining the neighbourhoods of j vertices, then $|S_{i+1}| = |S_i| + 1 - j$. Since $|S_1| \leq \frac{n}{\log\log m}$, the total number of neighbourhoods joined is at most $\frac{2n}{\log\log m}$.

Let $\Gamma(X')$ be the set of vertices in Y which are adjacent to at least one vertex in X'. The neighbourhoods of the vertices in S_i are disjoint for each i; so the sum of their sizes is at most $|\Gamma(X')| \leq n$. Observe that whether S_{i+1} is obtained from S_i by letting x_i continue a degree sequence process or by letting it join some neighbourhoods,

$$\sum_{s \in S_{i+1}} |\Gamma(s)| = |V_{x_i}| - |U_{x_i}| + \sum_{s \in S_i} |\Gamma(s)| \ .$$

Now $|U_{x_i}| \leq k-1$ if x_i continues a degree sequence process; if x_i joins some r neighbourhoods then $|U_{x_i}| \leq r(k-1)$. Since at most $\frac{2n}{\log\log m}$ neighbourhoods are joined in total,

$$\sum_{x \in X'} |V_x| \leq |\Gamma(X')| + (k-1)|X'| + \frac{2kn}{\log\log m} \ .$$

It follows that the total length of the lists of vertices is at most $|\Gamma(X')| + k|X'| + \frac{2kn}{\log\log m}$, and the total length of the lists of removals is at most $(k-1)|X'| + \frac{2kn}{\log\log n}$. The total number of vertices whose neighbourhoods are joined (and which are therefore listed on some `JOIN` : line in the recording) is at most $\frac{2n}{\log\log m}$.

This is already sufficient to give

$$\begin{aligned} Forb_{m,n}(JS(k,l)) &\leq 2^{m+n} m^{k|Q|+|\Gamma(P)|-k|P|+o(m)} (m+n)^{|\Gamma(X')|+k|X'|+\frac{2kn}{\log\log m}} \\ &\qquad \ldots n^{(k-1)|X'|+\frac{2kn}{\log\log n}} m^{\frac{2n}{\log\log m}+o(m)} \\ &= m^{k|Q|+|\Gamma(P)|-k|P|+|\Gamma(X')|+(2k-1)|X'|+o(m)} \leq m^{(2k-1)m+2n+o(m)} \ . \end{aligned}$$

As in Theorem 4.6, we expect to find that vertices appearing in U_{x_i} are likely to appear in V_{x_j} for some x_j close to x_i in the same degree sequence process. We can make this precise by applying a virtually identical compression argument. We define the reference tag $R_{x_i,j}$ in the same way as in that theorem, with reference to the list of vertices which contains x_i.

We can again divide X' into blocks, with each block containing vertices in just one degree sequence process. If a block starts at a vertex x which joins the neighbourhoods of r vertices, then it may contain at most $k-1+r(k-1)$ bad reference tags; otherwise a block may contain at most $2(k-1)$ bad reference tags.

The total length of the lists of vertices is less than $2(km+n)$, so that there are at most $\frac{2(km+n)}{\log m} + \frac{2n}{\log\log m}$ blocks, the extra $\frac{2n}{\log\log m}$ coming from possible 'short' blocks at the ends of degree sequence processes. Therefore there are at most $\frac{3n}{\log\log m}$ bad reference tags in total.

As in Theorem 4.6, we can now write the *compressed recording* of G, where instead of writing the lists of removals $[U_x, \ldots]$ et cetera, we write lists of reference tags $[(R_{x,1}, \ldots), \ldots]$ et cetera.

This allows us to improve our bound for $Forb_{m,n}(JS(k,l))$; instead of bounding above the choices for the lists of removals by $m^{(k-1)|X'|+o(m)}$, we can now bound above the choices for the lists of removals by $m^{o(m)}$. We find that

$$Forb_{m,n}(JS(k,l)) \leq m^{k|Q|+|\Gamma(P)|-k|P|+|\Gamma(X')|+k|X'|+o(m)} \leq m^{km+2n+o(m)} \ .$$

Finally, we wish to obtain the claimed bound. We use our knowledge of the neighbours of vertices in P to produce an *extra-compression* of the lists of vertices.

For each $p \in P$, either we can find an $x_p \in X'$ which is the first vertex in the lists of vertices to share a common neighbour with p, or $\Gamma(p) \cap \Gamma(X') = \emptyset$.

Let P_1 be the set of vertices $p \in P$ for which x_p exists, and $P_2 = P - P_1$ be the vertices whose neighbourhoods are disjoint from $\Gamma(X')$.

For each $p \in P_1$, let $I_{p,x_p} = \Gamma(p) \cap \Gamma(x_p)$. Since $d(x_p) > \log\log m \geq d(p)$, $|I_{p,x_p}| \geq |\Gamma(p)| - (k-1)$.

For each $x \in X'$, if $x \neq x_p$ for every $p \in P_1$, let $V'_x = V_x$. If $x = x_p$ for at least one p, let

$$V'_x = V_x - \bigcup_{p:x=x_p} I_{p,x_p} \ .$$

We write down the *extra-compressed recording* of G as in the following example.

$\{X, Y\}$
Recording of the low degree vertices and their neighbours
$[I_{p_1,x_{p_1}}, x_{p_1}, I_{p_2,x_{p_2}}, x_{p_2}, \ldots]$
$[V'_{s_1}, s_1, V'_{x_1}, x_1, \ldots]$
$[(R_{s_1,1}, \ldots), \ldots]$
$\ldots$
$[V'_{s_{|S_1|}}, s_{|S_1|}, \ldots]$
$[(R_{s_{|S_1|},1}, \ldots), \ldots]$
`JOIN` $:b, \ldots, d$
$[V'_y, y, \ldots]$
$[(R_{y,1}, \ldots), \ldots]$
`JOIN` $: \ldots$
$\ldots$

where $P_1 = \{p_1, p_2, \ldots\}$ with $p_1 < p_2 < \ldots$ in the standard order. We can clearly recover the compressed recording of G from this; we have only to insert each of the sets $I_{p_i,x_{p_i}}$ into the identified $V'_{x_{p_i}}$. Therefore $Forb_{m,n}(JS(k,l))$ is bounded above by the number of possible extra-compressed recordings.

We now wish to find the total length of the lists of vertices in the extra-compressed recording of $G[X, Y]$. Recall that

$$\sum_{x \in X'} |V_x| \leq |\Gamma(X')| + (k-1)|X'| + \frac{2kn}{\log\log m} \ .$$

Observe that

$$\sum_{x\in X'} |V'_x| = \sum_{x\in X'} |V_x| - \sum_{p\in P_1} |I_p| \le \sum_{x\in X'} |V_x| + (k-1)|P_1| - \sum_{p\in P_1} |\Gamma(p)| \ ,$$

and

$$|\Gamma(X')| \le n - \sum_{p\in P_2} |\Gamma(p)| \ .$$

Then the total length of the lists of vertices in the extra-compressed recording is at most $|X'| + n + (k-1)|P| - \sum_{p\in P} |\Gamma(p)| + (k-1)|X'| + \frac{2kn}{\log\log m}$.

The *list of insertions* $[I_{p_1,x_{p_1}}, x_{p_1}, \ldots]$ can be chosen in at most $\left(2\binom{\log\log n}{k-1}m\right)^{|P|} = m^{|P|+o(m)}$ ways.

Finally, we can obtain the claimed bound:

$$Forb_{m,n}(JS(k,l)) \le$$

$$2^{m+n} m^{k|Q|+|\Gamma(P)|-k|P|+o(m)} m^{|P|+o(m)} m^{k|X'|+n+(k-1)|P|-|\Gamma(P)|+o(m)}$$

$$\le m^{km+n+o(m)} \ .$$

□

As $DS(k,k)$ is an induced subgraph of $JS(k,k)$, the family of graphs given in Theorem 4.7 provides a lower bound for $JS(k,k)$ which matches the upper bound, so $Forb_{m,n}(JS(k,k)) = m^{km+n+o(m)}$.

Corollary 4.9. *For $n = \Theta(m)$, $Forb_{m,n}(DS^*(k,l)) \le m^{km+n+o(m)}$.*

Proof. The bipartite complement of $DS^*(k,l)$ is $JS(k,l)$. It follows that we have $Forb_{n,m}(DS^*(k,l)) = Forb_{n,m}(JS(k,l)) \le m^{km+n+o(m)}$. □

Again we observe that $Forb_{m,n}(DS^*(k,k)) = m^{km+n+o(m)}$.

Corollary 4.10. *For $n = \Theta(m)$, $Forb_{m,n}(JS^*(k,l)) \le m^{km+2n+o(m)}$.*

Proof. Let $G = G[X,Y]$ be a bipartite graph not containing $JS^*(k,l)$.

Let Y' be Y if $|Y|$ is odd, and $Y - \{y\}$, some $y \in Y$, if $|Y|$ is even.

Let X' be the vertices in X with less than $\frac{|Y'|}{2}$ neighbours in Y', and X'' those with more than $\frac{|Y'|}{2}$ neighbours in Y'. Let $m' = |X'|$, and $m'' = |X''|$.

Observe that the neighbourhoods of any two vertices in X' cover at most $|Y'|-1$ vertices. Therefore the subgraph of $G[X,Y]$ induced by $X' \cup Y'$ contains no copy of $JS(k,l)$. Similarly, the subgraph of $G[X,Y]$ induced by $X'' \cup Y'$ contains no copy of $DS^*(k,l)$. We obtain

$$Forb_{m,n}(H) \leq m2^n 2^m (m')^{km'+n+o(m')}(m'')^{km''+n+o(m'')} \leq m^{km+2n+o(m)} .$$

□

4.3 Unfixed bipartitions

We have now established good bounds on $Forb_{m,n}(H[U,V])$ for every bipartite graph $H[U,V]$ except for the six exceptional graphs. It is convenient to use these bounds to find good bounds on $Forb_n(H[U,V])$ and $Free_n(H)$ at this point.

Note that if G is a bipartite graph which has a bipartition (X,Y), then the statement that no bipartition of G contains a copy of $H[U,V]$ is certainly at least as strong as the statement that both $G[X,Y]$ contains no copy of $H[U,V]$ and $G[Y,X]$ contains no copy of $H[U,V]$. Then it is trivial that

$$\begin{aligned} Free_n(H) &\leq Forb_n(H[U,V]) \\ &\leq 2^n \max_{r<n} \min(Forb_{r,n-r}(H[U,V]), Forb_{n-r,r}(H[U,V])) , \end{aligned} \tag{4.1}$$

so that we can obtain an upper bound for $Forb_n(H[U,V])$ by finding the worst case of $\min(Forb_{r,n-r}(H[U,V]), Forb_{n-r,r}(H[U,V]))$, and an upper bound for $Free_n(H)$ by finding the best case of $Forb_n(H[U,V])$. We will see that these cases are, respectively, the case $r = \frac{n}{2}$ and H as drawn in Table 4.2.

Observe that the condition 'G with any bipartition does not contain a copy of H with any bipartition' is in general significantly stronger than '$G[X,Y]$ does not contain a copy of $H[U,V]$', so we might expect the upper bounds obtained from the above inequality to be poor. This is not the case.

Theorem 4.11. *If H contains a cycle, or all of its bipartite complements contain a cycle, then $Free_n(H) = 2^{\Omega(n^{\frac{6}{5}})}$. If $H[U,V]$ or its bipartite complement contains a cycle then $Forb_n(H[U,V]) = 2^{\Omega(n^{\frac{6}{5}})}$.*

Proof. If H contains a cycle or all of its bipartite complements contain a cycle, then either it has girth at most eight, or all of its bipartite complements have girth at most eight. By Corollary 4.3 there are $2^{\Omega(n^{\frac{6}{5}})}$ bipartite graphs on n vertices which are connected, have connected bipartite complement and girth at least 12. In the first case, all of these graphs contain no copy of H; in the second case, the unique connected bipartite complement of each of these graphs contains no copy of H. $\square$

We now have only to establish appropriate lower bounds on $\mathit{Free}_n(H)$ for the five infinite families $SI(k,l)$, $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and $JS^*(k,l)$ to match those we have for $\mathit{Forb}_{m,n}(H[U,V])$. Again, we observe that both $SI(k,0)$ and $DS(l,l)$ are induced subgraphs of each of $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and $JS^*(k,l)$ for $l \leq k$; so it suffices to find lower bounds on $\mathit{Free}_n(H)$ for $SI(k,l)$ and $DS(k,k)$.

Theorem 4.12. *For any fixed r, there are at least $n^{\frac{rn}{2}+o(n)}$ bipartite graphs whose maximum degree is r and in which no two vertices have three or more common neighbours.*

Proof. Let G be a bipartite graph on n vertices obtained by choosing uniformly at random r matchings from $\{1,\ldots,\lfloor\frac{n}{2}\rfloor\}$ to $\{\lfloor\frac{n}{2}\rfloor+1, 2\lfloor\frac{n}{2}\rfloor\}$ and putting an edge in G whenever that edge is present in any of the r matchings.

We call x, y a *problem pair* if x and y have at least three common neighbours.

The probability that a given pair of vertices $x, y \leq \frac{n}{2}$ is a problem pair is at most

$$\sum_{i=3}^{r} \binom{r}{i} \left(\frac{2r}{n}\right)^{i},$$

so the expected number of problem pairs in G is at most $2\binom{\lfloor\frac{n}{2}\rfloor}{2}$ times that. For sufficiently large n, this quantity is smaller than $\frac{1}{2}$, so there are at least

$$\frac{\lfloor\frac{n}{2}\rfloor!^{r}}{2} = n^{\frac{rn}{2}+o(n)}$$

choices of r matchings which give rise to graphs G with no problem pairs. Now given such a graph G, each of the at most $\frac{rn}{2}$ edges is present in some subset of the r matchings, so there are at most $(2^r)^{rn/2} = n^{o(n)}$ distinct sets of r matchings giving rise to G. Thus we have at least $n^{\frac{rn}{2}+o(n)}$ distinct graphs G with maximum degree r and no two vertices having three or more common neighbours, as required. $\square$

Corollary 4.13.

$$\mathit{Free}_n(SI(k,l)) = n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)} ,$$

and

$$\mathit{Forb}_n(SI(k,l)) = n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)} .$$

Proof. The upper bound follows from Theorem 4.5 and the inequality (4.1), since

$$\mathit{Forb}_{\frac{n}{2},\frac{n}{2}}(SI(k,l)) = n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)}$$

is the worst case.

For the lower bound, if $k \geq l$, by Theorem 4.12 we can find $n^{\frac{(k-1)n}{2}+o(n)}$ bipartite graphs which have maximum degree $k-1$ and which therefore do not contain a copy of $SI(k,l)$ with any bipartition. If on the other hand $k < l$, then again by Theorem 4.12 we can find $n^{\frac{(l-1)n}{2}+o(n)}$ bipartite graphs which have maximum degree $l-1$ and in which no two vertices have three or more common neighbours. Now observe that although $SI(k,l)$ has several bipartitions, and hence several bipartite complements, all of the bipartite complements of $SI(k,l)$ have either a vertex of degree l or two vertices sharing three common neighbours. So there are $n^{\frac{(l-1)n}{2}+o(n)}$ bipartite graphs which do not contain a copy of any of the bipartite complements of $SI(k,l)$, and there must be $n^{\frac{(l-1)n}{2}+o(n)}$ bipartite graphs which do not contain a copy of $SI(k,l)$. This gives us the required inequality

$$n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)} \leq \mathit{Free}_n(SI(k,l))$$
$$\leq \mathit{Forb}_n(SI(k,l)) \leq n^{\max(\frac{(k-1)n}{2},\frac{(l-1)n}{2})+o(n)} .$$

$\square$

We now have only to bound $\mathit{Free}_n(DS(k,k))$. We use a similar construction to that in Theorem 4.7.

Theorem 4.14. $Free_n(DS(k,k)) = n^{\frac{(k+1)n}{2}+o(n)}$.

Proof. The upper bound follows from the inequality (4.1) and Theorem 4.6.

For the lower bound, let $X = \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $Y = \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. Let $X_0 = \{1, \ldots, \lfloor \frac{n}{2\log n} \rfloor\}$. Let $Y_0 = \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2\log n} \rfloor\}$.

Partition $X - X_0$ into sets $X_1, X_2, \ldots$, each (except possibly the last) of size $\lfloor \log n \rfloor$. As in Theorem 4.7, there are $n^{\frac{n}{2}-o(n)}$ such distinct partitions.

Partition $Y - Y_0$ into sets $Y_1, Y_2, \ldots$, each (except possibly the last) of size $\lfloor \log n \rfloor$. Similarly, there are $n^{\frac{n}{2}-o(n)}$ ways to do this.

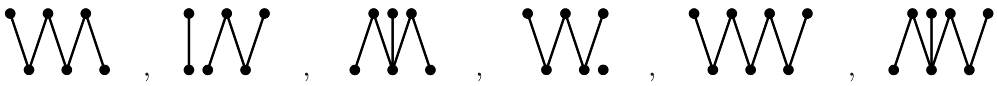
By Theorem 4.12 there are $n^{\frac{(k-1)n}{2}+o(n)}$ bipartite graphs with bipartition $\{X - X_0, Y - Y_0\}$ whose maximum degree is $k-1$.

Construct a bipartite graph G as follows. Put an edge from each $i \in X_0$ to each vertex in $Y_0 \cup Y_1 \cup \ldots \cup Y_{i-1}$. Put an edge from each $m + i \in Y_0$ to each vertex in $X_0 \cup X_1 \cup \ldots \cup X_{i-1}$. Put edges between $X - X_0$ and $Y - Y_0$ in any way such that the maximum degree of the subgraph induced by $(X - X_0) \cup (Y - Y_0)$ is at most $k-1$.

Observe that, whatever choices were made, G does not contain a copy of $DS(k,k)$ with any bipartition. Furthermore, different choices imply different G. Therefore $Free_n(DS(k,k)) = n^{\frac{(k+1)n}{2}+o(n)}$ as required. $\square$

4.4 Exceptional graphs

The only bipartite graphs which we have not yet covered are those with three vertices in the smaller part which are acyclic and whose bipartite complements are acyclic. These are graphs on either six or seven vertices, shown in Table 4.5.



P_6 $\overline{P_6}$ $Star_{1,2,2}$ $\overline{Star_{1,2,2}}$ P_7 $Star_{1,2,3}$

Figure 4.5 Exceptional bipartite graphs

Note that the first pair of these are bipartite complements of each other, as are the second pair; the last two are both self-complementary.

Each of these exceptional bipartite graphs contains the graph P_4, so that we have trivial lower bounds for each exceptional H. The graph P_7 contains $DS(2,2)$, so that for this graph we have slightly better lower bounds.

For the first problem, if H is any exceptional graph except P_7 we have $Forb_{m,n}(H) \geq Forb_{m,n}(P_4) = m^{m+n+o(m)}$. We will show that these lower bounds are correct for the graphs P_6 and $\overline{P_6}$. We have

$$Forb_{m,n}(P_7) \geq m^{2m+n+o(m)} \ .$$

For the second and third problems, if H is any exceptional graph except P_7 we have $Forb_n(H) \geq Forb_n(P_4) = n^{n+o(n)}$ and $Free_n(H) \geq n^{n+o(n)}$; and we will show these lower bounds are correct. For P_7 we have the better lower bound $Forb_n(P_7) = Free_n(P_7) \geq n^{\frac{3n}{2}+o(n)}$, but we are unable to find any non-trivial upper bounds.

Thus far, we have examined ways to record a bipartite graph one vertex at a time. An alternative method is to consider breaking a graph down into smaller pieces by specified operations. Results along these lines are called decomposition results, and there exist several relating to bipartite graphs.

Giakoumakis and Vanherpe [38] considered the two operations of bipartite complement and disjoint union. They defined the class of *bi-cographs* to be the class of bipartite graphs which can be fully decomposed using only these two operations: a single vertex in either part is a bi-cograph, the bipartite complement of a bi-cograph is a bi-cograph, and the disjoint union of two bi-cographs is a bi-cograph. They were able to prove that the class of bi-cographs is exactly the class of bipartite graphs which contain no induced P_7, $Star_{1,2,3}$ or Sun_4, where the graph Sun_4 is the bipartite graph on eight vertices given by taking a copy of C_4 and adding a matching from the vertices of the C_4 to the other four vertices.

