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Universality for degenerate hypergraphs

Peter Allen[a], Julia Böttcher[a], Jasmin Katz[a,*]

[a] Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, UK

Abstract

A graph Γ is said to be *universal* for a class of graphs $\mathcal{H}$ if Γ contains a copy of every $H \in \mathcal{H}$ as a subgraph. The number of edges required for a host graph Γ to be universal for the class of D-degenerate graphs on n vertices has been shown to be $O((\log n)^{2/D}(\log\log n)^5 n^{2-1/D})$. We generalise this result to r-uniform hypergraphs, showing the following. Given $D, r \geq 2$ and n sufficiently large, there exists a constant $C = C(D, r)$ such that there exists a graph with at most

$$Cn^{r-1/D}(\log n)^{2/D}(\log\log n)^{2r+1}$$

edges, which is universal for the class of D-degenerate r-uniform hypergraphs on n vertices. This is tight up to the multiplicative constant and polylogarithmic term.

Keywords: Universality ; Probabilistic Combinatorics ; Extremal Combinatorics ; Degenerate Graphs ; Hypergraphs

1. Introduction

A graph Γ is said to be *universal* for a class of graphs $\mathcal{H}$ if Γ contains a copy of every $H \in \mathcal{H}$ as a (not necessarily induced) subgraph. We say such a graph is $\mathcal{H}$-universal. The number of edges required for a host graph to be universal for various classes of graphs such as trees, bounded degree graphs and degenerate graphs is an active area of research in extremal graph theory.

The class $\mathcal{H}^{(r)}_{\Delta}(n)$ of n vertex r-graphs with maximum degree Δ were the first class of hypergraphs for which this question was asked. The minimum number of edges required for a graph to be $\mathcal{H}^{(r)}_{\Delta}(n)$-universal can be shown via a counting argument to be $\Omega(n^{r-r/\Delta})$. In the $r = 2$ case, an explicit construction was given by Alon and Capalbo [2], which matches this bound. This result was extended to $r \geq 2$ by Hetterich, Parczyk and Person in 2016 [6] for r even, $r|\Delta$ or $\Delta = 2$. Recently, Nenadov [7] extended this result to general $r \geq 2$, but with a polylogarithmic error term. The

* Corresponding author.
E-mail address: j.katz3@lse.ac.uk

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proofs of both of these hypergraph results utilise the original construction of Alon and Capalbo. These results give deterministic constructions (although some of the proofs of universality utilise the probabilistic method).

Another active area of research is determining the threshold probability for a random r-graph $H^{(r)}(n,p)$ to be universal for $\mathcal{H}^{(r)}_{\Delta}(n)$. The random graph $H^{(r)}(n,p)$ is the probability space of all labelled r-graphs on n vertices, where each set of r vertices is chosen as an edge independently with probability p. For $r = 2$, Alon and Capalbo [3] showed that for every $\varepsilon > 0$ the random graph $H^{(2)}(n,p)$ is asymptotically almost surely (a.a.s) $\mathcal{H}^{(2)}_{\Delta}((1-\varepsilon)n)$-universal for the natural bound $p = O((\log n/n)^{1/\Delta})$. This result was further improved and generalised to hypergraphs by Parczyk and Person [8], who showed that for $p = O((\log n/n)^{1/\Delta})$ the random r-graph $H^{(r)}(n,p)$ is $\mathcal{H}^{(r)}_{\Delta}(n)$-universal. However, this bound is not tight. In the $r = 2$ case, Ferber and Nenadov [5] have shown that $H^{(2)}(n,p)$ is $\mathcal{H}^{(2)}_{\Delta}(n)$-universal for $p = O(n^{-1/(\Delta-0.5)}\log^3 n)$.

We turn our attention to the focus of this paper — universality for degenerate r-graphs. An r-graph G is *D-degenerate* if every induced r-subgraph has a vertex of degree at most D. Equivalently, there exists an ordering of the vertices $v_1,\dots,v_n$ such that every vertex v_i is adjacent to at most D edges contained entirely in the set $\{v_1,\dots v_i\}$. We will denote the class of all n-vertex, D-degenerate r-graphs by $\mathcal{H}^{(r)}(n,D)$, and the subclass of D-degenerate r-graphs with maximum degree Δ by $\mathcal{H}^{(r)}_{\Delta}(n,D)$.

