MAXIMAL IDEAL SPACE OF SOME BANACH ALGEBRAS OF DIRICHLET SERIES

AMOL SASANE

Abstract. Let $\mathscr{H}^\infty$ be the set of all Dirichlet series $f = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (where $a_n \in \mathbb{C}$ for each n) that converge at each $s \in \mathbb{C}_+$, such that $\|f\|_\infty := \sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. Let $\mathscr{B} \subset \mathscr{H}^\infty$ be a Banach algebra containing the Dirichlet polynomials (Dirichlet series with finitely many nonzero terms) with a norm $\|\cdot\|_{\mathscr{B}}$ such that the inclusion $\mathscr{B} \subset \mathscr{H}^\infty$ is continuous. For $m \in \mathbb{N} = \{1, 2, 3, \cdots\}$, let $\partial^{-m}\mathscr{B}$ denote the Banach algebra consisting of all $f \in \mathscr{B}$ such that $f', \cdots, f^{(m)} \in \mathscr{B}$, with pointwise operations and the norm $\|f\|_{\partial^{-m}\mathscr{B}} = \sum_{\ell=0}^{m} \frac{1}{\ell!}\|f^{(\ell)}\|_{\mathscr{B}}$. Assuming that the Wiener $1/f$ property holds for $\mathscr{B}$ (that is, $\inf_{s\in\mathbb{C}_+} |f(s)| > 0$ implies $\frac{1}{f} \in \mathscr{B}$), it is shown that for all $m \in \mathbb{N}$, the maximal ideal space $\mathrm{M}(\partial^{-m}\mathscr{B})$ of $\partial^{-m}\mathscr{B}$ is homeomorphic to $\overline{\mathbb{D}}^{\mathbb{N}}$, where $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$. Examples of such Banach algebras are $\mathscr{H}^\infty$, the subalgebra $\mathscr{A}_u$ of $\mathscr{H}^\infty$ consisting of uniformly continuous functions in $\mathbb{C}_+$, and the Wiener algebra $\mathscr{W}$ of Dirichlet series with $\|f\|_{\mathscr{W}} := \sum_{n=1}^{\infty} |a_n| < \infty$. Some consequences (existence of logarithms, projective freeness, infinite Bass stable rank) are given as applications.

1. Introduction

The aim of this article is to determine the maximal ideal space of a particular family $\{\partial^{-m}\mathscr{B}\}_{m\in\mathbb{N}}$ (defined below) of Banach algebras that are contained in the Hardy algebra $\mathscr{H}^\infty$ of Dirichlet series. The motivation is twofold: there has been old and recent interest in studying various Banach algebras of Dirichlet series (see e.g. [6], [10], [21]), and the Banach algebras $\partial^{-m}\mathscr{B}$ we study are also the 'Dirichlet series analogue' of the Banach algebras $\partial^{-m}H^\infty$ previously studied in [19] in the context of the classical Hardy algebra H^∞ of the disc.

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Set $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Let $\mathscr{H}^\infty$ be the set of all Dirichlet series $f = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $a_n \in \mathbb{C}$ for each $n \in \mathbb{N}$, that converge for all $s \in \mathbb{C}_+$, such that $\|f\|_\infty := \sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. With pointwise operations and the supremum norm, $\mathscr{H}^\infty$ is a Banach algebra.

The Banach algebras $\partial^{-m}\mathscr{B}$. Throughout, $\mathscr{B} \subset \mathscr{H}^\infty$ will denote a Banach algebra with a norm $\|\cdot\|_{\mathscr{B}}$. For $m \in \mathbb{N} = \{1, 2, 3, \cdots\}$, let $\partial^{-m}\mathscr{B}$ denote the Banach algebra consisting of all $f \in \mathscr{B}$ such that $f', \cdots, f^{(m)} \in \mathscr{B}$, with pointwise operations and the norm

$$\|f\|_{\partial^{-m}\mathscr{B}} = \sum_{\ell=0}^{m} \frac{1}{\ell!} \|f^{(\ell)}\|_{\mathscr{B}}.$$

We note that if $f = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for all $s \in \mathbb{C}_+$, then it converges uniformly on compact sets contained in $\mathbb{C}_+$, and hence by Weierstrass's theorem on uniform limits of holomorphic functions, $f^{(\ell)}$ is obtained by termwise differentiation, so that for all $\ell \in \mathbb{N}$, we have

$$f^{(\ell)} = \sum_{n=1}^{\infty} (-1)^\ell (\log n)^\ell \frac{a_n}{n^s} \text{ in } \mathbb{C}_+.$$

Let $\mathscr{P}$ denote the set of *Dirichlet polynomials*, that is, Dirichlet series with finite support,

$$\mathscr{P} = \Big\{p = \sum_{n=1}^{N} \frac{a_n}{n^s} : N \in \mathbb{N},\ a_1, \cdots, a_N \in \mathbb{C}\Big\} \subset \mathscr{H}^\infty.$$

Let $\mathscr{A}_u$ be the subset of $\mathscr{H}^\infty$ of Dirichlet series that are uniformly continuous in $\mathbb{C}_+$. Another description of $\mathscr{A}_u$ is that it is the closure of Dirichlet polynomials in the $\|\cdot\|_\infty$-norm, see, e.g., [1, Theorem 2.3].

