Post-trade netting and contagion

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Abstract

We analyse how post-trade netting in over-the-counter derivatives markets affects systemic risk. In particular, we focus on two post-trade netting services that rely on multilateral netting techniques: portfolio rebalancing and portfolio compression. First, we provide mathematical characterisations of their netting mechanisms and explain their relationship. Then, we analyse the effects of post-trade netting from a network perspective by considering contagion arising from defaults on variation margin payments. We provide sufficient conditions for post-trade netting to reduce systemic risk and show that post-trade netting can be harmful. We also explore the implications particularly when institutions strategically react to liquidity stress by delaying their payments.

1 Introduction

In the last decade, regulatory reforms to enhance the resilience of over-the-counter (OTC) derivatives markets provide incentives for market participants to use *post-trade risk reduction (PTRR)* services, which apply multilateral netting techniques to help mitigate risks and manage collateral obligations efficiently.[1] More recently, the European Securities and Markets Authority (ESMA) summarises that "PTRR transactions are successfully undertaken today and have reduced a considerable amount of risks in the market", (ESMA, 2020b, p. 3).

The main services used in this context are portfolio compression and portfolio rebalancing, see IOSCO (2024)[2]. Portfolio compression consists of replacing, removing or adding new trades usually with the aim to reduce gross positions while keeping net positions for each counterparty unchanged. Portfolio rebalancing, also known as counterparty risk rebalancing and counterparty risk optimisation, consists of adding new trades usually with the aim to reduce bilaterally netted gross positions while keeping net positions for each counterparty unchanged.

PTRR services typically consist of the following three steps (see ESMA (2020b)): First, participants submit their portfolio information to a third-party service provider (not a party to the transactions) and specify risk tolerances. Second, the service provider runs its optimisation algorithms and informs each participant of their new portfolio positions. Third, the new positions are established once all participants agree (otherwise, the exercise is void).

These services aim to mitigate operational and counterparty risks in existing derivatives portfolios while not materially changing the market risk. They reduce the complexity of the

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[1] The incentives are partly attributable to the Uncleared Margin Rules and the Leverage Ratio requirements. See Duffie (2018) for an overview of the post-crisis regulatory reforms.

[2] Recently, a third service has been considered in this context, namely *basis risk mitigation* (IOSCO, 2024) which is also called *basis risk optimisation* (FCA, 2024). Since, basis risk optimisation is concerned with risks such as strike risk and fixing risk (IOSCO, 2024) which are different types of risk, we will not consider it here.

intermediation chains (ESRB, 2020), a crucial factor affecting contagion. But this does not automatically imply that they reduce systemic risk. We will use a network approach to investigate the consequences of these services for systemic risk.

The main contributions of this paper are twofold. First, we define the notion of a *post-trade netting (PTN-) exercise* and then show that *portfolio rebalancing* and *portfolio compression* are special cases of a PTN-exercise. To be able to provide this unifying perspective, we propose what we believe is the first formal mathematical characterisation of portfolio rebalancing. We also formally characterise the relationship between portfolio rebalancing and portfolio compression (Theorem 2.10).

Second, we extend the existing literature on the relationship between post-trade netting services and systemic risk in several dimensions. Our analysis is based on the clearing framework of Veraart (2020), a generalisation of the framework of Eisenberg & Noe (2001) and Rogers & Veraart (2013). We consider variation margin (VM) payment networks in OTC derivatives markets as e.g. in Paddrik et al. (2020); Veraart (2022). Previous literature has focused on portfolio compression and implications for systemic risk (Veraart, 2022). We provide what we believe are the first results on the consequences of a general PTN-exercise that includes both portfolio rebalancing and portfolio compression for systemic risk. A key insight is that several implications for systemic risk are solely driven by the fact that PTN-exercises do not change the net positions of individual counterparties. We also discuss the implications of adding additional constraints (such as a reduction in gross positions, or a reduction in bilaterally netted positions, or a reduction in total payment obligations, etc.) for systemic risk.

Another extension of our work is that we consider the implications of PTN-exercises for systemic risk both from an ex post and an ex ante perspective. The ex post analysis compares the outcome in the original and in the PTN-network after a shock; the ex ante analysis considers both networks prior to a shock and investigates the resilience of the networks to a potential future shock.

Our ex post analysis builds on Veraart (2022) and extends some results to general PTN-exercises and to general fixed points that characterise the equilibrium (i.e., fixed points that are not necessarily the greatest fixed point). Our main result, Theorem 4.1, shows that no defaults among participants of a PTN-exercise is sufficient for reducing systemic risk when considering the greatest fixed point. We show that this result carries over to the least fixed point only under some additional conditions. We also show that no defaults among participants in the PTN-network is also a sufficient condition for systemic risk reduction (Proposition 4.3).

It has been shown in Veraart (2022) that under zero recovery rates, portfolio compression with certain constraints always leads to a reduction in systemic risk in the compressed system compared to the original system when considering the greatest fixed point. We show that this result carries over to the least fixed point under the same conditions, but it does not carry over to general PTN-exercises, in which case they can be harmful, i.e., lead to a default caused by contagion that only arises in the PTN-network but not in the original network (Theorem 4.4).

The consideration of other fixed points (not restricted to the greatest) is another contribution of our work. The least fixed point is of particular interest here. It has been shown by Csóka & Herings (2018) that decentralised clearing procedures usually result in the least fixed point. Also, Bardoscia et al. (2019) develop a framework in the spirit of decentralised clearing, in which institutions react strategically to liquidity stress by delaying their payments. We show that the approach by Bardoscia et al. (2019) is mathematically equivalent to considering the least clearing vector in the Rogers & Veraart (2013) model under zero recovery rates (Theorem 4.6).

Our ex ante analysis builds on the work by Glasserman & Young (2015) and applies it to both the original and the PTN-network. Specifically, we show that if the sufficient conditions identified by Glasserman & Young (2015) for contagion to be impossible or weak hold in the original network, then they also hold in the PTN-network that satisfies an additional condition

(Proposition 4.7).

The rest of the paper is organised as follows. In the remainder of this section, we discuss the related literature. Section 2 introduces a unifying mathematical characterisation of post-trade netting exercises and provides a novel mathematical characterisation of a portfolio rebalancing exercise. It then shows that both portfolio rebalancing and portfolio compression are special types of a post-trade netting exercise and we establish the mathematical relationship between portfolio rebalancing and portfolio compression. Section 3 describes the framework for assessing systemic risk. Section 4 presents the main results on when post-trade netting exercises reduce systemic risk. Section 5 concludes. The Appendix contains all the proofs of the results as well as some additional discussions and examples.

1.1 Related literature

Post-trade netting mechanisms have been developed throughout Europe since the thirteenth century—they were used by merchants in the early modern fairs to clear bills of exchange; we refer to Börner & Hatfield (2017) for some historical background.

The literature on recent multilateral netting activities has mainly been focused on centralised netting via the central counterparties (CCPs). For example, Duffie & Zhu (2011) provide a framework for multilateral netting by CCPs to point out a trade-off in netting efficiency between central clearing and bilateral clearing; Glasserman et al. (2016) analyse illiquidity associated with netting by multiple CCPs; and Amini et al. (2016) show that partial multilateral netting can have adverse effects on network contagion. Meanwhile, there has not been much literature on post-trade netting services. We are not aware of any work that examines the effects of portfolio rebalancing and possible implications for systemic risk.

For portfolio compression, there are two strands of literature. The first mainly focuses on compression algorithms. O'Kane (2017) proposes several optimisation-based algorithms for portfolio compression and analyses their performance on exposure reduction. Similarly, D'Errico & Roukny (2021) study the efficiency of portfolio compression with different levels of preference and use a transaction-level data set to learn empirically how much market excess portfolio compression can eliminate.

The second strand explores the risk implications of portfolio compression using network modelling. Veraart (2022) derives necessary conditions for portfolio compression to be harmful to the system and shows how the potential harmfulness can arise. Schuldenzucker & Seuken (2020) apply the Rogers & Veraart (2013) model to investigate the compression incentives and when compression can bring negative effects to the detriment of the system. Amini & Feinstein (2023) consider an optimal network compression problem where systemic risk measures are adopted in the objective function. Amini & Minca (2020) have looked further into the change of seniority structure of claims that implicitly occurs when networks are changed either by central clearing or portfolio compression.

2 A unifying characterisation of post-trade netting exercises

2.1 The financial market

We consider a financial system comprised of financial institutions $\mathcal{N} = \{1, 2, ..., N\}$ with $N \geq 3$, which are typically major dealer banks that are active in the global derivatives markets. For simplicity, we refer to them as "banks". A wide range of non-banks that trade in derivatives markets use post-trade netting services,[3] and all our results also apply to them.

