Frontiers of Mathematical Finance
Vol. 7, 2025, pp. 78-94 (December 11, 2025)
doi:10.3934/fmf.2025013

RISK PREMIUM AND ROUGH VOLATILITY

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(Communicated by Wim Schoutens)

Abstract. On the one hand, rough volatility has been shown to provide a consistent framework to capture the properties of stock price dynamics both under the historical measure and for pricing purposes. On the other hand, market price of volatility risk is a well-studied object in financial economics, and empirical estimates show it to be stochastic rather than deterministic. Starting from a rough volatility model under the historical measure, we take up this challenge and provide an analysis of the impact of such a non-deterministic risk for pricing purposes.

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Introduction. Rough volatility is a recent paradigm proposed by Gatheral, Jaisson, and Rosenbaum [14], which has attracted the attention of many academics and practitioners thanks to its numerous attractive properties [7]. Despite some debate about whether volatility should be rough [1, 2, 16, 25], this class of models provides a general framework to analyze both time series of the instantaneous volatility (under the historical measure $\mathbb{P}$) and prices of financial derivatives (under the pricing

2020 *Mathematics Subject Classification.* 60G15, 60G22, 91G20.
Key words and phrases. Risk premium, fractional Brownian motion, rough volatility.
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measure $\mathbb{Q}$). Starting from a rough version of the Bergomi model [8] under $\mathbb{P}$, Bayer, Friz, and Gatheral [6] showed that a deterministic market price of risk preserved its structure under $\mathbb{Q}$ (somehow akin to the Heston model [17] specification).

However, the financial economics literature has long shown that this market price of risk, monitoring the transition from $\mathbb{P}$ to $\mathbb{Q}$ via Girsanov's transform, is neither constant nor deterministic, but instead stochastic. Its estimation has been the source of long academic discussions, outside the scope of the present paper, and for such discussions we refer the interested reader to [4, 9, 10, 12, 20, 22]. This of course has serious practical implications for risk management, for which [15] is a fascinating source of information. We focus here on this particular bridge between $\mathbb{P}$ and $\mathbb{Q}$ and show, not surprisingly, that the required stochasticity of the market price of risk unfortunately breaks the structure of the rough Bergomi model under $\mathbb{Q}$. However, we link the Hölder regularity of the volatility process (lower in this class of rough models) with that of the change of measure, and design several specifications making the model tractable under $\mathbb{Q}$. While the rough Bergomi model tracks the behavior of the historical volatility well, it is less powerful for option prices, especially when considering VIX smiles (which are more or less flat under this model). Our new setup allows for more flexibility there, while preserving the $\mathbb{P}$-tractability of the model.

Section 1 provides the technical setup and analysis of the market price of risk, while the design of useful continuous-time rough stochastic volatility models with non-deterministic market prices of risk are detailed in Section 2. Finally, in Section 3, we perform an empirical analysis, estimating risk premia from historical options data.

1. **Rough volatility models and change of measure.**

Rough volatility models are a natural extension of classical stochastic volatility models. Starting from such a model under the historical measure $\mathbb{P}$, we charaterize below its dynamics under martingale measures $\mathbb{Q}$ equivalent to $\mathbb{P}$, which then, by the fundamental theorem of asset pricing, allows for arbitrage-free option pricing. Following [14] for example, we consider a rather general class of (rough) stochastic volatility models under $\mathbb{P}$, where the stock price process admits the following dynamics:

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= \mu_t \mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{P}}, \\ v_t &= \psi(t, Y_t), \\ Y_t &= \displaystyle\int_0^t k(t,s)\mathrm{d}Z_s^{\mathbb{P}}, \end{cases} \tag{1.1}$$

starting from $S_0, v_0 > 0$, over a fixed time interval $\mathbb{T} := [0, T]$, for $T > 0$. Here, μ is a $(\mathcal{F}_t)_{t\in\mathbb{T}}$-measurable process, $\psi : \mathbb{T} \times \mathbb{R} \to (0, \infty)$ a continuous function, $\boldsymbol{W}^{\mathbb{P}} = (W^{\mathbb{P}}, W^{\mathbb{P},\perp})$ a two-dimensional standard Brownian motion on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F} := \mathcal{F}^{W^{\mathbb{P}}} \vee \mathcal{F}^{W^{\mathbb{P},\perp}}$ and $Z^{\mathbb{P}} := \rho W^{\mathbb{P}} + \overline{\rho} W^{\mathbb{P},\perp}$, where $\rho \in [-1, 1]$ and $\overline{\rho} := \sqrt{1-\rho^2}$. We assume that for each $t \in \mathbb{T}$, the kernel $k(t, \cdot)$ is null on $\mathbb{T} \setminus [0, t]$, and $\int_0^\cdot k(\cdot, s)\mathrm{d}Z_s^{\mathbb{P}}$ is a well-defined continuous Gaussian process. In particular, $k(t, \cdot) \in L^2([0, t])$ for each $t \in \mathbb{T}$ ensures that the process is well defined, and we refer to [Section 5][3] or [11, Section 3] for conditions of continuity criteria. For example, the Gamma kernel, commonly used to model turbulence, and pioneered by Barndorff-Nielsen and Schmiegel [5], is given by

$$k(t,s) = (t-s)^{H-\frac{1}{2}}\mathrm{e}^{-\beta(t-s)}\mathbb{1}_{\{t\geq s\}}, \qquad \text{with } H \in (0,1), \quad \beta \geq 0.$$

We further introduce the set $\mathbb{F}_b$ of $\mathbb{P}$-bounded and $(\mathcal{F}_t)_{t\in\mathbb{T}}$-progressively measurable processes, namely $X \in \mathbb{F}_b$, if there exists a constant $c > 0$ such that $\mathbb{P}\left(\sup_{t\geq 0}|X_t| \leq c\right) = 1$, and recall the Doléans-Dade stochastic exponential of a square integrable continuous process X:

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right), \qquad \text{for } t \in \mathbb{T}.$$

Finally, we introduce a progressively measurable interest rate process $(r_t)_{t\in\mathbb{T}}$, define the corresponding Sharpe ratio by $\chi_t := \frac{r_t - \mu_t}{\sqrt{v_t}}$ for $t \in \mathbb{T}$, and consider the following assumption.

Assumption 1.1. The processes χ and γ are $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted with càdlàg paths.

In order to state the main result, define the Radon-Nikodym derivative, for each $t \in \mathbb{T}$,

$$\mathcal{D}_t^\gamma := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t^{\mathbb{P}}} = \mathcal{E}\left(\int_0^\cdot \chi_u \mathrm{d}W_u^{\mathbb{P}} + \int_0^\cdot \gamma_u \mathrm{d}W_u^{\mathbb{P},\perp}\right)_t. \tag{1.2}$$

Proposition 1.2. *Under Assumption 1.1, $\mathcal{D}^\gamma$ is a locally integrable $\mathbb{P}$-local martingale on $\mathbb{T}$.*

Proof. Since $W^{\mathbb{P}}$ and $W^{\mathbb{P},\perp}$ are independent Brownian motions (and so locally square-integrable local martingales), by [24, Section II, Theorem 20], the processes $X^1 := \int_0^\cdot \chi_u \mathrm{d}W_u^{\mathbb{P}}$ and $X^2 := \int_0^\cdot \gamma_u \mathrm{d}W_u^{\mathbb{P},\perp}$ are locally square-integrable local martingales. The sum of two locally square-integrable local martingales is itself, and hence so is $X := X^1 + X^2$. Finally, notice that the Radon-Nikodym derivative $\mathcal{D}^\gamma$ defined in (1.2) is precisely the stochastic exponential of X, and thus a non-negative local martingale itself. This implies that $\mathcal{D}^\gamma$ is a positive super-martingale, which together with the fact that $\mathcal{D}_0^\gamma = 1$ yields $\mathcal{D}_t^\gamma \in L^1$ for all $t \in \mathbb{T}$. □

Finally, weconsider the following set of assumptions, in place for the rest of the paper.

Assumption 1.3.