Let $\mathscr{W}$ denote the set of all Dirichlet series $f = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ such that

$$\|f\|_1 := \sum_{n=1}^{\infty} |a_n| < \infty.$$

With pointwise operations and the $\|\cdot\|_1$ norm, $\mathscr{W}$ is a Banach algebra. It is clear that

$$\mathscr{W} \subset \mathscr{A}_u \subset \mathscr{H}^\infty.$$

In the case of $\mathscr{W}$, an analogue of the classical Wiener $1/f$ lemma ([20, p.91]) for the unit circle holds, that is, if $f \in \mathscr{W}$ is such that $\inf_{s\in\mathbb{C}_+} |f(s)| > 0$, then $\frac{1}{f} \in \mathscr{W}$ (see, e.g., [12, Thm. 1], and also [9] for an elementary proof).

We say that a Banach algebra $\mathscr{B} \subset \mathscr{H}^\infty$ has the *Wiener property* if

(W) For all $f \in \mathscr{B}$ satisfying $\inf_{s\in\mathbb{C}_+} |f(s)| > 0$, we have $\frac{1}{f} \in \mathscr{B}$.

The Banach algebra $\mathscr{H}^\infty$ also possesses the Wiener property (W) (see, e.g., [3, Theorem 2.6]).

Lemma 1.1. *$\mathscr{A}_u$ possesses the Wiener property (W).*

Proof. Let $f \in \mathscr{A}_u$ satisfy $d := \inf_{s\in\mathbb{C}_+} |f(s)| > 0$. As $\mathscr{A}_u \subset \mathscr{H}^\infty$, it follows that $\frac{1}{f} \in \mathscr{H}^\infty$. Moreover, $\frac{1}{f}$ is uniformly continuous in $\mathbb{C}_+$: for all $z, w \in \mathbb{C}_+$, we have

$$|\tfrac{1}{f}(w) - \tfrac{1}{f}(z)| = \tfrac{|f(z)-f(w)|}{|f(z)||f(w)|} \leqslant \tfrac{1}{d^2}|f(w) - f(z)|,$$

and f is uniformly continuous in $\mathbb{C}_+$. □

Let A be a commutative unital complex semisimple Banach algebra. The dual space A^* of A consists of all continuous linear complex-valued maps defined on A. The *maximal ideal space* $\mathrm{M}(A)$ of A is the set of all nonzero multiplicative elements in A^* (the kernels of which are then in one-to-one correspondence with the maximal ideals of A). As $\mathrm{M}(A)$ is a subset of A^*, it inherits the weak-$*$ topology of A^*, called the *Gelfand topology* on $\mathrm{M}(A)$. The topological space $\mathrm{M}(A)$ is a compact Hausdorff space, and is contained in the unit sphere of the Banach space A^* with the operator norm, $\|\varphi\| = \sup_{a\in A,\, \|a\|\leqslant 1} |\varphi(a)|$ for all $\varphi \in A^*$. Let $C(\mathrm{M}(A))$ denote the Banach algebra of complex-valued continuous functions on $\mathrm{M}(A)$ with pointwise operations and the supremum norm, $\|f\|_\infty = \sup_{\varphi\in\mathrm{M(A)}} |f(\varphi)|$ for all $f \in C(\mathrm{M}(A))$. The *Gelfand transform* $\widehat{a} \in C(\mathrm{M}(A))$ of an element $a \in A$ is defined by $\widehat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \mathrm{M}(A)$.

Main result. The main result in this article is the following.

Theorem 1.2. *Let $m \in \mathbb{N}$, and let the Banach algebra $\mathscr{B}$ be such that*

- $\mathscr{P} \subset \mathscr{B} \subset \mathscr{H}^\infty$
- *there exists a $C > 0$ such that for all $f \in \mathscr{B}$, $\|f\|_\infty \leqslant C\|f\|_{\mathscr{B}}$*
- *$\mathscr{B}$ possess the Wiener property (W).*

Then the maximal ideal space of $\partial^{-m}\mathscr{B}$ is homeomorphic to $\overline{\mathbb{D}}^{\mathbb{N}}$.