Fouquet, Giakoumakis and Vanherpe [34] then introduced a further decomposition operation. Suppose that the bipartite graph $G[X,Y]$ is such that there exist non-trivial induced subgraphs $G_1[X_1,Y_1]$ and $G_2[X_2,Y_2]$, with $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$, and the subgraph of $G[X,Y]$ induced by (X_1, Y_2) is a complete bipartite graph, while that induced by (X_2, Y_1) is an empty graph. Then we say that $G_1[X_1, Y_1]$, $G_2[X_2, Y_2]$ is a *$K+S$-decomposition* of $G[X, Y]$. They called decomposition by using the three operations of taking bipartite complement, disjoint union and $K+S$-decomposition the *canonical decomposition* of a bipartite graph, and defined the class of *weak-bisplit graphs* to be those graphs which can be fully decomposed using the canonical decomposition. They proved that the weak-bisplit graphs are exactly those bipartite graphs containing no induced P_7 or $Star_{1,2,3}$.

A *prime* bipartite graph is one in which $\Gamma(x) = \Gamma(y)$ if and only if $x = y$.

Lozin [55] was able to prove that the class of prime bipartite graphs which can be decomposed to $K_{1,3}$-free graphs using the canonical decomposition is exactly the class of $Star_{1,2,3}$-free prime bipartite graphs. We will use this result to obtain our remaining upper bounds, so we state it explicitly and give a short proof, based on that of Lozin.

Theorem 4.15. *Any prime bipartite graph $G[X, Y]$ that does not contain an induced copy of $Star_{1,2,3}$ can be decomposed, using the three canonical decomposition operations, to the class of $K_{1,3}$-free graphs (equivalently, to the class of paths and cycles).*

Proof. We observe that if $G[X, Y]$ is a bipartite graph, then $G[X, Y]$ has a $K+S$-decomposition if and only if the bipartite complement of $G[X, Y]$ has one.

Suppose the theorem is false. Then there is a prime bipartite graph $G[X, Y]$ of minimal order which does not contain an induced $Star_{1,2,3}$ and which does not satisfy the conditions of the theorem.

Since, by the result of Fouquet, Giakoumakis and Vanherpe, every bipartite graph that is both $Star_{1,2,3}$-free and P_7-free can be fully decomposed by the canonical decomposition, $G[X, Y]$ cannot be P_7-free. In particular, we may assume without loss of generality that the vertices $U = (1, 2, 3, 4, 5, 6, 7)$ induce a P_7, in that order, with $1, 3, 5, 7 \in Y$ and $2, 4, 6 \in X$. We show, following the method of Lozin, that since $G[X, Y]$ is prime and $Star_{1,2,3}$-free, the possibilities for edges from a vertex z to any induced P_7 are very limited.

Let V be any ordered set of seven vertices of $G[X, Y]$ inducing a P_7 in that order. Let T be a subset of $1, 2, 3, 4, 5, 6, 7$. Then let $S_V(T)$ be the set of vertices $z \in V(G[X, Y]) - V$ which are adjacent to the vertices in V at positions T, and not adjacent to those in V at any other position. We will write $S(T)$ for $S_U(T)$.

Now we can easily check that $S(3) = S(4) = S(5) = S(1, 5) = S(3, 7) = S(2, 6) = S(1, 3, 5) = S(3, 5, 7) = \emptyset$, for if there were a vertex z in any of these sets we would find an induced $Star_{1,2,3}$ within $\{1, 2, 3, 4, 5, 6, 7, z\}$.

Suppose $x \in S(3, 5)$. Since $G[X, Y]$ is prime, x and 4 may not have the same neighbourhoods, and we can assume without loss of generality that there is y, not in U, which is adjacent to 4 but not to x. Since $S(4)$ is empty y is adjacent also to at least one of $\{2, 6\}$, and we can quickly check that in any case $\{1, 2, 3, 4, 5, 6, 7, x, y\}$ contains an induced $Star_{1,2,3}$. Then $S(3, 5) = \emptyset$, and by similar methods we can check that $S(1, 3) = S(5, 7) = \emptyset$.

Now if x is adjacent to 3 or 5 or both, it must be adjacent also to both 1 and 7.

Suppose $y \in S(2)$. Since $G[X, Y]$ is prime, without loss of generality we can find x adjacent to 1 but not to y. Observe that $(y, 2, 3, 4, 5, 6, 7)$ also induces a P_7, and so x may not be adjacent to 3 or 5 by the previous results applied to this new P_7. But then $\{1, 2, 3, 4, 5, x, y\}$ induces a copy of $Star_{1,2,3}$. So $S(2) = \emptyset$, and similarly $S(4) = S(2, 4) = S(4, 6) = \emptyset$.

This leaves only $S(\emptyset)$, $S(2, 4, 6)$, $S(1)$, $S(1, 7)$, $S(7)$, $S(1, 3, 7)$, $S(1, 5, 7)$ and $S(1, 3, 5, 7)$ as possible non-empty sets. These adjacency results hold for every induced P_7 in $G[X, Y]$.

Observe that if $x \in S(1) \cup S(1, 7)$ then $V = (x, 1, 2, 3, 4, 5, 6)$ induces a P_7, so $S(1, 3, 7) = S_V(2, 4) = \emptyset = S_V(2, 6) = S(1, 5, 7)$; similarly if $x \in S(7)$ then $S(1, 3, 7) = S(1, 5, 7) = \emptyset$.

We consider two cases.

Case 1: $S(1, 3, 7) = S(1, 5, 7) = \emptyset$.

We define a set $A \subset V(G[X, Y])$ recursively as follows. First, $U \subset A$. Second, if $V \subset A$ induces a P_7, and $z \in S_V(1) \cup S_V(1, 7) \cup S_V(7)$, then

$z \in A$. We can think of the set A as being the vertices of $G[X, Y]$ which are covered by starting with the P_7 induced by U and moving it along one vertex at a time.

If $z \in A$ has two or more neighbours in A, by construction there is an induced P_7 with vertices in A which includes z and two of its neighbours. Then z has exactly two neighbours in A, or the adjacency results would be violated with respect to this P_7. Thus A induces a subgraph of $G[X, Y]$ which is either a path or a cycle on at least seven vertices.

Now if $x \in X - A$ is adjacent to some $a \in A$, then there is a set $V \subset A$ which contains a and induces a P_7. Since $x \notin A$, $x \notin S_V(1) \cup S_V(1,7) \cup S_V(7)$. Therefore $x \in S_V(1,3,5,7) \cup S_V(2,4,6)$. By applying the adjacency results to each P_7 contained in A, we see that x is adjacent to every vertex in $A \cap Y$.

Similarly, if $y \in Y - A$, then y is adjacent either to every vertex, or to no vertex, in $A \cap X$. We let the set of vertices in $V(G[X, Y]) - A$ which are not adjacent to any vertices in A be E, and let $K = S(1,3,5,7) \cup S(2,4,6)$. Then $V(G[X, Y]) = A \cup E \cup K$.

Since $G[X, Y]$ is not $K_{1,3}$-free, there must be a vertex of $G[X, Y]$ not in A.

Since $G[X, Y]$ is not disconnected, if there is a vertex in E there must also be a vertex in K.

Since $G[X, Y]$ does not have disconnected bipartite complement, if there is a vertex in K there must also be a vertex in E. Thus there is at least one vertex in both E and K.

Recall that $G[X, Y]$ is a counterexample of minimal order. Then the subgraph G' of $G[X, Y]$ induced by removing the vertex 1 is not a counterexample. Since vertices of $G[X, Y]$ not in A which are adjacent to 1 are also adjacent to 3, G' must be prime, and G' is certainly $Star_{1,2,3}$-free.

Observe that G' is not $K_{1,3}$-free, for there is a vertex of degree at least three in K. Its bipartite complement is not $K_{1,3}$-free, for there is a vertex in E of co-degree at least three. It is connected, since there is a vertex in K, and its bipartite complement is connected, since there is a vertex in E. Thus G' must have a $K{+}S$-decomposition G'_1, G'_2. But it is not hard to check that the bipartite graph P_6 does not have a $K{+}S$-decomposition, and the vertices $2, 3, 4, 5, 6, 7$ in G' induce a P_6. So either G'_1 contains all of these vertices, or G'_2 does; in either case, we can find G_1, G_2 containing G'_1, G'_2 respectively which are a $K{+}S$-decomposition of $G[X, Y]$. This is a contradiction.

Case 2: there is a vertex x in $S(1, 3, 7) \cup S(1, 5, 7)$.

Consider the bipartite complement $J[X, Y]$ of $G[X, Y]$. This contains an induced P_7 on the vertices $(3, 6, 1, 4, 7, 2, 5)$ in that order, and the vertex x in $S(1, 3, 7) \cup S(1, 5, 7)$ is adjacent to either 5 or to 3, and to no other vertex in U. Thus $J[X, Y]$ is a counterexample of minimal order which fulfills the conditions of case 1, so it does not exist. □

This now allows us to count the number of $Star_{1,2,3}$-free bipartite graphs on n vertices.

Corollary 4.16. $Forb_n(Star_{1,2,3}) = n^{n+o(n)}$.

Proof. Since $JS(1, 1) = P_5$ is an induced subgraph of $Star_{1,2,3}$, we have the claimed lower bound.

Now suppose we have a bipartite graph G on n vertices which does not contain a copy of $Star_{1,2,3}$.

We can find a bipartite $Star_{1,2,3}$-free graph G' which is prime by identifying sets of vertices with identical neighbourhoods in G.

By Theorem 4.15, either G' is disconnected, or has a $K{+}S$-decomposition, or is a path (on at least two vertices) or cycle, or one of these is true of its bipartite complement.

We can record G in the following way. First,we record a bipartition X, Y of G. Then we find sets of vertices with identical neighbourhoods and replace each set with single vertices with that neighbourhood, labelled $d_1, \ldots$. This gives a bipartite graph G' which is prime. If G' is disconnected, we write `UNION(`, followed by the recordings of each of the components of G', then the closed bracket. If G' has a $K{+}S$-decomposition, we write `K + S(`, followed by the recordings of the decomposition graphs G_1 and G_2 (where G_1 is the subgraph of G' induced by $X_1 \cup Y_1$ and G_2 that induced by $X_2 \cup Y_2$, $X_1 \cup Y_2$

induces a bipartite clique and $X_2 \cup Y_1$ an independent set), followed by the closed bracket. We write the recording of G_1 before that of G_2. If G' is a path, we write `PATH(`, followed by the vertices of the path, in the path order, then the closed bracket. If G' is a cycle, we write `CYCLE(`, followed by the vertices of the cycle, in an order of the cycle, then the closed bracket. If we cannot do any of the previous we write `COMPLEMENT(`, then the recording of the bipartite complement of G', then the closed bracket. Finally, we replace the vertices d_i in the recording of G' by `IDENTIFY(`, followed by the set of vertices with identical neighbourhoods which were identified to give d_i, then the closed bracket.

Now the total number of appearances in the recording of `UNION(`, $\mathtt{K}+\mathtt{S}($, `PATH(`, `CYCLE(` and `IDENTIFY(` is at most $n-1$, and the total number of appearances of `COMPLEMENT(` is also at most this number. Thus the whole recording consists of the bipartition, a linear order on the n vertices of $G[X, Y]$, and at most $4n$ insertions of seven different strings (including the closed bracket). Thus there are at most $2^n n^n 8^{5n} = n^{n+o(n)}$ possible recordings of bipartite graphs with n vertices not containing $\mathit{Star}_{1,2,3}$, and $\mathit{Forb}_n(\mathit{Star}_{1,2,3}) = n^{n+o(n)}$. □

Since each of the exceptional graphs except for P_7 is an induced subgraph of $\mathit{Star}_{1,2,3}$, we see that $\mathit{Forb}_n(H) = n^{n+o(n)}$ and $\mathit{Free}_n(H) = n^{n+o(n)}$ for each of P_6, $\overline{P_6}$, $\mathit{Star}_{1,2,2}$, $\overline{\mathit{Star}_{1,2,2}}$ and $\mathit{Star}_{1,2,3}$.

Observe that if $G[X, Y]$ does not contain a copy of $P_6[U, V]$, then certainly $G[X, Y]$ does not contain an induced $\mathit{Star}_{1,2,3}$ with any bipartition, so by Corollary 4.16 $\mathit{Forb}_{m,n}(P_6) = m^{m+n+o(m)}$. This immediately gives $\mathit{Forb}_{m,n}(\overline{P_6}) = m^{m+n+o(m)}$. But we cannot use Corollary 4.16 to bound $\mathit{Forb}_{m,n}(H[U, V])$ for H any of $\mathit{Star}_{1,2,2}$, $\overline{\mathit{Star}_{1,2,2}}$ or $\mathit{Star}_{1,2,3}$: there are graphs $G[X, Y]$ which do not contain a copy of $H[U, V]$ but which do have an induced $\mathit{Star}_{1,2,3}$. For example, $\mathit{Star}_{1,2,3}[V, U]$ does not contain a copy of $\mathit{Star}_{1,2,3}[U, V]$.

4.5 Remaining problems

There are still some unresolved problems. The most important is to explain what happens with speeds greater than n^{cn} for any c; when we forbid graphs containing cycles or their bipartite complements. We have (almost) shown that there is a gap between the factorial range of speeds and $2^{\Omega(n^{\frac{6}{5}})}$ containing none of these bipartite properties—but are there more gaps above this speed? Which graph parameters control the speed? This seems to be a very hard problem.

A more accessible problem would be to find good bounds on $\mathit{Free}_n(P_7) = \mathit{Forb}_n(P_7)$. A possible approach to finding such bounds would be to find a decomposition result for P_7-free bipartite graphs, perhaps in a manner similar to Lozin's result for $\mathit{Star}_{1,2,3}$-free graphs: but the 'basic graphs' will need to be a large class (as compared to the small class of paths and cycles Lozin was able to use), since the lower bound on $\mathit{Free}_n(P_7)$ is $n^{\frac{3n}{2}+o(n)}$. We conjecture that in fact $\mathit{Free}_n(P_7)$ has speed $n^{cn+o(n)}$ for some c, and $c = \frac{3}{2}$ seems likely. It is worth observing that P_5 can be formed by adding a vertex and two edges to $DS(1,1)$; and the P_5-free bipartite graphs turn out to be precisely disjoint unions of $DS(1,1)$-free bipartite graphs. In the same way P_7 can be formed from $DS(2,2)$ by adding a vertex and two edges, and perhaps there is some similar (though certainly not identical) relationship between P_7-free graphs and $DS(2,2)$-free graphs.

For completeness, it would be nice to find more accurate bounds for $\mathit{Forb}_{m,n}(H[U,V])$ for each of the four infinite families $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and $JS^*(k,l)$. We know that the upper bound for $JS(1,0) = P_4$ is correct, but we conjecture that in every other case the lower bound is accurate (and so also for $\mathit{Forb}_n(H[U,V])$ and $\mathit{Free}_n(H)$).

It would be of some interest to find good bounds on $\mathit{Forb}_{m,n}(H[U,V])$ for the three exceptional graphs $\mathit{Star}_{1,2,2}$, $\overline{\mathit{Star}_{1,2,2}}$ and $\mathit{Star}_{1,2,3}$. It seems likely that the lower bounds should be correct.

Finally, we recall that Brightwell, Grable and Prömel left unclassified the speed of partial orders without certain induced sub-orders: those corresponding to the four infinite families $DS(k,l)$, $DS^*(k,l)$, $JS(k,l)$ and

$JS^*(k,l)$, and the six exceptional graphs. They conjectured that in each case the correct speed should be $n^{O(n)}$. Our results certainly support this conjecture: but there is no obvious way to extend them to prove the conjecture.

5

Speed and clique-width

5.1 Introduction

In 1993 Courcelle, Engelfriet and Rozenberg [24] considered the following set of (vertex-coloured) graph construction operations:

- Create a vertex v with colour i, denoted $i(v)$.
- Take the disjoint union of two previously constructed graphs G and H (preserving the vertex colours): $G \oplus H$.
- Put an edge from each vertex of colour i to each vertex of colour j: $\eta_{i,j}$.
- Recolour all vertices of colour i to colour j: $\rho_{i \to j}$.

For example we may construct a path P_2 on 2 vertices v_1, v_2 using these operations with two colours:

$$\eta_{1,2}(2(v_1) \oplus 1(v_2)) \ ,$$

and then permitting a third colour and iterating

$$P_n = \eta_{1,2}(\rho_{1\to 2}(\rho_{2\to 3}(P_{n-1})) \oplus 1(v_n))$$

allows the construction of a path of length n.

On the other hand, it is easy to see that two colours do not allow the construction of paths on more than six vertices. Suppose that there were some method of constructing P_7 with two colours. The last operation would have to be an edge creation, since P_7 is connected; since any partition of seven vertices into two parts contains a part with four vertices, and P_7 has maximum degree two, the edge creation operation could not be $\eta_{1,2}$. Thus we can assume that the operation was $\eta_{1,1}$, and therefore prior to this operation there were precisely two vertices with colour one, each of which was the end of a shorter path whose other vertices were coloured two. Now it is not possible to construct a four-vertex path with one end vertex given colour one and the rest colour two: the final operation in such a construction could not have been either a disjoint union (since P_4 is connected) or a recolouring (since both colours are present), so it would again have needed to be an edge creation operation. But with three vertices of colour two and one of colour one, the operation $\eta_{1,1}$ does nothing, the operation $\eta_{1,2}$ creates a vertex of degree three, and the operation $\eta_{2,2}$ creates a triangle. Thus none of these could have been the putative final operation: it is not possible to construct this colouring of P_4 with two colours, and so it is not possible to construct P_7 at all.

It is clear that, when there is no limit on the number of colours permitted, we may construct any graph following these rules: we simply create each vertex with its own unique colour and put edges between vertices as desired. On the other hand, if only k colours are permitted then we may not be able to construct some graphs.

Courcelle, Engelfriet and Rozenberg then defined the *clique-width* of a graph $cwd(G)$ to be the least number of colours required to construct (a vertex-colouring of) G using these four operations. Observe that if H is an induced subgraph of a graph G, with $cwd(G) = k$, then we may take the expression demonstrating that G has clique-width k and omit the vertex creation statements $i(v)$ for each $v \in V(G) - V(H)$. This yields an expression constructing H, demonstrating that $cwd(H) \leq cwd(G)$.

The concept of clique-width is much studied in complexity theory: many problems which are in general NP-hard become soluble in polynomial time on classes of graphs of bounded clique-width. It is thus interesting to determine which classes have bounded and unbounded clique-width. In light of the fact that clique-width does not increase on taking induced subgraphs, we should study hereditary graph classes.

Recently Lozin and Volz [58] studied clique-width on hereditary classes of bipartite graphs defined by one forbidden induced subgraph (precisely the classes we considered in Chapter 4). They were able to describe exactly which classes do and do not have bounded clique-width. Lozin [57] observed that in this case (and, trivially, also in the case of hereditary classes of simple graphs defined by one forbidden induced subgraph) the classes with bounded clique-width are exactly those whose speed is bounded above by a function of the form $n^{n+o(n)}$, and asked whether this correspondence between speed and clique-width could possibly hold for other hereditary graph classes.

We will see that essentially the answer to this question is no: there is a hereditary class of graphs each of clique-width at most 14 whose speed is at least $n!\left(\frac{2^6}{15}\right)^n$, and a hereditary class of graphs on which clique-width is unbounded, whose speed is at most $n!2^{2n}$. However the correspondence can only fail for the (admittedly large number of) hereditary classes $\mathcal{H}$ whose speed is a function of the form $n^{n+o(n)}$ (specifically, the speed of $\mathcal{H}$ must be greater than the Bell number $\mathcal{B}_n$ but smaller than $n!c^n$ for some c depending only on $\mathcal{H}$).

5.2 Positive results

There does exist some correspondence between the speed of a hereditary class and the existence of a bound on clique-width in that class. On the one hand, Scheinerman and Zito [67] and Balogh, Bollobás and Weinreich [6], [8] gave descriptions of the hereditary graph classes whose speeds are eventually bounded by the Bell number $\mathcal{B}_n$. We deduce bounds on clique-width.

We need a routine preliminary lemma.

Lemma 5.1. *Let G be any graph on at least two vertices, and $G' = G - x$ be the graph obtained by removing the vertex x from G. Then*

$$cwd(G) \leq 2cwd(G') + 1 \ .$$

Proof. Let G' have clique-width k, and let I be an expression using k colours which constructs G'. Let V_1 be the neighbours in G of x, and V_2 the non-neighbours.