When $D \ll \Delta$, it is possible to find tighter bounds on the values of p for which the random r-graph is universal for the class $\mathcal{H}^{(r)}_{\Delta}(n,D)$ than for $\mathcal{H}^{(r)}_{\Delta}(n)$ alone. The class $\mathcal{H}^{(r)}_{\Delta}(n,D)$ has mostly been studied in the $r = 2$ case, for which Ferber and Nenadov [5] showed that for $p \geq (\log^3 n/n)^{1/2D}$, the graph $H^{(2)}(n,p)$ is $\mathcal{H}^{(2)}_{\Delta}(n,D)$-universal. For $r > 2$, Nenadov gave an upper bound on the number of edges required for universality for hypergraphs of bounded *density*. Noting that a D-degenerate graph has density at most D, Nenadov [7] showed that there exists an $\mathcal{H}^{(r)}_{\Delta}(n,D)$-universal r-graph with at most $Cn^{r-1/D}\log^{1/D} n$ edges. When we remove the requirement for bounded degree, the class $\mathcal{H}^{(r)}(n,D)$ may contain graphs with an $(r-1)$-set in $n-r+1$ edges. For any reasonable p, the random r-graph does not contain any such set of vertices. Nonetheless, for $r = 2$ and $D > 1$ a random construction was used to prove the existence of $\mathcal{H}^{(r)}(n,D)$-universal graphs on $O(n^{2-1/D}(\log^{2/D} n)(\log\log n)^5)$ edges by Allen, Böttcher and Liebenau [1], and this is tight up to the polylogarithmic factor. In this paper, we extend this random construction to hypergraphs to give the bound stated in Theorem 1.1.

Theorem 1.1. *Given $D, r \geq 2$ and n sufficiently large, there exists a constant $C = C(D,r)$ such that there exists a graph with at most*

$$Cn^{r-1/D}(\log n)^{2/D}(\log\log n)^{2r+1}$$

edges which is $\mathcal{H}(n,D)$-universal.

This result is tight up to the polylogarithmic factor, as shown by the following result. In fact, we prove the following slightly stronger statement: if Γ contains a copy of every connected n-vertex D-degenerate graph with maximum degree bounded by $rD+1$, then $e(\Gamma) \geq \frac{1}{100r^2D}n^{r-1/D}$.

Theorem 1.2. *Suppose $r \geq 2$, $D \geq 1$ and let n be sufficiently large. Suppose Γ is a $\mathcal{H}^{(r)}_{rD+1}(n,D)$-universal r-graph. Then $e(\Gamma) \geq \frac{1}{100r^2D}n^{r-1/D}$.*

2. Sketch of Proof

2.1. Block Model and Embedding Strategy

Throughout, all logarithms are base 2. For a set S, we let $\binom{S}{k}$ denote the set of k-element subsets of S. In this notation, an r-graph $H = (V,E)$ is a hypergraph where each edge $s \in E$ is an element of $\binom{V}{r}$. Let $H = (V,E)$ be an r-graph. For some $v \in V(H)$, we denote the *link* of v by $L(v) = \{X \in \binom{V}{r-1} : X \cup v \in E(H)\}$. Suppose we have an ordering of vertices $v_1,\dots,v_n$ of an r-graph H. Denote by $L^-(v_i) = L(v_i) \cap \binom{\{v_1,\dots,v_{i-1}\}}{r-1}$ the *back-link* of v_i, that is, the subset of the link of v_i consisting of sets containing only vertices that precede v_i in our ordering of $V(H)$.

By definition, for any D-degenerate r-graph there exists a *D-degenerate ordering* of the vertices $v_1, \ldots, v_n$ such that for each $i \in [n]$ we have $|L^-(v_i)| \leq D$. Hence, for any D-degenerate r-graph H on n vertices, it is clear that H has less than Dn edges. This implies the following observation.