(i) The function $\psi : \mathbb{T} \times \mathbb{R} \to (0, \infty)$ is continuous, bounded, and bounded away from the origin on $\mathbb{T} \times (-\infty, a]$ for each $a > 0$;

(ii) $K_{\mathbb{T}} > 0$ such that $\sup_{t\in\mathbb{T}}\left\{\int_0^t k(t,u)\lambda_u \mathrm{d}u\right\} \leq K_{\mathbb{T}}$, $\mathbb{P}$-almost surely, where λ denotes the market price of volatility risk defined by

$$\lambda_t := \rho\chi_t + \overline{\rho}\,\gamma_t; \tag{1.3}$$

(iii) The correlation is negative: $\rho \leq 0$;

(iv) The Radon-Nikodym derivative satisfies $\mathbb{E}[\mathcal{D}_t^\gamma] = 1$ for all $t \in \mathbb{T}$.

Remark 1.4. Note that Assumption 1.3(ii) also implies that, for all $n \in \mathbb{N}$,

$$\sup_{t\in\mathbb{T}}\left\{\int_0^t k(t,u)\lambda_u \mathrm{d}u\right\} \leq K_{\mathbb{T}}, \qquad \widehat{\mathbb{P}}_n\text{-almost surely,}$$

with

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}_n}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^\cdot \chi_u \mathrm{d}W_u^{\mathbb{P}} + \int_0^\cdot \gamma_u \mathrm{d}W_u^{\mathbb{P},\perp}\right)_{t\wedge\tau_n} \quad \text{and} \quad \tau_n := \inf\{t \geq 0,\ Y_t = n\}. \tag{1.4}$$

Indeed, given an event A such that $\mathbb{P}(A) = 1$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} 1 = \mathbb{P}(A) &= \mathbb{E}[\mathbf{1}_A] = \widehat{\mathbb{E}}_n[\mathbf{1}_A \mathcal{E}(X)^{-1}] \\ &\le \widehat{\mathbb{E}}_n[\mathbf{1}_A]^{\frac{1}{2}} \widehat{\mathbb{E}}_n[\mathcal{E}(X)^{-2}]^{\frac{1}{2}} = \widehat{\mathbb{P}}_n(A)^{\frac{1}{2}} \mathbb{E}[\mathcal{E}(X)^{-2}\mathcal{E}(X)]^{\frac{1}{2}} \\ &= \widehat{\mathbb{P}}_n(A)^{\frac{1}{2}} \mathbb{E}[\mathcal{E}(X)^{-1}]^{\frac{1}{2}} = \widehat{\mathbb{P}}_n(A)^{\frac{1}{2}} \mathbb{E}\left[\mathcal{E}(-X) \exp\left(\int_0^{t\wedge\tau_n} \chi_u^2 + \gamma_u^2 \mathrm{d}u\right)\right]^{\frac{1}{2}} \le \widehat{\mathbb{P}}_n(A)^{\frac{1}{2}}, \end{aligned}$$

where in the last step we have exploited the fact that $\mathcal{E}(-X)$ is a non-negative supermartingale since $-X$ is a continuous local martingale. Thus, we have proved $\widehat{\mathbb{P}}_n(A) = 1$. An analogous argument shows that the same holds for $\mathbb{Q}$.

Remark 1.5. Assumption 1.3(iv) is guaranteed under different sets of stronger assumptions on γ, μ, χ, r and v, in particular,

- if $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T |X_t|^2 \mathrm{d}t\right\}\right] < \infty$, namely X satisfies the Novikov condition;
- if Assumption 1.1 and 1.3(i)-(ii) hold and processes $\mu, \gamma, andr$ belong to $\mathbb{F}_b$, as detailed in Appendix A.

Proposition 1.2 justifies the use of a Doléans-Dade exponential in the definition of $\mathcal{D}^\gamma$.

Theorem 1.6. *Under Assumptions 1.1 and 1.3, the following hold:*

(I) the Radon-Nikodym derivative process $\mathcal{D}^\gamma$ in (1.2) is a true $\mathbb{Q}$-martingale;

(II) under the (arbitrage-free) equivalent risk-neutral martingale measure $\mathbb{Q}$,

$$\left\{\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= r_t \mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{Q}}, \\ v_t &= \psi\left(t, \widehat{Y}_t + \int_0^t k(t,s)\lambda_s \mathrm{d}s\right), \\ \widehat{Y}_t &= \int_0^t k(t,s)\mathrm{d}Z_s^{\mathbb{Q}}, \end{aligned}\right. \tag{1.5}$$

with $S_0, v_0 > 0$, λ is the market price of volatility risk in (1.3) and where $W^{\mathbb{Q}}$, and $Z^{\mathbb{Q}}$ are $\mathbb{Q}$-Brownian motions defined as

$$W^{\mathbb{Q}} := W^{\mathbb{P}} + \int_0^\cdot \chi_u \mathrm{d}u \qquad \text{and} \qquad Z^{\mathbb{Q}} := Z^{\mathbb{P}} + \int_0^\cdot \lambda_u \mathrm{d}u; \tag{1.6}$$

(III) the discounted stock price $\widetilde{S} := \frac{S}{B}$ with $\mathrm{d}B_t = r_t B_t \mathrm{d}t$, $B_0 = 1$, is a true $\mathbb{Q}$-martingale.

Proof. To satisfy the no-arbitrage conditions, the change of measure for $W^{\mathbb{P}}$ is constrained by the martingale restriction on the discounted spot dynamics, while the Brownian motion $Z^{\mathbb{P}}$ gives freedom to the model and makes the market incomplete by the free choice of the process γ. Consequently, the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ and the corresponding Radon-Nikodym derivative directly follow from Girsanov's Theorem via (1.2), provided that $\mathcal{D}_t^\gamma \in L^1$ and $\mathcal{D}^\gamma$ is a true martingale. Thus, once we have shown (I), then (II) automatically follows. By Proposition 1.2, $\mathcal{D}_t^\gamma \in L^1$, and, being a non-negative local martingale, is a super-martingale, and a true martingale on $\mathbb{T}$ if and only if $\mathbb{E}\left[\mathcal{D}_T^\gamma\right] = 1$. This is guaranteed by Assumption 1.3(iv), hence (I) holds, and therefore (II) as well.

We now prove (III): *The discounted price* $\widetilde{S} = \frac{S}{B}$ *is a true martingale for* $\rho \le 0$. Itô's formula under $\mathbb{Q}$ yields

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{S}_t}{\widetilde{S}_t} &= \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{Q}}, \\ v_t &= \psi\left(t, \widehat{Y}_t + \displaystyle\int_0^t k(t,s)\lambda_s \mathrm{d}s\right), \\ \widehat{Y}_t &= \displaystyle\int_0^t k(t,s)\mathrm{d}Z_s^{\mathbb{Q}}. \end{cases}$$

Define the stopping time $\iota_n := \inf\{t \ge 0,\ \widehat{Y}_t = n\}$. For any $t \in \mathbb{T}$, the random function $g(x) := \psi\left(t, x + \int_0^t k(t,s)\lambda_s \mathrm{d}s\right)$ is bounded $\mathbb{Q}$-almost surely on $(-\infty, a]$ by Assumption 1.3(i)-(ii), with λ in (1.3), so that, since $\widetilde{S}$ is a $\mathbb{Q}$-local martingale,

$$\widetilde{S}_0 = \mathbb{E}^{\mathbb{Q}}[\widetilde{S}_{T\wedge\iota_n}] = \mathbb{E}^{\mathbb{Q}}[\widetilde{S}_T \mathbb{1}_{\{T<\iota_n\}}] + \mathbb{E}^{\mathbb{Q}}[\widetilde{S}_{\iota_n}\mathbb{1}_{\{T>\iota_n\}}].$$

The first term converges to $\mathbb{E}^{\mathbb{Q}}[\widetilde{S}_T]$ as n tends to infinity, hence

$$\widetilde{S}_0 - \mathbb{E}^{\mathbb{Q}}[\widetilde{S}_T] = \lim_{n\uparrow\infty} \mathbb{E}^{\mathbb{Q}}[\widetilde{S}_{\iota_n}\mathbb{1}_{\{T>\iota_n\}}].$$