We construct an expression I' using $2k$ colours by modifying I. First, whenever a vertex $v \in V_2$ is created with colour i in I, we modify the creation colour to $i+k$ in I'. Second, we replace a single edge creation operation $\eta_{i,j}$ in I with four edge creation operations $\eta_{i,j}$, $\eta_{i+k,j}$, $\eta_{i,j+k}$ and $\eta_{i+k,j+k}$. Third, we replace a single recolouring operation $\rho_{i\to j}$ with two recolouring operations $\rho_{i\to j}$ and $\rho_{i+k\to j+k}$.

Observe that the expression I' creates a coloured graph which is identical to that created by I, except that the vertices V_2 all have colours k greater in I' than I.

Finally we obtain a $(2k+1)$-colour expression creating G:

$$\eta_{1,2k+1}(\eta_{2,2k+1}(\ldots \eta_{k,2k+1}(I' \oplus (2k+1)(x))\ldots)) \ ,$$

so that G has clique-width at most $2k+1$, as desired. $\square$

Now we can prove our theorem.

Theorem 5.2. *If the speed of a hereditary graph class $\mathcal{H}$ is at most $\mathcal{B}_n$ for infinitely many values of n then there is a constant $c = c(\mathcal{H})$ such that the clique-width of any graph in $\mathcal{H}$ is at most c.*

Proof. We recall the results of Scheinerman and Zito [67] and Balogh, Bollobás and Weinreich [6], [8] on speeds of hereditary properties. Putting them together:

If the speed of $\mathcal{H}$ is smaller than $\mathcal{B}_n$ for infinitely many values of n, there are two possibilities.

Case 1: The speed of $\mathcal{H}$ is bounded by some exponentially growing function. In this case there exists a c such that whenever G is in $\mathcal{H}$, $V(G)$ may be partitioned into c parts, each of which is either a clique or an independent set, and such that between any pair of parts either no edge is present or every possible edge is present. It is immediate that the clique-width of G is at most c.

Case 2: There exists an integer $k > 1$ such that the speed of $\mathcal{H}$ is given by a function of the form $n^{\left(1-\frac{1}{k}\right)n+o(n)}$. In this case the structure of graphs in $\mathcal{H}$ is a little more complex.

We describe first some specific hereditary classes (those given in Theorem 29 of [6]). Let K be a graph-with-loops on vertex set $[k]$, and G_k a simple graph on the same vertex set $[k]$. We construct an infinite graph H' on $\mathbb{N}$ by partitioning $\mathbb{N}$ into intervals of k consecutive vertices, then setting two vertices i and j adjacent if they are in the same interval and $(i \mod k)(j \mod k)$ is an edge of G_k. We now construct from H' an infinite graph H on the same vertex set, with ij an edge of H if either $ij \in E(H')$ or i and j are in different intervals and $(i \mod k)(j \mod k) \in E(K)$. Finally, let $\mathcal{P}(K, G_k)$ be the hereditary class consisting of all the finite induced subgraphs of H.

It is clear that if $G' \in \mathcal{P}(K, G_k)$ then we can perform the appropriate steps of this construction using the clique-width expressions with $2k$ colours to construct G', so that these graphs have clique-width at most $2k$.

Now (by Theorem 30 of [6]) there is a constant $r = r(\mathcal{H})$ such that whenever $G \in \mathcal{H}$ we can remove r vertices from G and obtain a graph G' in one of these hereditary classes. Since G' has clique-width at most $2k$, by applying Lemma 5.1 r times, it follows that G has clique-width at most $2 \cdot 3^r k$. This is the desired bound on clique-width in $\mathcal{H}$.

Case 3: if the speed of $\mathcal{H}$ is equal to $\mathcal{B}_n$ for infinitely many values of n then there is n_0 such that it is equal to $\mathcal{B}_n$ for all $n \geq n_0$, and either for $n \geq n_0$ every n-vertex graph is a disjoint union of cliques, or for $n \geq n_0$ every n-vertex graph is the complement of a disjoint union of cliques. In either case there are no graphs in $\mathcal{H}$ with clique-width exceeding $\max(n_0, 2)$. $\square$

The approximation of $\mathcal{B}_n$

$$\mathcal{B}_n \sim (\ln w)^{\frac{-1}{2}} w^{n-w} e^w \ ,$$

where w is the solution to $n = w \ln(w+1)$, was given by de Bruijn [16].

Note that this gives us

$$\mathcal{B}_n = n^{n - \left(\frac{\ln\ln(n)+1+o(1)}{\ln n}\right)n} \ .$$

On the other hand, if for all C a graph class has speed greater than $n!C^n$, then it must have unbounded clique-width.

Theorem 5.3. *The number of graphs on n vertices with clique-width at most k is bounded above by $n!C^n$ for some constant C depending on k.*

Proof. If a graph on n vertices has clique-width at most k, then there is an expression using k colours which constructs it. We simply bound above the number of expressions which could possibly give different graphs. We insist on a convenient form for these expressions.

Suppose that we are in the process of constructing a graph, and have just taken a disjoint union of two coloured graphs. We may now apply edge creation or recolouring operations. We may assume that we perform any edge creation operations first, and then do any necessary recolouring; the number of edge creation operations immediately following a disjoint union operation is thus at most $\binom{k}{2} + k$. It is also clear that the recolouring operation $\rho_{i \to j}$ does nothing if there are no vertices of colour i, and is redundant if there are no vertices of colour j (although in many constructions it makes notation simpler to perform some redundant recolourings): so each recolouring operation decreases the number of colour classes containing vertices by one. Thus at most $k-1$ recolouring operations may be performed between disjoint unions.

Now each disjoint union operation joins together two graphs of size at least one: so the number of disjoint union operations is $n-1$, and the number of vertex creations is n.

Finally, given an expression E, we can choose to record it in the modified form E' as follows: E' consists of first a list of the vertex labels in the order they appear in E, followed by a simplified expression obtained by taking the symbols of E in order and replacing every occurrence of $i(v)$ with i, for each colour i and vertex label v. It is clear that we can reconstruct E given E', so we will count the number of modified forms.

The simplified expressions contain the symbols

i (for each $1 \leq i \leq k$),

$\oplus$,

$\eta_{i,j}($ for each $1 \leq i, j \leq k$,

$\rho_{i \to j}($ for each $1 \leq i, j \leq k$, and

$)$,

for a total of $k + 1 + \binom{k}{2} + k + 2\binom{k}{2} + 1 < 2k^2$ distinct symbols.

There are at most $n + n - 1 + 2(n-1)(\binom{k}{2} + k) + 2(n-1)(k-1) < 2k^2 n$ symbols in the entire simplified expression. Together with the $n!$ ways in which the n vertex labels may be ordered, there are at most

$$n! \left(2k^2\right)^{2k^2 n} = n! C^n$$

modified expresssions, where

$$C = \left(2k^2\right)^{2k^2} \quad .$$

This is an upper bound on the number of n-vertex graphs with clique-width at most k. □

Although the constant C in the above theorem clearly grows faster than necessary, it should be at least exponential in k.

Theorem 5.4. *The number of graphs on n vertices with clique-width $k+3$ is at least $n! c^n$ where $c = \frac{2^{\frac{k-2}{2}}}{k+1}$.*

Proof. First observe that using colours $4, 5, \ldots, k+3$ we may construct any graph we choose on k vertices, with each vertex given its own unique colour. We may add a *special vertex* given colour 1 and put this adjacent to all the other vertices. Now we partition $[n]$ into an ordered sequence of $\left\lfloor \frac{n}{k+1} \right\rfloor$ sets of size $k+1$ and if necessary one smaller set. Let $G_1, \ldots, G_{\lfloor \frac{n}{k+1} \rfloor}$ be coloured graphs on the sets of $k+1$ vertices.

Now we can construct a graph G on vertex set $[n]$ by joining the special vertices of the G_i into a path, in the given order, by using the path construction expression above, replacing each vertex creation operation $1(v_i)$ with the expression for G_i, and $2(v_1)$ with $\rho_{1\to 2}(G_1)$, and finally adding as isolated vertices any vertices in the smaller set.

Since a path has only two automorphisms, this process can construct the same labelled graph G in just two ways: the other being of course to take the ordered partition and the sequence of graphs in the reversed orders.

It follows that the number of distinct graphs that can be constructed in this way is at least

$$\frac{n!}{(k+1)!^{\frac{n}{k+1}}} 2^{\binom{k}{2} \lfloor \frac{n}{k+1} \rfloor -1} > n! \left(\frac{2^{\frac{k-2}{2}}}{k+1} \right)^n ,$$

as required. □

An interesting observation is that, in the above construction, the number of recolouring operations used is only $\frac{2n}{k+1}$; and this can be halved by choosing a better path construction (avoiding the redundant recolourings). This seems like an incredibly small number of recolourings; one might wonder whether it is possible to obtain a similar result without using any recolourings. Specifically: does there exist an unbounded function of k, $c(k)$, such that there are at least $n!c^n$ graphs on n vertices which may be constructed using the clique-width operations, with k vertex colours, and without performing any recolouring operations? It is certainly possible to find $\mathcal{B}_n$ such graphs—there are this many disjoint unions of cliques, all of which have clique-width one and so can be constructed without recolourings.

5.3 Negative result

Unfortunately speed does not determine whether clique-width is bounded or not. There is no sharp threshold function $f(n)$ such that the clique-width of a hereditary class is bounded if and only if its speed is less than $f(n)$.

A graph G is a unit interval graph if it is possible to choose for each vertex x of G an interval I_x in $\mathbb{R}$ of unit length, such that xy is an edge of G if and only if the intervals I_x and I_y intersect.

Let $\mathcal{X}$ be the class of unit interval graphs. Golumbic and Rotics [41] showed that this class has unbounded clique-width (and Lozin [56] showed that it is an inclusion-minimal class with unbounded clique-width). It is clear that if we have a unit interval representation of G, and G' is an induced subgraph of G, then taking the unit interval representation of G and removing intervals corresponding to vertices in $V(G) - V(G')$ yields a unit interval representation of G', so $\mathcal{X}$ is a hereditary class.

Theorem 5.5. *The class $\mathcal{X}$ has speed at most $n!4^n$.*

Proof. Let G be any unit interval graph on vertex set $[n]$, and fix a unit interval representation of G. Each interval has a start point and an end point (moving along $\mathbb{R}$), so we can record a string of length $2n$ consisting of n symbols `S` and n symbols `E` giving the order along $\mathbb{R}$ in which the starts and ends of intervals occur, and separately record the permutation σ, where the start point of the interval corresponding to vertex i is the $\sigma(i)$-th start point encountered. Since the intervals are of unit length the same permutation gives the order in which the end points appear. Although we cannot reconstruct the unit interval representation from this recorded information, we can reconstruct the intersections of intervals and hence G.

It follows that there are at most as many unit interval graphs on n vertices as there are choices of permutations of n and $2n$-element strings using two symbols: namely $n!2^{2n}$. □

We note that Hanlon [46] gave a much more precise result, by means of a generating function method.

The class of graphs with clique-width at most 14 has speed at least $n!\left(\frac{2^6}{15}\right)^n$, which is definitely faster than the speed of the class $\mathcal{X}$ of unit interval graphs.

6

Counting 2-SAT functions

6.1 k-SAT functions

Given a collection of n Boolean variables $x_1, \ldots, x_n$, a *satisfying assignment* for a Boolean function S on the n variables is an assignment of True or False to each variable such that $S(x_1, \ldots, x_n)$ is True. A Boolean function is defined by its set of satisfying assignments. We identify one special function: the *trivial* function is that function with no satisfying assigments.

Associated with a Boolean variable x is a *positive literal* x and a *negative literal* $\overline{x}$. The positive literal is True exactly when the variable is True; the negative literal is True exactly when the variable is False.

A *k-clause* is a collection of k literals, no two of which are associated with the same variable. A k-clause is satisfied if and only if at least one of its literals is True. A *SAT formula* is a collection of clauses of any size; a *k-SAT formula* is a collection of k-clauses, and the formula is satisfied if and only if all of its clauses are satisfied. A Boolean function S_F whose satisfying assignments are exactly those which satisfy a k-SAT formula F is called a *k-SAT function*.

A great deal of work has been done on SAT formulae in general. The problem of deciding whether a given SAT formula is satisfiable is the archetypal NP-complete problem of Cook [22]; for fixed $k \geq 3$ the problem of deciding whether a given k-SAT formula has a satisfying assignment is a classic NP-complete problem (Karp [47]). On the other hand there exist polynomial time algorithms deciding whether a given 2-SAT formula is satisfiable. Even thirty-five years after these results, many papers are written discussing variants of SAT with restricted clauses, restricted solutions, and so on, and finding computational complexities, algorithms which run in polynomial expected time, et cetera.

The question of whether a random k-SAT formula is satisfiable has been studied in detail. A sharp threshold between satisfiability and unsatisfiability for 2-SAT was found by Goerdt [39], [40], Chvátal and Reed [19] and Fernandez de la Vega [33], who showed that a 2-SAT formula with n variables and rn clauses chosen uniformly at random is almost surely satisfiable when $r < 1$, and almost surely unsatisfiable for $r > 1$. Bollobás, Borgs, Chayes, Kim and Wilson [11] were even able to describe the behaviour around the threshold $r = 1$. Less is known about the problem for $k > 2$: Achlioptas and Peres [1] found that (assuming a similar threshold for k-SAT exists) it is located at $(2^k \log_e 2 - O(k))n$.

Of course, in a sense the purpose of a k-SAT formula is to give a k-SAT function, so it seems reasonable that we should make some attempt to understand k-SAT functions. But relatively little is known about k-SAT functions (for fixed k). These objects are considerably harder to handle: while one can easily write down a sequence of k-clauses and obtain a k-SAT formula, and quickly check whether two such sequences correspond to the same k-SAT formula, there is no obvious way to do the same with k-SAT functions. Most k-SAT formulae give rise to the trivial function; many k-SAT functions are represented by several different formulae, and it is not generally obvious when two formulae in fact give the same function (indeed, this is yet another NP-hard problem); while lists of satisfying assignments, although easy to compare, do not in general correspond to k-SAT functions.

However there is an obvious large class of k-SAT functions whose members can be easily generated and distinguished.

We call a k-SAT formula *monotone* if all of its clauses contain only positive literals, and a k-SAT function is called monotone if it is given by a monotone formula.

Given a k-SAT formula F on n variables, and a subset R of those variables, we can define a k-SAT formula $\rho_R(F)$ by replacing each positive literal associated to a variable in R with the corresponding negative literal, and each negative literal associated to a variable in R with the corresponding positive literal, in every clause of F. We call this process *relabelling the literals*, and say R is the set of variables that were relabelled. If S is a k-SAT function, then for any formula F which gives rise to S, $\rho_R(F)$ gives rise to the same k-SAT function, which we call $\rho_R(S)$, and whose satisfying assignments are exactly the satisfying assignments of S with the truth values of the variables R reversed. The operation ρ_R is self-inverse.

We say that any k-SAT function which is the result of relabelling some set of variables on a monotone function is *unate*. Of course it is easy to generate unate k-SAT functions by writing down a monotone formula and relabelling some variables; and the following lemma shows that it is also easy to distinguish them:

Lemma 6.1. *Any unate k-SAT function is given by exactly one k-SAT formula.*

Proof. Suppose not: let F_1 be a monotone k-SAT formula, and S the unate function given by the formula $\rho_R(F_1)$ which is also given by a formula $\rho_R(F_2) \neq \rho_R(F_1)$. Then $F_2 \neq F_1$, and $\rho_R(S)$ is a monotone k-SAT function defined by two distinct formulae F_1 and F_2.

If F_2 contains a clause C not in F_1, then the assignment A of True to all variables except those whose positive literals appear in C is a satisfying assignment for F_1: either C contains some negative literals, in which case there are at most $k-1$ variables assigned False in A and every k-clause of positive literals is satisfied by A, or C contains only positive literals, in which case exactly k variables are assigned False in A, but the only k-clause of positive literals that is not satisfied by A is C, and C is not in F_1. But A does not satisfy the clause C, so A does not satisfy F_2, which contradicts the assumption that F_1 and F_2 both define $\rho_R(S)$. □

Since it is easy to generate and distinguish unate k-SAT functions it is easy to count them.

Theorem 6.2. *The number of unate k-SAT functions on n variables is between*

$$2^{\binom{n}{k}+n} - n2^{\binom{n-1}{k}+n} \textit{ and}$$

$$2^{\binom{n}{k}+n} .$$

Proof. The upper bound is immediate: there are $2^{\binom{n}{k}}$ monotone formulae on n variables and 2^n choices of R.

To obtain the lower bound, we first observe that if F_1 and F_2 are monotone formulae and R_1 and R_2 are sets of variables, then $\rho_{R_1}(F_1) = \rho_{R_2}(F_2)$ is only possible when $F = F_1 = F_2$ and R_1 and R_2 differ only on variables whose positive literals do not appear in F. Since the number of monotone formulae on n variables which do not use any given positive literal is $2^{\binom{n-1}{k}}$, at least $2^{\binom{n}{k}} - n2^{\binom{n-1}{k}}$ monotone formulae use all the literals. Their relabellings are distinct formulae, and by Lemma 6.1 these give the desired number of distinct unate k-SAT functions on n variables. $\square$

It is conjectured that for every k, almost every k-SAT function is unate.

Bollobás, Brightwell and Leader [13] proved some upper bounds on the number of k-SAT functions on n variables for $k \leq \frac{n}{2}$, and Bollobás and Brightwell [12] gave upper bounds for $k \geq \frac{n}{2}$. The case $k = 1$ is trivial.

Theorem 6.3. *There are $3^n + 1$ 1-SAT functions on n variables.*

Proof. If a 1-SAT formula contains both the clauses (x_i) and $(\overline{x_i})$ then it is the trivial function. If it does not, then for each $1 \leq i \leq n$ there are three possibilities: that the formula contains (x_i), that it contains $(\overline{x_i})$, or that it contains neither clause. These 3^n different formulae correspond to distinct non-trivial 1-SAT functions. $\square$

For $k \geq 3$ they proved a weak upper bound (in that the coefficient of the dominant term $\binom{n}{k}$ in the exponent is significantly larger than the conjectured value 1).

Theorem 6.4. *For $k \geq 3$, there are at most $2^{\sqrt{\pi(k+1)\binom{n}{k}}}$ k-SAT functions on $n \geq 2k$ variables.*

For the case $k = 2$ they proved a much stronger upper bound.

Theorem 6.5. *The number of 2-SAT functions on n variables is $2^{\binom{n}{2}+o(n^2)}$.*

We will now examine the 2-SAT functions in more detail, among other things proving their conjecture that almost every 2-SAT function is unate.

6.2 Almost every 2-SAT function is unate

Let $G(n)$ be the number of 2-SAT functions on n variables. In light of Theorem 6.2 we know that $G(n) \geq 2^{\binom{n}{2}+n} - n2^{\binom{n-1}{2}+n} > \left(1 - 2^{-\frac{3n}{4}}\right) 2^{\binom{n}{2}+n}$ for sufficiently large n.

In this section we prove a matching upper bound.

Theorem 6.6. *For all sufficiently large n,*

$$\left(1 - 2^{-\frac{3n}{4}}\right) 2^{\binom{n}{2}+n} < G(n) < 2^{\binom{n}{2}+n}\left(1 + 2^{-\frac{n}{25}}\right) ,$$

and almost every 2-SAT function is unate.

The upper bound argument of Bollobás, Brightwell and Leader [13] goes through three steps. First they argue that only a subclass of 2-SAT functions need to be considered, then they establish a bijection between the n-variable elements of this subclass and a class of $2n$-element posets, each of which gives rise to a two-coloured graph on n vertices which does not contain a certain structure. Finally they apply a two-colour version of the Szemerédi Regularity Lemma to the n-vertex graph, and show that the forbidden structure restricts the possibilities for the pairs of parts. Their upper bound then follows from an argument similar to Theorem 1.8.

Our argument will follow theirs up to the point of constructing two-coloured graphs, when we choose a different (and in this case more natural) object to study. We will then apply several enumerating techniques. The 2-SAT functions which are unate we have already counted. Those which are in some sense 'close to' unate we will be able to enumerate by identifying the small flaws which make them fail to be unate and which also restrict the number of possible choices. For those which are far from being unate, we need to return to the argument of Bollobás, Brightwell and Leader, borrowing their use of the Regularity Lemma to obtain an estimate. Finally we make use of an inductive argument in the style of Kleitman and Rothschild [48] to cover the remaining possibilities.

Bollobás, Brightwell and Leader observed that $G(n)$ can also be interpreted as the number of subsets of the n-cube that are the union of subcubes of codimension 2. However, we will not make use of this viewpoint.