Observation 2.1. *Let H be a D-degenerate r-graph. It is clear that H has less than Dn edges. Let X be the set of vertices in H of degree at least k, we have $\frac{|X|k}{r} \leq \frac{\sum_{x\in X} d(x)}{r} \leq Dn$, and hence $|X| \leq \frac{rDn}{k}$.*

The proof of Theorem 1.2 follows from a standard counting argument, and we omit the proof here. The proof of Theorem 1.1 generalises the random block model of Allen, Böttcher and Liebenau [1]. That is, we will define an r-graph Γ on $\Theta(n)$ vertices, whose vertices are split into $\Theta(\log\log n)$ *blocks* with sizes growing from $n^{1-1/D}$ up to n, and with edges placed between and in blocks randomly with probability depending on the size of the blocks they lie between. We will then show that any D-degenerate graph can be embedded in Γ, putting the vertices of highest degree in the smallest block down to those of constant degree in the largest block.

Definition 2.2 (Random block model, $\Gamma(r, n, D)$)**.** *Given any natural numbers $r, D, n \geq 2$, we define N to be the smallest integer such that $n^{D^{1-N}} \leq 3^{D^2}$. For each $0 \leq k \leq N$, we define the variable*

$$\Delta_k := \begin{cases} n^{D^{-k}} & \text{for } 0 < k \leq N \\ Dn & \text{if } k = 0. \end{cases}$$

The vertex set of $\Gamma := \Gamma(r, n, D)$ is the disjoint union $W = W_1 \sqcup \ldots \sqcup W_N$, where each W_k is called a block *and has order*

$$|W_k| = 100 \cdot 3^D rn/\Delta_k.$$

For any subset $s \subseteq W$, we define an intersection pattern *$\pi \in \{0, \ldots, |s|\}^N$, where for each $j \in [N]$, $\pi_j(s) = |s \cap W_j|$. We define*

$$p^* := 2^{(r-1)}(2(r-1)D)^{1/D}(\log n)^{2/D}(\log\log n)^{r+1}/\Delta_1.$$

For each $s \in \binom{W}{r}$, define

$$p_s := \min\left\{1, p^* \prod_{i=1}^{N} \Delta_i^{\pi_i(s)}\right\}.$$

The edge set $E(\Gamma)$ is then defined by letting each set $s \in \binom{W}{r}$ be an edge of our hypergraph independently with probability p_s.

We further partition each block W_k into sub-blocks *$W_{k,1} \ldots W_{k,\log n}$, where $W_{k,1} = \frac{1}{2}|W_k|$ and for $j \geq 2$, $|W_{k,j}| \geq \frac{|W_k|}{2\log n}$.*

We give some properties of the model which will be used in the proof.

Proposition 2.3 (Properties of the Model)**.** *Let n be sufficiently large and $r, D \geq 2$. For N, Δ_{N-1} as in Definition 2.2, the following holds.*

(a) $\frac{\log\log n}{2\log D} \leq N < \log\log n$ *and* $3^D \leq \Delta_{N-1} \leq 3^{D^2}$,

(b) $\Gamma(r,n,D)$ *has at most* $200\cdot 3^D rn$ *vertices and a.a.s. at most*

$$2(20\cdot 3^D r)^r(\log\log n)^{2r+1}(\log n)^{2/D}n^{r-1/D}$$

edges.

The properties given by Proposition 2.3 (a) can be calculated from the definition of N. The upper bound on the number of edges given in (b) is calculated via the Chernoff bound. We omit the details. We now describe how we embed a graph in this random block model.

