Integr. Equ. Oper. Theory (2026) 98:3
https://doi.org/10.1007/s00020-026-02825-1

Integral Equations and Operator Theory

On Banach subalgebras of the Dirichlet Hardy algebra $\mathscr{H}^\infty$ consisting of lacunary Dirichlet series

Amol Sasane

Abstract. Let $\mathscr{H}^\infty$ be the set of all Dirichlet series $f=\sum_{n=1}^{\infty} a_n n^{-s}$ (where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$) that converge at each s in the half-plane $\mathbb{C}_0 := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$, such that $\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)| < \infty$. Then $\mathscr{H}^\infty$ is a Banach algebra with pointwise operations and the supremum norm $\|\cdot\|_\infty$, and has been studied in earlier works. The article introduces a new family of Banach subalgebras $\mathscr{H}^\infty_S$ of $\mathscr{H}^\infty$. For $S \subset \mathbb{N}$, let $\mathscr{H}^\infty_S$ be the set of all elements $\sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$ such that for all $n \in \mathbb{N} \setminus S$, $a_n = 0$. Then $\mathscr{H}^\infty_S$ is a unital Banach subalgebra of $\mathscr{H}^\infty$ with the $\|\cdot\|_\infty$ norm if and only if S is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. It is shown that for such S, $\mathscr{H}^\infty_S$ is the multiplier algebra of $\mathscr{H}^2_S$, where $\mathscr{H}^2_S$ is the Hilbert space of all $f = \sum_{n\in S} a_n n^{-s}$ such that $\|f\|_2 := (\sum_{n\in S} |a_n|^2)^{\frac{1}{2}} < \infty$. A characterisation of the group of units in $\mathscr{H}^\infty_S$ is given, by showing an analogue of the Wiener $1/f$ theorem for $\mathscr{H}^\infty_S$. If S has an infinite set of generators allowing a unique representation of each element of S, then it is shown that the Bass stable rank of $\mathscr{H}^\infty_S$ is infinite.

Mathematics Subject Classification. Primary 30B50, Secondary 46J15.

Keywords. Dirichlet series, Banach algebras, Bass stable rank.

1. Introduction

The aim of this article is to introduce and study some algebraic-analytic properties of a particular family $\mathscr{H}^\infty_S$ (defined in §1.2) of Banach algebras that are contained in the Hardy algebra $\mathscr{H}^\infty$ of Dirichlet series (recalled in §1.1). The motivation is twofold: there has been old and recent interest in studying various Banach algebras of Dirichlet series (see, e.g., [10], [14]), and the Banach algebras $\mathscr{H}^\infty_S$ we study are also the 'Dirichlet series analogue' of the previously studied (see, e.g., [8]) Banach subalgebra $H^\infty_1 = \{f \in H^\infty :$

Published online: 20 January 2026

$f'(0)=0\}$ of H^∞ (the classical Hardy algebra consisting of bounded and holomorphic functions on $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$, with pointwise operations and the supremum norm, $\|f\|_\infty=\sup_{z\in\mathbb{D}}|f(z)|$ for $f\in H^\infty$).

1.1. The Banach algebra $\mathscr{H}^\infty$

Let $\mathscr{H}^\infty$ be the set of all Dirichlet series $f=\sum\limits_{n=1}^{\infty}a_n n^{-s}$ (where $a_n\in\mathbb{C}$ for each $n\in\mathbb{N}$) that converge at each $s\in\mathbb{C}_0:=\{s\in\mathbb{C}:\text{Re } s>0\}$, such that $\|f\|_\infty:=\sup_{s\in\mathbb{C}_0}|f(s)|<\infty$. We call a_n the n^{th} *coefficient of* f. With pointwise operations and the supremum norm, $\mathscr{H}^\infty$ is a Banach algebra (introduced in [10]). Multiplication in $\mathscr{H}^\infty$ is also given by

$$\Big(\sum_{n=1}^{\infty}a_n n^{-s}\Big)\cdot\Big(\sum_{n=1}^{\infty}b_n n^{-s}\Big)=\sum_{n=1}^{\infty}\Big(\sum_{(\mathbb{N}\ni)\,d|n}a_d b_{\frac{n}{d}}\Big)n^{-s},$$

where the notation $d\,|\,n$ means $d\in\mathbb{Z}$ divides $n\in\mathbb{Z}$. The unit element is $\mathbf{1}:=\sum\limits_{n=1}^{\infty}\delta_{n1}n^{-s}$, where $\delta_{n1}=0$ for $n\neq1$ and $\delta_{11}=1$. The Banach algebra $\mathscr{H}^\infty$ is precisely the multiplier space of the Hilbert space $\mathscr{H}^2$, where

$$\mathscr{H}^2=\Big\{f=\sum_{n=1}^{\infty}a_n n^{-s}:\text{such that }\|f\|_2:=\sqrt{\sum_{n=1}^{\infty}|a_n|^2}<\infty\Big\}$$

(see [10, Thm. 3.1]). The inner product in $\mathscr{H}^2$ is given by $\langle f,g\rangle=\sum\limits_{n=1}^{\infty}a_n\overline{b_n}$, where $f=\sum\limits_{n=1}^{\infty}a_n n^{-s}$ and $g=\sum\limits_{n=1}^{\infty}b_n n^{-s}\in\mathscr{H}^2$. Each element $f\in\mathscr{H}^2$ defines a holomorphic function in $\mathbb{C}_{\frac{1}{2}}=\{s\in\mathbb{C}:\text{Re}\,s>\frac{1}{2}\}$. If ζ denotes the Riemann zeta function, $\zeta(s)=\sum\limits_{n=1}^{\infty}n^{-s}$, ($\text{Re } s>1$), then for each $a\in\mathbb{C}$ such that $\text{Re}\,a>\frac{1}{2}$, $\zeta_a(s)=\sum\limits_{n=1}^{\infty}n^{-a}n^{-s}$ belongs to $\mathscr{H}^2$. For $f=\sum\limits_{n=1}^{\infty}a_n n^{-s}\in\mathscr{H}^\infty$, we have $\|f\|_2\le\|f\|_\infty$ (so that $\mathscr{H}^\infty\subset\mathscr{H}^2$), and in particular, $|a_n|\le\|f\|_\infty$ for all $n\in\mathbb{N}$ (see, e.g., [7, Prop. 1.19]). The set $\{e_n:=n^{-s}:n\in\mathbb{N}\}$ forms an orthonormal basis for $\mathscr{H}^2$. For $a\in\mathbb{C}_{\frac{1}{2}}$, with $K_a(s):=\sum\limits_{n=1}^{\infty}e_n(s)\overline{e_n(a)}=\zeta_{\overline{a}}(s)=\zeta(s+\overline{a})$, we have $f(a)=\langle f,K_a\rangle$ for all $f\in\mathscr{H}^2$. The Hilbert space $\mathscr{H}^2$ is a reproducing kernel Hilbert space with kernel function given by $K_{\mathscr{H}^2}(s,a)=\zeta(s+\overline{a})$, for $s,a\in\mathbb{C}_{\frac{1}{2}}$.

1.2. The set $\mathscr{H}_S^\infty$

For $S\subset\mathbb{N}$, define $\mathscr{H}_S^\infty$ to be the set of all elements $f=\sum\limits_{n=1}^{\infty}a_n n^{-s}\in\mathscr{H}^\infty$ such that for all $n\in\mathbb{N}\setminus S$, $a_n=0$.

1.3. Organisation of the article

We show in Section 2 that $\mathscr{H}_S^\infty$ is a unital Banach subalgebra of $\mathscr{H}^\infty$ with the supremum norm if and only if S is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. In Section 3, we show that just as $\mathscr{H}^\infty$ is exactly the multiplier algebra of $\mathscr{H}^2$, the Banach algebra $\mathscr{H}_S^\infty$ is exactly the multiplier algebra of a certain Hilbert subspace $\mathscr{H}_S^2$ of $\mathscr{H}^2$. In Section 4, we characterise the group of units in $\mathscr{H}_S^\infty$. In Section 5 we relate $\mathscr{H}_S^\infty$ to a natural Banach subalgebra of the Hardy algebra $H^\infty(B_{c_0})$ on the unit ball B_{c_0} of c_0 (space of complex sequences converging to 0 with termwise operations and the supremum norm), with vanishing derivatives of certain orders at $\mathbf{0} \in c_0$. Finally, in Section 6, we prove that if S has an infinite set of generators allowing a unique representation of each element of S, then the Bass stable rank of $\mathscr{H}_S^\infty$ is infinite, and also state a related conjecture.