Banks are connected by derivatives contracts, described by a graph represented by the corresponding *notional matrix* $C \in [0, \infty)^{N \times N}$, where the ij-th entry C_{ij} denotes the notional

[3]See `https://osttra.com/news/trioptima-named-best-compression-and-optimization-service/`.

amount of liabilities from bank i to bank j that arise from trading derivatives, and $C_{ii} = 0$ for all $i \in \mathcal{N}$. The *bilaterally netted notional matrix* corresponding to C is given by C^{bi}, where $C^{bi}_{ij} = \max(C_{ij} - C_{ji}, 0)$ for all $i, j \in \mathcal{N}$. The *net exposures* of C are given by $C^\top \mathbf{1} - C\mathbf{1} \in \mathbb{R}^N$, where $\mathbf{1}$ is the column vector containing only 1s, and the *gross exposures* of C are $C^\top \mathbf{1} + C\mathbf{1}$. In particular, the net exposures of C and C^{bi} coincide for all C (but the same does not usually hold for the gross exposures).

We assume that the derivative contracts are traded over-the-counter (OTC) and are fungible, such as single-name Credit Default Swap (CDS) contracts written on the same reference entity with the same maturity date—in which case C_{ij} represents the notional amount that bank i has promised to bank j if a credit event of the underlying reference entity occurs.[4]

2.2 Mathematical characterisation of post-trade netting exercises

We now introduce the mathematical characterisation of post-trade netting exercises, which embed the main features of PTRR services. In particular, we will see that both portfolio rebalancing and portfolio compression can be interpreted as a post-trade netting exercise.

Definition 2.1 (Post-trade netting exercise)**.** *Let* $L, L^{\mathcal{P}} \in [0,\infty)^{N\times N}$ *and* $\mathcal{P} \subseteq \mathcal{N}$. *We refer to* $(L, \mathcal{P}, L^{\mathcal{P}})$ *as a* post-trade netting exercise (PTN-exercise) *and to the elements of* $\mathcal{P}$ *as the* participants *if they satisfy* $L^{\mathcal{P}}_{ij} = L_{ij}\ \forall\,(i,j) \notin \mathcal{P} \times \mathcal{P}$ *and*

$$\sum_{j\in\mathcal{P}} (L^{\mathcal{P}}_{ji} - L^{\mathcal{P}}_{ij}) = \sum_{j\in\mathcal{P}} (L_{ji} - L_{ij}) \quad \forall\, i \in \mathcal{P}. \tag{1}$$

We refer to (1) *as the* PTN-contraint*.*

Derivatives positions can only change between pairs where both parties are participants; the PTN-constraint ensures that the net positions remain the same after the PTN-exercise.

Remark 2.2. $(L, \mathcal{P}, L^{\mathcal{P}})$ is a PTN-exercise if and only if $(L^{\mathcal{P}}, \mathcal{P}, L)$ is a PTN-exercise.

While post-trade netting takes place on notional positions, it is mathematically convenient to define the PTN-exercise for matrices that are not necessarily notional matrices but represent the associated liabilities that arise from these notional positions, e.g., variation margin payment obligations. We will discuss this mapping from notional positions to liabilities in Section 3.

Often additional constraints are imposed on PTN-exercises. The following constraints will later be important for our theoretical results.

Definition 2.3. *Let* $(L, \mathcal{P}, L^{\mathcal{P}})$ *be a post-trade netting exercise. We refer to the PTN-matrix* $L^{\mathcal{P}}$ *as* super-conservative*, if*

$$(L^{\mathcal{P}})_{ij} \le L^{bi}_{ij} \quad \forall i, j \in \mathcal{N}, \tag{2}$$

conservative*, if*

$$(L^{\mathcal{P}})_{ij} \le L_{ij} \quad \forall i, j \in \mathcal{N}, \tag{3}$$

aggregate-conservative*, if*

$$\sum_{j\in\mathcal{N}} L^{\mathcal{P}}_{ij} \le \sum_{j\in\mathcal{N}} L_{ij} \quad \forall i \in \mathcal{N}, \tag{4}$$

[4] The derivative contracts are comparable in the sense that they have the same fundamental characteristics such as maturity and underlying. CDS contracts have been standardised in terms of coupons and maturity dates; trade positions on these contracts can be bucketed by the reference entity (single name or index) and maturity.

system-conservative*, if*

$$\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} L^{\mathcal{P}}_{ij} \le \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} L_{ij}, \tag{5}$$

net-conservative*, if*

$$(L^{\mathcal{P}})^{bi}_{ij} \le L^{bi}_{ij} \quad \forall i,j\in\mathcal{N}. \tag{6}$$

Here, the super-conservative constraint (2) implies the conservative constraint (3) which implies the aggregate-conservative constraint (4) which implies the system-conservative constraint (5). The net-conservative constraint (6), however, does not imply any of the constraints (3), (4), (5).

A conservative PTN-matrix is in line with the constraint used in the notion of *conservative compression* introduced by D'Errico & Roukny (2021). The idea is that individual position sizes can only be decreased by a PTN-exercise. An aggregate-conservative PTN-exercise ensures that total payment obligations of each institution can only decrease by the PTN-exercise. We will look into this in more detail in Section 4.2.

A system-conservative PTN-exercise can only decrease the total payment obligations in the system. The system-conservative approach can also be considered as part of an optimisation problem, that uses the total payment obligations as the objective function.

Definition 2.4 (Optimal post-trade netting exercise)**.** *Let* $L\in[0,\infty)^{N\times N}$ *and* $\mathcal{P}\subseteq\mathcal{N}$*. The* PTN-optimisation problem *is given by*

$$\begin{aligned}
&\min_{L^{\mathcal{P}}} \sum_{i,j\in\mathcal{P}} L^{\mathcal{P}}_{ij} \\
&\quad\textit{subject to} \\
&\sum_{j\in\mathcal{P}}(L^{\mathcal{P}}_{ji}-L^{\mathcal{P}}_{ij}) = \sum_{j\in\mathcal{P}}(L_{ji}-L_{ij}) && \forall\, i\in\mathcal{P}, \\
&\qquad L^{\mathcal{P}}_{ij}\ge 0 && \forall\,(i,j)\in\mathcal{P}\times\mathcal{P}, \\
&\qquad L^{\mathcal{P}}_{ij} = L_{ij} && \forall\,(i,j)\notin\mathcal{P}\times\mathcal{P}.
\end{aligned}$$

Let $(L^{\mathcal{P}})^*$ *be a solution to the PTN-optimisation problem. We refer to the triple* $(L,\mathcal{P},(L^{\mathcal{P}})^*)$ *as an* optimal *PTN-exercise.*

Note that the PTN-optimisation problem is a linear programming problem, and it admits a solution since the feasible region is non-empty (since the matrix L satisfies all constraints and the objective function is bounded from below by 0). Also, note that the PTN-optimisation problem is identical to the non-conservative compression problem in D'Errico & Roukny (2021) if $L=C$ and $\mathcal{P}=\mathcal{N}$.

Post-trade netting services can include other constraints as well. For example, constraints could be linked to information from credit ratings or other external sources. And it is likely that in practice, a bank imposes a stricter requirement on individual positions for some counterparties but not for all of them. This is beyond the scope of our paper, see ESMA (2020a) for further discussion.

2.3 Portfolio rebalancing and portfolio compression

We now move from the general mathematical characterisation to the concrete examples in the real world, namely, portfolio rebalancing and portfolio compression.

2.3.1 Portfolio rebalancing

We first provide what we believe is the first mathematical characterisation of portfolio rebalancing. Portfolio rebalancing, also known as counterparty risk rebalancing and counterparty risk optimisation, consists of injecting market risk-neutral transactions such that for each participant, the net exposures remain unchanged. Usually, the objective is to reduce the gross exposures of the bilaterally netted positions at the same time.

Definition 2.5 (Portfolio rebalancing)**.** *1. A* rebalancing exercise $(C, \mathcal{P}, C+R)$ *consists of a notional matrix* C*, a non-empty set of participants* $\mathcal{P}$*, and a* rebalancing notional matrix $R \in [0,\infty)^{N\times N}$ *such that*

$$\begin{aligned} \sum_{j\in\mathcal{P}} R_{ji} &= \sum_{j\in\mathcal{P}} R_{ij} \quad \forall\, i \in \mathcal{P}, \\ R_{ij} &= 0 \quad \forall\,(i,j) \notin \mathcal{P}\times\mathcal{P}. \end{aligned} \tag{7}$$

We refer to $C^{\mathcal{P}} = C + R$ *as the* rebalanced notional matrix*.*

2. *If the rebalancing matrix* R *satisfies*

$$(C+R)^{bi}_{ij} \le C^{bi}_{ij} \quad \forall i, j \in \mathcal{N}, \tag{8}$$

then we refer to $(C, \mathcal{P}, C+R)$ *as a* net-conservative rebalancing exercise*.*

3. *We refer to the optimisation problem that minimises* $\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}(C+R)^{bi}_{ij}$ *over all* $R \in [0,\infty)^{N\times N}$ *subject to* (7) *as the* rebalancing optimisation problem *and if the additional constraint* (8) *is included, we refer to it as the* net-conservative rebalancing optimisation problem*.*

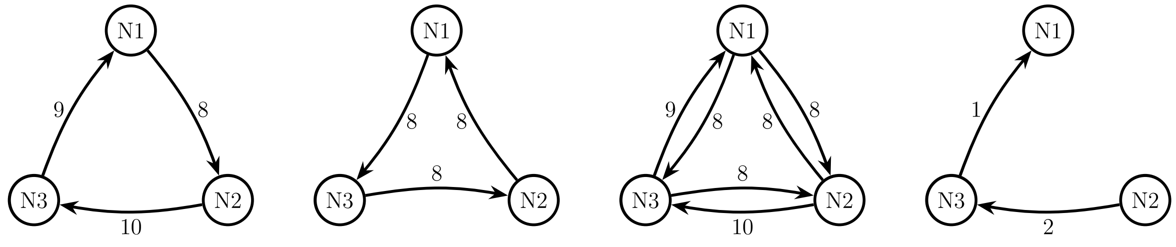
It follows directly from this definition, that a rebalancing exercise is a PTN-exercise. Since rebalancing can only add new trades, $R \ge 0$ and in particular $C^{\mathcal{P}} = C + R \ge C$.