6.2.1 Elementary functions

Given a 2-SAT function, we define its *spine* to be the set of literals which are true in all satisfying assignments; obviously any non-trivial function cannot have both x and $\overline{x}$ in its spine. We will refer to the variable x being in the spine of the function.

Suppose that for some pair of literals u, v, in every satisfying assignment $(u \iff v)$. Then we say that the literals are *associated*; clearly u, v are associated if and only if $\overline{u}, \overline{v}$ are associated. Then we can say that the corresponding variables are associated, and trivially this is an equivalence relation.

We call a 2-SAT function *elementary* if it has no variables in its spine and no associated pairs of variables. Let there be $H(n)$ elementary 2-SAT functions on n variables.

Given any non-trivial 2-SAT function S on n variables, we can reduce it to an elementary 2-SAT function by ignoring all variables in the spine of S and all but the lowest numbered in each equivalence class of associated variables, then compressing the labels to obtain an elementary function on $n - l$ variables ($l \geq 0$). This reduction is at worst $\binom{n}{l}(2n - 2l + 2)^l$-to-1, since for each of the l variables removed ($\binom{n}{l}$ choices of label) we can choose to associate its positive literal to any of the $2n - 2l$ remaining literals, or to put either its positive or negative literal in the spine. Thus

$$H(n) \leq G(n) \leq 1 + \sum_{l=0}^{n} H(n-l)\binom{n}{l}(2n-2l+2)^l$$

where the 1 is counting the trivial function.

Since every unate function is certainly elementary, we have

$$2^{\binom{n}{2}+n}\left(1-2^{-\frac{3n}{4}}\right) < H(n) \ .$$

It is clear that $H(n)$ is monotone increasing, and we will prove that $H(n) < 2^{\binom{n}{2}+n}\left(1+2^{-\frac{n}{24}}\right)$ for all sufficiently large n. This will be enough to prove Theorem 6.6.

Proof. Let N be sufficiently large that for all $n \geq N$, $H(n) < 2^{\binom{n}{2}+n}\left(1+2^{-\frac{n}{24}}\right)$. Then we have

$$\begin{aligned} G(n) &< 1 + \sum_{l=0}^{n-N} H(n-l)\binom{n}{l}(2n-2l+2)^l + \sum_{l=n-N+1}^{n} H(N)\binom{n}{l}(2n-2l+2)^l \\ &< \sum_{l=0}^{n-N} 2^{\binom{n-l}{2}+n-l}\left(1+2^{-\frac{n-l}{24}}\right)\binom{n}{l}(2n-2l+2)^l + NH(N)2^n(2n)^n \\ &< 2^{\binom{n}{2}+n}\left(1+2^{-\frac{n}{25}}\right) \end{aligned}$$

for sufficiently large n. $\square$

We define a bijection between the elementary 2-SAT functions on n variables and a particular class of partial orders on $2n$ elements: given any formula F giving rise to an elementary 2-SAT function S_F, let $P_1(F)$ be the relation on the $2n$ elements$\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ given by $a < b$ if the clause $(\overline{a}, b)$ appears in F.

Suppose that there were a sequence $a_1 < a_2 < \ldots < a_r < a_1$ in $P_1(F)$. Suppose we have a satisfying assignment for S_F with a_1 True. Then as $(\overline{a}_1, a_2)$ must contain a True literal, a_2 is also True in any such assignment. Suppose we have a satisfying assignment for S_F with a_1 False. Then as $(\overline{a}_r, a_1)$ must contain a True literal, a_r must also be False, and by induction a_i must be False for each $1 \leq i \leq r$ in any such assignment. But then a_1 and a_2 are associated, contradicting S_F being elementary. So no such sequence exists. Then let $P(F)$ be the transitive closure of $P_1(F)$; we see that this is a partial order.

The relation $P(F)$ must satisfy $(u < v \iff \overline{v} < \overline{u})$, since a sequence of clauses giving the first relation also gives the second. It cannot have $\overline{u} < u$ for any u, since if u is False in a satisfying assignment for S_F, the sequence of clauses certifying $\overline{u} < u$ certify u True, which is a contradiction, so that u is in the spine of S_F.

A satisfying assignment for S_F is an up-set in $P(F)$ containing exactly one of each pair of literals. Furthermore, suppose $u \not< v$ in $P(F)$, then let U be the smallest up-set containing both u and $\overline{v}$; there is no x with $u < x$ and $\overline{v} < \overline{x}$, as this implies $x < v$ so $u < v$, so that U contains at most one of each pair of literals. There is no y with $y < v$, $\overline{y} < v$, for this implies $\overline{v} < y < v$. So we can add in to U one literal at a time to obtain an up-set containing exactly one of each pair of literals, which is a satisfying assignment for S_F with u and $\overline{v}$ true. Thus $u < v$ in $P(F)$ if and only if $(u \implies v)$ is True in every satisfying assignment for S_F. So $P(F)$ depends only on the function S_F and not on the specific formula F giving S_F, and the satisfying assignments of S_F can be found given $P(F)$. Thus there is a 1-1 correspondence between elementary 2-SAT functions on n variables $\{x_1, \ldots, x_n\}$ and partial orders on $2n$ elements $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ such that $(u < v \iff \overline{v} < \overline{u})$ and there is no u with $\overline{u} < u$. We will write $P(S)$ for the partial order corresponding to the elementary function S.

6.2.2 Reduction to diagrams

At this point we depart from the method of proof in [13]; we develop a line-and-arrow representation, which turns out to be more amenable to detailed analysis than the coloured graph representation studied in [13].

The trivial function is not elementary, so that every elementary function must have a satisfying assignment. For each elementary 2-SAT function S, choose a satisfying assignment, and let R be the set of variables assigned False. Let $M(S) = \rho_R(S)$; then assigning True to all variables is a satisfying assignment of $M(S)$. Since there are only 2^n ways to relabel the literals, the restriction of M to the elementary 2-SAT functions on n variables is at worst 2^n-to-1. Observe that whatever elementary S was chosen, $P(M(S))$ cannot have $x_i < \overline{x}_j$ for any i, j, as this is equivalent to $(x_i \implies \overline{x}_j)$ holding in every satisfying assignment of $M(S)$, contradicting all True being a satisfying assignment. Furthermore, $M(S)$ is unate if and only if S is unate.

Call a 2-SAT function *nonnegative* if it is elementary and all True is a satisfying assignment. Then M maps from the class of elementary 2-SAT functions to the nonnegative 2-SAT functions.

Given a nonnegative 2-SAT function S on n variables $\{x_1, \ldots, x_n\}$, we construct a *diagram* $D(S)$, which will be a graph on n points $\{x_1, \ldots, x_n\}$ in which some edges are directed (which we call *arrows*) and some are not (which we call *lines*). The directed edges will form a partial order, the *partial order within* D, and we will use equivalently '$a < b$' and 'there is an arrow from a to b'. We do this as follows. First take $P(S)$, the partial order on $2n$ elements associated with S. Then whenever $x_i < x_j$ in $P(S)$, we put $x_i < x_j$ in $D(S)$. Whenever $\overline{x}_i < x_j$ is a covering relation in $P(S)$ we put a line $x_i x_j$ in $D(S)$. Observe that if $\overline{x}_i < x_j$ so also $\overline{x}_j < x_i$, and that given $D(S)$ we can certainly recover $P(S)$ and hence S.

The diagram $D(S)$ is a natural way of representing S in the following sense: if S is a nonnegative function, and F is a formula for S containing every possible clause $(\overline{x_i}, x_j)$ and a minimal number of clauses (x_i, x_j), then F is unique (since the covering graph of a partial order is unique) and the diagram $D(S)$ is obtained by replacing each clause (x_i, x_j) with a line between x_i and x_j, and each clause $\overline{x_i}, x_j)$ with an arrow from x_i to x_j.

Observe that, in $D(S)$, no pair of points a and b are joined by both a line and an arrow: ab and $a < b$. For this would imply that in $P(S)$ we had $\overline{b} < a < b$. We also cannot find each of the following *forbidden structures*, shown in Figure 1:

(1) a, b, c such that $b < a, c < a$ and there is a line bc.

(2) a, b, c such that $a < b$ and there are lines ac, bc.

(3) a, b, c, d such that $a < b, c < d$ and there are lines ac, bd.

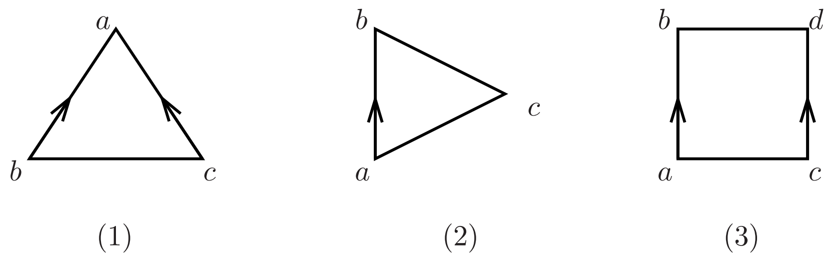


Figure 6.1 Forbidden structures

For if (1) existed, then in $P(S)$, $\overline{a} < \overline{b} < c < a$. If (2) existed, then in $P(S)$, $\overline{c} < a < b$ so that $\overline{c} < b$ cannot be a covering relation. If (3) existed, then in $P(S)$, $\overline{b} < \overline{a} < c < d$ and $\overline{b} < d$ cannot be a covering relation.

A *valid diagram* is any diagram in which no two points are joined by both an arrow and a line, and which does not contain any of the three forbidden structures. Then certainly the set of valid diagrams on n points contains the set of all diagrams $D(S)$ for S a nonnegative 2-SAT function on n variables, and it is equally clear that given a valid diagram D the 2-SAT formula F obtained by replacing the lines and arrows of D with clauses as above is a formula for a nonnegative function S whose diagram $D(S)$ is D: so that $S \to D(S)$ is a bijection between the nonnegative 2-SAT functions and the valid diagrams.

At this point it is worth observing that a monotone 2-SAT function is always a nonnegative function; it corresponds to a diagram in which there are no arrows, only lines between points. A unate 2-SAT function, by contrast, is not usually nonnegative: we get a unate function by relabelling the literals on a monotone function, and the result is only nonnegative if there were no clauses consisting of any two of the relabelled literals. So we expect to find that M is in some sense 'nearly' 2^n-to-1 on the set of unate 2-SAT functions on n variables, and we expect to find that most unate nonnegative 2-SAT functions correspond to diagrams in which there is a large set of points within which no arrows are found, a small set of points within which no lines or arrows are found, and between the small and large set no lines are found and no arrows go from the large set to the small set.

We will bound both the number $F(n)$ of valid diagrams on n points and the number of valid diagrams on n points not corresponding to unate functions. This latter bound will be significantly smaller than $F(n)$, enough so that the unfortunate fact that M is not exactly 2^n-to-1 everywhere does not cause us problems.

In the rest of this chapter, all diagrams will consist of a *top set* T, containing all points which are maximal in the partial order within the diagram, and a *bottom set* B, all other points (see Figure 2).

Note that there are no arrows within T, and that every member of B is below at least one point of T.

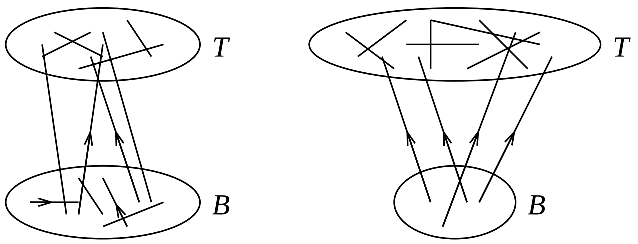


Figure 6.2 A general valid diagram and a diagram arising from a unate 2-SAT function

We will at times want to say that between a and b in a diagram there is no line or arrow in either direction, when we will simply say that there is *nothing* between a and b; we will also sometimes want to say that there is either a line or an arrow in one or the other direction between a and b, when we will say there is *something* between a and b.

Lemma 6.7. *A valid diagram corresponds to a nonnegative unate* 2*-SAT function if and only if there are no lines between its top and bottom sets, and no lines or arrows within its bottom set.*

Proof. If S is a nonnegative unate 2-SAT function, then let R be a minimal set of variables such that $\rho_R(S)$ is a monotone function. If a is a point in $D(S)$, and $a \notin R$, then there certainly is no clause $(\overline{a}, b)$ in the 2-SAT formula for S, and so a cannot be below any other point in D, i.e. $a \in T$. If on the other hand $a \in R$, then the clause $(\overline{a}, b)$ must appear in the formula for S (since R is minimal) and $a \in B$. If there is a line tb in D, with $t \in T, b \in B$, then the clause (t, b) is in the formula for S, so that $b, \overline{b}$ are both mentioned in the formula, which is impossible since S is unate. Similarly, if $a < b$ or ab is in D, with $a, b \in B$, then $b, \overline{b}$ are in the formula for S which is impossible. Hence $D(S)$ is of the required form. Now suppose we are given a diagram $D(S)$ of the required form, corresponding to the nonnegative function S. Since there is no arrow $a < b$ with a line ac, we see that in $P(S)$ every relation $\overline{x_i} < x_j$ between positive and negative literals is a covering relation, so $\rho_B(S)$ is a unate function. □

We observe that valid diagrams corresponding to nonnegative unate 2-SAT functions are in $1-1$ correspondence with graphs with a specified independent set B.

Suppose that we attempt to construct a valid diagram. We first choose the set B, then we choose where to put lines within T, then where to put lines between B and T, and where to put arrows from B up to T. Then we can choose where to put lines and arrows within B. But this last choice is already restricted. Suppose that we have two points $a, b \in B$. Then either there is, or there is not, a point in T above both a and b. In the former case, we cannot put a line ab in without creating the forbidden structure (1) and hence an invalid diagram. In the latter case, we cannot put an arrow from a to b, for there exists $x \in T$ with $b < x$ and hence if $a < b$ then $a < x$ also; by symmetry, we cannot put an arrow from b to a either. Thus between any two points in b we have either a choice of nothing or a line between a and b, or a choice of nothing or an arrow in one or the other direction.

We can now put a crude upper bound on the number of valid diagrams on n points:

$$\sum_{|B|=0}^{n-1} \binom{n}{|B|} 2^{\binom{|T|}{2}} 3^{|T||B|} 2^{\binom{|B|}{2}} |B|!$$

simply by following the above construction, noting that there are $|B|!$ ways to order the points in B, so that there are at most that many ways to decide the directions of any arrows we might choose in B.

We have now reduced the problem to examining a class of combinatorial structures which are relatively easy to handle; we will now develop and apply tools to handle these structures.

Since we want to find that there are about $2^{\binom{n}{2}}$ valid diagrams, we must find a way of dealing with the possibilities for lines and arrows between B and T, and with the requirement to order B. We will do this by dividing into two cases, when B is small and when it is not. We will need only some simple approximations to deal with the first case, even though it will turn out to be the large case. We will apply the Szemerédi Regularity Lemma followed by an induction argument to dispose of the second case.

6.2.3 Some useful tools

The following will be used frequently to restrict choice, relying on the forbidden structure (2) which says that if a point is comparable with one set of points and connected by lines to another set, then there can exist no lines between these two sets.

Let V be a set of points, and I be an index set. Suppose that there exist for each $i \in I$ sets $S_i, L_i \subset V$, with $S_i \cap L_i = \emptyset$. Call the set

$$\bigcup_{i \in I} \big\{\{a,b\} : a \in L_i, b \in S_i\big\}$$

the *forbidden set* (for $(S_i, L_i)_{i \in I}$).

Lemma 6.8. *Suppose that there is a constant l such that for each i, $|L_i| \geq l$. Let $S = \cup_{i \in I} S_i$. Then the forbidden set has size at least $\frac{l|S|}{2}$.*

Proof. Observe that for each $s \in S$, at least l members of the forbidden set contain s. But as a given member of the forbidden set can contain at most two elements of S, this counts each member at most twice, and the forbidden set has size at least $\frac{l|S|}{2}$. $\square$

We will usually refer to the S_i as the *small sets* and the L_i as the *large sets*.

When we use this, the set I will be some points in the top or bottom of a diagram, the sets L_i and S_i will be points in the other layer connected to i by arrows or lines (in no particular order), and the forbidden structure (2) will dictate that there are no lines between pairs in the forbidden set. This places two restrictions on the available choices: firstly we have to choose the S_i within S, and secondly we will be unable to choose lines between at least $\frac{l|S|}{2}$ pairs.

Some simple bounds will also be useful.

Whenever $k \leq \frac{n}{3}$, we have $\sum_{i=0}^{k} \binom{n}{i} < 2\binom{n}{k}$, since

$$\binom{n}{i} = \frac{i+1}{n-i} \cdots \frac{k}{n-k+1}\binom{n}{k} < \left(\frac{1}{2}\right)^{k-i}\binom{n}{k}$$

when $i \leq \frac{n}{3}$.

In [48], Kleitman and Rothschild show that the number of partial orders on n points is asymptotically $2^{\frac{n^2}{4}+O(n\log n)}$ (this result is sharpened and the proof simplified by Brightwell, Prömel and Steger in [15]). Hence there exists J such that for all $n > J$, there are fewer than $2^{\frac{3n^2-5n}{10}}$ partial orders on n points. We use J for this number throughout.

6.2.4 The large case

In this subsection, we discuss the case $|B| \leq \frac{n}{100}$. Note that we expect to find many valid diagrams in this case, as it covers all the diagrams corresponding to monotone 2-SAT functions and the vast majority of those corresponding to nonnegative unate functions.

All the diagrams in this case are in some sense 'close to' being unate, and we will establish all our bounds by describing structural flaws which simultaneously cause diagrams to not actually be unate and restrict the available choices. We will in several cases obtain significantly better bounds than we really need for the proof of our theorem: this will make the proofs of the later refinements easier.

Theorem 6.9. *For all sufficiently large n, there are at most $2^{\binom{n}{2}+(\log n)^2}$ valid diagrams on n points with $|B| \leq \frac{n}{100}$, and at most $2^{\binom{n}{2}-\frac{n}{23}}$ of these do not correspond to unate functions. Furthermore, there are at most $2^{\binom{n}{2}-2n^{\frac{5}{4}}}$ valid diagrams on n points with $3n^{\frac{19}{20}} < |B| \leq \frac{n}{100}$.*

Proof. Given a diagram and a point $x \in B$, let $\Gamma_{arr}(x) = \{t \in T : t > x\}$ be the *arrow neighbours* of x. Let $\Gamma_{line}(x) = \{t \in T : xt \text{ is a line}\}$ be the *line neighbours* of x. Note that there might be points in B connected to x by arrows or lines, but we do not include them in these sets.

Let $P \subset B$ be the set of points $x \in B$ with both $|\Gamma_{arr}(x)| < \frac{n}{10}$ and $|\Gamma_{line}(x)| < \frac{n}{10}$. Let $I = B - P$.

Now for each point $i \in I$, at least one of $|\Gamma_{arr}(x)|$, $|\Gamma_{line}(x)|$ is at least $\frac{n}{10}$. We apply Lemma 6.8, with this I, with $V = T$, with L_i the larger of $\Gamma_{arr}(i)$ and $\Gamma_{line}(i)$, and with S_i the smaller. Then $l \geq \frac{n}{10}$, and we have $S = \cup_{i \in I} S_i$.

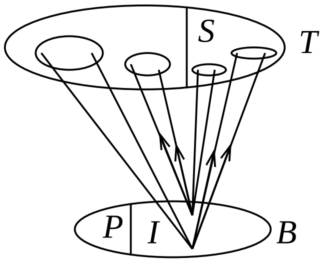


Figure 6.3 A typical valid diagram with $|B| \leq \frac{n}{100}$

Let $D(B, P, S)$ be the number of valid diagrams with sets B, P, S (see Figure 6.3). Then the number of valid diagrams with $|B| \leq \frac{n}{100}$ is at most

$$\sum_{|B|,|P|,|S|} \binom{n}{|B|} \binom{|B|}{|P|} \binom{|T|}{|S|} D(B, P, S) \ . \tag{6.1}$$

We will attempt to construct valid diagrams, obtaining an upper bound by counting the number of choices at each stage, as follows:

First, we choose the sets B, P, S. Then we choose for each point in P its arrow and line neighbours. Note that $\sum_{j=0}^{\frac{n}{10}} \binom{|T|}{j} < 2\binom{|T|}{\frac{n}{10}}$, and when n is sufficiently large,

$$\left(2\binom{|T|}{\frac{n}{10}}\right)^2 < 4(10e)^{\frac{n}{5}} < 2^{|T|}2^{-\frac{n}{21}}$$

which approximation will be used to bound the number of choices for lines and arrows connecting to points in P. We choose for each point in I whether its large set is its arrow or line neighbours, then we choose its large and small sets.