2. When is $\mathscr{H}_S^\infty$ an algebra?

We show that $\mathscr{H}_S^\infty$ is a unital Banach subalgebra of $\mathscr{H}^\infty$ with the $\|\cdot\|_\infty$ norm if and only if S is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. A subset $\mathcal{S}$ of a semigroup Σ is a *subsemigroup* of Σ if $\mathcal{S}\mathcal{S} := \{s_1 s_2 : s_1, s_2 \in \mathcal{S}\} \subset \mathcal{S}$. By a subalgebra B of a complex algebra A (see, e.g., [15, Def. 10.1]), we simply mean that B is closed under the algebraic operations inherited from A.

Proposition 2.1. *The following are equivalent*:
(1) *S is a multiplicative subsemigroup of* $\mathbb{N}$.
(2) *$\mathscr{H}_S^\infty$ is a subalgebra of $\mathscr{H}^\infty$.*
(3) *$\mathscr{H}_S^\infty$ is a Banach algebra with the supremum norm.*

Proof. (1)⇒(2): Let S be a multiplicative subsemigroup of $\mathbb{N}$. We just show closure under multiplication. Let $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}_S^\infty$ and $g = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathscr{H}_S^\infty$. The n^{th} coefficient of fg is given by $c_n := \sum_{d|n} a_d b_{\frac{n}{d}}$. If $c_n \neq 0$, then at least one of the summands, say $a_d b_{\frac{n}{d}}$ is nonzero, implying that $d, \frac{n}{d} \in S$, and so $n = d\frac{n}{d} \in S$. Thus $fg \in \mathscr{H}_S^\infty$. Consequently, $\mathscr{H}_S^\infty$ is a subalgebra of $\mathscr{H}^\infty$.
(2)⇒(1): Let $\mathscr{H}_S^\infty$ be a subalgebra of $\mathscr{H}^\infty$. If $m, n \in S$, then since for $f = n^{-s}$ and $g = m^{-s}$ in $\mathscr{H}_S^\infty$, we have $(nm)^{-s} = fg \in \mathscr{H}_S^\infty$, we get $nm \in S$. Thus S is a multiplicative subsemigroup of $\mathbb{N}$.
(3)⇒(2) is trivial. We now show (2)⇒(3), i.e., $\mathscr{H}_S^\infty$ is a closed subset of $\mathscr{H}^\infty$. Let $(f_m)_{m\in\mathbb{N}}$ be Cauchy in $\mathscr{H}_S^\infty$. Write $f_m = \sum_{n=1}^{\infty} a_n^{(m)} n^{-s}$. Then $(f_m)_{m\in\mathbb{N}}$ is Cauchy in $\mathscr{H}^\infty$, and so converges to some $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$. Let $n \in \mathbb{N}\setminus S$. Then $a_n^{(m)} = 0$ for all $m \in \mathbb{N}$. So $|a_n| = |a_n - a_n^{(m)}| \leq \|f - f_m\|_\infty$ for all $m \in \mathbb{N}$. Passing to the limit as $m \to \infty$, $|a_n| \leq 0$, i.e., $a_n = 0$. So $f \in \mathscr{H}_S^\infty$.

For a multiplicative subsemigroup S of $\mathbb{N}$, $\mathscr{H}_S^\infty$ is unital if and only if $1 \in S$.

Example 2.2. We list some multiplicative subsemigroups of $\mathbb{N}$ containing 1.

(1) For a prime p, let $S := \{p^k : k \in \mathbb{N} \cup \{0\}\}$.
($\mathscr{H}_S^\infty \ni \sum_{n=1}^{\infty} a_n n^{-s} \mapsto \sum_{n=1}^{\infty} a_n z^n \in H^\infty$ is then a Banach algebra isomorphism from $\mathscr{H}_S^\infty$ to H^∞.)
More generally, for any $n \in \mathbb{N}$, define $S := \{n^k : k \in \mathbb{N} \cup \{0\}\}$.
(2) For primes $p_{i_1} < \cdots < p_{i_m}$, let $S = \{p_{i_1}^{k_1} \cdots p_{i_m}^{k_m} : k_1, \cdots, k_m \in \mathbb{N} \cup \{0\}\}$.
(3) Let χ_0 be the principal Dirichlet character of modulus $m \in \mathbb{N}$, i.e., $\chi_0(n) = 1$ if $\gcd(n, m) = 1$, and $\chi(n) = 0$ if $\gcd(n, m) > 1$, where $\gcd(n, m)$ denotes the greatest common divisor of $n, m \in \mathbb{N}$.

Then $S_m = \{n \in \mathbb{N} : \chi_0(n) \neq 0\}$ is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. Let $p_1 < p_2 < p_3 < \cdots$ be all the prime numbers. For $n \in \mathbb{N}$, the fundamental theorem of arithmetic gives a unique compactly supported sequence $(\nu_k(n))_{k\in\mathbb{N}}$ in $\mathbb{N} \cup \{0\}$ such that $n = \prod_{k=1}^{\infty} p_k^{\nu_k(n)}$. As $\gcd(n, m) = \prod_{k=1}^{\infty} p_k^{\min\{\nu_k(n),\nu_k(m)\}}$, $\gcd(n, m) = 1$ if and only if for all $k \in \mathbb{N}$, $\min\{\nu_k(n), \nu_k(m)\} = 0$.
So $S_m = \{n = \prod_{k=1}^{\infty} p_k^{\nu_k(n)}$: for all $k \in \mathbb{N}$, $\nu_k(n) = 0$ if $\nu_k(m) > 0\}$. Since the intersection of multiplicative subsemigroups of $\mathbb{N}$ is a multiplicative subsemigroup of $\mathbb{N}$, for any subset $F \subset \mathbb{N}$, $S_F := \bigcap_{m\in F} S_m$ is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. If F is a finite nonempty set $F = \{m_1, \cdots, m_n\} \subset \mathbb{N}$, then $S_F := \bigcap_{m\in F} S_m = S_{\mathrm{lcm}(m_1,\cdots,m_n)}$, where $\mathrm{lcm}(m_1, \cdots, m_n)$ is the least common multiple of $m_1, \cdots, m_n$.

(5) $S = \mathbb{N}$. (Then $\mathscr{H}_S^\infty = \mathscr{H}^\infty$.)

(6) $S = \{1\}$. (Then $\mathscr{H}_S^\infty$ is isomorphic to the Banach algebra $\mathbb{C}$.)

(7) For $m \in \mathbb{N}$, $S := \{n^m : n \in \mathbb{N}\}$. (E.g. for $m = 2$, S is the set of all squares.)

(8) The set $S = \{1\} \cup \{n \in \mathbb{N} : \text{there exist } x, y \in \mathbb{N} \text{ such that } n = x^2 + y^2\}$ is a multiplicative subsemigroup of $\mathbb{N}$ containing 1. The sum of two squares theorem (see, e.g., [6, Thm. 7, §3, Chap. IV]) provides an alternative description of S: $n \in S$ if and only if in the prime factorisation of n, all prime factors of the form $4k+3$ ($k \in \mathbb{N} \cup \{0\}$) have an even exponent. $\Diamond$

3. $\mathscr{H}_S^\infty$ as the multiplier algebra of $\mathscr{H}_S^2$

It was shown in [10] that $\mathscr{H}^\infty$ is the multiplier algebra of $\mathscr{H}^2$, i.e., a function $f : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}$ satisfies $fg \in \mathscr{H}^2$ for all $g \in \mathscr{H}^2$ if and only if f has an extension to $\mathbb{C}_0$ which is an element of $\mathscr{H}^\infty$. Moreover, $\|f\|_\infty = \sup_{g\in\mathscr{H}^2, \|g\|_2 \leq 1} \|fg\|_2$.