The rebalancing constraint captures the fact that "new transactions are entered into to reduce counterparty risk by reducing the exposure between two counterparties", (ESMA, 2020a, p. 7). To see how the net exposures are maintained, note that condition (7) requires that the net exposures of injected transactions are zero, i.e., $R^\top \mathbf{1} - R\mathbf{1} = \mathbf{0}$, where $\mathbf{0}$ is the column vector containing only 0s. The net-conservative constraint (8) requires that bilaterally netted positions do not increase by the exercise. It reflects the notion that counterparty relationships should be preserved in the manner that if a bank i is a net-seller to (or a net-buyer from) bank j then, this remains after the exercise. The following example gives an illustration of portfolio rebalancing. It is similar to an example provided in ESMA (2020a, Section 3.2).

Example 2.6 (Portfolio rebalancing)**.** Figure 1 illustrates an example of portfolio rebalancing in a network of three banks. The notional matrix C, the rebalancing notional matrix R, the rebalanced notional matrix $C + R$, and the bilaterally netted notional matrix $(C + R)^{bi}$ are given, respectively, by

$$C = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 10 \\ 9 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 8 \\ 8 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix},$$

$$C + R = \begin{pmatrix} 0 & 8 & 8 \\ 8 & 0 & 10 \\ 9 & 8 & 0 \end{pmatrix}, \qquad (C+R)^{bi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$



(a) C (b) R (c) $C + R$ (d) $(C + R)^{bi}$

Figure 1: Example of portfolio rebalancing.

Note. Starting from the notional matrix C (Figure 1(a)), injecting the rebalancing notional matrix R (Figure 1(b)) results in the rebalanced notional matrix $C + R$ (Figure 1(c)). The bilaterally netted positions after rebalancing are $(C + R)^{bi}$ (Figure 1(d)). Here, the bilaterally netted positions prior to rebalancing are $C^{bi} = C$.

Two observations are in order. First, the net exposures of the three banks in the original network are given by $C^\top \mathbf{1} - C\mathbf{1} = (1, -2, 1)^\top$, and they coincide with the net exposures $(C + R)^\top \mathbf{1} - (C + R)\mathbf{1}$ after portfolio rebalancing.

Second, portfolio rebalancing increases the gross exposures of the notional positions from $\mathbf{1}^\top(C^\top \mathbf{1} + C\mathbf{1}) = 54$ to $\mathbf{1}^\top((C + R)^\top \mathbf{1} + (C + R)\mathbf{1}) = 102$, but it decreases the gross exposures of the bilaterally netted positions from $\mathbf{1}^\top((C^{bi})^\top \mathbf{1} + C^{bi}\mathbf{1}) = 54$ to $\mathbf{1}^\top((C + R)^{bi})^\top \mathbf{1} + ((C + R)^{bi})\mathbf{1}) = 6$. In this sense, portfolio rebalancing can decrease aggregate variation margin requirements, since these are bilaterally netted.

We provide an additional example comparing optimal rebalancing and net-conservative optimal rebalancing in the Appendix (Example C.1).

2.3.2 Portfolio compression

Next, we define formally portfolio compression. It consists of one or more of the following: injecting, removing, and replacing positions such that for each participant, the net exposures remain unchanged. Usually, the objective is to reduce the gross exposures of the positions at the same time.

Definition 2.7 (Portfolio compression). 1. *A* compression exercise $(C, \mathcal{P}, K)$ *consists of a notional matrix* C*, a non-empty set of participants* $\mathcal{P}$*, and a* compressed notional matrix $K \in [0, \infty)^{N \times N}$ *such that*

$$\sum_{j \in \mathcal{P}} (K_{ji} - K_{ij}) = \sum_{j \in \mathcal{P}} (C_{ji} - C_{ij}) \quad \forall\, i \in \mathcal{P}, \qquad K_{ij} = C_{ij}\, \forall\, (i, j) \notin \mathcal{P} \times \mathcal{P}. \tag{9}$$

2. *If the compression matrix* K *satisfies*

$$0 \leq K_{ij} \leq C_{ij}^{bi} \quad \forall i, j \in \mathcal{N}, \tag{10}$$

then we refer to K *as* super-conservative compression matrix *and to* $(C, \mathcal{P}, K)$ *as a* super-conservative compression exercise*.*

3. *We refer to the optimisation problem that minimises* $\sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K_{ij}$ *over all* $K \in [0, \infty)^{N \times N}$ *subject to* (9) *as the* compression optimisation problem *and if the additional constraint*

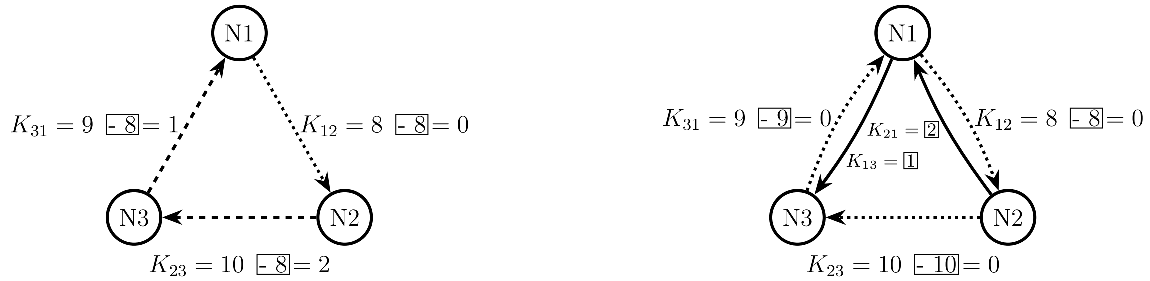
(10) *is included, we refer to it as the* super-conservative compression optimisation problem.

It is clear from the definition that a compression exercise is a PTN-exercise. Since, compression can add, remove and replace trades, K_{ij} can be greater, equal, or less than C_{ij} for $i, j \in \mathcal{P}$.

The compression optimisation problem captures the fact that portfolio compression "aims to reduce the number of contracts and/or the notional amounts of derivatives contracts in a particular asset class/product without changing the market risk of the portfolio", (ESMA, 2020a, p. 7).

Our notion of a compression optimisation problem is the same as the notion of non-conservative compression by D'Errico & Roukny (2021) if $\mathcal{P} = \mathcal{N}$. If the additional condition $0 \leq K_{ij} \leq C_{ij}$ $\forall i, j \in \mathcal{N}$ is included in the optimisation problem (which is weaker than (10)), this is conservative compression (again for $\mathcal{P} = \mathcal{N}$) considered by D'Errico & Roukny (2021). We will refer to any compression matrix satisfying $0 \leq K_{ij} \leq C_{ij}$ $\forall i, j \in \mathcal{N}$ as a conservative compression matrix (even if it is not included in an optimisation problem). D'Errico & Roukny (2021) also suggest that one could consider an additional lower bound on compression by requiring that $K_{ij} \geq \Gamma_{ij}$, for some $\Gamma_{ij} \geq 0$, meaning that the derivatives positions between banks i and j are not compressed too much. Condition (10) will be useful to establish a relationship to a bilaterally netted rebalancing matrix later (Theorem 2.10).

Example 2.8 (Portfolio compression)**.** To provide intuition about the idea of compression, we present two examples in Figure 2. The example in Figure 2(a) follows the idea of compression that eliminates cycles considered in D'Errico & Roukny (2021) and Veraart (2022). In contrast to rebalancing, which can only add positions, compression can also remove and replace positions. Here, a cycle with an edge weight of 8 is removed. Therefore, the edge from $N1$ to $N2$ is eliminated by compression, and the weights along the edges from $N2$ to $N3$ and from $N3$ to $N1$ are reduced. The example in Figure 2(b) shows that compression can also add new positions in addition to deleting or replacing positions.



(a) Compression satisfying $0 \leq K \leq C$.