We choose where to put lines within T; by Lemma 6.8 there is a set of size at least $\frac{n|S|}{20}$ in which we cannot choose to put lines.

We choose a total order for B, which will orient any arrows we choose to put in B. For any pair in $B^{(2)}$ we can choose to put either nothing or something, but we have no choice over whether 'something' is a line or an arrow one way or the other.

This gives us the following upper bound on $D(B, P, S)$.

$$D(B,P,S) < \left(2\binom{|T|}{\frac{n}{10}}\right)^{2|P|} 2^{|I|}2^{|I||T-S|}3^{|I||S|}2^{\binom{|T|}{2}-\frac{n|S|}{20}}|B|!2^{\binom{|B|}{2}}$$

$$< 2^{\binom{n}{2}}2^{-\frac{|P|n}{21}}2^{-\frac{n|S|}{20}+|I||S|\log\frac{3}{2}}2^{|I|}|B|! \tag{6.2}$$

Observe that $|I| \leq |B|$, so $|I|\log\frac{3}{2} < \frac{7n}{1000} < \frac{n}{20}$. Then $2^{-\frac{n|S|}{20}+|I||S|\log\frac{3}{2}} \leq 1$, so in the sum (6.1) giving an upper bound on the number of diagrams, the only terms which multiply the $2^{\binom{n}{2}}$ term by an amount greater than 1 are the choices for B, P, S, $2^{|I|}$, and $|B|!$. Together with the sum - over at most n^3 summands - this means that an upper bound for the number of valid diagrams will be $n^3 2^{3n} D(B, P, S)$, taking the worst case for $D(B, P, S)$. We split the proof into cases, of which only the last will be large:

(i) $|P| \geq 84n^{\frac{1}{4}}$.

(ii) $|P| < 84n^{\frac{1}{4}}$ and $|S| \geq 100n^{\frac{1}{4}}$.

(iii) $|P| < 84n^{\frac{1}{4}}$, $|S| < 100n^{\frac{1}{4}}$ and $|B| \geq 3n^{\frac{19}{20}}$.

(iv) $|B| < 3n^{\frac{19}{20}}$ and at least one of $|S| > 0$, $|P| > 0$ holds.

(v) $|S| = |P| = 0$, $|B| < 3n^{\frac{19}{20}}$ and there is a line within B.

(vi) $|S| = |P| = 0$, $5\sqrt{n} \leq |B| < 3n^{\frac{19}{20}}$ and there is no line within B.

(vii) $|S| = |P| = 0$, $|B| < 5\sqrt{n}$, there is no line within B and there are at least two arrows within B.

(viii) $|S| = |P| = 0$, $|B| < 5\sqrt{n}$, there is no line within B and there is exactly one arrow within B.

(ix) $|S| = |P| = 0$, $|B| < 5\sqrt{n}$ and there are no lines or arrows within B. By Lemma 6.7, all these diagrams correspond to unate 2-SAT functions.

Case (i):
$2^{-\frac{|P|n}{21}} \leq 2^{-4n^{\frac{5}{4}}}$, so that using (6.2) we can see that

$$D(B,P,S) < 2^{\binom{n}{2}} 2^{-\frac{|P|n}{21}} 2^{|I|} |B|! < 2^{\binom{n}{2} - 3\log n - 3n - 3n^{\frac{5}{4}}}$$

for sufficiently large n, so that in this case (6.1) is bounded above by $2^{\binom{n}{2} - 3n^{\frac{5}{4}}}$. △

Case (ii):
$|I| \log \frac{3}{2} < \frac{7n}{1000}$, so $2^{-\frac{n|S|}{20} + |I||S| \log \frac{3}{2}} < 2^{-4n^{\frac{5}{4}}}$, so that (6.2) gives us

$$D(B,P,S) < 2^{\binom{n}{2} - 4n^{\frac{5}{4}}} 2^n |B|! < 2^{\binom{n}{2} - 3\log n - 3n - 3n^{\frac{5}{4}}}$$

for sufficiently large n, so that in this case (6.1) is bounded above by $2^{\binom{n}{2} - 3n^{\frac{5}{4}}}$. △

Case (iii):
Since $|P| < 84n^{\frac{1}{4}}$, so $|I| > 2n^{\frac{19}{20}}$, and there must be either $n^{\frac{19}{20}}$ points in I all of whose arrow sets are their large sets, or that many points all of whose arrow sets are their small sets. In the first case, there are at least $\frac{n}{10} n^{\frac{19}{20}}$ arrows going up from I to the less than n points in T, so one of these points must be the target of at least $\frac{1}{10} n^{\frac{19}{20}}$ arrows. In the second case, recall that every point in B must be below at least one point in T, so that all of the given $n^{\frac{19}{20}}$ points must be below points in S, and one point in S must be above at least $\frac{n^{\frac{19}{20}}}{|S|}$ of them. Let C be a maximal set of points in B such that there is one point $t \in T$ with $t > c$ for all $c \in C$.
Then $|C| \geq \min(\frac{n^{\frac{19}{20}}}{10}, \frac{n^{\frac{19}{20}}}{|S|}) \geq \min(\frac{n^{\frac{19}{20}}}{10}, \frac{n^{\frac{19}{20}}}{100n^{\frac{1}{4}}}) = \frac{n^{\frac{14}{20}}}{100}$.
Now we see that within $C^{(2)}$, we can find no lines, since structure (1) is forbidden. Therefore the structure within C is simply a partial order, and we recall that for $|C| > J$, there are at most $2^{\frac{3|C|^2 - 5|C|}{10}}$ partial orders on $|C|$ points. As a result, in this case, when we choose lines and arrows within B,

we find that there are at most

$$2^{\binom{|B|}{2}-\binom{|C|}{2}}|B|!2^{\frac{3|C|^2-5|C|}{10}} = 2^{\binom{|B|}{2}-\frac{|C|^2}{5}}|B|!$$

possible choices.

Observe that $\frac{|C|^2}{5} \geq \frac{n^{\frac{7}{5}}}{50000}$. Consider constructing a diagram in this case: given B, P, S we choose lines and arrows from P to T, from I to $T-S$ and to S, within T and finally within B.

We see that we have in in this case an upper bound

$$D(B,P,S) < \left(2\binom{|T|}{\frac{n}{10}}\right)^{2|P|} 2^{|I|}2^{|I||T-S|}3^{|I||S|}2^{\binom{|T|}{2}-\frac{n|S|}{20}}2^{\binom{|B|}{2}-\frac{|C|^2}{5}}|B|!$$

$$< 2^{\binom{n}{2}}2^{-\frac{n^{\frac{7}{5}}}{50000}}2^n n! < 2^{\binom{n}{2}-3\log n-3n-3n^{\frac{5}{4}}}$$

for n sufficiently large that both $\frac{n^{\frac{14}{20}}}{100} > J$ and the above approximations hold. Then in this case (6.1) is bounded above by $2^{\binom{n}{2}-3n^{\frac{5}{4}}}$. △

At this point, note that we have the required upper bound

$$3.2^{\binom{n}{2}-3n^{\frac{5}{4}}} < 2^{\binom{n}{2}-2n^{\frac{5}{4}}}$$

on the number of valid diagrams with $3n^{\frac{19}{20}} < |B| \leq \frac{n}{100}$.

We also observe that we know that from now $|B|$, $|S|$, $|P|$ and $|I|$ are all much smaller than n, so that for all sufficiently large n, $n^3\binom{n}{|B|}\binom{|B|}{|P|}\binom{|T|}{|S|} < 2^{n^{0.99}}$, $2^{|I|}|B|! < 2^{\frac{n}{1000}}$, and $|I|\log\frac{3}{2} < \frac{n}{1000}$. We will use these bounds in the following cases.

Case (iv): Applying (6.2) gives us the bound

$$D(B,P,S) < 2^{\binom{n}{2}}2^{-\frac{|P|n}{21}}2^{-\frac{|S|n}{20}+|S|\frac{n}{1000}}2^{|I|}|B|!$$

$$< 2^{\binom{n}{2}}2^{-\frac{|P|n}{21}}2^{-\frac{|S|n}{21}}2^{|I|}|B|! < 2^{\binom{n}{2}-\frac{n}{22}-n^{0.99}}$$

for sufficiently large n, and (6.1) is bounded above by $2^{\binom{n}{2}-\frac{n}{22}}$ for all sufficiently large n. △

Case (v):

Every point in B is below at least one point in T by definition, and since $|S| = |P| = 0$ every point in B is in I and has empty small set, hence every point in B has its large set its set of arrow neighbours and has no line neighbours in T. So there are no lines between B and T. This is of course also true for the following four cases.

There is a line ab within B; there are $\binom{|B|}{2}$ ways to choose this line. Then L_a and L_b both have size at least $\frac{n}{10}$, and do not intersect. Hence $L_a \times L_b$ is a set of size at least $\frac{n^2}{100}$ within T, and any line in it would cause the forbidden structure (3) to exist. So we have

$$D(B,P,S) < \binom{|B|}{2} 2^{|B||T|} 2^{\binom{|T|}{2} - \frac{n^2}{100}} 2^{\binom{|B|}{2}} |B|! < 2^{\binom{n}{2} - n^{0.99} - n^{\frac{5}{4}}}$$

for sufficiently large n.

In this case (6.1) is bounded above by $2^{\binom{n}{2} - n^{\frac{5}{4}}}$. △

Case (vi):

Since B contains no lines the diagram structure within B is simply a partial order. Since $|B| \geq 5\sqrt{n}$, whenever $n > J^2$ we can say that there are at most $2^{\frac{3|B|^2 - 5|B|}{10}} = 2^{\binom{|B|}{2} - \frac{|B|^2}{5}}$ ways to choose arrows within B. Then given B, P, S we can construct diagrams in this case by choosing the arrows from $I = B$ to T, the partial order within B, and the lines within T. For sufficiently large n, this gives us the bound

$$D(B,P,S) < 2^{|B||T|} 2^{\binom{|B|}{2} - \frac{|B|^2}{5}} 2^{\binom{|T|}{2}} < 2^{\binom{n}{2} - 5n} < 2^{\binom{n}{2} - n - n^{0.99}}$$

so that in this case (6.1) is bounded above by $2^{\binom{n}{2} - n}$. △

Case (vii):

There are four possibilities for the two arrows that are guaranteed to exist in B: either $a < b < c$, or $a < b, a < c$, or $a > b, a > c$, or $a < b, c < d$. Observe that if $a < b$, then $\Gamma_{arr}(b) \subset \Gamma_{arr}(a)$, so that in each of these four cases we have restrictions on the choices of arrows going from these points upwards; $4^{|T|}$, $5^{|T|}$, $5^{|T|}$ and $9^{|T|}$ choices, respectively. This means that the choice for arrows between B and T is $2^{|B||T|-|T|}$, $2^{|B||T|-|T|\log\frac{8}{5}}$, $2^{|B||T|-|T|\log\frac{8}{5}}$, $2^{|B||T|-|T|\log\frac{16}{9}}$ respectively; so for sufficiently large n we get

$$D(B,P,S) < 4.2^{\binom{n}{2}-|T|\log\frac{8}{5}}|B|! < 2^{\binom{n}{2}-\frac{3n}{5}-n^{0.99}}$$

so that in this case (6.1) is bounded above by $2^{\binom{n}{2}-\frac{3n}{5}}$. $\triangle$

Case (viii):

Suppose the arrow in B is $a<b$. Then $\Gamma_{arr}(b)\subset\Gamma_{arr}(a)$, so that there are only $3^{|T|}$ choices for the arrow neighbours of a, b in T. Then we obtain the bound

$$D(B,P,S) < 2\binom{|B|}{2}2^{\binom{|T|}{2}+(|B|-2)|T|}3^{|T|} < 2^{\binom{n}{2}-\frac{7n}{20}-n^{0.99}}$$

for sufficiently large n, so that here (6.1) is bounded above by $2^{\binom{n}{2}-\frac{7n}{20}}$. $\triangle$

Case (ix):

We bound directly the number of valid diagrams in this case: we choose B, lines within T and arrows from B to T. This gives us the bound

$$\sum_{|B|<5\sqrt{n}}\binom{n}{|B|}2^{\binom{n}{2}-\binom{|B|}{2}} < 2^{\binom{n}{2}+\frac{3\log^2 n}{4}}$$

for sufficiently large n, since the largest term in the above sum occurs when $\frac{3\log n}{4}\le|B|\le\log n$, so that the largest term is at most $2^{\binom{n}{2}+\log^2 n-\frac{9\log^2 n}{32}}$.$\triangle$

Adding up the bounds on each case, we see that, for all sufficiently large n, Case (ix) dominates and there are at most $2^{\binom{n}{2}+\log^2 n}$ valid diagrams on n points with $|B|\le\frac{n}{100}$. Adding up the bounds on Cases (i)-(viii), we see that, for all sufficiently large n, Case (iv) dominates and there are at most $2^{\binom{n}{2}-\frac{n}{23}}$ valid diagrams on n points with $|B|\le\frac{n}{100}$ which do not correspond to unate 2-SAT functions. $\square$

6.2.5 The small case

In this subsection, we apply (a version of) the Szemerédi Regularity Lemma to restrict the possible partial orders, in much the same way as it is applied in [13], then use an induction argument to show that there really are very few valid diagrams with $|B| > \frac{n}{100}$. We will need a theorem of Füredi.

Theorem 6.10. *Let G be a graph on n vertices. Then the proper square of G, the graph $G^{(2)}$ on $V(G)$ with $ab \in E(G^{(2)})$ if and only if $ac, bc \in E(G)$ for some c, has at least $|E(G)| - \lfloor \frac{n}{2} \rfloor$ edges.*

A proof of this is found in Füredi [36]; see also [13].

Lemma 6.11. *When $|B| > \frac{n}{100}$, for any fixed δ, there exists N such that for all $n > N$, there are at most $2^{\binom{n}{2} - 2n^{\frac{5}{4}}}$ valid diagrams containing a point x such that $|\{y : y < x\}| > \delta n$.*

Proof. Note that if there is a point with δn points below it, then there is a point in T with δn points below it.

There are two cases to consider:

(1) $|T| < \frac{\delta^2}{5} n$.

(2) $|T| \geq \frac{\delta^2}{5} n$.

Case (1): We count the number of diagrams with $|T| < \frac{\delta^2}{5} n$ and a point in T having at least δn points below it as follows. Choose a top set. Choose lines within the top set, and arrows and lines from the top set to the bottom set such that there exists a point x in the top set above at least δn points in the bottom set. Let C be some set of δn points below x. Now choose arrows and lines within $B^{(2)} - C^{(2)}$. Within $C^{(2)}$ we can choose only arrows, so that the structure on C is simply a partial order. This gives an upper bound on the number of diagrams in this case:

$$\sum_{|T|}\binom{n}{|T|}2^{\binom{|T|}{2}}3^{|B||T|}2^{\binom{|B|}{2}-\binom{\delta n}{2}}2^{\frac{3\delta^2 n^2-5\delta n}{10}}|B|!$$

$$< 2^{\binom{n}{2}}2^{n^2\frac{\delta^2}{5}\log\frac{3}{2}-\frac{\delta^2 n^2}{5}+n\log n+n+\log n}$$

$$< 2^{\binom{n}{2}-3n^{\frac{5}{4}}}$$

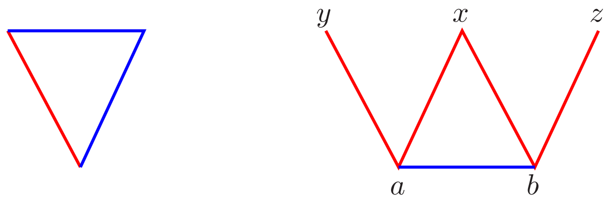
for all n sufficiently large that both the above approximations hold and $\delta n > J$.

Case (2): Since both $|T| \geq \frac{\delta^2}{5}n, |B| \geq 10^{-2}n$, we will be able to apply a version of the Szemerédi Regularity Lemma to count the number of valid diagrams in this case as follows.

Given a valid diagram, draw a coloured graph G on n vertices corresponding to the points of the diagram as follows. Whenever a point in B is below a point in T, connect them with a red edge in the coloured graph. Whenever two points are connected by lines, or two points in B are connected by arrows, connect them with a blue edge in the coloured graph.

Observe that if two vertices in T are connected by a blue edge, so no vertex in B is connected to one by a red edge and to the other by a blue edge since this would be the forbidden structure (2) (see Figure 6.4).

If two vertices a, b in B are connected by a blue edge, so there cannot be three vertices in T, x, y, z, with xa, xb red edges, ya a red edge but yb not (it does not matter whether yb is a blue edge or not an edge at all), zb a red edge but za not (again za could be a blue edge or not an edge at all). This is because if the edge ab corresponds to a line, then a, b, x corresponds to the forbidden structure (1), while if it corresponds to an arrow from a to b there would also have to be an arrow from a to z since the partial order is transitive, and vice versa. Since this coloured graph encodes all the information contained in the original diagram except the choice of T and the order on B, there are at most $2^n n!$ times as many valid diagrams in this case as coloured graphs, with the red edges forming a bipartite graph, not containing either of these two structures.



(α) (β)

Figure 6.4 Forbidden coloured subgraphs

Suppose $0 < \varepsilon < \frac{1}{1000}$. If $A, B \subset V(G)$, then let the *red-density* of the pair A, B, $d_r(A, B)$, be the number of red edges between A and B divided by $|A||B|$, and define the *blue-density* $d_b(A, B)$ similarly. Call such a pair *ε-uniform* if for every $A' \subset A, B' \subset B$ with $|A'| \geq \varepsilon|A|, |B'| \geq \varepsilon|B|$, so $|d_b(A, B) - d_b(A', B')| < \varepsilon$ and $|d_r(A, B) - d_r(A', B')| < \varepsilon$.

Let $V(G)$ be partitioned into m sets $X_1, \ldots, X_t, Y_1, \ldots, Y_{m-t}$ where all the X_i are subsets of T and all the Y_i are subsets of B. Suppose that each part has size $q > (1 - 2\varepsilon)\frac{n}{m}$, except for X_1 and Y_1 which each have size at most εn. Call a pair (X_i, Y_j) *rich* if it is ε-uniform, $d_r(X_i, Y_j) > 2\varepsilon^{\frac{1}{3}}$ and $d_b(X_i, Y_j) > 2\varepsilon^{\frac{1}{3}}$. Call any pair which is not ε-uniform *bad*, and any ε-uniform pair with both $d_r(U, V) \leq 2\varepsilon^{\frac{1}{3}}$ and $d_b(U, V) \leq 2\varepsilon^{\frac{1}{3}}$ *sparse*. Call the remaining pairs *normal*.

The following two constructions are standard arguments (see for example §1.5.3 or [13]) whose proof we omit.

Suppose that (X_i, X_j) is a normal pair, and there exists Y_k such that both (X_i, Y_k) and (X_j, Y_k) are rich. Then there is a vertex y in Y_k connected to a vertex a in X_i by a red edge and to a vertex b in X_j by a blue edge, with ab a blue edge. This is the forbidden coloured graph (α), so that if Y_k is such that (X_i, Y_k), (X_j, Y_k) are rich pairs then (X_i, X_j) must be either a sparse or a bad pair.

Suppose that (Y_i, Y_j) is a normal pair, so $d_b(Y_i, Y_j) \geq 2\varepsilon^{\frac{1}{3}}$, and there exists X_k such that both $(X_k, Y_i), (X_k, Y_j)$ are rich. Then we can find vertices $a \in Y_i, b \in Y_j, x, y, z \in X_k$ forming the forbidden coloured graph (β). Again, if X_k is such that (X_k, Y_i) and (X_k, Y_j) are rich pairs, then (Y_i, Y_j) must be either a sparse or a bad pair.

Now draw a graph H on m vertices corresponding to the parts of $V(G)$. Draw an edge between two vertices in $V(H)$ if and only if they correspond to a rich pair. Then H is bipartite with the X_i making one part and the Y_j the other. Hence the graph $H^{(2)}$ has edges only between vertices X_i, X_j or Y_i, Y_j. No edge of $H^{(2)}$ can correspond to a normal pair in G, so all these edges correspond to bad or sparse pairs in G. We now apply the theorem of Füredi (as in [13]): if there are r edges in H, there are at least $r - \lfloor \frac{m}{2} \rfloor$ edges in $H^{(2)}$, hence at least that many bad or sparse pairs must be in G.