Here each factor $\overline{\mathbb{D}}$ has the usual Euclidean topology inherited from $\mathbb{C}$, and $\overline{\mathbb{D}}^{\mathbb{N}}$ is given the product topology.

In [19, Proposition 1.3], it was shown that the maximal ideal space of the Banach algebra $\partial^{-m}H^\infty$ is homeomorphic to $\overline{\mathbb{D}}$ for $m \in \mathbb{N}$, where H^∞ is the classical Hardy algebra of bounded and holomorphic functions on the open unit disk $\mathbb{D}$, and $\partial^{-m}H^\infty = \{f \in H^\infty : f', \cdots, f^{(m)} \in H^\infty\}$. Theorem 1.2 is the 'Dirichlet series analogue' of this result.

Examples. Examples of such Banach algebras $\mathscr{B}$ are $\mathscr{H}^\infty$, $\mathscr{A}_u$ and $\mathscr{W}$. Given a subset $S \subset i\mathbb{R}$, the Banach algebra

$$\mathscr{H}_S^\infty := \{f \in \mathscr{H}^\infty : f \text{ has a continuous extension to } S\},$$

with pointwise operations and the norm $\|\cdot\|_\infty$, is also one that satisfies the assumptions of Theorem 1.2. The Wiener property (W) for $\mathscr{H}_S^\infty$ is an immediate consequence of that for $\mathscr{H}^\infty$.

Organisation of the article. In Section 2, we will prove Theorem 1.2, and in Section 3, some corollaries (existence of logarithms, projective freeness, infinite Bass stable rank) are given as applications.

2. Proof of the main result

We first show the following, which will be used to prove Theorem 1.2.

Lemma 2.1. *If $m \in \mathbb{N}$, then $\partial^{-m}\mathscr{B} \subset \mathscr{A}_u$.*

Proof. Let $f \in \partial^{-m}\mathscr{B}$. As $m \geqslant 1$, $f' \in \mathscr{B} \subset \mathscr{H}^\infty$. For $z, w \in \mathbb{C}_+$, let $[z, w]$ denote the straight line segment joining z to w. By the fundamental theorem of contour integration, $f(w) - f(z) = \int_{[z,w]} f'(\zeta)d\zeta$. By the ML-inequality,

$$|f(w) - f(z)| \leqslant |w - z| \max_{\zeta \in [z,w]} |f'(\zeta)| \leqslant |w - z| \|f'\|_\infty.$$

Thus f is uniformly continuous in $\mathbb{C}_+$. Also $f \in \mathscr{H}^\infty$. So $f \in \mathscr{A}_u$. □

Let $p_1 < p_2 < p_3 < \cdots$ be the sequence of all primes arranged in increasing order. By the fundamental theorem of arithmetic, every $n \in \mathbb{N}$ can be written uniquely in the form

$$n = \prod_{k=1}^{\infty} p_k^{\nu_{p_k}(n)},$$

where $\nu_{p_k}(n) \in \mathbb{N} \cup \{0\}$ denotes the largest integer m such that p_k^m divides n.

Proof of Theorem 1.2. For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots) \in \overline{\mathbb{D}}^{\mathbb{N}}$, define $\varphi_{\boldsymbol{\lambda}} : \mathscr{P} \to \mathbb{C}$ by

$$\varphi_{\boldsymbol{\lambda}}(p) = \sum_{n=1}^{N} a_n \prod_{k=1}^{\infty} \lambda_k^{\nu_{p_k}(n)}, \quad \text{for } p = \sum_{n=1}^{N} \frac{a_n}{n^s} \in \mathscr{P}.$$

For each $n \in \mathbb{N}_*$, since

$$\Big|\prod_{k=1}^{\infty} \lambda_k^{\nu_{p_k}(n)}\Big| \leqslant 1$$

we have that $|\varphi_{\boldsymbol{\lambda}}(p)| \leqslant \|p\|_1 \leqslant \|p\|_\infty$. As $m \geqslant 1$, it follows from Lemma 2.1 that $\mathscr{B} \subset \mathscr{A}_u$. So $\mathscr{P}$ is dense in $\partial^{-m}\mathscr{B}$ in the $\|\cdot\|_\infty$-norm. Given $f \in \partial^{-m}\mathscr{B}$, let $(p_n)_{n \in \mathbb{N}}$ be a sequence in $\mathscr{P}$ that converges to f in the $\|\cdot\|_\infty$-norm. Then $(\varphi_{\boldsymbol{\lambda}}(p_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ (as $|\varphi_{\boldsymbol{\lambda}}(p_n)-\varphi_{\boldsymbol{\lambda}}(p_m)|\leqslant\|p_n-p_m\|_\infty$), and hence convergent. Define

$$\varphi_{\boldsymbol{\lambda}}(f)=\lim_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(p_n).$$