(b) Compression satisfying $0 \leq K \nleq C$.

Figure 2: Two examples of portfolio compression for the same notional matrix C given by $C_{12} = 8$, $C_{23} = 10$, $C_{31} = 9$ and $C_{ij} = 0$ otherwise.

Note. Dashed edges indicate that contracts are replaced, dotted edges indicate that contracts are deleted, solid edges indicate that new contracts are injected. The numbers in rectangles indicate the changes in the notional positions. In the example in Figure 2(a), all changes are in $(-\infty, 0]$, so notional positions are only reduced. In the example in Figure 2(b), the changes are in $\mathbb{R}$, so some notional positions are reduced and some are increased.

We provide an additional example of the compression optimisation problem and compression that follows the idea of O'Kane (2017), who proposes a loop compression algorithm by finding and eliminating all closed loops on the bilaterally netted notional matrix, in the Appendix (Example C.2).

2.4 Relationship between portfolio rebalancing and portfolio compression

Having introduced the mathematical characterisation of post-trade netting and its real-world counterparts, we provide more details about the relationship between portfolio rebalancing and portfolio compression.

Definition 2.9 (Rebalancing-compression-parity)**.** *Let C be a notional matrix, $\mathcal{P} \subseteq \mathcal{N}$, and let K be a compression matrix. The K-*compression-rebalancing-parity-matrix *is the matrix $R \in [0,\infty)^{N\times N}$ given by*

$$R_{ij} = \max\{0, (K_{ij} - K_{ji}) - (C_{ij} - C_{ji})\} \; \forall i, j \in \mathcal{N}. \tag{11}$$

The next theorem establishes that the K-compression-rebalancing-parity-matrix is indeed a rebalancing matrix. It also formalises the relationship between (net-conservative) rebalancing and (super-conservative) compression.

Theorem 2.10. *Let $C \in [0,\infty)^{N\times N}$ be a notional matrix and let $\mathcal{P} \subseteq \mathcal{N}$.*

1. *If $K \in [0,\infty)^{N\times N}$ is a compression matrix, then the K-compression-rebalancing-parity-matrix R is a rebalancing matrix and satisfies*

$$(C+R)^{bi}_{ij} = K^{bi}_{ij} \quad \forall i, j \in \mathcal{N}. \tag{12}$$

   Furthermore, if K is a super-conservative compression matrix, then the corresponding K-compression-rebalancing-parity-matrix is a net-conservative rebalancing matrix.

2. *If $R \in [0,\infty)^{N\times N}$ is a rebalancing matrix, then $K = (C+R)^{bi} \in [0,\infty)^{N\times N}$ is a compression matrix. Furthermore, if R is a net-conservative rebalancing matrix, then $K = (C+R)^{bi}$ is a super-conservative compression matrix.*

3. *If $K^* \in [0,\infty)^{N\times N}$ is a solution to the (super-conservative) compression optimisation problem, then the K^*-compression-rebalancing-parity-matrix denoted by R^* is a solution to the (net-conservative) rebalancing optimisation problem and*

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}(C+R^*)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij}.$$

4. *If $R^* \in [0,\infty)^{N\times N}$ is a solution to the (net-conservative) rebalancing optimisation problem, then $K^* = (C+R^*)^{bi}$ is a solution to the (super-conservative) compression optimisation problem and*

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}(C+R^*)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij}.$$

Hence, we see that there is a strong mathematical connection between portfolio rebalancing and portfolio compression. Theorem 2.10 shows that there are situations in which portfolio rebalancing combined with bilateral netting achieves the same outcome as portfolio compression, i.e., both services are then mathematically equivalent. This, however, does not mean that they are used interchangeably in practice.

Portfolio compression and portfolio rebalancing are applied to different types of underlying portfolios (ESMA, 2020b, p. 14). In particular, portfolio compression is typically used for homogeneous portfolios whereas portfolio rebalancing is typically used for heterogeneous portfolios. In particular, portfolio compression is available for portfolios that are either fully non-centrally cleared or fully centrally cleared (IOSCO, 2024, p. 14). Portfolio rebalancing is typically used in situations where the underlying portfolio is of a mixed type[5], i.e., only partially centrally cleared or centrally cleared at more than one CCP, see ISDA et al. (2018, p. 11–12). In such a situation, portfolio rebalancing allows one to adjust positions across different parts of the portfolio.[6] For portfolio rebalancing to work, however, it is essential that the trades that are added as part of the rebalancing exercise remain in the correct part of the portfolio, i.e., "the rebalancing transaction needs to remain in the portfolio it manages the risk, hence to manage risk in the uncleared portfolio the rebalancing trade would need to remain in the uncleared portfolio, i.e. it cannot be cleared and novated to the CCP", (ESMA, 2020b, p. 14). These considerations recently lead to the recommendation that trades resulting from post-trade risk reduction services should not be subject to the central clearing obligation (ESMA, 2020b).

An example for portfolio rebalancing for a portfolio that is partially centrally cleared and partially non-centrally cleared is provided in ESMA (2020b, Appendix 1) and we report it in the Appendix B. We also provide an additional example in the Appendix (Example C.3).

The key idea is that the rebalancing exercise adds trades both to the non-centrally cleared and to the centrally cleared part of the portfolio and then after bilateral netting achieves a reduction in counterparty exposures. Portfolio compression would typically only be applied to a subset of the network, i.e., the centrally cleared or the non-centrally cleared part or two separate exercises would be applied to the two different parts of the portfolio which imposes additional constraints on the exercise compared to applying portfolio rebalancing to the whole portfolio. Hence, in such a situation portfolio rebalancing could achieve a larger reduction in bilaterally netted positions compared to running two separate compression exercises.

3 Assessing systemic risk for PTN-exercises

In this section, we explain how we will apply the contagion model of Veraart (2020) to the PTN-exercise characterised in Section 2 building on Veraart (2022).

To analyse systemic risk implications from a network perspective, we consider payment obligations in the form of variation margins, as in Paddrik et al. (2020).[7] We define the *liabilities matrix* L associated with the notional matrix C by $L = f(C)$, where $f : [0,\infty)^{N\times N} \to [0,\infty)^{N\times N}$ and either $f(C) = \psi \cdot C^{bi}$ or $f(C) = \psi \cdot C$, i.e., the liabilities matrix is either proportional to the bilaterally netted notional matrix or to the notional matrix, where $\psi > 0$.[8] Typically, variation margin payment obligations are bilaterally netted, but since our results also hold for the non-bilaterally netted case, we consider both cases.[9] The proportionality parameter ψ allows for simple modelling of different magnitudes of variation margin payment obligations that are

[5]Portfolio rebalancing can also be used for uncleared portfolios, see IOSCO (2024, p. 14).

[6]As outlined in ESMA (2020b, p. 14), "ESMA understands that rebalancing has been developed to manage risks across cleared and non-cleared portfolios as today (after the clearing obligation was introduced) parties may no longer offset their risks in the non-cleared part of the market and the cleared part of the market i.e. the credit exposure of cleared trades can no longer be netted against bilateral trades across different asset classes that are not eligible for clearing.

[7]Variation margins arise due to mark-to-market valuations of contracts and are exchanged on short notice to protect counterparties from the current exposures (BCBS & IOSCO, 2020). We have in mind, for example, the case of American International Group, Inc. (AIG) in the CDS market during the Global Financial Crisis. The protection seller faced tremendous pressure on the margin calls from the protection buyers after a sudden shock to the credit markets.

[8]Our analysis could be extended to situations with $k > 1$ assets, for example, CDSs on the same reference entity but with different maturity dates, and corresponding notional matrices $C_1, \dots, C_k$. Then, a liabilities matrix could be constructed by $L = f(C_1, \dots, C_k)$, where $f : [0,\infty)^{N\times N} \times \dots \times [0,\infty)^{N\times N} \to [0,\infty)^{N\times N}$ is a suitable function that maps the notional positions to variation margin payment obligations. One simple example would be that f is again a linear function in all k arguments.

[9]Our results are formulated for a PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$. For the special case of portfolio rebalancing or compression, note that if $(C, \mathcal{P}, C+R)$ is a rebalancing exercise then both $(\psi C^{bi}, \mathcal{P}, \psi(C+R)^{bi})$ and $(\psi C, \mathcal{P}, \psi(C+R))$ are PTN-exercises for all $\psi \geq 0$; similarly, if $(C, \mathcal{P}, K)$ is a compression exercise, then both $(\psi C^{bi}, \mathcal{P}, \psi K^{bi})$ and $(\psi C, \mathcal{P}, \psi K)$ are PTN-exercises for all $\psi \geq 0$, see the Appendix (Corollary A.1).

caused by changes in the underlying market conditions. So, larger changes in underlying market conditions will lead to larger variation margin payment obligations which can be captured by larger values of ψ.