Definition 2.4. *Given any D-degenerate r-graph H, let $v_1,\dots,v_n$ be a D-degeneracy ordering of $V(H)$, and let Γ satisfy Definition 2.2. Our embedding strategy is defined by the iterative construction of a* partial embedding $\psi_t : \{v_1,\dots,v_t\}\to\Gamma$. *We start with ψ_0, the trivial partial embedding of no vertices into Γ.*

For each $1\le t\le n$, we say a vertex $u\in\Gamma$ is a candidate *for v_t if $\psi_{t-1}(L^-(v_t))\subseteq L(u)$. We say such a vertex $u\in\Gamma$ is an* available candidate *if in addition $u\notin Im(\psi_{t-1})$. Let k be such that $\Delta_k<\deg(v_t)\le\Delta_{k-1}$ and choose j minimal such that there is some $u\in W_{k,j}$ which is an available candidate for v_t. We define $\psi_t=\psi_{t-1}\cup\{v_t\to u\}$. If there is no available candidate for v_t in W_k, we say ψ_t and the subsequent partial embeddings do not exist and that the embedding strategy* fails at time *t. If the embedding strategy does not fail, then $\psi:=\psi_n$ gives an embedding of H into Γ.*

2.2. Proofs

There are two details which require verification to prove Theorem 1.1. The first that given by Proposition 2.3 (b), and the second is that Γ is $\mathcal{H}^{(r)}(n,D)$-universal. As outlined above, the former is easily checked, and most of the paper consists of verifying the latter by showing that our embedding strategy never fails. Specifically, we show that the blocks satisfy a certain pseudorandomness property (which is explicitly stated in Lemma 2.7), and that a sufficient proportion of each sub-block always remains vacant. Together, these properties ensure that at each time t there is a suitable vertex in the appropriate block to embed v_t. This completes the proof that an instance of our random block model a.a.s. has the correct number of edges and contains a copy of every $H\in\mathcal{H}^{(r)}(n,D)$.

Given any partial embedding ψ_t and any vertex $v\in H$ whose back-link is in the domain of ψ_t (i.e. is embedded), we define the *embedded back-link of v* by $\psi_t(L^-(v))\subseteq\binom{W}{r-1}$. To simplify notation for any subset $B\subseteq\binom{W}{r-1}$ we let $V(B)=\bigcup B\subseteq W$ denote the set of vertices contained in B.

In our proof that this embedding strategy does not fail, we analyse the multi-set of embedded back-links of all the vertices in some sub-block, rather than just the embedded back-link of a single vertex. This may indeed be a multi-set, as two vertices may share the same back-links and for our proof we need to count with multiplicity. We will consider a multi-set $\mathcal{B}$, where each $B\in\mathcal{B}$ is a subset of $\binom{W}{r-1}$. Let $\mathcal{B}$ be some such multi-set. We denote by $|\mathcal{B}|$ the number of (not necessarily distinct, so counting with multiplicity) sets contained in $\mathcal{B}$, and call this the *size* of $\mathcal{B}$. We let $V(\mathcal{B})\subseteq W$ denote the set of vertices contained in the union $\bigcup_{B\in\mathcal{B}}V(B)$ (note that this is just a subset of W, and is not counted with multiplicity). We now define three key properties a multi-set of the vertices of Γ should exhibit if they are the embedded back-links of vertices.

Definition 2.5 (Well-behaved multi-set)**.** *Given $r,n,D\ge 2$, let $W=\bigsqcup_{k=1}^N W_k$ and $W_k=\bigsqcup_{j=1}^{\log n}W_{k,j}$ be the vertex set of $\Gamma=\Gamma(r,n,D)$ as given by Definition 2.2. For $1\le t\le n$, let $\mathcal{B}=\{B_i\}_{i=1}^t$ be a multi-set, where each $B_i\subseteq\binom{W}{r-1}$. Then $\mathcal{B}$ is called* well-behaved *if*

(WB1) *$|B_i|\le D$ for all $1\le i\le t$,*
(WB2) *for all $1\le k\le N$ and for all $u\in W_k$ we have $\left|\{i\in[t]:u\in V(B_i)\}\right|\le\Delta_{k-1}$, and*
(WB3) *for each $1\le k\le N$ and each $1\le j\le\log n$, we have $\left|V(\mathcal{B})\cap W_{k,j}\right|\le\frac{1}{2}|W_{k,j}|$.*

Note that (WB 1) and (WB 2) are consequences of our embedding strategy, while (WB 3) will require some proof. In particular, we will prove the following deterministic condition that $\Gamma=\Gamma(r,n,D)$ is likely to satisfy, and we will show this condition implies that our embedding strategy also maintains the third property.