Analogously, we will now show that more generally, $\mathscr{H}_S^\infty$ is exactly the multiplier algebra of $\mathscr{H}_S^2$, where for any subset $S \subset \mathbb{N}$, we define $\mathscr{H}_S^2$ to be the set of all $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^2$ such that for all $n \in \mathbb{N} \setminus S$, $a_n = 0$. Then $\mathscr{H}_S^2$ is a closed subspace of $\mathscr{H}^2$, and $\{n^{-s} : n \in S\}$ forms an orthonormal basis for $\mathscr{H}_S^2$. Define lacunary zeta function ζ_S by $\zeta_S(s) = \sum_{n\in S} n^{-s}$, $(\mathrm{Re}\, s > 1)$. For $a \in \mathbb{C}_{\frac{1}{2}}$, $\sum_{n\in S} e_n(s)\overline{e_n(a)} = \zeta_S(s + \overline{a})$, and we have $f(a) = \langle f, \zeta_S(\cdot + \overline{a})\rangle$ for all $f \in \mathscr{H}_S^2$. The Hilbert space $\mathscr{H}_S^2$ is a reproducing kernel Hilbert space with kernel function given by $K_{\mathscr{H}_S^2}(s, a) = \zeta_S(s + \overline{a})$ for $s, a \in \mathbb{C}_{\frac{1}{2}}$. In particular, in Example 2.2(2), $\mathscr{H}_{S_m}^\infty$ is a reproducing kernel Hilbert space with the kernel given by $K_{\mathscr{H}_{S_m}^\infty}(s, a) = L(s + \overline{a}, \chi_0)$, where L is the Dirichlet L-series given by $L(s, \chi_0) = \sum_{n=1}^{\infty} \chi_0(n) n^{-s}$ for $\mathrm{Re}\, s > 1$. (Peripherally, a natural question is: Is there a characterisation of the multiplicative subsemigroups S of $\mathbb{N}$ containing

1, for which the lacunary zeta functions ζ_S arise from modular forms? See, e.g., [1], for background on modular forms and their link to Dirichlet series.) We have the following (shown along the same lines as the proof for $\mathscr{H}^\infty$-$\mathscr{H}^2$ case given in [14, Theorem 6.4.7]).

Proposition 3.1. *Let S be a multiplicative subsemigroup of $\mathbb{N}$ containing 1. Then $\mathscr{H}^\infty_S$ is exactly the multiplier algebra of $\mathscr{H}^2_S$, that is, a function f defined on $\mathbb{C}_{\frac{1}{2}}$ satisfies $fg \in \mathscr{H}^2_S$ for all $g \in \mathscr{H}^2_S$ if and only if f has an extension to $\mathbb{C}_0$ which is an element of $\mathscr{H}^\infty_S$. Moreover, $\|f\|_\infty = \sup_{g\in\mathscr{H}^2_S,\, \|g\|_2\le 1} \|fg\|_2$.*

Let $\mathcal{P}$ denote the set of all *Dirichlet polynomials* (namely, $f = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ for which there exists an $N \in \mathbb{N}$ such that for all $n > N$, $a_n = 0$). For $r \in \mathbb{N}$, let $N_r = \{n = p_1^{k_1} \cdots p_r^{k_r} : k_1, \cdots, k_r \in \mathbb{N} \cup \{0\}\}$. Define $\mathcal{P}_r$ to be the set of all $f = \sum\limits_{n\in N_r} a_n n^{-s} \in \mathcal{P}$, where for all $n \in N_r$, $a_n \in \mathbb{C}$. We recall [14, Lemma 6.4.9].

Lemma 3.2. *For all $s \in \mathbb{C}_0$, and for all $r \in \mathbb{N}$, there exists a constant $C_{s,r} > 0$ such that for all $f \in \mathcal{P}_r$, we have $|f(s)| \le C_{s,r}\|f\|_2$.*

For $\varphi \in \mathrm{L}^1(\mathbb{R})$, let $\widehat{\varphi}$ be the Fourier transform of φ: $\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(t) e^{-i\xi t} dt$ for all $\xi \in \mathbb{R}$. Let $E = \{\varphi \in \mathrm{L}^1(\mathbb{R}) : \widehat{\varphi}$ has compact support$\}$. If $f = \sum\limits_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^2$, and $\varphi \in E$, then following 'vertical convolution identity' holds (see, e.g., [14, Proof of Theorem 6.4.7]): $\sum\limits_{n=1}^{\infty} a_n \widehat{\varphi}(\log n) n^{-s} = \int_{\mathbb{R}} f(s+it)\varphi(t)dt$, $s \in \mathbb{C}_{\frac{1}{2}}$.

Proof of Proposition 3.1. If $f \in \mathscr{H}^\infty_S$, and $g \in \mathscr{H}^2_S$, then $fg \in \mathscr{H}^2$, and $\|\varphi f\|_2 \le \|\varphi\|_\infty \|f\|_2$. Let $f = \sum\limits_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty_S$ and $g = \sum\limits_{n=1}^{\infty} b_n n^{-s} \in \mathscr{H}^2_S$. The n^{th} coefficient of fg is $c_n = \sum\limits_{d|n} a_d b_{\frac{n}{d}}$. If $c_n \ne 0$, then at least one of the summands, say $a_d b_{\frac{n}{d}}$ is nonzero, implying $d, \frac{n}{d} \in S$, and so $n = d\frac{n}{d} \in S$. Thus if $n \notin S$, then $c_n = 0$. So $fg \in \mathscr{H}^2_S$. Hence if $f \in \mathscr{H}^\infty_S$, then the multiplication map $\mathscr{H}^2_S \ni g \mapsto M_f g := fg \in \mathscr{H}^2_S$ is well-defined, and $\|M_f\| \le \|f\|_\infty$.
Next, let $f : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}$ be such that $fg \in \mathscr{H}^2_S$ for all $g \in \mathscr{H}^2_S$. Let $M_f : \mathscr{H}^2_S \to \mathscr{H}^2_S$ be the linear map of pointwise multiplication by f. As $\mathbf{1} \in \mathscr{H}^2_S$, we have $f = M_f(\mathbf{1}) \in \mathscr{H}^2_S$. By the closed graph theorem, M_f is a bounded operator. Denote the operator norm of M_f by $\|M_f\|$. Let $f = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ for all $s \in \mathbb{C}_{\frac{1}{2}}$.
Step 1. First let f be a Dirichlet polynomial. We claim $\|f\|_\infty = \|M_f\|$. Fix $r \in \mathbb{N}$ such that $f \in \mathcal{P}_r$ and let $s \in \mathbb{C}_0$. By induction, for all $k \in \mathbb{N}$, $\|f^k\|_2 \le \|M_f\|^k$. Lemma 3.2 applied to $f^k \in \mathcal{P}_r$ gives $|f(s)|^k \le C_{s,r}\|f^k\|_2 \le C_{s,r}\|M_f\|^k$ for $s \in \mathbb{C}_0$, and so $|f(s)| \le C_{s,r}^{\frac{1}{k}}\|M_f\|$. Passing to the limit that $k \to \infty$ now yields $|f(s)| \le 1\|M_f\|$. As $s \in \mathbb{C}_0$ was arbitrary, $\|f\|_\infty \le \|M_f\|$. Also, as $f \in \mathscr{H}^2_S$ is a Dirichlet polynomial and $\|f\|_\infty < \infty$, $f \in \mathscr{H}^\infty_S$. Then f is a multiplier on $\mathscr{H}^2_S$ and $\|M_f\| \le \|f\|_\infty$ by the first part of the proof. So $\|f\|_\infty = \|M_f\|$.
Step 2. Now consider the general case when f need not be Dirichlet polynomial. For a function $\varphi \in E$, we define $P_\varphi(s) = \sum\limits_{n=1}^{\infty} a_n \widehat{\varphi}(\log n) n^{-s}$. As $\widehat{\varphi}$ has

compact support, and $\log n \to \infty$ as $n \to \infty$, it follows that P_φ is a Dirichlet polynomial. We claim that $\|M_{P_\varphi}\| \leq \|M_f\| \|\varphi\|_1$. For a $t \in \mathbb{R}$, define the vertical translation operator T_t by $(T_t g)(s) = g(s+it)$ for all $g \in \mathscr{H}^2_S$. Then $T_t : \mathscr{H}^2_S \to \mathscr{H}^2_S$ is a linear isometry on $\mathscr{H}^2_S$, and $T_t f$ is a multiplier on $\mathscr{H}^2_S$ satisfying $\|M_{T_t f}\| = \|M_f\|$. Indeed, for all $g \in \mathscr{H}^2_S$, we have $(T_t f)g = T_t(f(T_{-t}g))$, and $\|(T_t f)g\|_2 = \|T_t(f(T_{-t}g))\|_2 = \|f(T_{-t}g)\|_2 \leq \|M_f\| \|T_{-t}g\|_2 = \|M_f\| \|g\|_2$, giving $\|M_{T_t f}\| \leq \|M_f\|$. Then also $\|M_f\| = \|M_{T_{-t}(T_t f)}\| \leq \|M_{T_t f}\|$. The vertical convolution formula yields for $s \in \mathbb{C}_{\frac{1}{2}}$ and $g \in \mathscr{H}^2_S$ that:

$$(P_\varphi g)(s) = (\textstyle\int_{\mathbb{R}} f(s+it)\varphi(t)dt)g(s) = \int_{\mathbb{R}} (T_t f)(s)g(s)\varphi(t)dt = \int_{\mathbb{R}} ((T_t f)g)(s)\varphi(t)dt.$$

We have $P_\varphi g = \int_{\mathbb{R}} ((T_t f)g)(\cdot)\varphi(t)dt$ in $\mathscr{H}^2_S$, where the right-hand side is a vector-valued Pettis integral in $\mathscr{H}^2_S$, and

$$\begin{aligned} \|M_{P_\varphi} g\|_2 &= \|P_\varphi g\|_2 \leq \textstyle\int_{\mathbb{R}} \|(T_t f)g\|_2 |\varphi(t)|dt \leq \int_{\mathbb{R}} \|M_{T_t f}\| \|g\|_2 |\varphi(t)|dt \\ &= \|M_f\| \|g\|_2 \textstyle\int_{\mathbb{R}} |\varphi(t)|dt = \|M_f\| \|g\|_2 \|\varphi\|_1. \end{aligned}$$

Hence $\|M_{P_\varphi}\| \leq \|M_f\| \|\varphi\|_1$.