In the following, we consider a PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$ and compare systemic risk associated with the liabilities matrix L to systemic risk associated with the other liabilities matrix $L^{\mathcal{P}}$. Examples for $L^{\mathcal{P}}$ include $L^{\mathcal{P}} = f(C^{\mathcal{P}})$, where $C^{\mathcal{P}} = K$ for a compression exercise $(C, \mathcal{P}, K)$ or $C^{\mathcal{P}} = C + R$ for a rebalancing exercise $(C, \mathcal{P}, C + R)$.

It will sometimes be of interest, to consider not just liabilities between banks, but also liabilities from banks to an external node, which in our context could be interpreted as variation margin payments to banks' customers. Specifically, this can be captured by our setting by assuming that the notional matrix C satisfies $C_{Nj} = 0$ for all $j \in \mathcal{N}$, i.e., we use the index N to represent the external node. In other words, there are no obligations from N to any bank in the network, but banks in $\mathcal{N} \setminus \{N\}$ can have obligations to N; hence, N can be interpreted as an external node. When adopting this interpretation, we assume that the external node cannot participate in a PTN-exercise, i.e., $N \in \mathcal{N} \setminus \mathcal{P}$.

Besides the derivatives contracts, each bank $i \in \mathcal{N}$ holds a *liquidity buffer* $A_i^b \geq 0$, which may represent cash or high-quality liquid assets to the extent, they can be readily exchanged as variation margins. We summarise banks' liquidity buffers in the N-dimensional vector $A^b = (A_1^b, \ldots, A_N^b)^\top$. We shall typically refer to the pair (L, A^b) as the *original network* and to the pair $(L^{\mathcal{P}}, A^b)$ as the *PTN-network*.

3.1 Clearing equilibrium

To examine the contagion effects of PTN-exercises, we need to characterise the network equilibrium formally. We use a quantity referred to as *re-evaluated equity* introduced in Veraart (2020) and used in the context of portfolio compression in Veraart (2022), which intuitively represents the difference between actual assets and payment obligations in equilibrium. Below, we describe the model of Veraart (2020), which generalises the clearing mechanisms introduced in Eisenberg & Noe (2001) and Rogers & Veraart (2013).

Definition 3.1. *A* valuation function $\mathbb{V} : \mathbb{R} \to [0, 1]$ *is defined by* $\mathbb{V}(y) = \mathbb{I}_{\{y \geq 1+k\}} + \mathbb{I}_{\{y<1+k\}} r(y)$, *where* $k \geq 0$, $\mathbb{I}_{\{.\}}$ *denotes the indicator function, and* $r : (-\infty, 1+k) \to [0, 1]$ *is a non-decreasing and right-continuous function.*

A valuation function summarises various types of valuation rules for assets. The zero recovery rate valuation function is defined in Veraart (2022) as $\mathbb{V}^{\text{zero}}(y) = \mathbb{I}_{\{y \geq 1+k\}}$. As discussed in Veraart (2020, 2022), the model specification conveniently nests several network clearing models as special cases via taking particular functional forms of $\mathbb{V}$.[10] For example, the clearing vector in the Eisenberg & Noe (2001) model can be recovered from the re-evaluated equity if $\mathbb{V} = \mathbb{V}^{\text{EN}}$, and vice versa, where $\mathbb{V}^{\text{EN}}(y) = 1 \wedge y^+$. Since we often use a valuation function $\mathbb{V}$ to assess systemic risk associated with $(L, \mathcal{P}, L^{\mathcal{P}})$, we also call $(L, A^b; \mathbb{V})$ the original network and $(L^{\mathcal{P}}, A^b; \mathbb{V})$ the PTN-network.

Definition 3.2 (Re-evaluated equity in the original and in the PTN-network)**.** *Let* $(L, A^b; \mathbb{V})$ *and* $(L^{\mathcal{P}}, A^b; \mathbb{V})$ *be the original and the PTN-network, respectively.*

[10] Veraart (2022) shows how a suitable choice of $\mathbb{V}$ can model collateralised payment obligations that are (partially) protected by initial margins. Hence, by considering a general valuation function in our analysis, we can capture these characteristics of derivatives markets as well.

1. *Define the function* $\Phi = \Phi(\cdot;\mathbb{V}) : \mathcal{E} \to \mathcal{E}$ *as*

$$\Phi_i(E) = \Phi_i(E;\mathbb{V}) = A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_j>0} L_{ji}\mathbb{V}\left(\frac{E_j+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \;\forall\, i \in \mathcal{N}, \tag{13}$$

where $\mathcal{E} = [-\bar{L}, A^b + \bar{A} - \bar{L}]$, $\bar{L}_i = \sum_{j=1}^{N} L_{ij}$, *and* $\bar{A}_i = \sum_{j=1}^{N} L_{ji}$. *We refer to a vector* $E \in \mathcal{E}$ *satisfying* $E = \Phi(E)$ *as a* re-evaluated equity *in the original network.*

2. *Define the function* $\Phi^{\mathcal{P}} = \Phi^{\mathcal{P}}(\cdot;\mathbb{V}) : \mathcal{E}^{\mathcal{P}} \to \mathcal{E}^{\mathcal{P}}$ *as*

$$\Phi_i^{\mathcal{P}}(E) = \Phi_i^{\mathcal{P}}(E;\mathbb{V}) = A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_j^{\mathcal{P}}>0} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{E_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \;\forall\, i \in \mathcal{N}, \tag{14}$$

where $\mathcal{E}^{\mathcal{P}} = [-\bar{L}^{\mathcal{P}}, A^b + \bar{A}^{\mathcal{P}} - \bar{L}^{\mathcal{P}}]$, $\bar{L}_i^{\mathcal{P}} = \sum_{j=1}^{N} L_{ij}^{\mathcal{P}}$, *and* $\bar{A}_i^{\mathcal{P}} = \sum_{j=1}^{N} L_{ji}^{\mathcal{P}}$. *We refer to a vector* $E \in \mathcal{E}^{\mathcal{P}}$ *satisfying* $E = \Phi^{\mathcal{P}}(E)$ *as a* re-evaluated equity *in the PTN-network.*

The idea behind the equilibrium is that the actual assets of each bank—the liquidity buffer plus the incoming payments from its counterparties—depend on the payments that other banks in the network can make. Because of this, the re-evaluated equity is characterised as a fixed point.

As discussed in Veraart (2020), $(\mathcal{E}, \leq)$ and $(\mathcal{E}^{\mathcal{P}}, \leq)$ are complete lattices and the functions Φ and $\Phi^{\mathcal{P}}$ are non-decreasing. Hence, the existence of a re-evaluated equity is guaranteed by Tarksi's fixed-point theorem (Tarski, 1955, Theorem 1). Moreover, there exist a greatest and a least re-evaluated equity. In Section 4, we will discuss in detail the implications of post-trade netting for these fixed points.

The equilibria derived above provide ways of assessing systemic risk. We first define the fundamental default set, which characterises the set of all banks that default even if all banks in the system make their payments in full. For this, we need a notion of initial equity.

Definition 3.3 (Fundamental default)**.** *Consider a PTN-exercise* $(L, \mathcal{P}, L^{\mathcal{P}})$. *We define the* initial equity *in the original network* $(L, A^b;\mathbb{V})$ *and the PTN-network* $(L^{\mathcal{P}}, A^b;\mathbb{V})$*, respectively, by*

$$\begin{aligned} E_i^{(0)} &= A_i^b + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i, \\ E_i^{\mathcal{P}(0)} &= A_i^b + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}, \quad \forall\, i \in \mathcal{N}. \end{aligned} \tag{15}$$

The fundamental default set *for this PTN-exercise is* $\mathcal{F}(L, A^b;\mathbb{V}) = \{i \in \mathcal{N} \mid E_i^{(0)} < 0\} = \{i \in \mathcal{N} \mid E_i^{\mathcal{P}(0)} < 0\}$.

We show in the Appendix (Lemma A.4) that $E^{(0)} = E^{\mathcal{P}(0)}$, so the fundamental default sets are the same in both networks. We next define defaults more generally, in particular contagious defaults which depend on the fixed points.

Definition 3.4 (Default and contagious default)**.** *Let* $\mathcal{F}(L, A^b;\mathbb{V})$ *be the fundamental default set. Let* $\tilde{E}$ *be a fixed point of* Φ *and* $\tilde{E}^{\mathcal{P}}$ *be a fixed point of* $\Phi^{\mathcal{P}}$. *We define (i)* $\mathcal{D}(\tilde{E}, L, A^b;\mathbb{V}) = \{i \in \mathcal{N} \mid \tilde{E}_i < 0\}$ *and* $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b;\mathbb{V}) = \{i \in \mathcal{N} \mid \tilde{E}_i^{\mathcal{P}} < 0\}$ *as the* default set *in the original and in the PTN-network, respectively; and (ii)* $\mathcal{D}(\tilde{E}, L, A^b;\mathbb{V}) \setminus \mathcal{F}(L, A^b;\mathbb{V})$ *and* $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b;\mathbb{V}) \setminus$

$\mathcal{F}(L, A^b; \mathbb{V})$ *as the* contagious default set *in the original network and in the PTN-network, respectively.*

We show in the Appendix (Lemma A.5) that the fundamental default set is a subset of the default set.