But it follows from the Szemerédi Regularity Lemma on 2-coloured graphs (proof following the usual method as in e.g. Bollobás [10]) that in fact every 2-coloured graph has such a partition, for some $\frac{1}{\varepsilon} < m < K$, where K depends on ε but not on n, with all the parts of size $q \leq \frac{n}{m}$ except for X_1 and Y_1 which have size at most εn, and with at most εm^2 bad pairs. We will find that choosing ε small enough that $62\varepsilon^{\frac{1}{3}} \log \frac{e}{2\varepsilon^{\frac{1}{3}}} + 116\varepsilon < \frac{\delta^2}{10}$ will work.

Consider the number of possibilities for a coloured graph not containing either of the structures (α) or (β). We must choose T, and the parts $X_1, \ldots, X_t, Y_1, \ldots, Y_{m-t}$. We must choose which pairs are to be rich, sparse, bad and normal. We must allow 3^{q^2} possibilities for the edges within every rich or bad pair. We must allow $3^{2n\varepsilon n}$ possibilities for the edges with one end in either X_1 or Y_1. We must allow $3^{m\frac{q^2}{2}}$ possibilities for the edges within parts. Let $\eta = 2\varepsilon^{\frac{1}{3}} \log \frac{e}{2\varepsilon^{\frac{1}{3}}}$, then within normal pairs there are at most $2^{q^2} 2\binom{q^2}{2\varepsilon^{\frac{1}{3}} q^2} < 2^{q^2+1+\eta q^2}$ possibilities for the edges, and within sparse pairs there are at most $\left(2\binom{q^2}{2\varepsilon^{\frac{1}{3}} q^2}\right)^2 < 2^{2+2\eta q^2}$ possibilities. There are r rich pairs, and at most εm^2 bad pairs, hence there are $s \geq r - \varepsilon m^2 - \lfloor \frac{m}{2} \rfloor$ sparse pairs. We divide this into two cases and evaluate the number of valid diagrams corresponding to coloured graphs in each case:

First, if $r \geq 5(3\eta + 11\varepsilon)m^2$.

We can count the possible graphs in this case simply by enumerating all the possibilities to obtain an upper bound:

$$\sum_{|T|,m,r}\binom{n}{|T|}m^n4^{m^2}3^{q^2(r+\varepsilon m^2+\frac{m}{2})+2\varepsilon n^2}2^{(q^2+1+\eta q^2)(\frac{m^2}{2}-r-s)+(2+2\eta q^2)s}$$

$$< \sum_{|T|,m,r}2^nm^n4^{m^2}\left(\frac{3}{4}\right)^{rq^2}3^{3\varepsilon n^2+\frac{\varepsilon n^2}{2}}2^{(1+\eta q^2)(\frac{m^2}{2}-r-s)+\frac{n^2}{2}+\varepsilon n^2+\lfloor\frac{m}{2}\rfloor q^2+(2+2\eta q^2)s}$$

$$< \sum_{|T|,m,r}2^nm^n4^{m^2}\left(\frac{3}{4}\right)^{rq^2}2^{\frac{n^2}{2}}3^{3\varepsilon n^2+\frac{\varepsilon n^2}{2}}2^{m^2+\eta n^2+\varepsilon n^2+\frac{\varepsilon n^2}{2}+2m^2+2\eta n^2}$$

$$< nK^32^nK^n4^{K^2}2^{\frac{n^2}{2}}2^{-(3\eta+11\varepsilon)n^2}2^{3K^2}2^{(\frac{7}{2}\varepsilon\log 3+3\eta+\frac{3}{2}\varepsilon)n^2}$$

$$< 2^{\frac{n^2}{2}}2^{\log n+n+(n+3)\log K+5K^2-\varepsilon n^2}$$

$$< 2^{\binom{n}{2}-n-n\log n-3n^{\frac{5}{4}}}$$

for all sufficiently large n.

It follows that for sufficiently large n, at most $2^{\binom{n}{2}-3n^{\frac{5}{4}}}$ valid diagrams give rise to these coloured graphs.

Second, suppose $r < 5(3\eta+11\varepsilon)m^2$.

Here we do not count graphs directly: instead we count diagrams, using the information we now have about the arrows and lines between B and T to obtain an upper bound. We still need to allow for choice of T, r, m, the partition, which pairs are to be rich, sparse, bad and normal, and choices of arrows and lines between B and T corresponding to rich, bad and normal/sparse pairs. But now we can count the choices within T as $2^{\binom{|T|}{2}}$, and within B as $2^{\binom{|B|}{2}-\binom{\delta n}{2}}2^{\frac{3\delta^2n^2-5\delta n}{10}}|B|!$ (since by assumption there is a point in T above $\delta n > J$ points in B). Let the number of valid diagrams on n points which correspond to coloured graphs with less than $5(3\eta+11\varepsilon)m^2$ rich pairs in a Szemerédi partition be $D_g(n)$, then $D_g(n)$ is at most

$$\sum_{|T|,m,r}\binom{n}{|T|}m^n4^{m^2}2^{\binom{|T|}{2}}3^{rq^2+3\varepsilon n^2}2^{(q^2+1+\eta q^2)(t-1)(m-t-1)}2^{\binom{|B|}{2}-\binom{\delta n}{2}}2^{\frac{3\delta^2n^2-5\delta n}{10}}|B|!$$

since there are at most $(t-1)(m-t-1)$ pairs which are normal or sparse between B and T, and the given upper bound for the number of ways to choose a normal pair is also an upper bound for the number of ways to choose a normal or sparse pair. Simplifying:

$$D_g(n) < \sum_{|T|,m,r} 2^n m^n 4^{m^2} 2^{\binom{|T|}{2}} 3^{rq^2+3\varepsilon n^2} 2^{(q^2+1+\eta q^2)(t-1)(m-t-1)} 2^{\binom{|B|}{2}-\frac{\delta^2 n^2}{5}} |B|!$$

$$< nK^3 2^n K^n 4^{K^2} 2^{\binom{n}{2}} 2^{5(3\eta+11\varepsilon)n^2 \log 3+3\varepsilon n^2 \log 3+K^2+\eta n^2-\frac{\delta^2 n^2}{5}} n!\ .$$

Since $q^2(t-1)(m-t-1) \leq |T||B|$, so

$$D_g(n) < nK^3 2^n K^n 2^{3K^2} 2^{\binom{n}{2}} 2^{(31\eta+116\varepsilon)n^2-\frac{\delta^2 n^2}{5}} n!$$

$$< nK^3 2^n K^n 2^{3K^2} 2^{\binom{n}{2}} 2^{-\frac{\delta^2 n^2}{10}} n!\ .$$

Again this is less than $2^{\binom{n}{2}-3n^{\frac{5}{4}}}$ for all n sufficiently large that both the above approximations hold and $\delta n > J$.

So for all sufficiently large n, there are at most $3.2^{\binom{n}{2}-3n^{\frac{5}{4}}} < 2^{\binom{n}{2}-2n^{\frac{5}{4}}}$ valid diagrams with $|B| > \frac{n}{100}$ and with a point above more than δn others. $\square$

Suppose $\delta = 10^{-9}$. Then observe that, if $n > 10^8$, the number of partial orders where no point has more than δn points below it is at most

$$\left(2\binom{n}{\delta n}\right)^n < 2^n \left(\frac{10^9 en}{n}\right)^{10^{-9}n^2} < 2^{\frac{40}{10^9}n^2+n} < 2^{\frac{n^2}{2\cdot 10^7}}\ .$$

This will be all we need now to prove the following theorem.

Theorem 6.12. *For all sufficiently large n, there are at most $2^{\binom{n}{2}-n^{\frac{5}{4}}}$ valid diagrams on n points with $|B| > \frac{n}{100}$.*

Proof. Let $\delta = 10^{-9}$. Let $N > 10^{60}$ be large enough that both the conclusion of the previous lemma holds with $\delta = 10^{-9}$ and that for all $k > \frac{N}{2}$, we have at most $2^{\binom{k}{2}-2k^{\frac{5}{4}}}$ valid diagrams on k points with $3k^{\frac{19}{20}} < |B| \leq \frac{k}{100}$, as provided by Theorem 6.9. Let $F_s(n)$ be the number of valid diagrams on n points with $|B| > \frac{n}{100}$.

Let $A = F_s(N)$. We will prove by induction that for every n, $F_s(n) < A2^{\binom{n}{2}-n-n^{\frac{5}{4}}}$. When $n \leq N$, this is trivially true by the choice of A. Now suppose that $n > N$, and the induction hypothesis holds for every $k < n$.

Given a valid diagram D on n points with $|B| > \frac{n}{100}$, we can let $P \subset T$ be the set of points each of which is connected to at most $\frac{|B|}{200}$ points in B by lines. Then we can apply Lemma 6.8, with $I = T - P$ and $V = B$. For each $i \in I$, we let S_i be the set of points below i, and L_i be the points in B connected to i by lines (see Figure 5). Then $l \geq \frac{|B|}{200}$, and by Lemma 6.8 we have a forbidden set of size at least $\frac{|B||S|}{400}$.

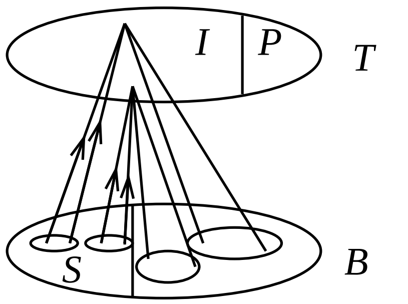


Figure 6.5 A typical valid diagram with $|B| > \frac{n}{100}$

We consider three cases:

(a) diagrams such that there is a point above at least δn others,

(b) diagrams such that there is no point above δn others, and $|S| > \frac{|B|}{2}$,

(c) diagrams such that there is no point above δn others, and $|S| \leq \frac{|B|}{2}$.

Case (a): By the previous lemma, there are at most $2^{\binom{n}{2}-2n^{\frac{5}{4}}}$ such valid diagrams.

Case (b): We count the number of valid diagrams by choosing the partial order within the diagram, then choosing the lines within the top set (now fixed by choice of the partial order) and between the top and bottom sets (fixing P and S); then we choose the lines within B, observing that there are at most $\binom{|B|}{2} - \frac{|B||S|}{400} \leq \binom{|B|}{2} - \frac{|B|^2}{800} \leq \binom{|B|}{2} - \frac{n^2}{8.10^6}$ places where we can choose to put lines within B, to obtain an upper bound:

$$2^{\frac{n^2}{2.10^7}} 2^{\binom{|T|}{2}+|B||T|+\binom{|B|}{2}-\frac{|B||S|}{400}} < 2^{\binom{n}{2}-\frac{n^2}{2.10^7}} < 2^{\binom{n}{2}-2n^{\frac{5}{4}}}$$

since $n > N > 10^{40}$.

Case (c): Since $|S| \leq \frac{|B|}{2}$, $|B - S| \geq \frac{|B|}{2}$. Every point in $B - S$ is under some point in T, and no point in $B - S$ is under any point in I, hence every point in $B - S$ is under some point in P. But no point in P can be above more than δn points, so that $|P| \geq \frac{|B|}{2\delta n} > \frac{1}{200\delta}$. Let $P' \subset P$ be a set with $|P'| = \frac{1}{200\delta}$. Now observe that the diagram D' on the $k = n - \frac{1}{200\delta} < n$ points $T \cup B - P'$, with $a < b$ in D' if and only if $a < b$ in D, and ab a line in D' if and only if ab is a line in D, must be a valid diagram.

Since $|B| > \frac{n}{100}$, and the points P' can be above at most $|P'|\delta n$ points of the top set of D', the diagram D' must have bottom set of size at least $\frac{n}{100} - |P'|\delta n = \frac{n}{200} > \frac{k}{200}$. Since $\frac{k}{200} > 3k^{\frac{19}{20}}$, either D' is a valid diagram on $k > \frac{n}{2} > \frac{N}{2}$ points with the size of its bottom set in $(3k^{\frac{19}{20}}, \frac{k}{100}]$, or D' is a valid diagram on $k < n$ points with bottom set larger than $\frac{k}{100}$. There are at most $2^{\binom{k}{2} - 2k^{\frac{5}{4}}}$ possible diagrams in the first case by Theorem 6.9, and by the induction hypothesis there are at most $A2^{\binom{k}{2} - k - k^{\frac{5}{4}}}$ possible diagrams in the second case. So there are at most $2^{\binom{k}{2} - 2k^{\frac{5}{4}}} + A2^{\binom{k}{2} - k - k^{\frac{5}{4}}} < (A+1)2^{\binom{k}{2} - k^{\frac{5}{4}}}$ possibilities for D'. Now the following construction includes every diagram in this case.

Choose a top set T, with $\frac{1}{200\delta} \leq |T| \leq \frac{99n}{100}$. Choose a set P' of $\frac{1}{200\delta}$ points in T. Choose lines within P' and from P' to $T - P'$. Choose at most δn points in B to be below each point in P'. Choose at most $\frac{|B|}{200}$ lines going from each point in P' to B. Choose any valid diagram with sufficiently large bottom set on $B \cup T - P'$. Hence an upper bound for the number of valid diagrams in this case is:

$$\sum_{|T|} \binom{n}{|T|} \binom{|T|}{\frac{1}{200\delta}} 2^{\binom{\frac{1}{200\delta}}{2} + \frac{1}{200\delta}\left(|T| - \frac{1}{200\delta}\right)} \left(2\binom{|B|}{\delta n} 2\binom{|B|}{\frac{|B|}{200}} \right)^{\frac{1}{200\delta}} (A+1)2^{\binom{k}{2} - k^{\frac{5}{4}}}$$

$$< n2^{2n}2^{\binom{n}{2}}2^{-\frac{|B|}{200\delta}}2^{\frac{1}{100\delta}} \left(\frac{200e|B|}{|B|}\right)^{\frac{|B|}{20000\delta}} (A+1)2^{-k^{\frac{5}{4}}}$$

$$< A2^{\binom{n}{2}}2^{1 + \frac{1}{100\delta} + \log n + 2n + \frac{|B|\log(200e)}{20000\delta} - \frac{|B|}{200\delta} - k^{\frac{5}{4}}}$$

$$< A2^{\binom{n}{2}}2^{1 + \frac{1}{100\delta} + \log n + 2n - \frac{|B|}{400\delta} - k^{\frac{5}{4}}}$$

$$< A2^{\binom{n}{2}}2^{1 + \frac{1}{100\delta} + \log n + 2n - \frac{n}{40000\delta} - n^{\frac{5}{4}} + \left(n^{\frac{5}{4}} - k^{\frac{5}{4}}\right)}$$

$$< A2^{\binom{n}{2}-2n-n^{\frac{5}{4}}}2^{1+\frac{1}{100\delta}+\log n+4n+\frac{5}{800\delta}n^{\frac{1}{4}}-\frac{n}{40000\delta}}$$
$$< A2^{\binom{n}{2}-2n-n^{\frac{5}{4}}}$$

since $n > 10^{60}$ and $\delta = 10^{-9}$.

Then there are at most $A2^{\binom{n}{2}-2n-n^{\frac{5}{4}}} + 2.2^{\binom{n}{2}-2n^{\frac{5}{4}}} < A2^{\binom{n}{2}-n-n^{\frac{5}{4}}}$ valid diagrams with $|B| > \frac{n}{100}$. Therefore the induction hypothesis holds for n.

By induction, $F_s(n) < A2^{\binom{n}{2}-n-n^{\frac{5}{4}}}$ for all n. If n is sufficiently large that $2^n > A$, then we have $F_s(n) < 2^{\binom{n}{2}-n^{\frac{5}{4}}}$, which completes the proof. □

6.2.6 The upper bound

Collecting results, from Theorem 6.9 we have that there are at most $2^{\binom{n}{2}-\frac{n}{23}}$ valid diagrams on n points with $|B| \leq \frac{n}{100}$ that do not correspond to unate 2-SAT functions, for all sufficiently large n. From Theorem 6.12, we have that there are at most $2^{\binom{n}{2}-n^{\frac{5}{4}}}$ valid diagrams on n points with $|B| > \frac{n}{100}$, for all sufficiently large n.

Since each nonnegative 2-SAT function on n variables can be obtained by applying M to at most 2^n elementary functions on n variables, we have that there are at most $2^n\left(2^{\binom{n}{2}-\frac{n}{23}} + 2^{\binom{n}{2}-n^{\frac{5}{4}}}\right)$ elementary 2-SAT functions which are not unate, for all sufficiently large n.

There are at most $2^{\binom{n}{2}+n}$ unate 2-SAT functions. So we obtain an upper bound for $H(n)$ valid for all sufficiently large n:

$$H(n) < 2^{\binom{n}{2}+n}\left(1 + 2^{-\frac{n}{23}} + 2^{-n^{\frac{5}{4}}}\right) < 2^{\binom{n}{2}+n}\left(1 + 2^{-\frac{n}{24}}\right)$$

for all sufficiently large n.

6.3 Error terms

Of course, not every 2-SAT function is unate. But we can, with not much more work, discover the next largest class of 2-SAT functions, which gives us the size of the first error term in the formula $G(n) = \big(1 + o(1)\big)2^{\binom{n}{2}+n}$. Our method is even powerful enough to bound the size of further error terms.

6.3.1 The first error term

It is well known that almost every triangle-free graph is bipartite (see Erdős, Kleitman and Rothschild [31]). Prömel, Schickinger and Steger [63] have shown that almost every triangle-free graph which is not bipartite can be made bipartite by removing just one vertex. We will now prove the equivalent result for our problem.

Let $W(n)$ be the set of 2-SAT functions which are not unate and which are given by taking a monotone function and changing exactly one positive literal x in one clause of its formula to the corresponding negative literal $\overline{x}$. Let $V(n)$ be the set of 2-SAT functions given by relabelling variables on the elements of $W(n)$.

Note that applying the above process to a monotone function results in a unate function if and only if x is mentioned just once in the formula for the monotone function.

We can easily find the size of $W(n)$. Observe that all elements of $W(n)$ are nonnegative; then they are in 1-1 correspondence with the diagrams on n points which have $|T| = n-1$, and $x \in B$ is linked to exactly one element $y \in T$ by an arrow, and to at least one other element of T by a line. As structure (2) is forbidden, for every other $z \in T$ there cannot be lines from z to both y and x. Therefore there are at most $n(n-1)2^{\binom{n-2}{2}}3^{n-2}$ such valid diagrams. There are at least

$$n(n-1)\left(2^{\binom{n-2}{2}} - n^2 2^{\binom{n-3}{2}}\right)\left(3^{n-2} - 2\left(\binom{n-2}{0} + \binom{n-2}{1}\right)2^{n-2}\right)$$

such valid diagrams in which every point has at least two lines connected to it.

Therefore

$$\begin{aligned}\binom{n}{2}2^{\binom{n}{2}}2^{-2(n-2)}\left(1-o(2^{-\frac{n}{2}})\right)3^{n-2}\left(1-o(2^{-\frac{n}{2}})\right) &< |W(n)| \\ &< \binom{n}{2}2^{\binom{n}{2}}2^{-2(n-2)}3^{n-2}\end{aligned}$$

so

$$\binom{n}{2}2^{\binom{n}{2}}\left(\frac{3}{4}\right)^{n-2}\left(1-o(2^{-\frac{n}{2}})\right) < |W(n)| < \binom{n}{2}2^{\binom{n}{2}}\left(\frac{3}{4}\right)^{n-2}$$

for sufficiently large n.

Now $|V(n)| \leq 2^n|W(n)|$, but also if R, R' are subsets of the n variables and w, w' are elements of $W(n)$ corresponding to diagrams in which every point has at least two lines connected to it, then the functions given by relabelling the variables R on w and R' on w' are distinct unless $R = R'$, $w = w'$. Thus

$$\binom{n}{2} 2^{\binom{n}{2}+n} \left(\frac{3}{4}\right)^{n-2} \left(1 - o(2^{-\frac{n}{2}})\right) < |V(n)| < \binom{n}{2} 2^{\binom{n}{2}+n} \left(\frac{3}{4}\right)^{n-2} .$$

Now we improve the bounds from Theorem 6.9 to show that $V(n)$ really is the next largest class of 2-SAT functions after the unate functions. The following proof is essentially a more precise, but much longer, replacement for the argument in Case (iv) of Theorem 6.9; it was left to this point to make that theorem more easily understood.