Step 3. Define the sequence $(\varphi_m)_{m\in\mathbb{N}}$ in $\mathrm{L}^1(\mathbb{R})$ by $\varphi_m(t) = \frac{m}{2\pi}\Big(\frac{\sin\frac{mt}{2}}{\frac{mt}{2}}\Big)^2$, $t \in \mathbb{R}$.

Then $\widehat{\varphi_m}(\xi) = \max\{1 - \frac{|\xi|}{m}, 0\}$, and as $\widehat{\varphi_m}$ has compact support, $\varphi_m \in E$ for all $m \in \mathbb{N}$. Since $\varphi_m \geq 0$, $1 = \widehat{\varphi_m}(0) = \int_{\mathbb{R}} \varphi_m(t)dt = \int_{\mathbb{R}} |\varphi_m(t)|dt = \|\varphi_m\|_1$. Then $P_{\varphi_m}(s) = \sum\limits_{n=1}^{\infty} a_n \widehat{\varphi_m}(\log n) n^{-s} = \int_{\mathbb{R}} (T_t f)(s)\varphi_m(t)dt$ for all $s \in \mathbb{C}$. Steps 1 and 2 give $\|P_{\varphi_m}\|_\infty = \|M_{P_{\varphi_m}}\| \leq \|M_f\| \|\varphi_m\|_1 = \|M_f\| 1 = \|M_f\|$ for all $m \in \mathbb{N}$. Taking a subsequence if necessary, one may assume, thanks to Montel's theorem, that P_{φ_m} tends to some F uniformly on compact subsets of $\mathbb{C}_0$, with $\|F\|_\infty := \sup_{s\in\mathbb{C}_0} |F(s)| \leq \|M_f\|$. Let $\sigma > \frac{1}{2}$, $t \in \mathbb{R}$, and $s = \sigma + it \in \mathbb{C}_{\frac{1}{2}}$. By the Cauchy-Schwarz inequality, $\sum\limits_{n=1}^{\infty} |a_n| n^{-\sigma} \leq (\sum\limits_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} (\sum\limits_{n=1}^{\infty} n^{-2\sigma})^{\frac{1}{2}} < \infty$. Also $\widehat{\varphi_m}(\log n) \to 1$ as $m \to \infty$, and $0 \leq \widehat{\varphi_m}(\log n) \leq 1$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum\limits_{n=N+1}^{\infty} |a_n| n^{-\sigma} < \frac{\epsilon}{4}$. Let m_n, $n \in \{1, \cdots, N\}$ be such that

$$|\widehat{\varphi_{m_n}}(\log n) - 1| \leq \epsilon (2N (\sum_{n=1}^{N} |a_n| n^{-\sigma} + 1))^{-1}.$$

Then for $m > \max\{m_1, \cdots, m_N\}$, we have

$$\begin{aligned} |P_{\varphi_m}(s) - \sum_{n=1}^{\infty} a_n n^{-s}| &= |\sum_{n=1}^{\infty} a_n (\widehat{\varphi_{m_n}}(\log n) - 1) n^{-s}| \\ &\leq \sum_{n=1}^{N} |a_n| |\widehat{\varphi_{m_n}}(\log n) - 1| n^{-\sigma} + \sum_{n=N+1}^{\infty} |a_n| 2n^{-\sigma} \leq N \frac{\epsilon}{2N} 1 + 2\frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Thus for each $s \in \mathbb{C}_{\frac{1}{2}}$, we have $P_{\varphi_m}(s) \to \sum\limits_{n=1}^{\infty} a_n n^{-s} = f(s)$ as $m \to \infty$. Hence $f = F$ on $\mathbb{C}_{\frac{1}{2}}$. But $f \in \mathscr{H}^2_S$, and so it is a Dirichlet series. We have shown that f has a Dirichlet series which converges in $\mathbb{C}_{\frac{1}{2}}$, and this f admits a bounded holomorphic extension F to $\mathbb{C}_0$. Thus it follows that $f \in \mathscr{H}^\infty$. As

$f \in \mathscr{H}_S^2 \cap \mathscr{H}^\infty$, we get $f \in \mathscr{H}_S^\infty$. Moreover, $\|f\|_\infty = \|F\|_\infty \leq \|M_f\|$. Since also $\|M_f\| \leq \|f\|_\infty$, we obtain $\|f\|_\infty = \|M_f\|$.

□

4. Characterisation of the group of units

In this section we will show that $f \in \mathscr{H}_S^\infty$ is invertible in $\mathscr{H}_S^\infty$ if and only if $\inf_{s\in\mathbb{C}_0} |f(s)| > 0$. Below, for a unital commutative complex Banach A, we denote by A^{-1} the multiplicative group of all invertible elements of A. For $\sigma \in \mathbb{R}$, let $\mathbb{C}_\sigma := \{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$. Recall that for a Dirichlet series $D = \sum_{n=1}^{\infty} a_n n^{-s}$, the *abscissa of convergence* is $\sigma_c(D) = \inf\{\sigma \in \mathbb{R} : D \text{ converges in } \mathbb{C}_\sigma\} \in [-\infty, \infty]$. Similarly, the abscissa of *absolute* convergence of the Dirichlet series D is defined by $\sigma_a(D) = \inf\{\sigma \in \mathbb{R} : D \text{ converges absolutely in } \mathbb{C}_\sigma\}$. Then we have $-\infty \leq \sigma_c(D) \leq \sigma_a(D) \leq \infty$. Also, $\sigma_a(D) \leq \sigma_c(D) + 1$, see, e.g., [7, Prop. 1.3].

Theorem 4.1. *Let S be a multiplicative subsemigroup of $\mathbb{N}$. Then $(\mathscr{H}_S^\infty)^{-1} = \{f \in \mathscr{H}_S^\infty : \inf_{s\in\mathbb{C}_0} |f(s)| > 0\}$.*

Proof. If $f \in (\mathscr{H}_S^\infty)^{-1}$, then there exists a $g \in \mathscr{H}_S^\infty$ such that for all $s \in \mathbb{C}_0$, $f(s)g(s) = 1$. In particular, $g \neq \mathbf{0}$, and so $\|g\|_\infty > 0$. Thus

$$\inf_{s\in\mathbb{C}_0} |f(s)| = \inf_{s\in\mathbb{C}_0} |g(s)|^{-1} = (\sup_{s\in\mathbb{C}_0} |g(s)|)^{-1} = \|g\|_\infty^{-1} > 0.$$

Conversely, let $f = \sum_{n\in S} a_n n^{-s} \in \mathscr{H}_S^\infty$ be such that $\delta := \inf_{s\in\mathbb{C}_0} |f(s)| > 0$. By [5, Thm. 2.6], it can be seen that $f \in (\mathscr{H}^\infty)^{-1}$, i.e., $\frac{1}{f} \in \mathscr{H}^\infty$. It remains to show $\frac{1}{f} \in \mathscr{H}_S^\infty$. Let $\epsilon > 0$. As $\sigma_a(f) - \sigma_c(f) \leq 1$, and $\sigma_c(f) \leq 0$, we get $\sigma_a(f) \leq 1$. Thus the Dirichlet series given by $f_{1+\epsilon}(s) := \sum_{n\in S} a_n n^{-(1+\epsilon+s)} = \sum_{n\in S} a_n n^{-(1+\epsilon)} n^{-s}$ converges absolutely for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$. In particular, if $s \in \mathbb{C}_0$ and $\sigma := \operatorname{Re} s$, then

$$\begin{aligned} |f_{1+\epsilon}(s) - a_1| \leq \sum_{n\in S\setminus\{1\}} \frac{|a_n|}{n^{1+\epsilon}} n^{-\sigma} &\leq \Big(\sum_{n\in S\setminus\{1\}} |a_n|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\in S\setminus\{1\}} \frac{n^{-2\sigma}}{n^{2(1+\epsilon)}}\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2^\sigma} \|f\|_2 \Big(\sum_{n=1}^{\infty} \frac{1}{n^2}\Big)^{\frac{1}{2}} \stackrel{\sigma\to\infty}{\longrightarrow} 0. \end{aligned}$$