3.2 Measuring contagion

We now describe how we assess the consequences of a PTN-exercise on systemic risk. While many different measures for systemic risk have been proposed in the literature (see, e.g., Bisias et al. (2012) for a survey), there is no commonly accepted choice. A possible criterion to distinguish between different risk measures is to consider the time dimension; for example, Bisias et al. (2012, Section 2.5) consider three temporal categories: "pre-event", "contemporaneous", and "post-event." In the following, we will consider two of these categories, the pre-event and post-event categories that we refer to as ex ante and ex post, respectively.

In both cases, we assume that the PTN-exercise takes place before a shock and we consider what happens both to the original network and to the PTN-network. In the ex post analysis, we study the outcome of the two networks after the shock. The underlying modelling framework is deterministic and follows Veraart (2022). In the ex ante analysis, we compare the vulnerabilities of the two networks prior to the shock. In particular, we include a random shock vector in our framework. This analysis is based on Glasserman & Young (2015).

We emphasise that our analysis in this paper is based on the re-evaluated equities, from which we derive risk measures that receive attention in the literature. Since PTN-exercises can increase or decrease total payment obligations in a network, we need to ensure that our risk measures are suitably normalised.

First, we derive the default sets in the original and the PTN-network as in Veraart (2022).[11] For any re-evaluated equities $\tilde{E}$ and $\tilde{E}^{\mathcal{P}}$ in the original and the PTN-network, we say that the PTN-exercise (i) *reduces systemic risk* (with respect to these re-evaluated equities) if $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$; (ii) *strongly reduces systemic risk* (with respect to these re-evaluated equities) if $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subsetneq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$; and (iii) is *harmful* (with respect to these re-evaluated equities) if $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \setminus \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V}) \neq \emptyset$. Since there can be more than one fixed point of Φ and $\Phi^{\mathcal{P}}$, a cautionary note is that these notions make sense only if the re-evaluated equities in the original and the PTN-network are comparable; this is for example the case if both re-evaluated equities correspond to the greatest fixed points or both correspond to the least fixed points.

The advantage of considering default sets is that they are indeed normalised and allow for a meaningful comparison between two networks associated with potentially different gross positions. They also capture who defaults and not just how many banks default; this is in line with taking a Pareto view on systemic risk measurement. As already discussed in Veraart (2022), a strong reduction in systemic risk can be interpreted as a Pareto improvement because it reflects the fact that a PTN-exercise does not cause any new defaults and at least one bank is no longer in default in the PTN-network compared to the original network. Similarly, we say that a PTN-exercise is harmful if at least one non-defaulting bank in the original network defaults in the PTN-network. In particular, we do not allow for a trade-off between banks in the sense that we would accept a new default with a small loss so that the aggregate loss in the system is smaller. Since, market participants choose which post-trade netting services they use and which new proposals they accept, we believe that a risk measure that is in the spirit of a Pareto improvement is suitable for this purpose.

[11] We note that Veraart (2022) defines the default set in $(L, A^b; \mathbb{V})$ as $\{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\}$, where $k \geq 0$ coincides with that used in the valuation function. In what follows, we set $k = 0$. We interpret this choice as primarily modelling illiquidity in the case of variation margin payments instead of solvency contagion; the total assets of each bank consist of its liquidity buffer and received margin payments, which are highly liquid assets. We refer Veraart (2020, Remark A.1) for further discussions about the role of k.

Second, as an alternative risk measure, we will also sometime consider the actual payments made to the external node (see, e.g., Amini & Feinstein (2023)). This measure does not just consider the number of defaults but also accounts indirectly for the magnitude of losses in the system. Since the external node does not participate in a PTN-exercise, the total liabilities to the external node cannot change, therefore the payments it receives can also be used as a normalised measure. Mathematically, this boils down to comparing $\mathbb{V}\left(\frac{\tilde{E}_i+\bar{L}_i}{\bar{L}_i}\right)$, which models the proportion of its debt that bank i repays in the original network under re-evaluated equity $\tilde{E}$, to $\mathbb{V}\left(\frac{\tilde{E}_i^{\mathcal{P}}+\bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right)$ which models the proportion of its debt that bank i repays in the PTN-network under re-evaluated equity $\tilde{E}^{\mathcal{P}}$; see Veraart (2020, 2022). Correspondingly, the total payments made to the external node are $\sum_{i\in\mathcal{N}:\bar{L}_i>0}\mathbb{V}\left(\frac{\tilde{E}_i+\bar{L}_i}{\bar{L}_i}\right)L_{iN}$ in the original network and $\sum_{i\in\mathcal{N}:\bar{L}_i^{\mathcal{P}}>0}\mathbb{V}\left(\frac{\tilde{E}_i^{\mathcal{P}}+\bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right)L_{iN}^{\mathcal{P}}=\sum_{i\in\mathcal{N}:\bar{L}_i^{\mathcal{P}}>0}\mathbb{V}\left(\frac{\tilde{E}_i^{\mathcal{P}}+\bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right)L_{iN}$ in the PTN-network.

One could consider other risk measures as well. For example, building on the concept of monetary risk measures, see Artzner et al. (1999); Föllmer & Weber (2015) and extensions thereof to measures of systemic risk, see e.g., Feinstein et al. (2017); Biagini et al. (2019), one could introduce a stochastic cash flow model associated with a financial system and define a suitable acceptance criterion so that then systemic risk is quantified by "the set of allocations of additional capital that lead to acceptable outcomes" (Feinstein et al., 2017). These extensions, however, are beyond the scope of the current analysis.

4 Consequences of post-trade netting for contagion

We now analyse when post-trade netting reduces systemic risk, first from an ex post perspective and then from an ex ante perspective.

4.1 An ex post analysis of post-trade netting and contagion

We begin with two results that identify sufficient conditions for post-trade netting to reduce systemic risk ex post, consistent with those in Veraart (2022). The first condition assumes that no participant defaults in the original network.

Theorem 4.1 (No participant defaults in the original network)**.** *Consider a PTN-exercise* $(L,\mathcal{P},L^{\mathcal{P}})$. *Let* $\tilde{E}$ *be a fixed point of* Φ *that satisfies*

$$\{i\in\mathcal{P}\mid\tilde{E}_i<0\}=\emptyset,\tag{16}$$

i.e., no participant defaults according to this fixed point in the original network.

1. *Then,* $\tilde{E}$ *is a fixed point of* $\Phi^{\mathcal{P}}$. *In particular, considering* $\tilde{E}$ *in both the original network and the PTN-network leads to the same defaults in the original and in the PTN-network and a reduction of systemic risk, but not a strong reduction of systemic risk.*
2. *If* $\tilde{E}$ *is the greatest fixed point of* Φ, *then* $\tilde{E}$ *is also the greatest fixed point of* $\Phi^{\mathcal{P}}$.
3. *If* $\tilde{E}$ *is the least fixed point of* Φ, *then* $\tilde{E}$ *does not have to be the least fixed point of* $\Phi^{\mathcal{P}}$.
4. *Suppose* $L^{\mathcal{P}}$ *is conservative, i.e., it satisfies (3) and* $\mathbb{V}=\mathbb{V}^{zero}$. *If* $\tilde{E}$ *is the least fixed point of* Φ, *then* $\tilde{E}$ *is the least fixed point of* $\Phi^{\mathcal{P}}$.

Veraart (2022) has shown that condition (16) is a sufficient condition for a reduction of systemic risk for conservative portfolio compression and the greatest fixed points, i.e., statement 2. above for condition (3). Theorem 4.1 establishes that the statement holds for general post-trade netting exercises. It also says that this result does not necessarily carry over to the least fixed point (statement 3.) except under special circumstances (statement 4.).

The fact that no participants' defaults in the original network implies that a PTN-exercise cannot be harmful is a reassuring result. It means that as long as only sound financial institutions engage in a PTN-exercise, such an exercise cannot be harmful.

The next corollary provides intuition about what Theorem 4.1 implies for the payments made in the financial system. Recall from Veraart (2020), that the payments made from bank i to bank j in a system with payment obligations L corresponding to a re-evaluated equity $\tilde{E}$ are

$$p_{ij} = \mathbb{V}\left(\frac{\tilde{E}_i - \bar{L}_i}{\bar{L}_i}\right) L_{ij} \quad \forall i, j \in \mathcal{N}.$$

Corollary 4.2. *Consider a post-trade netting exercise* $(L, \mathcal{P}, L^{\mathcal{P}})$*. Let* $\tilde{E}$ *be a fixed point of* Φ *that satisfies* (16)*. Then, the actual payments made from* i *to* j *corresponding to this fixed point in the original and the PTN-network that are given, respectively, by*

$$p_{ij}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) L_{ij},$$

$$p^{\mathcal{P}}_{ij}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}^{\mathcal{P}}_i}{\bar{L}^{\mathcal{P}}_i}\right) L^{\mathcal{P}}_{ij} \quad \forall i, j \in \mathcal{N},$$

satisfy

$$p^{\mathcal{P}}(\tilde{E}) + (L - L^{\mathcal{P}}) = p(\tilde{E}).$$

The corollary says, that for any fixed point that does not lead to defaults among participants in the original network, the clearing payments in the PTN-network can be obtained by clearing the original network and then subtracting $L - L^{\mathcal{P}}$ from the clearing payments in the original network. This essentially means that under condition (16) clearing the original network first and then adjusting the networks gives the same outcome as adjusting the network first and then clearing it.