Girsanov's theorem further gives $\mathbb{E}^{\mathbb{Q}}[\widetilde{S}_T \mathbb{1}_{\{T>\iota_n\}}] = \widetilde{S}_0 \widetilde{\mathbb{P}}_n(T > \iota_n)$, where $\widetilde{\mathbb{P}}_n$ is such that $\widetilde{W}_t^n = W_t^{\mathbb{Q}} - \int_0^{t\wedge\iota_n} v_s \mathrm{d}s$ is a $\widetilde{\mathbb{P}}_n$-Brownian motion. Note that, for $t < \iota_n$, $\widehat{Y}_t = \widetilde{Y}_t + \rho\int_0^t k(t,s)v_s\mathrm{d}s$, where $\widetilde{Y}_t = \int_0^t k(t,s)\mathrm{d}\widetilde{Z}_s^n$, and $\widetilde{Z}_t^n := Z_t^{\mathbb{Q}} - \rho\int_0^t k(t,s)v_s\mathrm{d}s$ is also a $\widetilde{\mathbb{P}}_n$-Brownian motion. We conclude that if $\rho \le 0$, then $\widehat{Y}_t \ge \widetilde{Y}_t$ and

$$\lim_{n\uparrow\infty} \widetilde{\mathbb{P}}_n(\iota_n \le T) \le \lim_{n\uparrow\infty} \widetilde{\mathbb{P}}_n(\widetilde{\iota}_n \le T) = \lim_{n\uparrow\infty} \mathbb{P}\left(\sup_{t\in[0,T]} Y_t \le n\right) = 0,$$

where $\widetilde{\iota}_n := \inf\{t \ge 0,\ \widetilde{Y}_t = n\}$, and hence $\widetilde{S}$ is a true martingale. □

Remark 1.7. Under Assumption 1.3, consider $\rho \le 0$, some valid function ψ and kernel k, and the constant values $\gamma_s = \overline{\gamma}$ and $\overline{\mu} = \mu_s \le r_s = \overline{r}$ for $\overline{\gamma}, \overline{\mu}, \overline{r} \in \mathbb{R}$ ensuring Assumption 1.3(iv) so that Theorem 1.6 applies. In this scenario, a sufficient condition for the change of measure to be well defined is that the physical drift must be smaller than the risk-free rate.

1.1. **Rough volatility models via generalized fractional operators.** Many rough volatility models can be represented [18] in terms of generalized fractional operators (GFOs), which are defined as follows [18, Definition 1.1]:

Definition 1.8. For any $\beta \in (0,1)$, $\alpha \in (-\beta, 1-\beta)$, and $h \in \mathcal{C}_b^1((0,\infty))$ such that $h'(\cdot) \le 0$, the GFO associated to the kernel $k(x) := x^\alpha h(x)$ applied to $f \in C^\beta(\mathbb{R})$ is defined as

$$(\mathcal{G}^\alpha f)(t) := \begin{cases} \displaystyle\int_0^t (f(s) - f(0))\frac{\mathrm{d}}{\mathrm{d}t}k(t-s)\mathrm{d}s, & \text{if } \alpha \in [0, 1-\beta), \\ \displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (f(s) - f(0))k(t-s)\mathrm{d}s, & \text{if } \alpha \in (-\beta, 0). \end{cases}$$

To simplify future notation, we let $H_\pm := H \pm \frac{1}{2}$ for $H \in (0, \frac{1}{2})$. We now introduce a specific setup that will drive the rest of our computations: Consider the power-law kernel

$$\mathtt{k}_\alpha(u) := u^\alpha \mathbb{1}_{\{u\ge 0\}}, \tag{1.7}$$

as well as the set
$\Lambda_{\beta,H} := \Big\{\lambda \in \mathcal{C}^{\beta}$ for some $\beta \in (0,1]$ such that $H_- \in (-\beta, 0)$ and $\lambda(0) = 0\Big\}$. To this particular power-law kernel, the GFO (from Definition 1.8, since $H_- \in (-\frac{1}{2}, 0)$) reads

$$(\mathcal{G}^{H_-} f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (f(s) - f(0))\mathtt{k}_{H_-}(t-s)\mathrm{d}s.$$

Further, denote

$$\mathtt{K}(t,s) := \int_0^t \mathtt{k}_{H_-}(u-s)\mathrm{d}u = \frac{\mathtt{k}_{H_+}(t-s)}{H_+},$$

so that the corresponding GFO is precisely $\frac{1}{H_+}\mathcal{G}^{H_+}$. To streamline notations and emphasize nice symmetries, we introduce the notations

$$\mathfrak{G}^- := \mathcal{G}^{H_-} \qquad \text{and} \qquad \mathfrak{G}^+ := \frac{1}{H_+}\mathcal{G}^{H_+}.$$

From the properties of the GFO [18, Proposition 1.2], $\mathfrak{G}^+\lambda \in \mathcal{C}^{\beta+H_+}$ as soon as $\lambda \in \Lambda_{\beta,H}$.

Corollary 1.9 (GFO representation)**.** *With the kernel $\mathtt{k}_{H_-}$ in (1.7) and $\lambda \in \Lambda_{\beta,H}$, system (1.5) under the risk-neutral measure $\mathbb{Q}$ can be rewritten as*

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= r_t \mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^{\mathbb{Q}}, \\ \quad v_t &= \psi\Big(t, (\mathfrak{G}^- Z^{\mathbb{Q}})(t) + (\mathfrak{G}^+\lambda)(t)\Big). \end{cases}$$

Proof. $\int_0^{\cdot} \mathtt{k}_{H_-}(\cdot - s)\mathrm{d}Z_s^{\mathbb{Q}} = \mathfrak{G}^- Z^{\mathbb{Q}}$ is straightforward by the properties of GFO in [18, Proposition 1.4]. Furthermore, for any $\lambda \in \Lambda_{\beta,H}$ and any $t \in \mathbb{T}$,

$$\int_0^t \mathtt{k}_{H_-}(t-s)\lambda_s \mathrm{d}s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\mathtt{K}(t,s)(\lambda_s - \lambda_0)\mathrm{d}s = \left(\mathfrak{G}^+\lambda\right)(t).$$

□

Note that since $\mathfrak{G}^+\lambda \in \mathcal{C}^{\beta+H_+}$, the risk premium has sample paths with Hölder regularity greater than $\frac{1}{2}$ regardless of the value of H.

2. **Modeling the risk premium process: A practical approach.** In practice, the process λ is directly modeled without resorting to a change of measure starting from γ. We now consider different modeling choices for the risk premium λ and analyze some of its properties. In spite of the formal derivation of Theorem 1.6, a numerical treatment of the integral $\int_0^t \bullet\, \mathrm{d}s$ is rather intricate. To overcome this issue, Bayer, Friz, and Gatheral [6] elegantly came up with the forward variance form of rough volatility in the spirit of Bergomi [8]. We shall restrict ourselves to this functional form (defined below in (2.1)) for the remainder of the section. Consider (1.5) with $\psi(t,x) = \xi_0(t)\mathrm{e}^{\nu x}$, $\xi_0(t) := \mathbb{E}[v_t|\mathcal{F}_0]$, and $\nu > 0$. Then, the risk-neutral dynamics in forward variance form read

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= r_t \mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{Q}}, \\ \quad v_t &= \xi_0(t)\exp\left(\nu\left(\displaystyle\int_0^t \mathtt{k}_{H_-}(t-s)\mathrm{d}Z_s^{\mathbb{Q}} + \int_0^t \mathtt{k}_{H_-}(t-s)\lambda_s \mathrm{d}s\right)\right). \end{cases} \tag{2.1}$$

In the remainder of this section, the process $X^{\mathbb{Q}}$ will denote a $\mathbb{Q}$-Brownian motion possibly correlated with $W^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$.

2.1. **Risk premium driven by Itô diffusion.** Generalized fractional operators provide a natural framework to model risk premium processes driven by diffusions. The statement below shows the details of such a construction. Recall that the Beta function is defined as $\mathfrak{B}(x,y) := \int_0^1 s^{x-1}(1-s)^{y-1}\mathrm{d}s$, for $x, y > 0$.