Then $\varphi_{\boldsymbol{\lambda}}:\partial^{-m}\mathscr{B}\to\mathbb{C}$ is well-defined: if $(\widetilde{p}_n)_{n\in\mathbb{N}}$ is another sequence of approximating Dirichlet polynomials, then

$$|\varphi_{\boldsymbol{\lambda}}(\widetilde{p}_n)-\varphi_{\boldsymbol{\lambda}}(p_n)|\leqslant\|\widetilde{p}_n-p_n\|_\infty\leqslant\|\widetilde{p}_n-f\|_\infty+\|f-p_n\|_\infty\to 0 \text{ as } n\to\infty,$$

and so $\lim\limits_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(\widetilde{p}_n)=\lim\limits_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(p_n)+\lim\limits_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(\widetilde{p}_n-p_n)=\lim\limits_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(p_n)+0.$

We claim that the map $\varphi_{\boldsymbol{\lambda}}$ is a complex homomorphism. It is enough to show linearity and multiplicativity on $\mathscr{P}$, since it then extends to $\partial^{-m}\mathscr{B}$ by the algebra of limits, and the continuity of addition, scalar multiplication and multiplication on $\mathscr{P}$ in the $\|\cdot\|_\infty$-norm. Linearity is clear, so we just show multiplicativity:

$$\begin{aligned}\varphi_{\boldsymbol{\lambda}}(pq)&=\sum_{n=1}^{N}\big(\sum_{d|n}a_d b_{\frac{n}{d}}\big)\prod_{k=1}^{\infty}\lambda_k^{\nu_{p_k}(n)}=\sum_{n=1}^{N}\big(\sum_{d|n}a_d b_{\frac{n}{d}}\big)\prod_{k=1}^{\infty}\lambda_k^{\nu_{p_k}(d)+\nu_{p_k}(\frac{n}{d})}\\&=\big(\sum_{d=1}^{N}a_d\prod_{k=1}^{\infty}\lambda_k^{\nu_{p_k}(d)}\big)\big(\sum_{\widetilde{d}=1}^{N}b_{\widetilde{d}}\prod_{k=1}^{\infty}\lambda_k^{\nu_{p_k}(\widetilde{d})}\big)=\varphi_{\boldsymbol{\lambda}}(p)\varphi_{\boldsymbol{\lambda}}(q)\end{aligned}$$

for all $p=\sum\limits_{n=0}^{N}\frac{a_n}{n^s}$, $q=\sum\limits_{n=0}^{N}\frac{b_n}{n^s}\in\mathscr{P}$. Finally, $\varphi_{\boldsymbol{\lambda}}$ is bounded, because

$$\begin{aligned}|\varphi_{\boldsymbol{\lambda}}(f)|=\big|\lim_{n\to\infty}\varphi_{\boldsymbol{\lambda}}(p_n)\big|=\lim_{n\to\infty}|\varphi_{\boldsymbol{\lambda}}(p_n)|\leqslant\lim_{n\to\infty}\|p_n\|_\infty=\|f\|_\infty\\ \leqslant C\|f\|_{\mathscr{B}}\leqslant C\|f\|_{\partial^{-m}\mathscr{B}},\end{aligned}$$

where $f\in\partial^{-m}\mathscr{B}$, and $(p_n)_{n\in\mathbb{N}}$ is an approximating sequence in $\mathscr{P}$ for f in the $\|\cdot\|_\infty$-norm. Note that in particular, we have $\|\varphi_{\boldsymbol{\lambda}}(f)|\leqslant\|f\|_\infty$.

Let $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots)$, $\boldsymbol{\mu}=(\mu_1,\mu_2,\cdots)$ be distinct elements of $\overline{\mathbb{D}}^{\mathbb{N}}$. Then there exists an $k_*\in\mathbb{N}$ such that $\lambda_{k_*}\neq\mu_{k_*}$. As $\partial^{-m}\mathscr{B}$ contains $\frac{1}{2^s},\frac{1}{3^s},\cdots\in\mathscr{P}$, we have

$$\varphi_{\boldsymbol{\lambda}}(\frac{1}{p_{k_*}^s})=1\cdot\lambda_1^0\cdots\lambda_{k_*-1}^0\lambda_{k_*}^1\lambda_{k_*+1}^0\cdots=\lambda_{k_*}\neq\mu_{k_*}=\varphi_{\boldsymbol{\mu}}(\frac{1}{p_{k_*}^s}).$$

Thus $\boldsymbol{\lambda}\mapsto\varphi_{\boldsymbol{\lambda}}$ embeds $\overline{\mathbb{D}}^{\mathbb{N}}$ in the maximal ideal space of $\partial^{-m}\mathscr{B}$.