Another direct implication of the corollary is that under condition (16) for a situation with an external node, the payments to the external node are the same in the original and the PTN-network. In particular, if N is the external node, then $N \notin \mathcal{P}$ and

$$p^{\mathcal{P}}_{iN}(\tilde{E}) = p_{iN}(\tilde{E}) \quad \forall i \in \mathcal{N}.$$

Theorem 4.1 also conveys a more subtle point. Under the condition of the theorem, the equilibria in both networks are the same, so the systemic risk is not strongly reduced. Theorem 4.1 therefore implies that if a PTN-exercise strongly reduces systemic risk, then at least one participant defaults in the original network.

We now consider a second sufficient condition for systemic risk reduction. It states that no participant defaults in the PTN-network.

Proposition 4.3 (No participant defaults in the PTN-network)**.** *Consider a PTN-exercise* $(L, \mathcal{P}, L^{\mathcal{P}})$*. Let* $\tilde{E}^{\mathcal{P}}$ *be a fixed point of* $\Phi^{\mathcal{P}}$ *that satisfies*

$$\{i \in \mathcal{P} \mid \tilde{E}^{\mathcal{P}}_i < 0\} = \emptyset, \tag{17}$$

i.e., no participant defaults according to this fixed point in the PTN-network.

1. *Then,* $\tilde{E}^{\mathcal{P}}$ *is a fixed point of* Φ*. In particular, considering* $\tilde{E}^{\mathcal{P}}$ *in both the original network and the PTN-network leads to the same defaults in the original and in the PTN-network and a reduction of systemic risk, but not a strong reduction of systemic risk.*

2. *If $\tilde{E}^{\mathcal{P}}$ is the greatest fixed point of $\Phi^{\mathcal{P}}$, then $\tilde{E}^{\mathcal{P}}$ is also the greatest fixed point of Φ.*

3. *If $\tilde{E}^{\mathcal{P}}$ is the least fixed point of $\Phi^{\mathcal{P}}$, then $\tilde{E}^{\mathcal{P}}$ does not have to be the least fixed point of Φ.*

Note that $(L, \mathcal{P}, L^{\mathcal{P}})$ is a PTN-exercise if and only if $(L^{\mathcal{P}}, \mathcal{P}, L)$ is a PTN-exercise (Remark 2.2). Proposition 4.3 therefore follows immediately by applying Theorem 4.1 to $(L^{\mathcal{P}}, \mathcal{P}, L)$, i.e., using $(L^{\mathcal{P}}, A^b; \mathbb{V})$ as the original network and $(L, A^b; \mathbb{V})$ as the PTN-network. We, therefore, omit the proof.[12]

Proposition 4.3 implies that if the post-trade netting exercise is harmful, then there has to be at least one participant that defaults in the post-trade netted network. In particular, no post-trade netting exercise exists that results in no defaults among the participants but at least one default among the non-participants only in the PTN-network.

We next investigate PTN-exercises under zero recovery rates. It has been shown in Veraart (2022, Proposition 4.12), that a PTN-exercise that satisfies (3) always reduces systemic risk under zero recovery rates when considering the greatest re-evaluated equity. We now show that this statement remains true when considering the least re-evaluated equity. Furthermore, we show that there exists a PTN-exercise that does not satisfy (3) and is harmful under both the greatest and the least re-evaluated equity.

Theorem 4.4 (Zero recovery rates)**.** *Let $(L, A^b; \mathbb{V})$ be a financial network with $\mathbb{V} = \mathbb{V}^{zero}$. Then,*

1. *for any conservative PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$ (i.e., it satisfies (3)), it holds that*

$$E_i^* \leq E_i^{\mathcal{P};*} \quad \forall i \in \mathcal{N}, \tag{18}$$

$$(E_*)_i \leq (E_*^{\mathcal{P}})_i \quad \forall i \in \mathcal{N}, \tag{19}$$

where E^*, $E^{\mathcal{P};*}$ are the greatest fixed points and E_*, $E_*^{\mathcal{P}}$ are the least fixed points of Φ and $\Phi^{\mathcal{P}}$, respectively. In particular, the PTN-exercise reduces systemic risk under both the greatest and the least re-evaluated equity.

2. *There exists a PTN-exercise that does not satisfy (3), for which neither (18) nor (19) hold. In particular, there exists a PTN-exercise that does not satisfy (3) that is harmful under both the greatest and the least re-evaluated equity.*

Finally, the next Proposition shows that an optimal PTN-exercise in which all banks in the system participate, i.e., $\mathcal{P} = \mathcal{N}$, eliminates all contagion. In practice, this is unlikely to happen but the result is useful to provide intuition about the underlying mechanisms of post-trade netting.

Proposition 4.5. *Consider an optimal PTN-exercise with $\mathcal{P} = \mathcal{N}$. Let $\tilde{E}$ be a fixed point of the corresponding Φ and let $\tilde{E}^{\mathcal{P}}$ be a fixed point of the corresponding $\Phi^{\mathcal{P}}$. Then, $\mathcal{F} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$.*

Hence, the proposition says that any default in the PTN-network (if it exists) is a fundamental default and hence this optimal PTN-exercise reduces systemic risk for all possible fixed points.

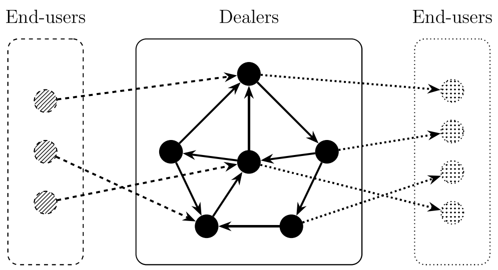
To prove the proposition, we exploit a result of D'Errico & Roukny (2021), who show that the graph corresponding to networks attaining the minimum gross exposures is bipartite (i.e., the situation in which bank i should pay bank j and bank j should pay a different bank k does not exist).

Proposition 4.5 assumes that $\mathcal{P} = \mathcal{N}$, i.e., all banks participate in the PTN-exercise. If we consider the interpretation of bank N as an external node, then this external node cannot

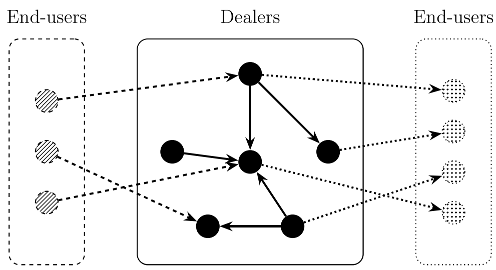
[12] In practice, the situation $L \leq L^{\mathcal{P}}$ does not arise, therefore we do not formulate the result corresponding to statement 4. in Theorem 4.1.

participate in the PTN-exercise and Proposition 4.5 cannot be applied. In particular, after an optimal PTN-exercise with $\mathcal{P} = \mathcal{N} \setminus \{N\}$, banks that are not in fundamental default can default on payments to the external node due to contagion.

Similarly, Figure 3 provides a stylised comparison between an original network and a PTN-network in which only parts of the nodes participate in an optimal PTN-exercise. In particular, we assume that the nodes consist of dealers and end-users and all dealers participate in an optimal PTN-exercise, but the end-users do not participate. The direction of the arrows indicates the net variation margin owed between counterparties. In Figure 3(a), feedback loops in the dealers' section may amplify default cascades, whereas they disappear in Figure 3(b) since this specific PTN-exercise breaks up possible contagion channels among dealers. Still, the contagious default of a dealer remains a possibility, since only the sub-network of dealers is bipartite, but not the whole network. So a dealer can default on payments to an end-user because they were hit by a shock from another end-user, from a dealer, or from both.



(a) Original network.



(b) Optimal PTN-network.

Figure 3: Stylised networks of variation margin flows in the derivatives market.

Note. In this example, only the dealers are participants.

4.1.1 Economic interpretation of the greatest and the least fixed points

We have seen that in situations where the fixed point is not unique, not all results that hold for the greatest fixed point also hold for the least fixed point. It is therefore of interest to know under which circumstances a system might settle either on the greatest or the least fixed point.