Proposition 2.1. *For $H \in (0, \frac{1}{2})$, $\alpha \in \left(-\frac{1}{2}, 0\right)$, let $\lambda := \mathfrak{b}\mathcal{G}^{\alpha}Y^{\mathbb{Q}} \in \mathcal{C}^{\alpha+\frac{1}{2}}$ with $\mathfrak{b} := \mathfrak{B}(H_+, \alpha+1)^{-1}$ and $Y_t^{\mathbb{Q}} = \int_0^t b(s, Y_s^{\mathbb{Q}})\mathrm{d}s + \int_0^t \sigma(s, Y_s^{\mathbb{Q}})\mathrm{d}X_s^{\mathbb{Q}}$, where $b(\cdot)$ and $\sigma(\cdot,\cdot)$ satisfy the Yamada-Watanabe conditions* [21, Section 5.2, Proposition 2.13] *for pathwise uniqueness ensuring a weak solution. Then, $\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}} \in \mathcal{C}^{H+\alpha+1}$ and*

$$v_t = \xi_0(t)\exp\left\{\nu\left((\mathfrak{G}^{-}Z^{\mathbb{Q}})(t) + (\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}})(t)\right)\right\}. \tag{2.2}$$

Furthermore, if $Y^{\mathbb{Q}} = X^{\mathbb{Q}}$ and $\mathrm{d}\langle Y^{\mathbb{Q}}, Z^{\mathbb{Q}}\rangle_t = \rho\,\mathrm{d}t$ with $\rho \le 0$, then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[v_t|\mathcal{F}_s\right] &= \xi_0(t)\exp\left\{\nu\left[\left(\mathfrak{G}^{-}Z^{\mathbb{Q}}\right)(s,t) + \left(\mathcal{G}^{\alpha+H_+}X^{\mathbb{Q}}\right)(s,t)\right]\right\} \\ &\quad\times\exp\left\{\frac{\nu^2}{2}\left(\frac{\mathtt{k}_{2H}(t-s)}{2H} + \frac{\mathtt{k}_{2(H+1)}(t-s)}{2H_+^2(H+1)} + \rho\frac{\mathtt{k}_{2H_+}(t-s)}{H_+^2}\right)\right\},\end{aligned} \tag{2.3}$$

where $\left(\mathcal{G}^{H_-}Z^{\mathbb{Q}}\right)(s,t) := \int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}}$ for $0 \le s \le t$, and similarly for $\left(\mathcal{G}^{\alpha+H_+}X^{\mathbb{Q}}\right)(s,t)$.

Proof. First, we prove (2.2), which follows from [18, Proposition 1.2] and the identities highlighted above in Corollary 1.9. Indeed, in view of Corollary 1.9, we only need to show that $\int_0^t \mathtt{k}_{H_-}(t-s)\lambda_s\mathrm{d}s = (\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}})(t)$. Replacing the expression for λ in the integral and using the stochastic Fubini theorem, we obtain

$$\begin{aligned}\int_0^t \mathtt{k}_{H_-}(t-s)\lambda_s\mathrm{d}s &= \mathfrak{b}\int_0^t \mathtt{k}_{H_-}(t-s)(\mathcal{G}^{\alpha}Y^{\mathbb{Q}})(s)\mathrm{d}s \\ &= \mathfrak{b}\int_0^t \mathtt{k}_{H_-}(t-s)\int_0^s \mathtt{k}_{\alpha}(s-u)\mathrm{d}Y_u^{\mathbb{Q}}\,\mathrm{d}s \\ &= \mathfrak{b}\int_0^t\int_u^t \mathtt{k}_{H_-}(t-s)\mathtt{k}_{\alpha}(s-u)\mathrm{d}s\,\mathrm{d}Y_u^{\mathbb{Q}}.\end{aligned}$$

Now, direct computations for the inner integral yield

$$\begin{aligned}\int_u^t \mathtt{k}_{H_-}(t-s)\mathtt{k}_{\alpha}(s-u)\mathrm{d}s &= \mathtt{k}_{\alpha+H_+}(t-u)\int_0^1(1-s)^{H_-}s^{\alpha}\mathrm{d}s \\ &= \mathfrak{B}(\alpha+1, H_+)\mathtt{k}_{\alpha+H_+}(t-u).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^t \mathtt{k}_{H_-}(t-s)\lambda_s\mathrm{d}s &= \mathfrak{b}\int_0^t \mathfrak{B}(\alpha+1, H_+)\mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}Y_u^{\mathbb{Q}} \\ &= \int_0^t \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}Y_u^{\mathbb{Q}} = (\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}})(t).\end{aligned}$$

We now move to the proof of the identity (2.3). Exploiting the representation of v_t in this specific case and the measurability and independence properties of the Brownian increments,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[v_t|\mathcal{F}_s\right] \\ &= \xi_0(t)\mathbb{E}_s^{\mathbb{Q}}\left[\exp\left\{\nu\left[\left(\mathcal{G}^{H_-}Z^{\mathbb{Q}}\right)(t) + \left(\mathcal{G}^{\alpha+H_+}X^{\mathbb{Q}}\right)(t)\right]\right\}\right]\end{aligned}$$

$$
\begin{aligned}
&= \xi_0(t)\mathbb{E}_s^{\mathbb{Q}}\left[\exp\left\{\nu\left[\int_0^t \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^t \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}\right]\right\}\right]\\
&= \xi_0(t)\exp\left\{\nu\left[\int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^s \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}\right]\right\}\\
&\qquad \times \mathbb{E}_s^{\mathbb{Q}}\left[\exp\left\{\nu\left[\int_s^t \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_s^t \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}\right]\right\}\right]\\
&= \xi_0(t)\exp\left\{\nu\left[\int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^s \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}\right]\right\}\\
&\qquad \times \exp\left\{\frac{\nu^2}{2}\left(\int_s^t \mathtt{k}_{2H_-}(t-u)\mathrm{d}u + \int_s^t \frac{\mathtt{k}_{2H+1}(t-u)}{H_+^2}\mathrm{d}u + \rho\int_s^t \frac{\mathtt{k}_{2H}(t-u)}{H_+}\mathrm{d}u\right)\right\}\\
&= \xi_0(t)\exp\left\{\nu\left[\int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^s \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}\right]\right\}\\
&\qquad \times \exp\left\{\frac{\nu^2}{2}\left(\frac{\mathtt{k}_{2H}(t-s)}{2H} + \frac{\mathtt{k}_{2(H+1)}(t-s)}{2H_+^2(H+1)} + \rho\frac{\mathtt{k}_{2H_+}(t-s)}{H_+^2}\right)\right\}.
\end{aligned}
$$

Thus, we only have to show that

$$
\left(\mathcal{G}^{H_-}Z^{\mathbb{Q}}\right)(s,t) = \int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} \text{ and } \left(\mathcal{G}^{\alpha+H_+}X^{\mathbb{Q}}\right)(s,t) = \int_0^s \mathtt{k}_{\alpha+H_+}(t-u)\mathrm{d}X_u^{\mathbb{Q}}.
$$

We prove the first identity, the second being analogous. It is a straightforward consequence of the definitions and the properties of Brownian increments:

$$
\begin{aligned}
\left(\mathcal{G}^{H_-}Z^{\mathbb{Q}}\right)(s,t) &:= \mathbb{E}_s^{\mathbb{Q}}\left[\int_0^t \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}}\right]\\
&= \int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \mathbb{E}_s^{\mathbb{Q}}\left[\int_s^t \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}}\right]\\
&= \int_0^s \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}}.
\end{aligned}
$$

□

Remark 2.2. Since the instantaneous variance in this model is log-Normal, the results in [19, Proposition 3.1] and numerical methods therein still apply for the VIX with minimal changes.