Lemma 2.6. *Let $D, r \geq 2$ and let n be sufficiently large. Let $\Gamma(n, D, r)$ be the random block model as in Definition 2.2. Let $\mathcal{B}$ be a well-behaved multi-set. Fix $1 \leq k \leq N$ and any $v \in W_k \setminus V(\mathcal{B})$. Let $\mathcal{E}$ be the event that there is some $B \in \mathcal{B}$ such that $B \subseteq L(v)$. Then*

$$\mathbb{P}(\mathcal{E}) \geq \min\{1/2, |\mathcal{B}|(\log n)^2(\log\log n)^D n^{D^{1-k}-1}\}.$$

We will give a sketch of the proof of this lemma.

Sketch of proof. If there is some $B \in \mathcal{B}$ such that $\mathbb{P}(B \subseteq L(v)) = 1$ then $\mathbb{P}(\mathcal{E}) = 1$ and we are done, so we assume $\mathbb{P}(B \subseteq L(v)) < 1$ for all $B \in \mathcal{B}$. We have that for each $B \in \mathcal{B}$, the probability that B is contained in the link of v is given by $\mathbb{P}(B \subseteq L(v)) = \prod_{b \in B} p_{v \cup b}$, where the empty product is equal to one (throughout this proof, for each $b \subseteq V(\Gamma)$ and $v \in V(\Gamma)$ we will slightly abuse notation and write $v \cup b$ to denote $\{v\} \cup b$). We further assume that for each $B \in \mathcal{B}$, we have $p_{v \cup b} < 1$ for all $b \in B$, as removing the sets with $p_{v \cup b} = 1$ from B does not change the probability $\mathbb{P}(B \subseteq L(v))$.

We generalise the intersection pattern π of a set $s \subseteq [W]$ to a *block intersection pattern* for each $B \in \mathcal{B}$, given by $\boldsymbol{\omega} := \boldsymbol{\omega}(B) \in \{0, \ldots, (r-1)D\}^N$, where $\omega_i(B) := \sum_{b \in B} \pi_i(b)$. We restrict our attention to the subset $\mathcal{B}' \subseteq \mathcal{B}$ of blocks with a most common block intersection pattern, $\boldsymbol{\omega}'$, and let $\mathcal{E}'$ be the event that there is some $B \in \mathcal{B}'$ such that $B \subseteq L(v)$. We will show

$$\mathbb{P}(\mathcal{E}') \geq \min\left\{\tfrac{1}{2}, |\mathcal{B}'|2^{(r-1)D}(\log n)^2(\log\log n)^{rD} n^{D^{1-k}-1}\right\}. \tag{1}$$

To show that this implies the lemma, we give an upper bound on the number of different block intersection patterns. Let $B \in \mathcal{B}$, where $B = \{b_1, \ldots, b_k\}$ for some $k \leq D$. To determine its intersection pattern, we need for each b_i a vector of $r-1$ elements of N that indicates which W_j the elements of b_i are contained in. We can arrange these vectors as the columns of an $(r-1) \times D$ matrix, where each entry is either from $[N]$, or empty (if $|B| < D$). Matrices of this form determine all possible intersection patterns, and there are therefore at most $(N+1)^{(r-1)D} \leq (2\log\log n)^{(r-1)D}$ possible intersection patterns. Hence, $|\mathcal{B}'| \geq |\mathcal{B}|(2\log\log n)^{-(r-1)D}$ and the lemma follows.