If $a_1 = 0$, then $\delta = \inf_{s\in\mathbb{C}_0} |f(s)| > 0$ and the above implies $0 < \delta \leq 0$, a contradiction. Thus $a_1 \neq 0$. The above also shows that there exists a $\sigma_0 > 0$ such that for all $s \in \mathbb{C}_{\sigma_0}$, we have with $\sigma := \operatorname{Re} s$ that

$$|f_{1+\epsilon}(s) - a_1| = \Big|\sum_{n\in S\setminus\{1\}} \frac{a_n}{n^{1+\epsilon}} n^{-s}\Big| \leq \sum_{n\in S\setminus\{1\}} \frac{|a_n|}{n^{1+\epsilon}} n^{-\sigma} < \frac{|a_1|}{2}. \qquad (*)$$

In the half-plane $\mathbb{C}_{\sigma_0}$,

$$\frac{1}{f_{1+\epsilon}(s)} = \Big(a_1 + \sum_{n\in S\setminus\{1\}} \frac{a_n}{n^{1+\epsilon}} n^{-s}\Big)^{-1} = a_1^{-1}\Big(1 + \sum_{m=1}^{\infty} (-1)^m \Big(a_1^{-1} \sum_{n\in S\setminus\{1\}} \frac{a_n}{n^{1+\epsilon}} n^{-s}\Big)^m\Big), \qquad (\star)$$

where the geometric series converges on account of (*). Thanks to the inequality $\sum_{m=1}^{\infty} \Big(|a_1|^{-1} \sum_{n\in S\setminus\{1\}} \frac{|a_n|}{n^{1+\epsilon}}\, n^{-\sigma}\Big)^m \leq \sum_{m=1}^{\infty} \frac{1}{2^m} < \infty$, it follows that we can rearrange the terms in ($\star$), and obtain a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that for all $s\in\mathbb{C}_{\sigma_0}$, we have $\frac{1}{f_{1+\epsilon}(s)}=\sum_{n\in S} c_n n^{-s}$. Note that we used the semigroup property of S here, since for $n\in S$, $(n^{-s})^m=(n^m)^{-s}$, and $n^m\in S$. But if the Dirichlet series for $\frac{1}{f}\in\mathscr{H}^\infty$ is given by $\frac{1}{f(s)}=\sum_{n=1}^{\infty} b_n n^{-s}$ for $s\in\mathbb{C}_0$, then we obtain from the above that for $s\in\mathbb{C}_{\sigma_0}$,

$$\sum_{n=1}^{\infty}\frac{b_n}{n^{1+\epsilon}}n^{-s}=\frac{1}{f(1+\epsilon+s)}=\frac{1}{f_{1+\epsilon}(s)}=\sum_{n\in S} c_n n^{-s}.$$

In particular, for $n\in\mathbb{N}\setminus S$, by the uniqueness of Dirichlet series coefficients (see, e.g., [6, Thm. 7, §5, Chap. X]), $\frac{b_n}{n^{1+\epsilon}}=0$, and so $b_n=0$. This shows that $\frac{1}{f(s)}=\sum_{n=1}^{\infty} b_n n^{-s}=\sum_{n\in S} b_n n^{-s}$, and so $\frac{1}{f}\in\mathscr{H}_S^\infty$, as wanted.

□

Let $\mathscr{A}_u$ be the subset of $\mathscr{H}^\infty$ of Dirichlet series that are uniformly continuous in $\mathbb{C}_0$. Alternatively, $\mathscr{A}_u$ is precisely the closure of Dirichlet polynomials in the $\|\cdot\|_\infty$ norm (see [2, Thm. 2.3]). For a multiplicative subsemigroup S of $\mathbb{N}$ containing 1, we introduce $\mathscr{A}_{u,S}=\mathscr{A}_u\cap\mathscr{H}_S^\infty$. Then $\mathscr{A}_{u,S}$ is a unital Banach algebra with pointwise operations and the $\|\cdot\|_\infty$ norm. Let $\mathscr{W}$ denote the set of all Dirichlet series $f=\sum_{n=1}^{\infty} a_n n^{-s}$ such that $\|f\|_1:=\sum_{n=1}^{\infty}|a_n|<\infty$. With pointwise operations and the $\|\cdot\|_1$ norm, $\mathscr{W}$ is a Banach algebra. Then $\mathscr{W}\subset\mathscr{A}_u\subset\mathscr{H}^\infty$. In the case of $\mathscr{W}$, an analogue of the classical Wiener $1/f$ lemma ([17, p.91]) for the unit circle holds, i.e., if $f\in\mathscr{W}$ is such that $\inf_{s\in\mathbb{C}_+}|f(s)|>0$, then $\frac{1}{f}\in\mathscr{W}$ (see, e.g., [11, Thm. 1], and also [9] for an elementary proof). For a multiplicative subsemigroup S of $\mathbb{N}$ containing 1, we introduce $\mathscr{W}_S=\mathscr{W}\cap\mathscr{H}_S^\infty$. Then $\mathscr{W}_S$ is a unital Banach algebra with pointwise operations and the $\|\cdot\|_1$ norm. We have $\mathscr{W}_S\subset\mathscr{A}_{u,S}\subset\mathscr{H}_S^\infty$.

Corollary 4.2. *Let S be a multiplicative subsemigroup of $\mathbb{N}$ containing 1, and $A\in\{\mathscr{A}_{u,S},\mathscr{W}_S\}$. Then $f\in A^{-1}$ if and only if $\delta:=\inf_{s\in\mathbb{C}_0}|f(s)|>0$.*

Proof. If $f\in A^{-1}$, then $f\in(\mathscr{H}_S^\infty)^{-1}$. So $\inf_{s\in\mathbb{C}_0}|f(s)|>0$ holds by Theorem 4.1. Conversely, let $f\in A$ satisfy $\delta:=\inf_{s\in\mathbb{C}_0}|f(s)|>0$. Theorem 4.1 implies $\frac{1}{f}\in\mathscr{H}_S^\infty$. For $A=\mathscr{W}_S$, the Wiener $1/f$ theorem for $\mathscr{W}$ gives $\frac{1}{f}\in\mathscr{W}$, and so $\frac{1}{f}\in\mathscr{W}\cap\mathscr{H}_S^\infty=\mathscr{W}_S$. For $A=\mathscr{A}_{u,S}$, as $\mathscr{A}_{u,S}\subset\mathscr{H}_S^\infty$, $\frac{1}{f}\in\mathscr{H}_S^\infty$. Also, $\frac{1}{f}$ is uniformly continuous in $\mathbb{C}_0$: $|\frac{1}{f}(w)-\frac{1}{f}(z)|=\frac{|f(z)-f(w)|}{|f(z)||f(w)|}\le\frac{1}{\delta^2}|f(w)-f(z)|$, for all $z,w\in\mathbb{C}_0$, and f is uniformly continuous in $\mathbb{C}_0$. Thus $\frac{1}{f}\in\mathscr{A}_u$, and so $\frac{1}{f}\in\mathscr{A}_u\cap\mathscr{H}_S^\infty=\mathscr{A}_{u,S}$. □