2.2. **A risk premium driven by a CIR process.** A second natural choice is to consider the Cox-Ingersoll-Ross (CIR) process

$$
\mathrm{d}Y_s^{\mathbb{Q}} = \kappa(\theta - Y_s^{\mathbb{Q}})\mathrm{d}s + \sigma\sqrt{Y_s^{\mathbb{Q}}}\,\mathrm{d}X_s^{\mathbb{Q}}, \tag{2.4}
$$

with $\kappa, \theta, \sigma > 0$. As tempting as this approach might seem, it is not trivial at all to compute the basic quantity $\mathbb{E}^{\mathbb{Q}}[v_t]$ here, as the following proposition shows.

Proposition 2.3. *Assume that the Brownian motions $Z^{\mathbb{Q}}$ and $X^{\mathbb{Q}}$ are independent and consider $\lambda = \mathcal{G}^{\alpha}Y^{\mathbb{Q}} \in \mathcal{C}^{\alpha+\frac{1}{2}}$, with $Y^{\mathbb{Q}}$ defined in (2.4). Then, for any $s \le t$,*

$$
\begin{aligned}
\mathbb{E}_s^{\mathbb{Q}}[v_t] = \xi_0(t)\exp&\left\{\nu\left[\left(\mathfrak{G}^{-}Z^{\mathbb{Q}}\right)(s,t) + \left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(s,t)\right]\right\}\\
&\exp\left\{\frac{\nu^2}{2}\int_s^t \mathtt{k}_{H_-}(t-u)^2\mathrm{d}u - Y_s^{\mathbb{Q}}C(s,T) - A(s,T)\right\},
\end{aligned}
$$

where $A(t,T) := -\kappa\theta \int_t^T C(u,T)\mathrm{d}u$, and C satisfies the Riccati equation

$$\nu \mathtt{k}_{H_-}(T,t) - \partial_t C(t,T) + C(t,T)\theta + \frac{\sigma^2}{2}C^2(t,T) = 0,$$

for $t \in [0,T)$, with boundary condition $C(T,T) = 0$.

Proof. By independence of the driving Brownian motions, we have, for any $u \le t$,

$$\begin{aligned}\mathbb{E}[v_t|\mathcal{F}_u] = \xi_0(t)\exp\left\{\nu\left(\left(\mathfrak{G}^- Z^{\mathbb{Q}}\right)(u,t) + \left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(u,t)\right)\right\}\\ \times\, \mathbb{E}\left[\exp\left\{\nu\left(\left(\mathfrak{G}^- Z^{\mathbb{Q}}\right)(t) - \left(\mathfrak{G}^- Z^{\mathbb{Q}}\right)(u,t)\right)\right\}\right]\\ \times\, \mathbb{E}\left[\exp\left\{\nu\left(\left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(t) - \left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(u,t)\right)\right\}\right],\end{aligned}$$

where the first expected value is the MGF of a Gaussian random variable, hence

$$\mathbb{E}\left[\exp\left\{\nu\Big(\left(\mathfrak{G}^- Z^{\mathbb{Q}}\right)(t) - \left(\mathfrak{G}^- Z^{\mathbb{Q}}\right)(u,t)\Big)\right\}\right] = \exp\left\{\frac{\nu^2}{2}\int_u^t \mathtt{k}_{H_-}(t,s)\mathrm{d}s\right\}.$$

We are now interested in computing the second expectation

$$\mathbb{E}\left[\exp\left\{\nu\left(\left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(t) - \left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(u,t)\right)\right\}\right],$$

where $\left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(t) - \left(\mathcal{G}^{\alpha+H_+}Y^{\mathbb{Q}}\right)(u,t) = \displaystyle\int_u^t \mathtt{k}_{H_-}(t,s)Y_s^{\mathbb{Q}}\mathrm{d}s$. This is, in spirit, similar to computing a bond price in the CIR model. To do so, define

$$B(t,T) := \mathbb{E}\left[\exp\left(\nu\int_t^T \mathtt{k}_{H_-}(T,s)Y_s^{\mathbb{Q}}\mathrm{d}s\right)\middle|\mathcal{F}_t\right], \tag{2.5}$$

where $t \le T$. We note that $B(\cdot,T)$ is a semimartingale as T is fixed, therefore, applying the conditional version of Feynman-Kac's formula, we obtain

$$\left(\nu r\mathtt{k}_{H_-}(T,t) + \partial_t + \kappa(\theta - y)\partial_r + \frac{\sigma^2}{2}r\partial_{yy}\right)\mathcal{B}(y,t,T) = 0. \tag{2.6}$$

where $\mathcal{B}(y,t,T)$ is such that $\mathcal{B}(Y_t^{\mathbb{Q}},t,T) = B(t,T)$ given in (2.5). With an ansatz of the type $\mathcal{B}(y,t,T) = \exp\{-yC(t,T) - A(t,T)\}$, we have at (y,t,T),

$$\begin{aligned}&\partial_t\mathcal{B}(y,t,T) = -(y\partial_t C(t,T) + \partial_t A(t,T))\mathcal{B}(y,t,T),\\ &\partial_y\mathcal{B}(y,t,T) = -C(t,T)\mathcal{B}(y,t,T), \qquad \partial_{yy}\mathcal{B}(y,t,T) = C(t,T)^2\mathcal{B}(y,t,T),\end{aligned}$$

and the PDE (2.6) becomes, with $r = Y_t^{\mathbb{Q}}$,

$$\left(\nu Y_t^{\mathbb{Q}}\mathtt{k}_{H_-}(T,t) - \left(Y^{\mathbb{Q}}\partial_t C + \partial_t A + \kappa\left(\theta - Y_t^{\mathbb{Q}}\right)C\right) + \frac{\sigma^2}{2}C^2 Y_t^{\mathbb{Q}}\right)\mathcal{B}(Y_t^{\mathbb{Q}},t,T) = 0,$$

which further simplifies to

$$\left(\nu\mathtt{k}_{H_-}(T,t) - \partial_t C - \kappa C + \frac{\sigma^2}{2}C^2\right)Y_t^{\mathbb{Q}}\mathcal{B}(Y_t^{\mathbb{Q}},t,T) - (\kappa\theta C + \partial_t A)\mathcal{B}(Y_t^{\mathbb{Q}},t,T) = 0.$$

The last term cancels for $A(t,T) = -\kappa\theta\int_t^T C(u,T)\mathrm{d}t$, and the Riccati equation $\nu\mathtt{k}_{H_-}(T,t) - \partial_t C(t,T) - \kappa C(t,T) + \frac{\sigma^2}{2}C^2(t,T) = 0$ remains, with $A(T,T) = C(T,T) = 0$. □

In the uncorrelated case, the computation of $\mathbb{E}^{\mathbb{Q}}[v_t]$ becomes very costly, having to solve a quadratic ODE (with time-dependent coefficients) for each time t. In the correlated case, there is no hope to obtain any semi-analytic result since one would need to compute cross terms, and there is no tool coming from Itô's calculus available in that case. The approach in this section was essentially *top-down*, meaning that we specified a form for the market price of volatility risk λ and deduced the shape of the model with this specification. Unfortunately, our analysis showed that this may ultimately not be so successful as the final form of the model is rather intricate, probably too much so for practical purposes. Alternatively, one may first infer some shape of λ from market data (short rate of interest, expected returns and instantaneous volatility) and then use it to price options under $\mathbb{Q}$.

3. **Roughly extracting the risk premium from the market.** We now consider the risk premium process λ to be deterministic, and obtain a formula linking $\mathbb{P}$ and $\mathbb{Q}$ market observable quantities. The following theorem shows how to infer the risk premium from the market using forecasts under $\mathbb{P}$ and variance swap prices under $\mathbb{Q}$.