For each $B \in \mathcal{B}'$, let X_B be the indicator random variable for the event $B \subseteq L(u)$ and let $X = \sum_{B \in \mathcal{B}} X_B$. Then by Chebyshev's inequality

$$\mathbb{P}(\mathcal{E}') = \mathbb{P}(X > 0) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}. \tag{2}$$

To find a lower bound for the right-hand side of (2), we will upper bound $\mathbb{E}X^2$. To that end, we write

$$\mathbb{E}X^2 = \sum_{B,A \in \mathcal{B}'} \mathbb{P}(X_B = 1, X_A = 1) = \sum_{B \in \mathcal{B}'} \Big(\mathbb{P}(B \subseteq L(v)) \cdot \sum_{A \in \mathcal{B}'} \mathbb{P}(A \setminus B \subseteq L(v))\Big). \tag{3}$$

Fix $B \in \mathcal{B}'$. We have

$$\begin{aligned}\sum_{A \in \mathcal{B}'} \mathbb{P}(A \setminus B \subseteq L(v)) &= \sum_{A \cap B = \emptyset} \mathbb{P}(A \subseteq L(v)) + \sum_{A \cap B \neq \emptyset} \mathbb{P}(A \setminus B \subseteq L(v)) \\ &\leq \mathbb{E}X + \sum_{A \cap B \neq \emptyset} \mathbb{P}(A \setminus B \subseteq L(v))\end{aligned} \tag{4}$$

We now find an upper bound for the expression on the right hand side of (4). To this end, we define a variable $\ell = \ell(B, B')$ to represent the maximal index $j \in [N]$ such that $V(B \cap B') \cap W_j \neq \emptyset$. Recall that each block in $\mathcal{B}'$ has only elements b with $p_{v\cup b} < 1$, so $\omega'_1 = 0$ and $\ell \geq 2$. Therefore,

$$\sum_{A\cap B\neq\emptyset} \mathbb{P}(A \setminus B \subseteq L(v)) = \sum_{\ell=2}^{N}\Bigg(\sum_{A:\ell(B,A)=\ell} \mathbb{P}(A \setminus B \subseteq L(v))\Bigg).$$

By definition of ω', we have that $V(B)$ contains at most ω'_ℓ vertices in W_ℓ. By (WB 2), each of these may be contained in $V(A)$ for at most $\Delta_{\ell-1}$ sets $A \in \mathcal{B}$. This gives at most $\omega'_\ell\Delta_{\ell-1}$ sets A such that $V(B\cap A)\cap W_\ell \neq \emptyset$. For each ℓ, we let $A_\ell \in \mathcal{B}$ be the block which maximises the probability $\mathbb{P}(A \setminus B \subseteq L(v))$ over all sets $A \in B$ with $\ell(B, A) = \ell$. We conclude that

$$\sum_{A\cap B\neq\emptyset} \mathbb{P}(A \setminus B \subseteq L(v)) = \sum_{\ell=2}^{N}\Bigg(\sum_{A:\ell(B,A)=\ell} \mathbb{P}(A \setminus B \subseteq L(v))\Bigg) \leq \sum_{\ell=2}^{N} \omega'_\ell\Delta_{\ell-1}\mathbb{P}(A_\ell \setminus B \subseteq L(v)).$$

We require our expression to be independent of our choice of B, so we let $A_*, B_* \in \mathcal{B}'$ be the blocks which maximise $\Delta_{\ell(B,A)-1}\mathbb{P}(A \setminus B \subseteq L(v))$ over all sets $A, B \in \mathcal{B}'$ and let $\ell_* = \ell(B_*, A_*)$. We also note $\sum_{\ell=2}^{N} \omega'_\ell \leq (r-1)D$. We therefore have

$$\sum_{A\cap B\neq\emptyset} \mathbb{P}(A \setminus B \subseteq L(v)) \leq \sum_{\ell=2}^{N} \omega'_\ell\Delta_{\ell_*-1}\mathbb{P}(A_* \setminus B_* \subseteq L(v)) \leq (r-1)D\Delta_{\ell_*-1}\mathbb{P}(A_* \setminus B_* \subseteq L(v)).$$