5. The image of $\mathscr{H}_S^\infty$ under the Bohr transform

In this section, we relate $\mathscr{H}_S^\infty$ to a natural Banach subalgebra of the Hardy algebra $H^\infty(B_{c_0})$ on the unit ball B_{c_0} of c_0 (space of complex sequences converging to 0 with termwise operations and the supremum norm), with vanishing derivatives of certain orders at $\mathbf{0} \in c_0$. We first introduce some notation. The Banach space ℓ^∞ is the set of all bounded complex sequences with termwise defined operations and the supremum norm: for $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$, $\|(a_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |a_n|$. We denote by c_0 the Banach subspace of ℓ^∞ of all sequences converging to 0, and c_{00} is the subset of c_0 of all sequences in ℓ^∞ with compact support. Let B_{c_0} be the open unit ball of c_0 with centre $\mathbf{0}$. Let $\boldsymbol{N}$ be the set of all compactly supported sequences that take values in the set of nonnegative integers, i.e., $\boldsymbol{N}$ is the subset of c_{00} consisting of sequences whose terms belong to $\mathbb{N}\cup\{0\}$. If $\boldsymbol{\nu}=(n_k)_{k\in\mathbb{N}} \in \boldsymbol{N}$ and $K \in \mathbb{N}$ is such that for all $k>K$, $n_k=0$, then $\boldsymbol{z}^{\boldsymbol{\nu}} := z_1^{n_1}\cdots z_K^{n_K}$ for all $\boldsymbol{z} \in B_{c_0}$, $\boldsymbol{\partial}^{\boldsymbol{\nu}} := \partial_{z_1}^{n_1}\cdots\partial_{z_K}^{n_K}$, $|\boldsymbol{\nu}| := n_1+\cdots+n_K$, and $\boldsymbol{\nu}! := n_1!\cdots n_K!$. If $\boldsymbol{\alpha}=(\alpha_k)_{k\in\mathbb{N}}, \boldsymbol{\beta}=(\beta_k)_{k\in\mathbb{N}} \in \boldsymbol{N}$, then $\boldsymbol{\beta}\preccurlyeq\boldsymbol{\alpha}$ if for all $k\in\mathbb{N}$, $\beta_k\le\alpha_k$. If $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{N}$ satisfy $\boldsymbol{\beta}\preccurlyeq\boldsymbol{\alpha}$ then

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!}.$$

By the fundamental theorem of arithmetic, for all $n\in\mathbb{N}$, $n=\prod_{k=1}^{\infty} p_k^{\nu_k(n)}$, where $\nu_k(n) \in \mathbb{N}\cup\{0\}$ and $(p_k)_{k\in\mathbb{N}}$ is the sequence of primes in ascending order. We have $\boldsymbol{\nu}(n) := (\nu_k(p))_{k\in\mathbb{N}} \in \boldsymbol{N}$. A seminal observation by H. Bohr [4], is that by putting $z_1 = 2^{-s}, z_2 = 3^{-s}, \cdots, z_n = p_n^{-s}, \cdots$, a Dirichlet series in $\mathscr{H}^\infty$ can be formally considered as a power series of infinitely many variables. So $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$ gives the formal power series $F(\boldsymbol{z}) = \sum_{n=1}^{\infty} a_n \prod_{k=1}^{\infty} z_k^{\nu_k(n)}$, where $\boldsymbol{z} = (z_1, z_2, z_3, \cdots)$. We recall the precise result below.

Let $H^\infty(B_{c_0})$ be the complex Banach algebra of bounded holomorphic (i.e., complex Fréchet differentiable) functions $F : B_{c_0} \to \mathbb{C}$, with pointwise operations, and the supremum norm. A function $P : c_0 \to \mathbb{C}$ is an *m-homogeneous polynomial* if there exists a continuous m-linear form $A : c_0^m \to \mathbb{C}$, such that $P(\boldsymbol{z})=A(\boldsymbol{z},\ldots,\boldsymbol{z})$ for every $\boldsymbol{z}\in c_0$. The 0-homogeneous polynomials are constant functions. We first recall that for a holomorphic $F : \mathbb{D}^N \to \mathbb{C}$, we have

$$F(\boldsymbol{z}) = \sum_{m=0}^{\infty} \sum_{\boldsymbol{\alpha}\in(\mathbb{N}\cup\{0\})^N, |\boldsymbol{\alpha}|=m} c_{\boldsymbol{\alpha}}(F)\boldsymbol{z}^{\boldsymbol{\alpha}} \quad \text{for all } \boldsymbol{z}\in\mathbb{D}^N,$$

where for each $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_N) \in (\mathbb{N}\cup\{0\})^N$, $|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_N$, and

$$c_{\boldsymbol{\alpha}}(F) = \frac{1}{(2\pi i)^N} \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_N|=r_N} \frac{f(\zeta_1,\cdots,\zeta_N)}{\zeta_1^{\alpha_1+1}\cdots\zeta_N^{\alpha_N+1}}\, d\zeta_N\cdots d\zeta_1,$$

and arbitrary $r_1,\cdots,r_N \in (0,1)$. Also,

$$c_{\boldsymbol{\alpha}}(F) = \frac{(\boldsymbol{\partial}^{\boldsymbol{\alpha}}F)(\mathbf{0})}{\boldsymbol{\alpha}!}.$$

Then for every m, the function $P_m : \mathbb{C}^N \to \mathbb{C}$ given by

$$P_m(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in(\mathbb{N}\cup\{0\})^N, |\boldsymbol{\alpha}|=m} c_{\boldsymbol{\alpha}}(F)\boldsymbol{z}^{\boldsymbol{\alpha}},$$

is an m-homogeneous polynomial, and we have $F = \sum_{m=0}^{\infty} P_m$ pointwise on $\mathbb{D}^N$. It was shown in [7, Prop. 2.28] that for a bounded function $F : B_{c_0} \to \mathbb{C}$, $F \in H^\infty(B_{c_0})$ if and only if there exists a unique sequence $(P_m)_{m\in\mathbb{N}_0}$ of m-homogeneous polynomials on c_0, such that $F = \sum_{m=0}^{\infty} P_m$ pointwise on B_{c_0}. Moreover, in this case,

$$P_m(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in(\mathbb{N}\cup\{0\})^N, |\boldsymbol{\alpha}|=m} c_{\boldsymbol{\alpha}}(F)\boldsymbol{z}^{\boldsymbol{\alpha}}$$

for all $\boldsymbol{z} \in B_{c_{00}}$, $f = \sum_{m=0}^{\infty} P_m$ uniformly on rB_{c_0} for every $0 < r < 1$, and $\|P_m\|_\infty \le \|F\|_\infty$. We also recall from [10]:

Proposition 5.1. *The map sending* $F \in H^\infty(B_{c_0})$ *to* $f = \sum_{n=1}^{\infty} \frac{1}{(\boldsymbol{\nu}(n))!}(\boldsymbol{\partial}^{\boldsymbol{\nu}(n)}F)(\boldsymbol{0})n^{-s}$, *is a Banach algebra isometric isomorphism from* $H^\infty(B_{c_0})$ *to* $\mathscr{H}^\infty$.

The set $\boldsymbol{N}$ is an additive semigroup with termwise addition. If $\boldsymbol{S}$ is an additive subsemigroup of $\boldsymbol{N}$ containing the zero sequence $\boldsymbol{0} := (0)_{k\in\mathbb{N}}$, then let $H^\infty_{\boldsymbol{S}}(B_{c_0})$ be the subalgebra of $H^\infty(B_{c_0})$ consisting of all $F \in H^\infty(B_{c_0})$ such that for all $\boldsymbol{\nu} \in \boldsymbol{N}\setminus\boldsymbol{S}$, $(\boldsymbol{\partial}^{\boldsymbol{\nu}}F)(\boldsymbol{0}) = 0$. The fact that $H^\infty_{\boldsymbol{S}}(B_{c_0})$ is an algebra follows immediately from the multivariable Leibniz rule, as follows. If $\boldsymbol{\alpha} \in \boldsymbol{N}\setminus\boldsymbol{S}$ and $\boldsymbol{\beta} \preccurlyeq \boldsymbol{\alpha}$, then either $\boldsymbol{\beta} \notin \boldsymbol{S}$ or $\boldsymbol{\alpha}-\boldsymbol{\beta} \notin \boldsymbol{S}$ (otherwise $\boldsymbol{\alpha} = \boldsymbol{\beta}+(\boldsymbol{\alpha}-\boldsymbol{\beta}) \in \boldsymbol{S}$, a contradiction), and so either $(\boldsymbol{\partial}^{\boldsymbol{\beta}}F)(\boldsymbol{0}) = 0$ or $(\boldsymbol{\partial}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}G)(\boldsymbol{0}) = 0$, showing that $(\boldsymbol{\partial}^{\boldsymbol{\alpha}}(FG))(\boldsymbol{0}) = \sum_{\boldsymbol{\beta}\preccurlyeq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}(\boldsymbol{\partial}^{\boldsymbol{\beta}}F)(\boldsymbol{0}) \cdot (\boldsymbol{\partial}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}G)(\boldsymbol{0}) = 0$, since each summand on the right-hand side is zero. The completeness is a consequence of the Taylor series expansion recalled above ([7, Proposition 2.28]). Thus $H^\infty_{\boldsymbol{S}}(B_{c_0})$ is a unital Banach subalgebra of $H^\infty(B_{c_0})$ with the supremum norm.

If S is a multiplicative subsemigroup of $\mathbb{N}$ containing 1, then the map $S \ni n \mapsto \boldsymbol{\nu}(n) := (\nu_k(n))_{k\in\mathbb{N}} \in \boldsymbol{N}$ is an injective semigroup homomorphism, and we denote its image by $\boldsymbol{\nu}(S)$. An immediate corollary of Proposition 5.1 is the following.