Theorem 3.1. *Consider the rough volatility model (1.1) under $\mathbb{P}$. If $\psi(t,x) = \xi_0(t)\mathrm{e}^{\nu x}$, $\mu_s = r_s$ for all $s \geq 0$, and $(\lambda_s)_{s\geq 0} \in L^2(\mathbb{R})$ is deterministic, then*

$$\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u \mathrm{d}u = \log\left(\frac{\mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_s]}{\mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_s]}\right) = \log\left(\frac{\xi_s(t)}{\mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_s]}\right). \tag{3.1}$$

Proof. If $\mu = r$ almost surely, the Radon-Nikodym derivative (1.2) in Theorem 1.6 reads $\mathcal{D}^\gamma = \mathcal{E}\left(\int_0^\cdot \gamma_s \mathrm{d}W_s^{\mathbb{P}\perp}\right)$, with $\lambda_s = \overline{\rho}\gamma_s$, and the inverse Radon-Nikodym derivative is given by $\mho^\gamma := \frac{1}{\mathcal{D}^\gamma} = \mathcal{E}\left(-\int_0^\cdot \gamma_s \mathrm{d}W_s^{\mathbb{Q}\perp}\right)$. Then, the conditional change of measure formula yields

$$\mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_s] = \frac{\mathbb{E}^{\mathbb{Q}}\left[v_t\mho_t^\gamma|\mathcal{F}_s\right]}{\mathbb{E}^{\mathbb{Q}}\left[\mho_t^\gamma|\mathcal{F}_s\right]}. \tag{3.2}$$

On the one hand, $\mathbb{E}^{\mathbb{Q}}\left[\mho_t^\gamma|\mathcal{F}_s\right] = \mathcal{E}\left(-\int_0^\cdot \gamma_u \mathrm{d}W_u^{\mathbb{Q}\perp}\right)_s$ by the properties of the stochastic exponential and Gaussian moment generating functions. On the other hand, since, for $t \in \mathbb{T}$, $Z_t^{\mathbb{Q}} = Z_t^{\mathbb{P}} + \int_0^t \lambda_s \mathrm{d}s$ and λ is deterministic, then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[v_t\mho_t^\gamma|\mathcal{F}_s\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\nu\left(\int_0^t \mathtt{k}_{H_-}(t,u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^t \lambda_u \mathtt{k}_{H_-}(t,u)\mathrm{d}u\right)\right\}\mathrm{e}^{-\int_0^t \gamma_u \mathrm{d}W_u^{\mathbb{Q}\perp} - \frac{1}{2}\int_0^t \gamma_u^2 \mathrm{d}u}\middle|\mathcal{F}_s\right]\\
&= \mathrm{e}^{\nu\int_0^t \lambda_u \mathtt{k}_{H_-}(t,u)\mathrm{d}u - \frac{1}{2}\int_0^t \gamma_u^2\mathrm{d}u}\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\nu\int_0^t \mathtt{k}_{H_-}(t,s)\mathrm{d}Z_s^{\mathbb{Q}} - \int_0^t \gamma_s \mathrm{d}W_s^{\mathbb{Q}\perp}\right\}\middle|\mathcal{F}_s\right],
\end{aligned}$$

where the second factor in the last term is just the conditional moment generating function of a Gaussian random variable. Applying Itô's isometry, then, conditionally on $\mathcal{F}_s$, the random variable $\nu\int_0^t \mathtt{k}_{H_-}(t,s)\mathrm{d}Z_s^{\mathbb{Q}} - \int_0^t \gamma_s \mathrm{d}W_s^{\mathbb{Q}\perp}$is distributed as $\mathcal{N}\left(\mu, \sigma^2\right)$ with

$$\begin{aligned}
\mu &:= \nu\int_0^s \mathtt{k}_{H_-}(t,u)\mathrm{d}Z_u^{\mathbb{Q}} - \int_0^s \gamma_u \mathrm{d}W_u^{\mathbb{Q}\perp},\\
\sigma^2 &:= \nu^2\int_s^t k^2(t,u)\mathrm{d}u + \int_s^t \gamma_u^2\mathrm{d}u - 2\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u\mathrm{d}u,
\end{aligned}$$

since $Z^{\mathbb{Q}} = \rho W^{\mathbb{Q}} + \overline{\rho} W^{\mathbb{Q}\perp}$. Exploiting the identities above and reordering terms,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[v_t \mathfrak{C}_t^{\gamma} | \mathcal{F}_s\right] \\
&= \exp\Bigg\{\nu \int_0^t \mathtt{k}_{H_-}(t,u)\lambda_u \mathrm{d}u + \overbrace{\nu \int_0^s \mathtt{k}_{H_-}(t,u)\mathrm{d}Z_u^{\mathbb{Q}} - \int_0^s \gamma_u \mathrm{d}W_u^{\mathbb{Q}\perp}}^{\mu} \\
&\quad + \frac{1}{2}\Bigg(\underbrace{\nu^2 \int_s^t k^2(t,u)\mathrm{d}u + \int_s^t \gamma_u^2 \mathrm{d}u - 2\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u \mathrm{d}u}_{\sigma^2} - \int_0^t \gamma_u^2 \mathrm{d}u\Bigg)\Bigg\} \\
&= \exp\left\{-\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u \mathrm{d}u\right\} \overbrace{\exp\left\{-\int_0^s \gamma_u \mathrm{d}W_u^{\mathbb{Q}\perp} - \frac{1}{2}\int_0^s \gamma_u^2 \mathrm{d}u\right\}}^{\mathbb{E}^{\mathbb{Q}}[\mathfrak{C}_t^{\gamma}|\mathcal{F}_s]} \\
&\quad \underbrace{+ \exp\left\{\nu\int_0^s \mathtt{k}_{H_-}(t,u)\mathrm{d}Z_u^{\mathbb{Q}} + \frac{\nu^2}{2}\int_s^t \mathtt{k}_{H_-}^2(t,u)\mathrm{d}u + \nu\int_0^t \mathtt{k}_{H_-}(t,u)\lambda_u \mathrm{d}u\right\}}_{\mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_s]},
\end{aligned}$$

by using the decomposition of σ^2 as the sum of three terms, and so

$$\mathbb{E}^{\mathbb{Q}}\left[v_t \mathfrak{C}_t^{\gamma} | \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathfrak{C}_t^{\gamma} | \mathcal{F}_s\right] \mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_s] \exp\left\{-\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u \mathrm{d}u\right\}. \tag{3.3}$$

Finally, going back to (3.2) and exploiting the identity in (3.3), the result follows from

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_s] &= \mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_s] \exp\left\{-\nu\overline{\rho}\int_s^t \mathtt{k}_{H_-}(t,u)\gamma_u \mathrm{d}u\right\} \\
&= \mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_s] \exp\left\{-\nu\int_s^t \mathtt{k}_{H_-}(t,u)\lambda_u \mathrm{d}u\right\}.
\end{aligned}$$

□

3.1. **Estimating the risk premium in rough Bergomi.** In this section, we work with the rough Bergomi model under $\mathbb{P}$ and its $\mathbb{Q}$-version:

$$\text{(under } \mathbb{P}) \quad \begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= \mu_t \mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{P}}, \\ v_t &= \exp\left\{\nu Z_t^H\right\}, \end{cases}$$

$$\text{(under } \mathbb{Q}) \quad \begin{cases} \dfrac{\mathrm{d}S_t}{S_t} &= r_t \mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^{\mathbb{Q}}, \\ v_t &= \xi_0(t)\exp\left\{\nu\left(\int_0^t \mathtt{k}_{H_-}(t-u)\mathrm{d}Z_u^{\mathbb{Q}} + \int_0^t \mathtt{k}_{H_-}(t-u)\lambda_u \mathrm{d}u\right)\right\}. \end{cases}$$

Assuming λ is deterministic, Theorem 3.1 gives an explicit procedure to compute the risk premium. In practice however, we are only able to observe variance swap quotes in discrete times, and hence it is natural to consider λ piecewise constant.

Assumption 3.2. Given a time partition $\{0 = T_0 < T_1, ..., < T_n = T\}$, the deterministic process λ admits the piecewise constant representation

$$\lambda(t) := \sum_{i=1}^{n} \lambda_i \mathbb{1}_{\{t \in [T_{i-1}, T_i)\}}, \quad \lambda_i \in \mathbb{R} \text{ for } i = 1, \dots, n. \tag{3.4}$$

Similarly, the forward variance admits the piecewise constant representation $\mathbb{E}_0^{\mathbb{Q}}[v_t] = \xi_0(t) := \sum_{i=1}^{n} \xi_i \mathbb{1}_{\{t\in[T_{i-1},T_i)\}}$ with $\xi_i \in \mathbb{R}$ for $i = 1, \ldots, n$, where $\xi_i := \frac{\mathfrak{V}_{T_i} T_i - \mathfrak{V}_{T_{i-1}} T_{i-1}}{T_i - T_{i-1}}$ and $\mathfrak{V}_T := \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T v_s \mathrm{d}s\right]$ is a market variance swap quote.