We claim that the inclusion $\overline{\mathbb{D}}^{\mathbb{N}}\subset\mathrm{M}(\partial^{-m}\mathscr{B})$ is continuous. Let $(\boldsymbol{\lambda}_i)_{i\in I}$ be a net in $\overline{\mathbb{D}}^{\mathbb{N}}$ which is convergent to $\boldsymbol{\lambda}\in\overline{\mathbb{D}}^{\mathbb{N}}$. Let $\epsilon>0$ and $f\in\partial^{-m}\mathscr{B}$. Then there exists a

$$p=\sum_{n=0}^{N}\frac{b_n}{n^s}\in\mathscr{P},$$

such that $\|f-p\|_\infty<\frac{\epsilon}{4}$. Let $p_1,\cdots,p_{k_N}$ be the only primes which appear in the prime factorisation of $1,\cdots,N$. If $\succcurlyeq$ denotes the order on the directed set I, then there exists an $i_* \in I$ such that for all $i \succcurlyeq i_*$,

$$\sum_{n=1}^{N} |b_n| \Big| \prod_{k=1}^{k_N} \lambda_{i,k}^{\nu_{p_k}(n)} - \prod_{k=1}^{k_N} \lambda_{i_*,k}^{\nu_{p_k}(n)} \Big| < \frac{\epsilon}{2},$$

and so $|\varphi_{\boldsymbol{\lambda}_i}(p) - \varphi_{\boldsymbol{\lambda}}(p)| \leqslant \sum_{n=1}^{N} |b_n| \Big| \prod_{k=1}^{k_N} \lambda_{i,k}^{\nu_{p_k}(n)} - \prod_{k=1}^{k_N} \lambda_{i_*,k}^{\nu_{p_k}(n)} \Big| < \frac{\epsilon}{2}$. Thus

$$\begin{aligned} |\varphi_{\boldsymbol{\lambda}_i}(f) - \varphi_{\boldsymbol{\lambda}}(f)| &\leqslant |\varphi_{\boldsymbol{\lambda}_i}(p) - \varphi_{\boldsymbol{\lambda}}(p)| + |\varphi_{\boldsymbol{\lambda}_i}(f-p)| + |\varphi_{\boldsymbol{\lambda}}(f-p)| \\ &\leqslant \tfrac{\epsilon}{2} + \|f-p\|_\infty + \|f-p\|_\infty \leqslant \tfrac{\epsilon}{2} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \epsilon \end{aligned}$$

for all $i \succcurlyeq i_*$. Hence $(\varphi_{\boldsymbol{\lambda}_i})_{i\in I}$ converges to $\varphi_{\boldsymbol{\lambda}}$ in the weak-$*$ topology, i.e., the Gelfand topology on the maximal ideal space of $\partial^{-m}\mathscr{B}$.

Next we will show that every complex homomorphism is of the form $\varphi_{\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \overline{\mathbb{D}}^{\mathbb{N}}$.

Let $\varphi \in M(\partial^{-m}\mathscr{B})$. Define

$$\boldsymbol{\lambda} = \Big(\varphi(\tfrac{1}{2^s}), \varphi(\tfrac{1}{3^s}), \varphi(\tfrac{1}{5^s}), \cdots\Big).$$

We first show that for all $f \in \partial^{-m}\mathscr{B}$, we have

$$|\varphi(f)| \leqslant \|f\|_\infty. \qquad (*)$$

Suppose first that f also satisfies

$$\inf_{s\in\mathbb{C}_+} |f(s)| > 0.$$

As $\mathscr{B}$ possesses the Wiener property (W), we have $\frac{1}{f} \in \mathscr{B}$. Differentiating, we get successively that

$$(\tfrac{1}{f})' = -\frac{f'}{f^2}, \quad (\tfrac{1}{f})'' = -\frac{f''f^2 - 2f(f')^2}{f^4}, \quad \cdots,$$

and so (since $f, f', \cdots, f^{(m)} \in \mathscr{B}$), we conclude that $\frac{1}{f} \in \partial^{-m}\mathscr{B}$. So we have shown that if 0 does not belong to the closure of the range of $f \in \partial^{-m}\mathscr{B}$, then $\frac{1}{f} \in \partial^{-m}\mathscr{B}$, and in particular $1 = \varphi(1) = \varphi(f)\varphi(\frac{1}{f})$, showing that $\varphi(f) \neq 0$. Replacing f by $f-c$, where $c \in \mathbb{C}$, we conclude that if c does not belong to the closure of the range of f, then $\varphi(f) \neq c$. Thus $\varphi(f)$ belongs to the closure of the range of f. In particular, $|\varphi(f)| \leqslant \|f\|_\infty$, as wanted.