We observe that three types of diagram correspond to functions in $V(n)$:

(p) the diagrams in which B contains no lines, there is a point x in B which is connected to exactly one point in T by an arrow and to at least one other point in T by a line, any arrows in B go to x, and all other points in B are connected to points in T by arrows only,

(q) the diagrams in which B contains no arrows or lines, there is a point x in B which is connected to exactly one point in T by a line and to at least one other point in T by an arrow, and all other points in B are connected to points in T by arrows only,

(r) the diagrams in which B contains exactly one arrow and no lines, and there are no lines between B and T.

Theorem 6.13. *For all sufficiently large n,*

$$1 + \binom{n}{2}\left(\frac{3}{4}\right)^{n-2} - 2^{-\frac{n}{2}} < \frac{G(n)}{2^{\binom{n}{2}+n}} < 1 + \binom{n}{2}\left(\frac{3}{4}\right)^{n-2} + 2^{-\frac{418n}{1000}} .$$

Proof. We find that the bound in Theorem 6.12 is already good enough, as are the bounds in Theorem 6.9 Cases (i), (ii), (iii), (v), (vi), (vii). We observe that the diagrams in Case (viii) already correspond to 2-SAT functions in $V(n)$ (in form (r)), and that those in Case (ix) correspond to the unate functions, which we have already enumerated with sufficient accuracy. Hence we only need to improve the bounds given in Case (iv) of Theorem 6.9.

Case (iv) of Theorem 6.9 covered diagrams such that $|P| < 84n^{\frac{1}{4}}$, $|S| < 100n^{\frac{1}{4}}$, $|B| < 3n^{\frac{19}{20}}$ and at least one of $|P| > 0$, $|S| > 0$ holds.

We now split this into several cases and analyse each.

(a) $|P| < 84n^{\frac{1}{4}}$, $1 < |S| < 100n^{\frac{1}{4}}$, $|B| < 3n^{\frac{19}{20}}$.

(b) $|S| \leq 1$, $0 < |P| < 84n^{\frac{1}{4}}$ and $|B| < 3n^{\frac{19}{20}}$.

(c) $|S| = 1$, $|P| = 0$, $|B| < 3n^{\frac{19}{20}}$ and there are at least two points in B with non-empty small set.

(d) $|S| = 1$, $|P| = 0$, $|B| < 3n^{\frac{19}{20}}$, only one point $b \in B$ has $|S_b| = 1$ and there is a line within B.

(e) $|S| = 1$, $|P| = 0$, $5\sqrt{n} \leq |B| < 3n^{\frac{19}{20}}$, only one point $b \in B$ has $|S_b| = 1$ and there is no line within B.

(f) $|S| = 1$, $|P| = 0$, $|B| < 5\sqrt{n}$, only one point $b \in B$ has $|S_b| = 1$, there is no line within B but there is an arrow within B.

(g) $|S| = 1$, $|P| = 0$, $|B| < 5\sqrt{n}$, only one point $b \in B$ has $|S_b| = 1$ and there are no lines or arrows within B.

As in the later parts of Theorem 6.9, we will use the bounds $|I| \log \frac{3}{2} < \frac{n}{1000}$, $n^3 \binom{n}{|B|}\binom{|B|}{|P|}\binom{|T|}{|S|} < 2^{n^{0.99}}$ and $2^{|I|}|B|! < 2^{\frac{n}{1000}}$ in the following cases.

Case (a):

We divide this into two sub-cases.

Subcase (1): $|S| > 40$.

We use (6.2) from Theorem 6.9 which gives us

$$D(B, P, S) < 2^{\binom{n}{2}} 2^{-\frac{|P|n}{21}} 2^{-\frac{40n}{21}} < 2^{\binom{n}{2} - n - n^{0.99}}$$

for all sufficiently large n, so that the sum (6.1) from Theorem 6.9 is bounded above by $2^{\binom{n}{2} - n}$.

Subcase (2): $2 \leq |S| \leq 40$.

We observe that as there are at least two points in S, there are at most $\binom{40}{2}$ pairs in $T^{(2)}$ with both points in S, and the forbidden set has size at least $2l - 800$, where l is the minimum of the $|L_i|$ for $i \in I$. We can construct diagrams in this subcase, given B, P, S, by choosing the lines and arrows from the points in P to T, choosing for each point in I lines and arrows to S and lines to T, using the fact that there are at most $\binom{|T|}{l}$ ways to choose lines from the point i with $|L_i| = l$ ($|I|$ ways to choose i), choosing lines and arrows within B and finally lines within T, taking account of the forbidden set. This gives us

$$D(B,P,S) < 2^{|P||T|}2^{-\frac{|P|n}{21}}3^{|S||I|}|I|2^{(|I|-1)|T|}\binom{|T|}{l}2^{\binom{|B|}{2}}|B|!2^{\binom{|T|}{2}-2l+800}$$

$$< 2^{\binom{n}{2}}2^{-\frac{|P|n}{21}}2^{40|I|\log\frac{3}{2}}2^{-|T|}\binom{|T|}{l}2^{-2l+800}|B|!\ .$$

Now $2^{40|I|\log\frac{3}{2}}2^{800}|B|! < 2^{n^{0.99}}$ for all sufficiently large n, so

$$D(B,P,S) < 2^{\binom{n}{2}}2^{-2l-\frac{99n}{100}}\binom{n}{l}2^{n^{0.99}} < 2^{\binom{n}{2}-\frac{n}{2}-n^{0.99}}$$

for sufficiently large n, and the sum (6.1) is bounded above by $2^{\binom{n}{2}-\frac{n}{2}}$. $\triangle$

Case (b):

We divide this into two subcases.

Subcase (1): There is a point $p \in P$ with $|S_p| > \sqrt{n}$.

Given B, P, S, we can construct the valid diagrams in this case by choosing the lines and arrows within B and between B and T, and then choosing the lines within T. But since $|L_p| > |S_p|$ there are at least n pairs in $T^{(2)}$ which cannot be chosen as lines, and we can bound above the number of possibilities by

$$D(B,P,S) < 2^{\binom{|B|}{2}}|B|!2^{|P||T|}2^{-\frac{|P|n}{21}}2^{|I|}2^{|I||T-S|}3^{|I||S|}2^{\binom{|T|}{2}-n}$$

$$< 2^{\binom{n}{2}-n+(1+\log\frac{3}{2})|I|}|B|! < 2^{\binom{n}{2}-\frac{n}{2}-n^{0.99}}$$

for all sufficiently large n, so that in this subcase (6.1) is bounded above by $2^{\binom{n}{2}-\frac{n}{2}}$.

Subcase (2): Every point $p \in P$ has $|S_p| \leq \sqrt{n}$.

Now instead of there being $\left(2\binom{|T|}{\frac{n}{10}}\right)^2$ ways to choose the arrows and lines from each point in P, there are only $4\binom{|T|}{\frac{n}{10}}\binom{|T|}{\sqrt{n}} < 2^{\frac{n}{2}}$ ways, for sufficiently large n.

Hence we can bound above the number of possibilities by

$$\begin{aligned} D(B,P,S) &< 2^{\binom{|B|}{2}}|B|!2^{|P||T|}2^{-\frac{49|P|n}{100}}2^{|I|}2^{|I||T-S|}3^{|I||S|}2^{\binom{|T|}{2}} \\ &< 2^{\binom{n}{2}-\frac{97|P|n}{200}} < 2^{\binom{n}{2}-\frac{48n}{100}-n^{0.99}} \end{aligned}$$

for sufficiently large n, so that in this subcase (6.1) is bounded above by $2^{\binom{n}{2}-\frac{48n}{100}}$. △

Case (c):

Let the two points in B with non-empty small set be a, b. Then the forbidden set has size at least $l = \max(|L_a|, |L_b|)$. Observe that there are at most $l\binom{|T|}{l}$ ways to choose the large sets of each of a, b, and that

$$\left(l\binom{n}{l}\right)^2 2^{-l}$$

is maximised at $l = (\sqrt{2}-1)n$. We construct the diagrams in this case by choosing B, l, S, the two points with non-empty small set, whether the large set of each point in B will be its arrow neighbours or line neighbours, the small sets of all the points in B, then we choose the large sets of both the points with non-empty small set, the large sets of the rest of the points in B, the lines and arrows within B and finally the lines within T. This allows us to bound (6.1) in this case by

$$\sum_{|B|,l}\binom{n}{|B|}\binom{|T|}{1}\binom{|B|}{2}2^{|B|}2^{|B||S|}\left(l\binom{|T|}{l}\right)^2 2^{(|B|-2)|T|}2^{\binom{|B|}{2}}|B|!2^{\binom{|T|}{2}-l}$$

$$< 2^{2n^{0.99}} 2^{\binom{n}{2} - \frac{198n}{100}} \binom{n}{n(\sqrt{2}-1)}^2 2^{-(\sqrt{2}-1)n}$$
$$< 2^{\binom{n}{2} - \frac{42n}{100}}$$

for all sufficiently large n. △

Case (d):

We know that every point in B except b has only arrow neighbours in T. If the line in B does not touch b, or if L_b is the set of arrow neighbours of b, then we will be able to write down exactly the same bound as in Case (v) of Theorem 6.9. If b has its large set its line neighbours, and the line in B is ab for some $a \in B$, then we find that there can be no arrows from a to L_b or S, since structures (2) and (1) are forbidden. If $|L_a| = l$, then we find a forbidden set of size l in T. We can construct these diagrams by choosing a, b, arrows from $B - \{a, b\}$ to T, lines from b to T and arrows from a to $T - S - L_b$, the arrow and line connections within B, and finally lines in T. We can bound this above by:

$$D(B, P, S) < \sum_l 2\binom{|B|}{2} 2^{(|B|-2)|T|} \binom{|T|}{l} 2^{|T|-l} 2^{\binom{|B|}{2}} |B|! 2^{\binom{|T|}{2} - l}$$
$$< 2^{\binom{n}{2} - \frac{n}{2} - n^{0.99}}$$

for sufficiently large n, so that in this case (6.1) is bounded by $2^{\binom{n}{2} - \frac{n}{2}}$. △

Case (e):

We use exactly the same argument as in Case (vi) of Theorem 6.9 and get the same bound. △

Case (f):

Subcase (1): If b has L_b the set of its arrow neighbours in T, or there is an arrow in B that does not go to or from b, then let the arrow be $c < d$. We note that if $d < t$, then $c < t$ and so there are at most $3^{|T|}$ choices for arrows between $\{c, d\}$ and T. We note that there is also a forbidden set in T of size at least $\frac{n}{10}$ between L_b and S_b, so that following the usual logic the number of diagrams in this subcase is bounded above by:

$$\sum_{|B|} \binom{n}{|B|} \binom{|T|}{1} 2 \binom{|B|}{1} 2^{\binom{|B|}{2}} |B|! 2^{(|B|-2)|T|} 3^{|T|} 2^{\binom{|T|}{2} - \frac{n}{10}}$$
$$< 2^{\binom{n}{2} - |T| \log \frac{4}{3} - \frac{n}{10}} < 2^{\binom{n}{2} - \frac{n}{2}} .$$

Subcase (2): If b has L_b the set of its line neighbours, and the only arrows in B go to b, then for every point a with $a < b$, L_a is the set of arrow neighbours of a, since S_a is empty. These diagrams correspond to 2-SAT functions in $V(n)$, in the form (p).

Subcase (3): If b has L_b the set of its line neighbours, and there is an arrow in B from b to a, then every point in L_a is above b, which contradicts $|S_b| = 1$. Thus there are no valid diagrams in this subcase. △

Case (g):

These diagrams correspond to 2-SAT functions in $V(n)$, in forms (p) and (q). △

We have now improved the bounds on Case (iv) of Theorem 6.9, so that we may say that for sufficiently large n there are at most $2^{\binom{n}{2} - \frac{419n}{1000}}$ valid diagrams which do not correspond either to unate 2-SAT functions or to 2-SAT functions in $V(n)$.

Following the same logic as was used in §6.2.1 to prove Theorem 6.6, we now obtain for sufficiently large n an upper bound $2^{\binom{n}{2} + n - \frac{418n}{1000}}$ on the number of 2-SAT functions which are neither unate nor in $V(n)$. □

6.3.2 Other large classes

Prömel, Schickinger and Steger [63] went on to show that almost every triangle-free graph which is not bipartite and cannot be made so by removing one vertex can be made bipartite by removing two vertices, and so on. We will establish an upper bound of $2^{\binom{n}{2} + n - kn}$ on the number of 2-SAT functions which cannot be made unate by removing $25k$ variables, where

$k = k(n) < n^{\frac{1}{4}}$, which is a first step towards proving the equivalent result for our problem.

Given a 2-SAT function S, let F be the unique maximal 2-SAT formula for S. If V is a subset of the domain of S, we can define the formula F' which contains exactly the clauses of F that do not contain literals associated to the variables in V. We say we can *remove the variables* V from S to get S', a 2-SAT function on $n - |V|$ variables given by the formula F'. Observe that if R is any subset of the domain of S, then removing the variables V from $\rho_R(S)$ gives the same result as removing the variables V from S then applying ρ_R.

If we can remove a set of k variables from a 2-SAT function S to obtain a unate 2-SAT function, then we say that S is *k-nearly-unate*. If S is k-nearly-unate but not $(k-1)$-nearly-unate, we say that S is *exactly k-nearly-unate*.

Observe that if S is an elementary function, and the set V of variables is removed from S to give S', then $P(S')$ is precisely the partial order on the $2n - 2|V|$ literals induced by $P(S)$. If also S is nonnegative, however, $D(S')$ is not in general the diagram given by simply removing the points V and lines and arrows meeting them from $D(S)$. The arrows in $D(S')$ correspond to the arrows in $D(S)$ not meeting V. But if ab is a line in $D(S)$, and $a \in V$, and $c > a$, then cb may be a line in $D(S')$ (if $\overline{c} < b$ is a covering relation in $P(S')$). If ab is a line in $D(S)$, and $a, b \in V$, and $a < c, b < d$, then cd may be a line in $D(S')$. However, if e, f are points in $D(S')$, then ef can only be a line if either it is a line in $D(S)$ or one of the two above situations occurs.

We will require the following simple lemma.

Lemma 6.14. *Let G be any graph, and k any integer. Then we can find either a set E consisting of at least $\frac{k}{2}$ independent edges of G, or a set Z consisting of at most k vertices of G which meets every edge of G.*

Proof. Given a graph G, let E be a maximal set of independent edges of G. Then either E has size at least $\frac{k}{2}$, or the set $Z = \bigcup E$ has size at most k. Z must meet every edge of G since E is maximal. $\square$

Theorem 6.15. *For any $k = k(n)$ such that $k(n) < n^{\frac{1}{4}}$ for all sufficiently large n, the set of 2-SAT functions on n variables which are not k-nearly-unate has size at most $2^{\binom{n}{2}+n-\frac{kn}{25}}$ for all sufficiently large n.*

Proof. We follow essentially the same logic as was used to prove Theorem 6.6.

We observe that for sufficiently large n, if $k(n) = 0$ then the conclusion certainly holds for that n by Theorem 6.6, so in the remainder of the proof we shall assume $k > 0$.

First we show that for any such k, for sufficiently large n, the set $H_k(n)$ of elementary 2-SAT functions on n variables which are not k-nearly-unate has size at most $2^{\binom{n}{2}+n-\frac{kn}{24}}$.

We let $D_k(n)$ be the set of valid diagrams corresponding to the nonnegative 2-SAT functions on n variables which are not k-nearly-unate.

As before, we divide $D_k(n)$ into two parts: the diagrams with $|B| > \frac{n}{100}$, and the diagrams with $|B| \leq \frac{n}{100}$. Theorem 6.12 tells us that the first part has size at most $2^{\binom{n}{2}-n^{\frac{5}{4}}}$ for sufficiently large n, so we only need to bound the second part.

We bound above the set of valid diagrams with $|B| \leq \frac{n}{100}$ corresponding to 2-SAT functions on n points which are not k-nearly-unate. For such a diagram, we define the sets $\Gamma_{arr}(b)$, $\Gamma_{line}(b)$ for $b \in B$, I, P, S_a, L_a for $a \in I$, S as in Theorem 6.9. We also define sets S'_p, L'_p, for some $p \in P$, and the set S' as follows. If $p \in P$, and there is $a \in I$ with ap a line in D and L_a is the set of arrow neighbours of a, then let $S'_p = \Gamma_{arr}(p) - S$, $L'_p = \Gamma_{arr}(a) - S$. Let S' be the union of the defined S'_p. We now split the set of these valid diagrams into five parts:

Case (i): $|P| \geq 84n^{\frac{1}{4}}$.

Case (ii): $|P| < 84n^{\frac{1}{4}}$ and $|S| \geq 100n^{\frac{1}{4}}$.

Case (iii): $|P| < 84n^{\frac{1}{4}}$, $|S| < 100n^{\frac{1}{4}}$ and $|B| \geq 3n^{\frac{19}{20}}$.

Case (iv): $|P| < 84n^{\frac{1}{4}}$, $|S| < 100n^{\frac{1}{4}}$, $|B| < 3n^{\frac{19}{20}}$ and $|P| + |S| + |S'| > k$.

Case (v): $|P| < 84n^{\frac{1}{4}}$, $|S| < 100n^{\frac{1}{4}}$, $|B| < 3n^{\frac{19}{20}}$ and $|P| + |S| + |S'| \leq k$.

By identical logic to that in Theorem 6.9, each of Cases (i), (ii), (iii) contains at most $2^{\binom{n}{2}-3n^{\frac{5}{4}}}$ valid diagrams, for sufficiently large n.

We now provide a bound for Case (iv).

Observe that, if $p \in P$ is connected by a line to $a \in I$, with L_a the set of arrow neighbours of a, then $L'_p = \Gamma_{arr}(a) - S$ cannot intersect S'_p since the structure (1) is forbidden. Furthermore, there can be no lines between S'_p and L'_p since structure (3) is forbidden. Now $|L'_p| \geq \frac{n}{10} - |S| > \frac{n}{11}$ for sufficiently large n. This means that by Lemma 6.8 there is a forbidden set in T of size $\frac{|S'|n}{22}$, which does not intersect the forbidden set between the sets L_i and S_i ($i \in I$) since no member of this new forbidden set has an end in S. So we can construct any diagram in this case in the usual way: given B, P, S, choose the lines and arrows from P to T, whether the members of I have as their large set their set of line or arrow neighbours, the lines and arrows from I to $T - S$, the lines and arrows from I to S, the lines and arrows within B, which fixes S', and finally the lines within T, taking account of both the forbidden sets. This allows us to use in this case the bound

$$\begin{aligned} D(B,P,S) &< 2^{|P||T|}2^{-\frac{|P|n}{21}}2^{|I|}2^{|I||T-S|}3^{|I||S|}2^{\binom{|B|}{2}}|B|!2^{\binom{|T|}{2}-\frac{|S|n}{20}-\frac{|S'|n}{22}} \\ &< 2^{\binom{n}{2}}2^{|I|}|B|!2^{-\frac{(|P|+|S|+|S'|)n}{22}} \\ &< 2^{\binom{n}{2}-n^{0.99}-\frac{kn}{23}} \end{aligned}$$

for sufficiently large n, so that in this case the number of valid diagrams is bounded above by $2^{\binom{n}{2}-\frac{kn}{23}}$. $\triangle$

Finally we bound Case (v):

Given a diagram D in Case (v), we draw a graph G with $V(G) = I$. If $a, b \in I$, $L_a = \Gamma_{line}(a)$, $L_b = \Gamma_{arr}(b)$, and either ab is a line or ab being a line would create one of the forbidden structures (2), (3) (so that ab could potentially be a line in some diagram D' obtained by removing variables), then we put $ab \in E(G)$. If $a < b$ in D, $L_a = \Gamma_{line}(a)$ and $L_b = \Gamma_{line}(b)$, then we put $ab \in E(G)$. If $a < b$ in D, $L_a = \Gamma_{arr}(a)$ and $L_b = \Gamma_{arr}(b)$, then we put $ab \in E(G)$. Otherwise we do not put $ab \in E(G)$.

We split this case into three sub-cases:

Subcase (a): There is a line ab in I, with both L_a the set of arrow neighbours of a and L_b the set of arrow neighbours of b.

In this case, there is a forbidden set of size at least $\frac{n^2}{100}$ in T, and we can bound above the number of valid diagrams in this case as in Case (v) of Theorem 6.9, obtaining a bound $2^{\binom{n}{2}-n^{\frac{5}{4}}}$ for sufficiently large n.