We now estimate $\{\lambda_1, \cdots, \lambda_n\}$. The dataset consists of daily Eurostoxx variance swap quotes for maturities {1M, 3M, 6M, 1Y, 2Y} (Figures 1 and 2), while Figure 3 shows the daily realized volatility obtained from Oxford-Man institute data.

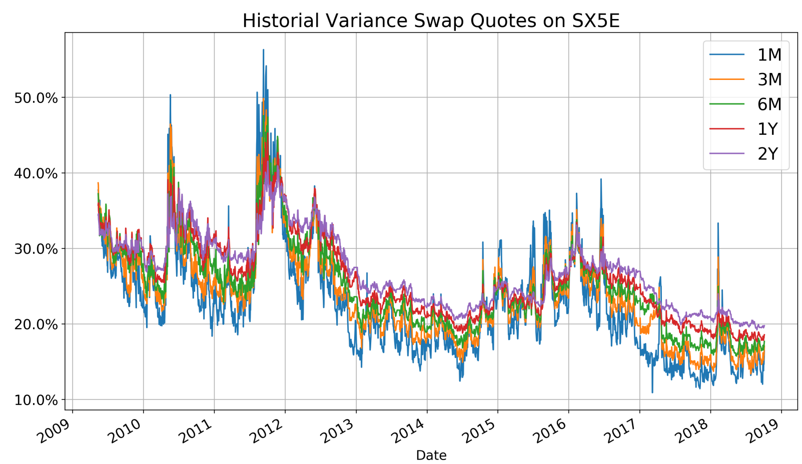

FIGURE 1. Variance swap volatility daily quotes on SX5E

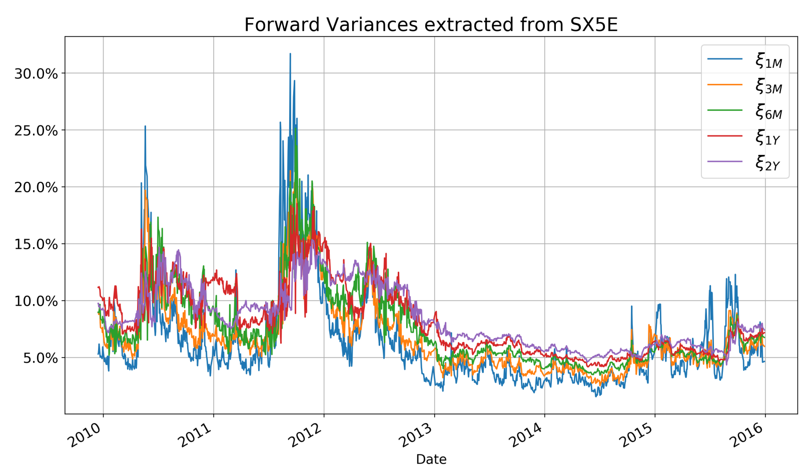

FIGURE 2. Forward variances extracted from variance swap quotes on SX5E

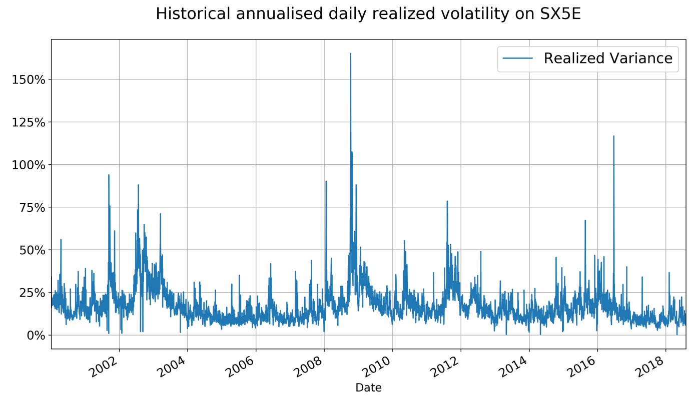

Figure 3. Annualized daily realized volatility on SX5E

In order to apply formula (3.1), we need the following ingredients:

$$\text{Parameters } H, \nu, \rho, \qquad \mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_0], \qquad \mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_0]. \tag{3.5}$$

So far we have obtained $\mathbb{E}^{\mathbb{Q}}[v_t|\mathcal{F}_0]$ from variance swap market quotes. The next step is to estimate (H, ν, ρ) using historical time-series. Gatheral, Jaisson, and Rosenbaum [14] explain how to estimate $\widehat{H}$ and $\widehat{\nu}$ from daily volatility data (Figure 3), and we follow their approach using a 100-day rolling window (Figure 4) and refer the reader to the original paper for details.

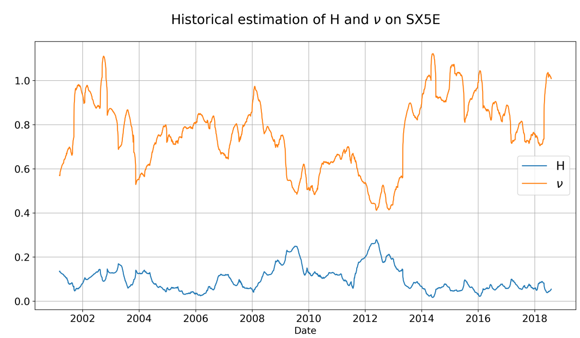

Figure 4. Estimated $\widehat{H}$ and $\widehat{\nu}$ on SX5E.

To estimate the correlation parameter, we use $\text{Corr}\left(Z_t^H - Z_s^H, \int_s^t \mathrm{d}W_s\right) = \frac{\rho\sqrt{2H}}{H_+}$, which allows us to estimate the correlation with the proxy

$$\widehat{\rho} = \frac{\widehat{H} + \frac{1}{2}}{\sqrt{2\widehat{H}}} \widehat{\text{Corr}} \left(\frac{\log(S_{t_i}) - \log(S_{t_{i-1}})}{\sqrt{v_{t_{i-1}}}}, \log(v_{t_i}) - \log(v_{t_{i-1}}) \right).$$

Figure 5 below displays the historical estimates using an estimation window of 100 days.

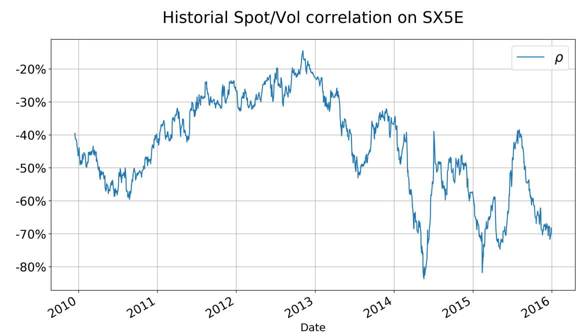

FIGURE 5. Daily correlation estimate on SX5E and realized volatility.

To forecast volatility and obtain $\mathbb{E}^{\mathbb{P}}[v_t|\mathcal{F}_0]$, we proceed as in [14], and use the forecasting formula for the fractional Brownian motion due to Nuzman and Poor [23]:

$$Z^H_{t+\Delta}|\mathcal{F}_t \sim \mathcal{N}\left(\frac{\cos(H\pi)}{\pi}\Delta^{H_+}\int_0^t \frac{Z^H_s \mathrm{d}s}{(t-s+\Delta)(t-s)^{H_+}}, \frac{C_H\Delta^{2H}}{2H}\right).$$

Finally, we orderly estimate λ_i for each $i = 1, \ldots, n$ using Theorem 3.1 and the piecewise constant assumption (3.4), as

$$\sum_{j=1}^{i} \lambda_j \int_{T_{j-1}}^{T_j} \mathtt{k}_{H_-}(t,u)\mathrm{d}u = \frac{1}{\nu(1-\rho^2)} \log\left(\frac{\xi_0(T_i)}{\mathbb{E}^{\mathbb{P}}[v_{T_i}|\mathcal{F}_0]}\right).$$

Figure 6 shows the historical evolution of the risk premium process.