By (4) and (3) we have

$$\mathbb{E}X^2 \leq \max\Big\{2(\mathbb{E}X)^2, 2\mathbb{E}X\cdot(r-1)D\Delta_{\ell_*-1}\mathbb{P}(A_* \setminus B_* \subseteq L(v))\Big\}.$$

If $2\mathbb{E}X \geq 2(r-1)D\Delta_{\ell_*-1}\mathbb{P}(A_* \setminus B_* \subseteq L(v))$, then $\mathbb{P}(\mathcal{E}') \geq 1/2$ by (2) and we are done. So we assume this is not the case. By (2) we have

$$\mathbb{P}(\mathcal{E}') \geq \frac{\mathbb{E}X}{2(r-1)D\Delta_{\ell_*-1}\mathbb{P}(A_* \setminus B_* \subseteq L(v))}. \tag{5}$$

As every $B \in \mathcal{B}'$ has the same intersection pattern, we can write

$$\mathbb{E}X = \sum_{B\in\mathcal{B}'} \mathbb{E}X_B = |\mathcal{B}'|\cdot\mathbb{P}(A_* \subseteq L(v)) = |\mathcal{B}'|\prod_{b\in A_*} p_{v\cup b}.$$

We substitute this into (5) and write $\mathbb{P}(A_* \setminus B_* \subseteq L(v)) = \prod_{b \in A_* \setminus B_*} p_{v \cup b}$ to give

$$\mathbb{P}(\mathcal{E}') \geq \frac{t' \prod_{b \in A_*} p_{v \cup b}}{2(r-1)D\Delta_{\ell_*-1} \prod_{b \in A_* \setminus B_*} p_{b \cup v}} = \frac{t' \prod_{b \in (A_* \cap B_*)} p_{v \cup b}}{2(r-1)D\Delta_{\ell_*-1}}.$$

We now let $b_* \in (A_* \cap B_*)$ minimise $p_{v \cup b_*}$. Note that $\pi_i(b_*) \neq 0$ for some $i \leq \ell_*$ by the definition of ℓ_*, and hence $p_{v \cup b_*} = p^* \prod_{i=1}^{N} \Delta_i^{\pi_i(v \cup b_*)} \geq p^* \Delta_k \Delta_{\ell_*}$. Noting $\Delta_{\ell_*}^D = \Delta_{\ell_*-1}$ we have

$$\mathbb{P}(\mathcal{E}') \geq |\mathcal{B}'| \frac{(p^* \Delta_k \Delta_{\ell_*})^D}{2(r-1)D\Delta_{\ell_*-1}} = |\mathcal{B}'| \frac{(p^* \Delta_k)^D}{2(r-1)D} \geq |\mathcal{B}'| 2^{(r-1)D} n^{D^{1-k}-1} (\log n)^2 (\log\log n)^{rD},$$

as required.

□

We use this lemma to prove the following pseudorandomness property of our block model.

Lemma 2.7. *Let $D, r \geq 2$ and let n be sufficiently large. Then $\Gamma(n, r, D)$ as given in Definition 2.2 a.a.s. satisfies the following. For every $t \in [n]$, every well-behaved multi-set $\mathcal{B}$ containing t subsets of $\binom{W}{r-1}$, and every $k \in [N]$ and $j \in [\log n]$, we have*

$$\Big| \{ u \in W_{k,j} : \exists B \in \mathcal{B} \text{ such that } B \subseteq L(u) \} \Big| \geq \min\Big\{ \frac{1}{16}, \frac{t}{4} (\log n)^2 (\log\log n)^D n^{D^{1-k}-1} \Big\} |W_{k,j}|. \quad (6)$$

The proof of Lemma 2.7 roughly follows fixing some $\mathcal{B}, k$ and j, and determining the probability that (6) doesn't hold via an application of the Chernoff bound and Lemma 2.6. A union bound over all choices of $\mathcal{B}, k$ and j then gives the result. We omit the details here.

We now give a brief overview of how Lemma 2.7 allows us to prove Theorem 1.1.