Subcase (b): There is no line between any two points $a, b \in I$ with both L_a the set of arrow neighbours of a and L_b the set of arrow neighbours of b. We cannot find any set of $q = k - |P| - |S| - |S'|$ points in I such that every edge in $E(G)$ touches the set.

In this case, by Lemma 6.14 there must be a set E of $\lceil \frac{q}{2} \rceil$ edges in $E(G)$, no two of which meet at any point.

Suppose we know for each point $i \in I$ whether its large set is its set of arrow or line neighbours. We have previously used the bound $2^{|I|}$ on the choices of lines and arrows between I and any given $t \in T - S$. But now observe that if $ab \in E$, we have only 3 choices for the lines and arrows from a, b to t; if ab is in E because $a < b$ in D with both $L_a = \Gamma_{line}(a)$ and $L_b = \Gamma_{line}(b)$, then we cannot choose to put lines from both a and b to t since structure (2) is forbidden, and so on. Since the edges in E are independent, we obtain a bound $2^{|I|-q}3^{\frac{q}{2}}$ on the number of choices of lines and arrows between I and t.

We can construct every diagram in this case as follows: Given B, P, S, we choose the $\lceil \frac{q}{2} \rceil$ edges in E, and whether they are to correspond to lines or arrows in one or the other direction in D. We choose for each $i \in I$ whether L_i is to be the set of line or arrow neighbours of i. We choose the lines and arrows from I to $T - S$, taking account of the above restriction, and from I to S. We choose the lines and arrows from P to T, the lines and arrows within B, and the lines within T, taking account of both the forbidden sets (note S' and the L'_p are already chosen).

This gives us the upper bound on $D(B,P,S)$

$$\left(3\binom{|I|}{2}\right)^{\lceil\frac{q}{2}\rceil} 2^{|I|+(|I|-q)|T-S|}3^{\frac{q}{2}|T-S|}3^{|I||S|}2^{|P||T|}2^{-\frac{|P|n}{21}}2^{\binom{|B|}{2}}|B|!2^{\binom{|T|}{2}-\frac{n|S|}{20}-\frac{n|S'|}{22}}$$

$$< \left(3|B|^2\right)^k 2^{|I|}|B|!2^{\binom{n}{2}}2^{-\frac{(|P|+|S|+|S'|)n}{22}-\frac{k-|P|-|S|-|S'|}{2}\frac{98n}{100}\log\frac{4}{3}}$$

$$< 2^{\binom{n}{2}-\frac{kn}{23}-n^{0.99}}$$

for sufficiently large n, so that the number of diagrams in this subcase is bounded above by $2^{\binom{n}{2}-\frac{kn}{23}}$.

Subcase (c): There is no line between points $a, b \in I$ with both L_a the set of arrow neighbours of a and L_b the set of arrow neighbours of b. We can find a set Z of $k - |P| - |S| - |S'|$ points in I which meets every edge in $E(G)$.

Let D' be the diagram corresponding to the nonnegative 2-SAT function obtained by removing the k variables $V = P \cup S \cup S' \cup Z$ from the 2-SAT function corresponding to D. Recall that the arrows in D' correspond to the arrows in D not meeting V, but there may be some lines in D' which do not correspond to lines in D, but exist because the covering relations have changed. We will continue to use T, B to refer to the top and bottom sets of D, and will use T', B' for the top and bottom sets of D'.

Suppose $a \in I$ has $L_a = \Gamma_{line}(a)$, and $a < b$ for some $b \in I$. Then b has $L_b = \Gamma_{line}(b)$, otherwise $S_a \supset L_b$, but $|S_a| < 100n^{\frac{1}{4}} < |L_b|$. So either a or b must be in Z. It follows that any point $a \in I - Z$ with $L_a = \Gamma_{line}(a)$ is maximal in D'.

Suppose that a point $b \in I - Z$ has $L_b = \Gamma_{arr}(b)$, and in D' there is a line db for some d. Certainly db cannot have existed in D, as that would imply either a line between two points whose large sets are their sets of arrow neighbours, $d \in S$, $d \in Z$ or $d \in P$. Furthermore, $d \notin B$, as that would imply either $d \in P$ or $d \in Z$, so $d \in T$. Recall that $d \notin V$. There are three possibilities.

First, there could be $e \in V$, with de a line in D and $e < b$. But then $e \in I$, since e has at least $|L_b|$ arrow neighbours in T, and this would imply $d \in S$.

Second, there could be $f \in V$ with $f < d$ and fb a line in D. But then $f \notin I$, since this would either imply $d \in S$ or fb would be a line between two points in I with large sets their sets of arrow neighbours. So $f \in P$, and then we have $d \in S'$.

Third, there could be $a, c \in V$ with $a < b$, $c < d$, and ac a line in D. Since $a < b$, so a has at least $|L_b|$ arrow neighbours, and $a \in I$. Since $|S| < 100n^{\frac{1}{4}} < |L_b|$, so L_a would be the set of arrow neighbours of a. Then if $c \in I$, we would have either $d \in S$ or ac would be a line between two points in I with large sets their sets of arrow neighbours. So $c \in P$, and we have $d \in S'$.

Thus we see that we cannot have any point $b \in I - Z$ with $L_b = \Gamma_{arr}(b)$ which meets a line in D'.

Now the top set T' of D' consists of $T - S - S'$ together with those points in $I - Z$ whose large set was their line set, while the bottom set B' of D' consists of those points in $I - Z$ whose large set was their arrow set. Then there are no lines within B'. There are no arrows within B', since Z was removed. There are no lines between B' and T' as above. Now, by Lemma 6.7, D' corresponds to an unate 2-SAT function on $n - k$ points, contradicting the original diagram on n points being in $D_k(n)$. Hence there are no diagrams in this subcase. △

Adding up the bounds from all the cases, we see that for any $k = k(n)$ with $k(n) < n^{\frac{1}{4}}$ for all sufficiently large n, we have

$$|D_k(n)| < 3.2^{\binom{n}{2} - 3n^{\frac{5}{4}}} + 2.2^{\binom{n}{2} - n^{\frac{5}{4}}} + 2.2^{\binom{n}{2} - \frac{kn}{23}}$$

$$< 2^{\binom{n}{2} - \frac{kn}{24}}$$

for all sufficiently large n, and $|H_k(n)| < 2^{\binom{n}{2} + n - \frac{kn}{24}}$.

Since every 2-SAT function which is not k-nearly-unate can be reduced to an elementary 2-SAT function on $n - l$ variables which is not $(k - l)$-nearly-unate if $k \geq l$, or to an elementary 2-SAT function on $n - l$ variables if $k < l$, we can follow the logic in §6.2.1 and bound above the size of the set of 2-SAT functions on n variables which are not k-nearly-unate by

$$1+\sum_{l=0}^{k}\binom{n}{l}(2n+2-2l)^l H_{k-l}(n-l)+\sum_{l=k+1}^{n}\binom{n}{l}(2n+2-2l)^l H(n-l)$$

$$< 1+\sum_{l=0}^{k}\binom{n}{l}(2n+2-2l)^l 2^{\binom{n-l}{2}+n-l-\frac{(k-l)(n-l)}{24}}$$

$$+\sum_{l=k+1}^{n}\binom{n}{l}(2n+2-2l)^l H(n-l)\ ,$$

which sum is, by the same logic as used to prove Theorem 6.6, dominated by the term $2^{\binom{n}{2}+n-\frac{kn}{24}}$, hence is, for sufficiently large n, bounded above by $2^{\binom{n}{2}+n-\frac{kn}{25}}$. □

6.4 Expected values

In this section we calculate the asymptotic values of the expectations of some random variables in the model of 2-SAT functions on n variables chosen uniformly at random.

Given a 2-SAT function S, let $A(S)$ be the set of satisfying assignments of S. In proving Theorem 6.6 we made use of a mapping M from elementary to nonnegative 2-SAT functions, which certainly does not map more than 2^n elementary functions to any nonnegative function. When we defined M we explained why it is reasonable to assume that most nonnegative functions are the target under M of nearly 2^n nonnegative functions, and in proving the upper bound on $F(n)$ we justified our belief.

But it is certainly not true that M is exactly 2^n-to-1: the number of elementary functions is less than 2^n times the number of nonnegative functions. This comes from the fact that in constructing M we had to choose one of several viable options for each elementary function. To be precise, for each elementary 2-SAT function S we could have chosen any member A of $A(S)$ and relabelled the variables assigned False in A to obtain the nonnegative 2-SAT function $M(S)$. This provides us with an easy route to a proof of the following theorem.

Theorem 6.16. *The expected value of* $|A(S)|$ *is*

$$\big(1+o(1)\big)\sum_{k=0}^{n}\binom{n}{k}2^{-\binom{k}{2}}=2^{\left(\frac{1}{2}+o(1)\right)\log^2 n}\ .$$

Let I_n be the random variable counting the number of independent sets in $\mathcal{G}_{n,\frac{1}{2}}$. Note that

$$\mathbb{E}I_n=\sum_{k=0}^{n}\binom{n}{k}2^{-\binom{k}{2}}\ ,$$

which is asymptotically the same as the expected value of $|A(S)|$. We should believe that I_n and $A(S)$ are related: whenever S is a monotone function, independent sets in the graph (as a diagram with no arrows) $D(S)$ correspond to satisfying assignments of S, and it seems reasonable that the unate functions should both dominate the expectation and behave similarly to the monotone functions.

Proof. The average number of satisfying assignments of a 2-SAT function on n variables is equal to the sum, over 2-SAT functions S on n variables, $\sum_S |A(S)|$, divided by the number of 2-SAT functions on n variables: as we know the latter we need only compute the former. This is equal to the number of pairs (S,A), where S is a 2-SAT function on n variables and A is a satisfying assignment of S. First we find the number of such pairs where S is an elementary 2-SAT function on n variables:

Observe that if (S,A) is a pair as above, with S elementary, then we can let the set of variables which are assigned False in A be X, and $\rho_X(S)$ is a nonnegative 2-SAT function, which corresponds to a unique valid diagram D. Thus there are as many pairs with S elementary as there are pairs (D,X) where D is a valid diagram and X is a subset of the n variables. There are 2^n possible subsets X, so there are $2^nF(n)$ such pairs. Following the logic in §6.2.1, the total number of pairs (S,A), where S is any 2-SAT function on n variables, is $(1+o(1))2^nF(n)$. Now recall that when the valid diagrams were split into several types, the largest was the class of diagrams in Case (ix) of Theorem 6.9. There are

$$2^{\binom{n}{2}}\sum_{k=0}^{5\sqrt{n}}\binom{n}{k}2^{-\binom{k}{2}}$$

diagrams in that case, so that there are

$$\big(1+o(1)\big)2^{\binom{n}{2}+n}\sum_{k=0}^{5\sqrt{n}}\binom{n}{k}2^{-\binom{k}{2}}$$

pairs (S,A) where S is a 2-SAT function on n variables and A is a satisfying assignment for F. It is straightforward to see that the sum is dominated by the terms with k approximately $\log n$, as with the sum

$$\sum_{k=0}^{n}\binom{n}{k}2^{-\binom{k}{2}}\ ,$$

so that the average number of satisfying assignments of a 2-SAT function on n variables is

$$\big(1+o(1)\big)\sum_{k=0}^{n}\binom{n}{k}2^{-\binom{k}{2}}$$

as required, and the approximation $2^{\left(\frac{1}{2}+o(1)\right)\log^2 n}$ is valid. □

There is another way of approximating $\sum_S |A(S)|$. We could divide the 2-SAT functions up into three classes: the unate functions (whose contribution we expect to dominate the sum), some large class of non-unate functions where we can control the number of satisfying assignments, and some small class of non-unate functions where we cannot control the satisfying assignments but which is too small to significantly affect the sum. In light of Theorem 6.15 we can make this precise: we know that the unate functions contribute at least $2^{\binom{n}{2}+n}$ to the sum over 2-SAT functions on n variables $S \sum|A(S)|$ (since each has at least one satisfying assignment), we might believe in our ability to control the number of satisfying assignments of the non-unate but 50-nearly-unate functions and thus show that their contribution is small, and we know that there are so few 2-SAT functions which are not 50-nearly-unate that even if each contributes 2^n to the sum, their contribution is still dominated by the unate 2-SAT functions.

This method is more general: we can, for example, use it to calculate moments of $|A(S)|$.

Theorem 6.17. *The kth moment of $|A(S)|$ is given by*

$$\mathbb{E}|A(S)|^k = \big(1+o(1)\big)\mathbb{E}I_n^k\ .$$

Proof. We evaluate $\Sigma = \sum_S |A(S)|^k$, where the sum is taken over all 2-SAT functions on n variables. Since $|A(S)|^k \leq 2^{kn}$ for any 2-SAT function S, and there are at most $2^{\binom{n}{2}-kn}$ 2-SAT functions on n variables which are not $(25k+25)$-nearly-unate, these functions do not contribute more than $2^{\binom{n}{2}}$ to Σ.

We need to show that the contribution of the non-unate $(25k+25)$-nearly-unate functions to Σ is similarly insignificant. From Theorem 6.15 we know that there are at most $2^{\binom{n}{2}+\frac{24n}{25}}$ of these functions. Now if S is a $(25k+25)$-nearly-unate 2-SAT function, then we can remove $25k+25$ variables to obtain an unate function S' on $n-25k-25$ variables. If $|A(S')| \leq 2^{\frac{n}{50}-25k-25}$ then S has at most $2^{\frac{n}{50}}$ satisfying assignments, and the contribution to Σ of these functions is at most $2^{\binom{n}{2}+\frac{49n}{50}}$.

If, on the other hand, $|A(S')| > 2^{\frac{n}{50}-25k-25}$ then there are only a few possibilities for S': suppose S' is such a formula, and let S'' be a monotone formula obtained by relabelling some set of variables. Choose any $\varepsilon > 0$ such that $\sum_{i=0}^{\varepsilon n}\binom{n}{i} < 2^{\frac{n}{50}-25k-25}$ is true for all sufficiently large n. Since $|A(S'')| \geq 2^{\frac{n}{50}-25k-25}$ there must be a satisfying assignment A of S'' in which at least εn variables are assigned False. It follows that the number of possibilities for S'' is at most $2^n 2^{\binom{n}{2}-\binom{\varepsilon n}{2}}$, where the 2^n counts the number of ways of picking the εn variables assigned False and the $\binom{\varepsilon n}{2}$ counts the number of clauses consisting of two positive literals drawn from the εn variables, which may not be present in the formula for S''. Now the number of possibilities for S' is $2^{\binom{n}{2}-\binom{\varepsilon n}{2}+2n}$, and the number of clauses including the $25k+25$ removed variables is less than $4(25k+25)n$, so that the number of possibilities for S is at most $2^{\binom{n}{2}-\binom{\varepsilon n}{2}+(25k+30)n}$. Thus their contribution to Σ is at most $2^{\binom{n}{2}-\binom{\varepsilon n}{2}+(26k+30)n}$.

The contribution of the monotone functions to Σ is easy to calculate: the satisfying assignments of a monotone function are given by the sets of variables assigned False, which in turn correspond to the independent sets in the diagram $D(S)$ (which has no directed edges). Thus their contribution is exactly $2^{\binom{n}{2}}\mathbb{E}I_n^k$.

Now if every unate function was obtained by relabelling some unique set of variables X on a unique monotone formula then the contribution to Σ of the unate 2-SAT functions would be exactly $2^{\binom{n}{2}+n}\mathbb{E}I_n^k$. But this is not the case: there are at most $n2^{\binom{n-1}{2}+n}$ unate functions for which there are two possibilities for the set of variables X to be relabelled, and so on. Nevertheless, the contribution to Σ of the unate functions is certainly at least

$$2^{\binom{n}{2}+n}\mathbb{E}I_n^k - \sum_{r=1}^{n}\binom{n}{r}2^{\binom{n-r}{2}+n}\mathbb{E}I_{n-r}^k 2^{rk} = \big(1-o(1)\big)2^{\binom{n}{2}+n}\mathbb{E}I_n^k$$

since $\mathbb{E}I_n^k$ is monotone increasing with n: and $2^{\binom{n}{2}+n}\mathbb{E}I_n^k$ remains an upper bound.

Finally since $\mathbb{E}I_n^k$ is certainly larger than 1 we have $\Sigma = \big(1+o(1)\big)2^{\binom{n}{2}+n}\mathbb{E}I_n^k$, and since the number of 2-SAT functions is $\big(1+o(1)\big)2^{\binom{n}{2}+n}$ we have $\mathbb{E}|A(S)|^k = \big(1+o(1)\big)\mathbb{E}I_n^k$ as required. $\square$

Since $\mathbb{E}|A(S)| = 2^{\left(\frac{1}{2}+o(1)\right)\log^2 n}$ is sub-exponential the same proof shows that the central moments of $|A(S)|$ also coincide asymptotically with those of I_n: $\mathbb{E}(|A(S)| - \mathbb{E}|A(S)|)^k = \big(1+o(1)\big)\mathbb{E}(I_n - \mathbb{E}I_n)^k$, provided that the latter is not exponentially small.

6.5 Further thoughts

We could certainly continue to extract further classes of 2-SAT functions in decreasing size order, either in the manner of Theorem 6.13 or by enumerating the large classes of k-nearly-unate functions for suitable k and appealing to Theorem 6.15 to show that there are no larger classes left uncounted.

It is obvious that Theorem 6.15 is not best possible: the constant $\frac{1}{25}$ is certainly too small. Also, we conjecture that in fact the result holds for any function $k(n)$. However, observe that the class of 2-SAT functions on n variables which consist of the first k variables all having their positive literals in the spine, and the remaining $n-k$ variables forming a monotone function, has size $2^{\binom{n-k}{2}} = 2^{\binom{n}{2}-kn+\frac{k^2+k}{2}}$. All functions in this class are exactly k-nearly-unate, so that we do not expect a sharp upper bound on the number of non-k-nearly-unate functions to be of a substantially different form to that given.

Prömel, Schickinger and Steger [63] found that almost every triangle-free graph which is not bipartite, and cannot be made bipartite by removing one vertex, can be made bipartite by removing two vertices, and so on; Theorem 6.15 suggests that the corresponding result holds for our problem. We conjecture that the class of 2-SAT functions on n variables which are exactly k-nearly-unate is larger than the class of those which are exactly $(k+1)$-nearly-unate by a factor of at least $2^{\varepsilon n}$ for all sufficiently large n, all k, and some constant $\varepsilon > \frac{1}{25}$.

Our results only hold for a number n of variables greater than some (very large) number N. It is worth considering whether we really needed to apply the Szemerédi Regularity Lemma, which was responsible for causing N to be so large; to obtain our results we did not make as much use of the lemma as in [13], where, (implicitly) the Regularity Lemma is applied for an infinite sequence of ε's tending to zero, and we did not need the lemma at all to deal with the large case. However, even if a method of avoiding it could be found, other parts of the proof require N to be so large as to be useless for practical application.

We have not attempted to attack the problem posed in [13] of determining the asymptotic behaviour of the number of k-SAT functions for $k > 2$; in [13], the upper bound $2^{\sqrt{\pi(k+1)}\binom{n}{k}}$ is given, and the conjecture offered that the exponent should in fact be $\binom{n}{k}\big(1 + o(1)\big)$. One could certainly follow the methods here: for 3-SAT functions, the first step would be to discard functions which in some way emulate 2-SAT functions (corresponding to our move to elementary functions) and then give some kind of 3-uniform hyperdiagram structure. At this point one would find a major difference between 2-SAT and k-SAT for $k \geq 3$. The computational problem of determining whether a 2-SAT formula is satisfiable has a simple polynomial time algorithm: and the set of forbidden structures is finite (and small). By contrast, for $k \geq 3$ the computational problem is NP-complete, so we should expect to find that the set of forbidden hyperdiagrams is both infinite and hard to describe. However, it seems likely that this would not cause insurmountable problems: even for 2-SAT, the reader will have observed that the forbidden structure (3) was used just once in the proof of Theorem 6.6, in case (v) of Theorem 6.9, and even then its use was unnecessary (the structure (2) would have produced an acceptable bound for Theorem 6.6, although not for the later Theorem 6.13). It seems very likely that for 3-SAT and higher a similar phenomenon will occur: finding the dominant term for the number of k-SAT function on n variables will only require knowledge of the smallest few forbidden structures. However the reader will also notice that in the course of proving Theorem 6.6 we made use of a considerable number of previous results (such as Kleitman and Rothschild's enumeration of partial orders and Füredi's result on proper squares of graphs) which do not yet have hypergraph analogues, so that following our methods to prove an upper bound for even the 3-SAT case would be a very difficult task.

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