Corollary 5.2. *Let* S *be a multiplicative subsemigroup of* $\mathbb{N}$ *containing* 1, *and let* $\boldsymbol{\nu}(S)$ *be the image of* S *under the map* $S \ni n \mapsto \boldsymbol{\nu}(n)$.
The map sending elements $F \in H^\infty_{\boldsymbol{\nu}(S)}(B_{c_0})$ *to* $f = \sum_{n=1}^{\infty} \frac{1}{(\boldsymbol{\nu}(n))!}(\boldsymbol{\partial}^{\boldsymbol{\nu}(n)}F)(\boldsymbol{0})n^{-s}$, *is a Banach algebra isometric isomorphism from* $H^\infty_{\boldsymbol{\nu}(S)}(B_{c_0})$ *to* $\mathscr{H}^\infty_S$.

Let A be a commutative unital complex semisimple Banach algebra. The dual space A^* of A consists of all continuous linear complex-valued maps on A. The *maximal ideal space* $\mathrm{M}(A)$ of A is the set of all nonzero multiplicative elements in A^* (the kernels of which are then in one-to-one correspondence with the maximal ideals of A). As $\mathrm{M}(A) \subset A^*$, it inherits the weak-$*$ topology of A^*. The topological space $\mathrm{M}(A)$ is a compact Hausdorff space, and is contained in the unit sphere of the Banach space A^* with the operator norm, $\|\varphi\| = \sup_{a\in A,\|a\|\le 1} |\varphi(a)|$ for all $\varphi \in A^*$. Let $C(\mathrm{M}(A))$ be the Banach algebra of complex-valued continuous maps on $\mathrm{M}(A)$ with pointwise operations and the norm $\|f\|_\infty$ where $\|f\|_\infty = \sup_{\varphi\in \mathrm{M}(A)} |f(\varphi)|$ for $f \in C(\mathrm{M}(A))$. The *Gelfand transform* $\widehat{a} \in C(\mathrm{M}(A))$ of $a \in A$ is defined by $\widehat{a}(\varphi)=\varphi(a)$ for $\varphi \in \mathrm{M}(A)$.

For $\boldsymbol{z}_* \in B_{c_0}$, the map $\varphi_{\boldsymbol{z}_*} : H^\infty_{\boldsymbol{\nu}(S)}(B_{c_0}) \to \mathbb{C}$ defined by $\varphi_{\boldsymbol{z}_*}(f) = f(\boldsymbol{z}_*)$ for all $f \in H^\infty_{\boldsymbol{\nu}(S)}(B_{c_0})$ is an element of $\mathrm{M}(H^\infty_{\boldsymbol{\nu}(S)}(B_{c_0}))$. We will use this observation to prove Theorem 6.2 in the next and final section.

6. Bass stable rank

In algebraic K-theory, the notion of 'stable rank' of a ring was introduced to facilitate K-theoretic computations (see [3]). We recall the pertinent definitions below.

Let A be a unital commutative ring with unit element denoted by 1. An element $(a_1, \cdots, a_n) \in A^n$ is *unimodular* if there exist $b_1, \cdots, b_n \in A$ such that $b_1a_1 + \cdots + b_na_n = 1$. The set of all unimodular elements of A^n is denoted by $U_n(A)$. We call $(a_1, \cdots, a_{n+1}) \in U_{n+1}(A)$ *reducible* if there exist $x_1, \cdots, x_n \in A$ such that $(a_1+x_1a_{n+1}, \cdots, a_n+x_na_{n+1}) \in U_n(A)$. The *Bass stable rank* of A is the least $n \in \mathbb{N}$ for which every element in $U_{n+1}(A)$ is reducible. The *Bass stable rank of A is infinite* if there is no such n.

What is the Bass stable rank of $\mathscr{H}^\infty_S$?

- If $S=\{1\}$, then $\mathscr{H}^\infty_S$ is $\mathbb{C}$ as a ring, and the Bass stable rank is 1.
- If $S = \{p^k : k \in \mathbb{N} \cup \{0\}\}$, p a prime, then $\mathscr{H}^\infty_S$ is isomorphic as a Banach algebra to the Hardy algebra H^∞, whose Bass stable rank is 1 (see [16]).
- If $S=\mathbb{N}$, the Bass stable rank of $\mathscr{H}^\infty_S = \mathscr{H}^\infty$ is infinite ([13, Thm. 1.6]).

It is natural to expect that the Bass stable rank of $\mathscr{H}^\infty_S$ ought to be related to an appropriate notion of 'rank/dimension' of the semigroup S, which perhaps gives lower or upper bounds on the Bass stable rank. There are several notions of the rank of a semigroup. For instance, we recall below the notion of 'lower rank' and the notion of 'upper rank' introduced in [12]. For every subset $\mathcal{S}$ of a semigroup Σ, there is at least one subsemigroup of Σ containing $\mathcal{S}$, namely Σ itself. So the intersection of all the subsemigroups of Σ containing $\mathcal{S}$ is a subsemigroup of Σ containing $\mathcal{S}$, and we denote it by $\langle \mathcal{S} \rangle$. For $\emptyset \neq \mathcal{S} \subset \Sigma$, the subsemigroup $\langle \mathcal{S} \rangle$ consists of all elements of Σ that can be expressed as finite products of elements of $\mathcal{S}$. Let $|\mathcal{S}|$ denote the cardinal number of $\mathcal{S}$. The *lower rank* of Σ is $r(\Sigma) := \inf\{|\mathcal{S}| : \mathcal{S} \subset \Sigma, \text{ and } \langle S \rangle = \Sigma\}$. A subset $\mathcal{S}$ of a semigroup Σ is *independent* if for all $s \in \mathcal{S}$, we have $s \notin \langle \mathcal{S} \setminus \{s\} \rangle$. The *upper rank of* Σ is $R(\Sigma) := \sup\{|\mathcal{S}| : \mathcal{S} \subset \Sigma, \text{ and } \mathcal{S} \text{ is independent}\}$. It was shown in [12] that $r(S) \leq R(S)$. We have the following:

Conjecture 6.1. *Let S be a multiplicative subsemigroup of $\mathbb{N}$ such that $1 \in S$, and $R(S)=\infty$. Then the Bass stable rank of $\mathscr{H}^\infty_S$ is infinite.*

Let S be a multiplicative subsemigroup of $\mathbb{N}$ containing 1, $Q \subset S$ be infinite, and $\langle Q \rangle = S$. As $Q \subset \mathbb{N}$, Q must be countable. Arrange its members in strictly increasing order as $q_1 < q_2 < q_3 < \cdots$. We have the following.

Theorem 6.2. *Let S be a multiplicative subsemigroup of $\mathbb{N}$ containing 1, and let $q_1 < q_2 < q_3 < \cdots$ be a sequence in S such that for all $n \in S$, $n = \prod_{k=1}^{\infty} q_k^{\alpha_k(n)}$ for a unique compactly supported sequence $(\alpha_k(n))_{k\in\mathbb{N}}$ of nonnegative integers. Then the Bass stable rank of $\mathscr{H}_S^\infty$ is infinite.*

Proof. We follow an approach similar to the one from [13, Thm. 1.6], except that the role of the primes is now replaced by $(q_k)_{k\in\mathbb{N}}$. Fix $n \in \mathbb{N}$. Define $f_1, \cdots, f_{n+1} \in \mathscr{H}_S^\infty$ by $f_1 = q_1^{-s}, \ \cdots, \ f_n = q_n^{-s}, \ f_{n+1} = \prod_{j=1}^{n} (\mathbf{1} - (q_j q_{n+j})^{-s})$. Then $(f_1, \cdots, f_{n+1}) \in U_{n+1}(\mathscr{H}_S^\infty)$ since expanding the product defining f_{n+1} gives $f_{n+1} = \mathbf{1} - q_1^{-s} \cdot g_1 - \cdots - q_n^{-s} \cdot g_n = \mathbf{1} - f_1 g_1 - \cdots - f_n g_n$, for suitable $g_1, \cdots, g_n \in \mathscr{H}_S^\infty$, and so with $g_{n+1} := 1$, we get $f_1 g_1 + \cdots + f_n g_n + f_{n+1} g_{n+1} = \mathbf{1}$. Let $(f_1, \cdots, f_{n+1})$ be reducible, and the elements $x_1, \cdots, x_n \in \mathscr{H}_S^\infty$ be such that $(q_1^{-s} + x_1 f_{n+1}, \ \cdots, \ q_n^{-s} + x_n f_{n+1}) \in U_n(\mathscr{H}_S^\infty)$. Let $y_1, \cdots, y_n \in \mathscr{H}_S^\infty$ be such that $(q_1^{-s} + x_1 f_{n+1}) y_1 + \cdots + (q_n^{-s} + x_n f_{n+1}) y_n = \mathbf{1}$. Denote the isomorphism from Corollary 5.2 by $\iota : \mathscr{H}_S^\infty \to H_{\boldsymbol{\nu}(S)}^\infty(B_{c_0})$. Then we have $(\iota(q_1^{-s}) + \iota(x_1)\iota(f_{n+1}))\iota(y_1) + \cdots + (\iota(q_n^{-s}) + \iota(x_n)\iota(f_{n+1}))\iota(y_n) = \mathbf{1}$. Taking the Gelfand transform, we obtain

$$(\widehat{\iota(q_1^{-s})} + \widehat{\iota(x_1)}\widehat{\iota(f_{n+1})})\widehat{\iota(y_1)} + \cdots + (\widehat{\iota(q_n^{-s})} + \widehat{\iota(x_n)}\widehat{\iota(f_{n+1})})\widehat{\iota(y_n)} = 1. \qquad (\star)$$