Applying this to $f := \frac{1}{p_k^s}$ yields $|\lambda_k| \leqslant 1$, $k \in \mathbb{N}_*$, and so $\boldsymbol{\lambda} \in \overline{\mathbb{D}}^{\mathbb{N}}$.

Since $\partial^{-m}\mathscr{B} \subset \mathscr{A}_u$, any $f \in \partial^{-m}\mathscr{B}$ can be approximated in the $\|\cdot\|_\infty$-norm by a sequence $(p_n)_{n\in\mathbb{N}}$ of Dirichlet polynomials. But $(*)$ shows that φ is continuous in the $\|\cdot\|_\infty$-norm, giving

$$\varphi(f) = \varphi(\lim_{n\to\infty} p_n) = \lim_{n\to\infty} \varphi(p_n) = \lim_{n\to\infty} \varphi_{\boldsymbol{\lambda}}(p_n) = \varphi_{\boldsymbol{\lambda}}(\lim_{n\to\infty} p_n) = \varphi_{\boldsymbol{\lambda}}(f).$$

We have seen that the Gelfand topology of the maximal ideal space of $\partial^{-m}\mathscr{B}$ is weaker/coarser than the product topology of $\overline{\mathbb{D}}^{\mathbb{N}}$. As the Gelfand topology is Hausdorff, and $\overline{\mathbb{D}}^{\mathbb{N}}$ is compact (Tychonoff's theorem), the two topologies coincide (see, e.g., [18, 14, §3.8], stating that if $\tau_1 \subset \tau_2$ are topologies on a set X, such that τ_1 is Hausdorff and τ_2 is compact, then $\tau_1 = \tau_2$). □

Remark 2.2. The theorem and its proof above also works for $m = 0$ if $\mathscr{B}$ is $\mathscr{A}_u$ or $\mathscr{W}$. The description of the maximal ideal space of $\mathscr{W}$ as being homeomorphic to $\overline{\mathbb{D}}^{\mathbb{N}}$ was shown in [21, Theorem 1.5].

3. Some consequences

Throughout this section, we will assume that $m \in \mathbb{N}$, and $\mathscr{B}$ is a Banach algebra such that

- $\mathscr{P} \subset \mathscr{B} \subset \mathscr{H}^\infty$
- there exists a $C > 0$ such that for all $f \in \mathscr{B}$, $\|f\|_\infty \leqslant C\|f\|_{\mathscr{B}}$
- $\mathscr{B}$ possess the Wiener property (W).

Contractibility of $\mathbf{M}(\partial^{-m}\mathscr{B})$. Recall that a topological space X is *contractible* if the identity map $\mathrm{id}_X : X \to X$ is null-homotopic, i.e., there exist an element $x_* \in X$ and a continuous map $\mathrm{H} : [0,1] \times X \to X$ such that $\mathrm{H}(0, \cdot) = \mathrm{id}_X$ and $\mathrm{H}(1, x) = x_*$ for all $x \in X$.

Corollary 3.1. *$\mathrm{M}(\partial^{-m}\mathscr{B})$ is contractible.*

Proof. It suffices to show $\overline{\mathbb{D}}^{\mathbb{N}}$ is contractible. Let $\boldsymbol{x}_* = \mathbf{0} = (0, 0, \cdots) \in \overline{\mathbb{D}}^{\mathbb{N}}$, and $\mathrm{H}(t, \boldsymbol{x}) = (1-t)\boldsymbol{x} = \big((1-t)x_1, (1-t)x_2, \cdots\big)$ for $\boldsymbol{x} = \big(x_1, x_2, \cdots\big) \in \overline{\mathbb{D}}^{\mathbb{N}}$ and $t \in [0, 1]$. Then H is continuous, $\mathrm{H}(0, \cdot) = \mathrm{id}_X$, and $\mathrm{H}(1, \boldsymbol{x}) = \boldsymbol{x}_*$ for all $\boldsymbol{x} \in \overline{\mathbb{D}}^{\mathbb{N}}$. □

Existence of logarithms. For a unital commutative complex Banach algebra A, the multiplicative group of all invertible elements of A is denoted by A^{-1}. Then $e^A := \{e^a : a \in A\}$ is a subgroup of A^{-1}. By the Arens-Royden theorem (see, e.g., [17, Theorem, p.295]), the group A^{-1}/e^A is isomorphic to the first Čech cohomology group $H^1(\mathrm{M}(A), \mathbb{Z})$ of $\mathrm{M}(A)$ with integer coefficients. For background on Čech cohomology, see, e.g., [8]. For a contractible space, all cohomology groups are trivial (see, e.g., [8, IX, Theorem 3.4]).

Corollary 3.2. $(\partial^{-m}\mathscr{B})^{-1} = e^{\partial^{-m}\mathscr{B}}$.