Sketch of proof of Theorem 1.1. Let $\Gamma(n, r, D)$ be as given in Definition 2.2. Then we have that Γ a.a.s. has at most

$$2(20 \cdot 3^D r)^r (\log\log n)^{2r+1} (\log n)^{2/D} n^{r-1/D}$$

edges, and satisfies the following by Lemma 2.7. For every $t \in [n]$, every well-behaved multi-set $\mathcal{B}$ of $\binom{W}{r-1}$, and every $k \in [N]$ and $j \in [\log n]$, we have

$$\Big| \{ u \in W_{k,j} : \exists B \in \mathcal{B} \text{ such that } B \subseteq L(u) \} \Big| \geq \min\Big\{ \frac{1}{16}, \frac{|\mathcal{B}|}{4} (\log n)^2 (\log\log n)^D n^{D^{1-k}-1} \Big\} |W_{k,j}|. \quad (7)$$

Let H be a D-degenerate graph on n vertices, and let $V(H) = v_1, \ldots, v_n$ be a an ordering of its vertices, such that for each $v_i, L^-(v_i) \leq D$. Let ψ_t be partial embedding of $\{v_1, \ldots, v_t\}$ as given by Definition 2.4. We define the following constants.

$$L_{k,j} = \begin{cases} \frac{1}{(4\log n)^{j-1}} \frac{rnD}{\Delta_k} & \text{for } k \in [N-1] \\ \frac{1}{(4\log n)^{j-1}} n & \text{for } k = N \end{cases}$$

We show inductively that for all $t \in [n]$, each $L_{k,j}$ is an upper bound on $|\text{Im}\psi_t \cap W_{k,j}|$. We suppose this upper bound is broken at some time t. Since it held at time $t-1$, it must have been broken when we embedded v_t into some block $W_{k,j}$. First, suppose $j = 1$. Then if $k = N$, then $|W_{N,1} \cap \text{Im}\psi_t| = L_{1,N} + 1 = n + 1$, a contradiction. If $k < N$, then $|W_{k,1} \cap \text{Im}\psi_t| = L_{k,1} + 1 > \frac{rnD}{\Delta_k}$, a contradiction to Observation 2.1.

So we can assume $j > 1$. In this case, we find a contradiction by showing that there is in fact some $x \in W_{k,j} \cap \text{Im}\psi_t$ which should have been embedded to $W_{k,j-1}$. For this, we let $\mathcal{B} = \{\psi_t(L^-(x)) : x \in W_{k,j}\}$ be the multi-set of embedded back neighbours of each $x \in W_{k,j}$. We show that they are well-behaved, and use Lemma 2.7 to show that there are a sufficient number of candidates $W_{k,j-1}$ for one of these vertices. The inductive hypothesis $|\text{Im}\psi_{t-1} \cap W_{k,j}| \leq L_{k,j}$ implies that they are not covered by $\text{Im}\psi_{t-1}$. This concludes the proof of Theorem 1.1.

□

3. Remarks

There are many open questions in this area of research. Firstly, while a counting argument shows that Theorem 1.1 is tight up to the polylogarithmic factor, it is possible that no polylogarithmic factor is necessary. Additionally, our model Γ contains more than n vertices. We can instead ask how many edges a graph on n vertices must have to be universal for the class $\mathcal{H}^{(r)}(n, D)$ (known as *spanning universality*). This question is generally more difficult, and has not yet been answered for $r = 2$.

Furthermore, there are many results for 2-graphs which are not known for r-graphs. In the $r = 2$ case, universality for many classes of sparse graphs has been studied, including planar graphs, trees, forests and graphs of bounded degree. One of the most general notions of sparsity is bounded density, where the *density*, d, of a graph H is given by $d = \max_{H' \subseteq H} \frac{e(H')}{v(H')}$. For these graphs, it has been shown [4] that the minimum number of edges required to be universal for the class of n-vertex graphs of density d is $O(n^{2-1/(\lceil d \rceil + 1)})$ (when $d \in \mathbb{Q}, d \geq 1$) and $\Omega(n^{2-1/d-o(1)})$. However, there is no known corresponding result for $r > 2$.

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