For $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let $\boldsymbol{z}_* = (z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n}, 0, \cdots) \in B_{c_0}$, and

$$\mathbf{\Phi}(\boldsymbol{z}) = \begin{cases} -\prod_{j=1}^{n} (1 - |z_j|^2) \big(\widehat{\iota(x_1)}(\varphi_{\boldsymbol{z}_*}), \cdots, \widehat{\iota(x_n)}(\varphi_{\boldsymbol{z}_*})\big) & \text{if } |z_j| < 1, \ j = 1, \cdots, n, \\ \mathbf{0} \ (\in \mathbb{C}^n) & \text{otherwise.} \end{cases}$$

Then $\mathbf{\Phi}$ is a continuous map from $\mathbb{C}^n$ into $\mathbb{C}^n$. We have that $\mathbf{\Phi}$ vanishes outside $\mathbb{D}^n$, and so $\max_{\boldsymbol{z} \in \mathbb{D}^n} \|\mathbf{\Phi}(\boldsymbol{z})\|_2 = \sup_{\boldsymbol{z} \in \mathbb{C}^n} \|\mathbf{\Phi}(\boldsymbol{z})\|_2$, where $\| \cdot \|_2$ denotes the usual Euclidean norm in $\mathbb{C}^n$. This implies that there must exist an $r \geqslant 1$ such that $\mathbf{\Phi}$ maps $K := r\overline{\mathbb{D}}^n$ into K. Since the set K is compact and convex, by Brouwer's Fixed Point Theorem (see, e.g., [15, Theorem 5.28]), it follows that there exists a $\boldsymbol{\zeta} \in K$ such that $\mathbf{\Phi}(\boldsymbol{\zeta}) = \boldsymbol{\zeta}$. Since $\mathbf{\Phi}$ is zero outside $\mathbb{D}^n$, we see that $\boldsymbol{\zeta} \in \mathbb{D}^n$. Let $\boldsymbol{\zeta} = (\zeta_1, \cdots, \zeta_n)$, and $\boldsymbol{\zeta}_* = (\zeta_1, \cdots, \zeta_n, \overline{\zeta_1}, \cdots, \overline{\zeta_n}, 0, \cdots) \in B_{c_0}$. Then for each $j \in \{1, \cdots, n\}$,

$$\begin{aligned} 0 &= \zeta_j + \prod_{k=1}^{n} (1 - |\lambda_k|^2) \widehat{\iota(x_j)}(\varphi_{\boldsymbol{\zeta}_*}) \prod_{k=1}^{n} (1 - |\lambda_k|^2) \\ &= \zeta_j + (\widehat{\iota(x_j)}\widehat{\iota(f_{n+1})})(\varphi_{\boldsymbol{\zeta}_*}). \end{aligned} \qquad (\star\star)$$

But from $(\star)$, we have $\sum_{j=1}^{n} (\widehat{\iota(q_j^{-s})} + \widehat{\iota(x_j)}\widehat{\iota(f_{n+1})})\widehat{\iota(y_j)}\big|_{\varphi_{\boldsymbol{\zeta}_*}} = 1$, which together with $(\star\star)$ yields $0 = 1$, a contradiction. As $n \in \mathbb{N}$ was arbitrary, it follows that the Bass stable rank of $\mathscr{H}_S^\infty$ is infinite. $\square$

E.g., consider $S = \{1\} \cup \{n : \text{there exist } x, y \in \mathbb{N} \text{ such that } n = x^2 + y^2\}$ from Example 2.2(8). Then the Bass stable rank of $\mathscr{H}_S^\infty$ is infinite, as S is generated by $P \cup Q$, where P consists of primes p that are not of the form

$4k + 3$ for some $k \in \mathbb{N} \cup \{0\}$, and Q is the set of elements $q = p^2$, where p is a prime of the form $4k + 3$ for some $k \in \mathbb{N} \cup \{0\}$.

Acknowledgements

Thanks are due to the referee for the careful review, in particular for shortening the proofs of Propositions 2.1 and 3.1, and for the suggestions on improving the overall presentation of the article.

Funding The author has no financial or proprietary interests in any material discussed in this article.

Data Availability Statement No data sets were generated or analysed during the current study.

Declarations

Conflicts of Interest The author declares that he has no conflict of interest.

References

[1] Apostol, T.: Modular functions and Dirichlet series in number theory. 2nd ed. Graduate Texts in Mathematics 41, Springer (1990)

[2] Aron, R., Bayart, F., Gauthier, P., Maestre, M., Nestoridis, V.: Dirichlet approximation and universal Dirichlet series. Proc. Am. Math. Soc. **145**(10), 4449–4464 (2017)

[3] Bass, H.: Algebraic K-Theory. Benjamin, W.A (1968)

[4] Bohr, H.: Über die bedeutung der potenzreihen unendlich vieler variabeln in der theorie der dirichletscher reihen $\sum a_n/n^s$. Nachr. Ges. Wiss. Göttingen Math. Phys. **441–488**, 1913 (1913)

[5] Bonet, J.: The Fréchet Schwartz algebra of uniformly convergent Dirichlet series. Proc. Edinb. Math. Soc. Series II **61**(4), 933–942 (2018)

[6] Chandrasekharan, K.: Introduction to analytic number theory. Die Grundlehren der mathematischen Wissenschaften 148, Springer (1968)

[7] Defant, A., García, D., Maestre, M., Sevilla-Peris, P.: Dirichlet series and holomorphic functions in high dimensions. New Mathematical Monographs 37, Cambridge University Press (2019)

[8] Davidson, K., Paulsen, V., Raghupathi, M., Singh, D.: A constrained Nevanlinna-Pick interpolation problem. Indiana Univ. Math. J. **58**(2), 709–732 (2009)

[9] Edwards, D.: On absolutely convergent Dirichlet series. Proc. Am. Math. Soc. **8**, 1067–1074 (1957)

[10] Hedenmalm, H., Lindqvist, P., Seip, K.: A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0, 1)$. Duke Math. J. **86**(1), 1–37 (1997)

[11] Hewitt, E., Williamson, J.: Note on absolutely convergent Dirichlet series. Proc. Am. Math. Soc. **8**, 863–868 (1957)

[12] Howie, J., Ribeiro, M.: Rank properties in finite semigroups. Commun. Algebra **27**(11), 5333–5347 (1999)

[13] Mortini, R., Sasane, A.: Some curiosities of the algebra of bounded Dirichlet series. Bull. Korean Math. Soc. **53**(1), 153–161 (2016)

[14] Queffelec, H., Queffelec, M.: Diophantine approximation and Dirichlet series. Second edition. Texts and Readings in Mathematics 80, Hindustan Book Agency (2020)

[15] Rudin, W.: Functional analysis. 2nd ed. International Series in Pure and Applied Mathematics, McGraw-Hill (1991)

[16] Treil, S.: The stable rank of the algebra H^∞ equals 1. J. Funct. Anal. **109**(1), 130–154 (1992)

[17] Wiener, N.: The Fourier integral and certain of its applications. Cambridge University Press. Reprinted by Dover Publications, New York (1933)

Amol Sasane (✉)
Department of Mathematics
London School of Economics
Houghton Street
WC2A 2AE London
United Kingdom
e-mail: `A.J.Sasane@LSE.ac.uk`

Received: September 2, 2025.
Revised: December 24, 2025.
Accepted: January 2, 2026.