Projective freeness. For a commutative unital ring A with unit element denoted by 1, $A^{n\times n}$ denotes the $n \times n$ matrix ring over A, and $\mathrm{GL}_n(A) \subset A^{n\times n}$ denotes the group of invertible matrices. A commutative unital ring A is *projective free* if every finitely generated projective A-module is free. If A-modules M, N are isomorphic, then we write $M \cong N$. If M is a finitely generated A-module, then (i) M is *free* if $M \cong A^k$ for some $k \in \mathbb{N} \cup \{0\}$, and (ii) M is *projective* if there exists an A-module N and an $n \in \mathbb{N} \cup \{0\}$ such that $M \oplus N \cong A^n$. In terms of matrices (see, e.g., [7, Proposition 2.6]), the ring A is projective free if and only if every idempotent matrix P is conjugate (by an invertible matrix S) to a diagonal matrix with elements 1 and 0 on the diagonal, i.e., for all $n \in \mathbb{N}$ and every $P \in A^{n\times n}$ satisfying $P^2 = P$, there exists an $S \in \mathrm{GL}_n(A)$ such that for some $k \in \mathbb{N} \cup \{0\}$, $S^{-1}PS = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

In 1976, it was shown independently by Quillen and Suslin that if $\mathbb{F}$ is a field, then the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ is projective free, settling Serre's conjecture from 1955 (see [13]). In the context of a commutative unital complex Banach algebra A, [5, Theorem 4.1] (see also [4, Corollary 1.4]) says that the contractibility of the maximal ideal space $\mathrm{M}(A)$ is sufficient for A to be projective free.

Corollary 3.3. *$\partial^{-m}\mathscr{B}$ is a projective free ring.*

Bass stable rank. In algebraic K-theory, the notion of stable rank of a ring was introduced to facilitate K-theoretic computations [2]. Let A be a unital commutative ring with unit element denoted by 1. An element $(a_1, \cdots, a_n) \in A^n$ is *unimodular* if there exist $b_1, \cdots, b_n \in A$ such that $b_1a_1 + \cdots + b_na_n = 1$. The set of all unimodular elements of A^n is denoted by $U_n(A)$. We call $(a_1, \cdots, a_{n+1}) \in U_{n+1}(A)$ *reducible* if there exist $x_1, \cdots, x_n \in A$ such that $(a_1 + x_1a_{n+1}, \cdots, a_n + x_na_{n+1}) \in U_n(A)$. The *Bass stable rank* of A is the least $n \in \mathbb{N}$ for which every element in $U_{n+1}(A)$ is reducible. The *Bass stable rank of A is infinite* if there is no such n. The fact that the Bass stable rank of the infinite polydisc algebra $A(\mathbb{D}^\infty)$ is infinite was shown in [14, Proposition 1]. Analogously, we show the following (see also [15, Theorem 1.6], where a similar idea was used to show that the Bass stable rank of $\mathscr{H}^\infty$ is infinite).

Corollary 3.4. *The Bass stable rank of $\partial^{-m}\mathscr{B}$ is infinite.*

Proof. Fix $n \in \mathbb{N}$. Let $f_1, \cdots, f_{n+1} \in \mathscr{P} \subset \partial^{-m}\mathscr{B}$ be given by

$$f_1 = \frac{1}{2^s}, \quad \cdots, \quad f_n = \frac{1}{p_n^s}, \quad f_{n+1} = \prod_{j=1}^{n} \Big(1 - \frac{1}{(p_j p_{n+j})^s}\Big).$$

Then $(f_1, \cdots, f_{n+1}) \in U_{n+1}(\partial^{-m}\mathscr{B})$ because by expanding the product defining f_{n+1}, we obtain

$$f_{n+1} = 1 - \frac{1}{2^s} \cdot g_1 - \cdots - \frac{1}{p_n^s} \cdot g_n = 1 - f_1 g_1 - \cdots - f_n g_n,$$

for suitably defined $g_1, \cdots, g_n \in \mathscr{P} \subset \partial^{-m}\mathscr{B}$, and so with $g_{n+1} := 1$, we get $f_1g_1 + \cdots + f_ng_n + f_{n+1}g_{n+1} = 1$. Let $(f_1, \cdots, f_{n+1})$ be reducible, and $x_1, \cdots, x_n \in \partial^{-m}\mathscr{B}$ be such that

$$\Big(\frac{1}{2^s} + x_1 f_{n+1}, \ \cdots \ , \ \frac{1}{p_n^s} + x_n f_{n+1}\Big) \in U_n(\partial^{-m}\mathscr{B}).$$

Let $y_1, \cdots, y_n \in \partial^{-m}\mathscr{B}$ be such that

$$\Big(\frac{1}{2^s} + x_1 f_{n+1}\Big) y_1 + \cdots + \Big(\frac{1}{p_n^s} + x_n f_{n+1}\Big) y_n = 1.$$