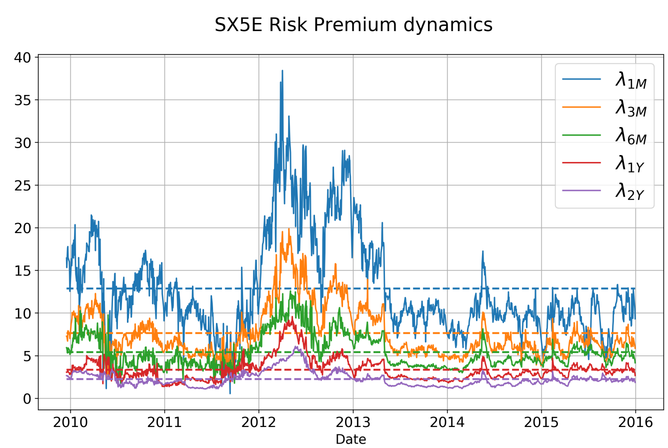

FIGURE 6. Daily SX5E risk premia; dashed lines represent means.

Remark 3.3. We would like to emphasize that assessing the best method to estimate (3.5) is beyond the scope of this paper. However, as highlighted in the introduction, we stress the importance of Theorem 3.1 toward which this empirical work provides a first step.

Appendix A. **Proof of Remark 1.5.** Following the ideas in [13, Proof of Theorem 1.1], we show that Assumption 1.3(iv) is guaranteed provided that Assumptions 1.1 and 1.3(i)-(ii) hold and the processes $\mu, \gamma, andr$ belong to $\mathbb{F}_b$. From the definition of the change of measure and the stopping time in (1.4), then $\mathcal{D}^{\gamma}_{t\wedge\tau_n} = \frac{\mathrm{d}\widehat{\mathbb{P}}_n}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}$.
For any $s \in \mathbb{T}$, the random function $f(x) := \frac{r_s-\mu_s}{\sqrt{\psi(s,x)}}$ is $\mathbb{P}$-bounded on $(-\infty, a]$ for any $a > 0$ since r and μ are $\mathbb{P}$-bounded, and $\psi(s,\cdot)$ is bounded away from zero on intervals of the form $(-\infty, a]$ by Assumption 1.3(i) together with the additional assumptions in Remark 1.5. Then, by Proposition 1.2,

$$1 = \mathbb{E}\left[\mathcal{D}^{\gamma}_{T\wedge\tau_n}\right] = \mathbb{E}\left[\mathcal{D}^{\gamma}_{T}\mathbb{1}_{\{T<\tau_n\}}\right] + \mathbb{E}\left[\mathcal{D}^{\gamma}_{\tau_n}\mathbb{1}_{\{\tau_n\le T\}}\right]. \tag{A.1}$$

The first term in (A.1) converges to $\mathbb{E}\left[\mathcal{D}^{\gamma}_{T}\right]$ as n tends to infinity, yielding

$$1 - \mathbb{E}\left[\mathcal{D}^{\gamma}_{T}\right] = \lim_{n\uparrow\infty} \mathbb{E}\left[\mathcal{D}^{\gamma}_{\tau_n}\mathbb{1}_{\{\tau_n\le T\}}\right].$$

Girsanov's theorem implies $\mathbb{E}\left[\mathcal{D}^{\gamma}_{\tau_n}\mathbb{1}_{\{\tau_n\le T\}}\right] = \widehat{\mathbb{P}}_n(\tau_n \le T)$, where $\widehat{\mathbb{P}}_n$ is defined such that $\widehat{W}^n_t = W^{\mathbb{P}}_t - \int_0^{t\wedge\tau_n}\chi_u \mathrm{d}u$ is a $\widehat{\mathbb{P}}_n$-Brownian motion. Then, under $\widehat{\mathbb{P}}_n$, the process Y becomes

$$Y_t = \widehat{Y}^n_t + \int_0^{t\wedge\tau_n} k(t,s)\Big(\rho\chi_u + \overline{\rho}\,\gamma_u\Big)\mathrm{d}u = \widehat{Y}^n_t + \int_0^{t\wedge\tau_n} k(t,s)\lambda_u \mathrm{d}u,$$

where $\widehat{Y}^n_t := \displaystyle\int_0^t k(t,s)\mathrm{d}\widehat{Z}^n_s$ and

$$\widehat{Z}^n_t = Z^{\mathbb{P}}_t - \int_0^{t\wedge\tau_n}\Big(\rho\chi_u + \overline{\rho}\,\gamma_u\Big)\mathrm{d}u = Z^{\mathbb{P}}_t - \int_0^{t\wedge\tau_n}\lambda_u \mathrm{d}u,$$

for $t \ge 0$, where $\widehat{Z}^n$ is a $\widehat{\mathbb{P}}_n$-Brownian motion. Furthermore, by Assumption 1.3(ii), we have

$$\begin{aligned}
\widehat{\mathbb{P}}_n\left(\sup_{t\in\mathbb{T}} Y_t \ge n\right) &= \widehat{\mathbb{P}}_n\left(\sup_{t\in\mathbb{T}}\left\{\widehat{Y}^n_t + \int_0^{t\wedge\tau_n} k(t,u)\lambda_u\mathrm{d}u\right\} \ge n\right) \\
&\le \widehat{\mathbb{P}}_n\left(\sup_{t\in\mathbb{T}}\widehat{Y}^n_t + \sup_{t\in\mathbb{T}}\left\{\int_0^{t\wedge\tau_n} k(t,u)\lambda_u\mathrm{d}u\right\} \ge n\right) \\
&\le \widehat{\mathbb{P}}_n\left(\sup_{t\in\mathbb{T}}\widehat{Y}^n_t \ge n - K_{\mathbb{T}}\right),
\end{aligned} \tag{A.2}$$

Inequality (A.2), in turn, implies $\widehat{\mathbb{P}}_n(\tau_n \le T) \le \widehat{\mathbb{P}}_n(\widehat{\tau}_n \le T)$ for $\widehat{\tau}_n := \inf\{t \ge 0,\ \widehat{Y}^n_t = n - K_{\mathbb{T}}\}$. Finally, since $\widehat{Z}^n$ is a $\widehat{\mathbb{P}}_n$-Brownian motion, we obtain

$$\lim_{n\uparrow\infty}\widehat{\mathbb{P}}_n(\tau_n \le T) \le \lim_{n\uparrow\infty}\widehat{\mathbb{P}}_n(\widehat{\tau}_n \le T) = \lim_{n\uparrow\infty}\mathbb{P}\left(\sup_{t\in\mathbb{T}} Y_t \ge n - K_{\mathbb{T}}\right) = 0,$$

and it follows that $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is indeed a true martingale and note that $\lim\limits_{n\uparrow\infty}\widehat{\mathbb{P}}_n = \mathbb{Q}$ in the sense that relation (1.6) holds between the $\mathbb{P}$ and $\mathbb{Q}$ Brownian motions.

The data underpinning this study were obtained from a combination of publicly and commercially available sources:

- *Oxford-Man Institute of Quantitative Finance Realized Library: Available for academic research use at https://realized.oxford-man.ox.ac.uk/ upon registration and agreement to the terms of use.*
- *Yahoo Finance: Market data were retrieved from https://finance.yahoo.com, a publicly accessible platform, under their data usage terms.*
- *Bloomberg Terminal: Data sourced from Bloomberg are subject to licensing restrictions and cannot be shared publicly. Access to Bloomberg data requires an institutional subscription.*

Some components of the dataset are restricted due to commercial licensing agreements, and therefore cannot be made openly available. The decision to restrict access is to comply with the contractual obligations agreed with commercial data providers. The present authors may be contacted for further information about the datasets used here.

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Received March 18, 2024; revised June 13, 2025.