Taking the Gelfand transform, and denoting the variable in the infinite polydisc $\overline{\mathbb{D}}^{\mathbb{N}}$ by $\boldsymbol{z} = (z_1, z_2, z_3, \cdots)$, we obtain

$$(z_1 + \widehat{x}_1 \widehat{f}_{n+1})\widehat{y}_1 + \cdots + (z_n + \widehat{x}_n \widehat{f}_{n+1})\widehat{y}_n = 1. \qquad (\star)$$

Let $\boldsymbol{x} := (\widehat{x}_1, \cdots, \widehat{x}_n)$. For $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we define

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \begin{cases} -\boldsymbol{x}(z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n}, 0, \cdots) \prod\limits_{j=1}^{n} (1 - |z_j|^2) \text{ if } |z_j| < 1, \ j = 1, \cdots, n, \\ \boldsymbol{0} \ (\in \mathbb{C}^n) \ \text{ otherwise.} \end{cases}$$

Then $\boldsymbol{\Phi}$ is a continuous map from $\mathbb{C}^n$ into $\mathbb{C}^n$. We have that $\boldsymbol{\Phi}$ vanishes outside $\mathbb{D}^n$, and so

$$\max_{\boldsymbol{z} \in \mathbb{D}^n} \|\boldsymbol{\Phi}(\boldsymbol{z})\|_2 = \sup_{\boldsymbol{z} \in \mathbb{C}^n} \|\boldsymbol{\Phi}(\boldsymbol{z})\|_2,$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm in $\mathbb{C}^n$. This implies that there must exist an $r \geqslant 1$ such that $\boldsymbol{\Phi}$ maps $K := r\overline{\mathbb{D}}^n$ into K. As K is compact and convex, by Brouwer's Fixed Point Theorem (see, e.g., [18, Theorem 5.28]) it follows that there exists a $\boldsymbol{z}_* \in K$ such that $\boldsymbol{\Phi}(\boldsymbol{z}_*) = \boldsymbol{z}_*$. Since $\boldsymbol{\Phi}$ is zero outside $\mathbb{D}^n$, we see that $\boldsymbol{z}_* \in \mathbb{D}^n$. Let $\boldsymbol{z}_* = (\lambda_1, \cdots, \lambda_n)$. Then for each $j \in \{1, \cdots, n\}$, we obtain

$$\begin{aligned} 0 &= \lambda_j + \widehat{x}_j(\lambda_1, \cdots, \lambda_n, \overline{\lambda_1}, \cdots, \overline{\lambda_n}, 0, \cdots) \prod_{k=1}^{n} (1 - |\lambda_k|^2) \\ &= \lambda_j + (\widehat{x}_j \widehat{f}_{n+1})(\lambda_1, \cdots, \lambda_n, \overline{\lambda_1}, \cdots, \overline{\lambda_n}, 0, \cdots). \qquad (\star\star) \end{aligned}$$

But from $(\star)$, we have

$$\sum_{j=1}^{n} (z_j + \widehat{x}_j \widehat{f}_{n+1})\widehat{y}_j \Big|_{(\lambda_1, \cdots, \lambda_n, \overline{\lambda_1}, \cdots, \overline{\lambda_n}, 0, \cdots)} = 1,$$

which together with $(\star\star)$ yields $0 = 1$, a contradiction. As $n \in \mathbb{N}$ was arbitrary, it follows that the Bass stable rank of $\partial^{-m}\mathscr{B}$ is infinite. $\square$

Remarks 3.5.

(1) For Banach algebras, an analogue of the Bass stable rank, called the topological stable rank, was introduced in [16]. Let A be a commutative complex Banach algebra with unit element 1. The least $n \in \mathbb{N}$ for which $U_n(A)$ is dense in A^n is called the *topological stable rank* of A. The *topological stable rank of A is infinite* if there is no such n. For a commutative unital semisimple complex Banach algebra, the Bass stable rank is at most equal to its topological stable rank (see, e.g., [16, Corollary 2.4]). It follows from Corollary 3.4 that the topological stable rank of $\partial^{-m}\mathscr{B}$ is infinite for all $m \in \mathbb{N}$.

(2) The *Krull dimension* of a commutative ring A is the supremum of the lengths of chains of distinct proper prime ideals of A. If a ring has Krull dimension d, then its Bass stable rank is at most $d + 2$ (see, e.g., [11]). It follows from Corollary 3.4 that the Krull dimension of $\partial^{-m}\mathscr{B}$ is infinite for all $m \in \mathbb{N}$.

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Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

Email address: A.J.Sasane@LSE.ac.uk