Often, the literature focuses on the greatest fixed point, since it reflects the best possible outcome for the system. It can be derived by considering a fixed point iteration starting with the assumption that every bank satisfies its payment obligations in full, and then tracking whether there are any fundamental defaults that then might cause contagious defaults. Hence, the greatest fixed point is the outcome of a spread of insolvency that started from the best possible situation (Rogers & Veraart, 2013). If one starts, however, with the assumption that initially no banks receive any payments from other nodes, then they can only use their liquidity buffers to make payments. If these are enough to satisfy the payment obligations, then solvency starts to spread through the network and potentially some banks can avoid default. Hence, the least fixed point is the outcome of a spread of solvency that started from the worst possible situation, see Rogers & Veraart (2013) for more details.

Another interesting interpretation for the occurrence of least fixed points has been provided by Csóka & Herings (2018). They show that decentralised clearing procedures usually result in the least fixed point. Also, Bardoscia et al. (2019); Paddrik et al. (2020); Bardoscia et al. (2021) discuss how financial institutions may take defensive actions not to fulfill their payment obligations on a timely basis when under stress. This can also be interpreted as a decentralised approach.

In particular, Bardoscia et al. (2019) introduce the Full Payment Algorithm (FPA) to embed the phenomenon of sequential payments in a realistic setting with strategic behaviours. They assume that banks with insufficient liquidity buffers wait for potential payments from their counterparties and only make payments in full once they receive enough liquid assets; this dictates that even if a bank can obtain extra liquidity (such as through repo borrowing) to fulfill its obligations, it would do nothing but wait. As a result, the payments are made in sequence: at each iteration, each bank either pays in full or pays nothing. The algorithm terminates if no more banks can make any payment, and the vector of cumulative payments of each bank is the *output of the FPA*. For completeness, we provide the formal definitions in the Appendix A.2.3.

It is possible to relate the outcome of the FPA to the least clearing vector in the model by Rogers & Veraart (2013) under zero recovery rates:

Theorem 4.6. *For any given financial network, the output of the FPA is the least clearing vector in the Rogers & Veraart (2013) model with default cost parameters equal to zero.*

We can interpret the least fixed point in the Rogers & Veraart (2013) model with zero recovery rates as the worst equilibrium. Theorem 4.6 implies that the worst equilibrium can arise from a strategic response to stress. Moreover, the corresponding worst re-evaluated equity, i.e., the least fixed point of Φ with $\mathbb{V} = \mathbb{V}^{\text{zero}}$, denoted by E^0, is given by $E_i^0 = \Phi_i(E^0; \mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j:\bar{L}_j>0} L_{ji} \frac{L_j^*}{\bar{L}_j} - \bar{L}_i \quad \forall i \in \mathcal{N}$, where L^* is the output of the FPA.[13]

Theorem 4.4 has shown that under zero recovery rates, any conservative PTN-exercise reduces systemic risk under both the greatest and the least fixed point, but this does not need to hold for non-conservative PTN-exercises. Combining this with the statement of Theorem 4.6 then implies, that even in networks where banks react strategically to liquidity stress by delaying their payments as described in Bardoscia et al. (2019), conservative PTN-exercises will reduce systemic risk. For non-conservative PTN-exercises this is not necessarily the case. The proof of Theorem 4.4 contains an example illustrating this point.

4.2 An ex ante analysis of post-trade netting and contagion

In the following, we consider a PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$ and analyse contagion from an ex ante perspective. Throughout this subsection, we will assume that the PTN-exercise is aggregate-conservative, i.e., condition (4) holds, so the PTN-exercise can only reduce the total liabilities of every bank. Our ex ante analysis is based on the framework by Glasserman & Young (2015) which we apply to both the original and the PTN-network.

This framework considers a generalised Eisenberg & Noe (2001) model with a random shock $X = (X_1, \ldots, X_N)^\top$ to what we refer to as liquidity buffers A^b, where $0 \leq X \leq A^b$. Hence, the shocked liquidity buffers are $A^b - X$. Given a realisation of shocks $x = (x_1, \ldots, x_N)^\top$, we can then compute an equilibrium in the original and in the PTN-network as before but with A^b replaced by $A^b - x$. We will set $\mathbb{V} = \mathbb{V}^{\text{EN}}$, to compute the equilibrium and assume that we are in a situation where the re-evaluated equity is unique.[14] Hence, we will consider the original network $(L, A^b - x; \mathbb{V}^{\text{EN}})$ and the PTN-network $(L^{\mathcal{P}}, A^b - x; \mathbb{V}^{\text{EN}})$.

The framework by Glasserman & Young (2015) considers explicitly the payment obligations of banks to counterparties external to the network, which in our context could be interpreted as variation margin payments to banks' customers. As mentioned before, this can be incorporated

[13] We omit the proof of this result, which is a minor variation on the proof of Veraart (2020, Theorem 2.9).

[14] Eisenberg & Noe (2001) show that the clearing vector is unique if the financial system is *regular* (Eisenberg & Noe, 2001, Definition 5). A sufficient condition for regularity here is that every bank has strictly positive (shocked) liquidity buffers. An alternative sufficient condition for uniqueness is provided by Glasserman & Young (2015, p. 386) who show that the clearing vector is unique if "from every node i there exists a chain of positive obligations to some node k that has positive obligations to the external sector". For the details on the relationship between the clearing vector and the re-evaluated equity we refer to Veraart (2020).

into our setting by assuming that the notional matrix C satisfies $C_{Nj} = 0$ for all $j \in \mathcal{N}$. We assume that $N \in \mathcal{N} \setminus \mathcal{P}$ for any PTN-exercise considered in this Subsection.

Glasserman & Young (2015) develop a framework for estimating or bound contagion and amplification effects based on three key characteristics of each node $i \in \mathcal{N}$ in the financial network: the net worth (which coincides with our initial equity) $w_i = E_i^{(0)} = A_i^b + \sum_{j \in \mathcal{N}} L_{ji} - \bar{L}_i$, the outside leverage $\lambda_i = A_i^b / w_i$, and the financial connectivity $\theta_i = \frac{\bar{L}_i - L_{iN}}{\bar{L}_i}$. The financial connectivity captures the proportion of bank i's total liabilities to other banks in the network. We assume that $\theta_i > 0$ for all $i \in \mathcal{N} \setminus \{N\}$ and $\theta_N = 0$, so each bank (excluding the external node) has payment obligations to banks (excluding the external node); $w_i > 0$ $\forall i \in \mathcal{N} \setminus \{N\}$, i.e., no defaults prior to the shock; and $w_i < A_i^b$ $\forall i \in \mathcal{N} \setminus \{N\}$, i.e., every bank can in principle suffer a fundamental default caused by the shock to the liquidity buffers.[15]

The financial connectivity is the only characteristic among the three that can be changed by a PTN-exercise, since the net worth is the same in the original and the PTN-network.[16] Since we assume that (4) holds and $L_{iN}^{\mathcal{P}} = L_{iN}$ for all $i \in \mathcal{N}$ since $N \notin \mathcal{P}$, the financial connectivities in the original and in the PTN-network denoted by θ and $\theta^{\mathcal{P}}$, respectively, satisfy

$$\theta_i^{\mathcal{P}} = \frac{\bar{L}_i^{\mathcal{P}} - L_{iN}^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}} = 1 - \frac{L_{iN}}{\bar{L}_i^{\mathcal{P}}} \leq 1 - \frac{L_{iN}}{\bar{L}_i} = \theta_i$$

for all $i \in \mathcal{N}$ with $\bar{L}_i^{\mathcal{P}} > 0$. For all $i \in \mathcal{N}$ with $\bar{L}_i^{\mathcal{P}} = 0$ we set $\theta_i^{\mathcal{P}} = 0$ which clearly satisfies $\theta_i^{\mathcal{P}} \leq \theta_i$.

This reduction in the financial connectivity also results in a smaller contagion index for all banks in the aggregate-conservative PTN-network compared to the original network. The contagion index of a bank in the original network is defined by Glasserman & Young (2015) as $w_i \theta_i (\lambda_i - 1)$, $i \in \mathcal{N}$. Hence, $\forall i \in \mathcal{N}$

$$w_i \theta_i^{\mathcal{P}} (\lambda_i - 1) \leq w_i \theta_i (\lambda_i - 1). \tag{20}$$

The following result is then an immediate consequence of Glasserman & Young (2015, Proposition 1).

Proposition 4.7. *Consider a shock $X = (X_1, \dots, X_N)^\top$ such that for only one $i \in \mathcal{N} \setminus \{N\}$ $X_i > 0$ and $X_j = 0$ for all $j \neq i \in \mathcal{N}$. Fix a set $\mathcal{S} \subseteq \mathcal{N}$ with $i, N \notin \mathcal{S}$. Consider an aggregate-conservative PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$, i.e., (4) holds. Suppose that $\mathcal{F}(L, A^b; \mathbb{V}^{EN}) = \emptyset$ and the following holds*

$$\sum_{j \in \mathcal{S}} w_j > w_i \theta_i (\lambda_i - 1). \tag{21}$$

Then,

1. *$\mathbb{P}(\mathcal{D}(E^*, L, A^b - X; \mathbb{V}^{EN}) \cap \mathcal{S} = \emptyset) = 1$, i.e., contagion from i to $\mathcal{S}$ is impossible in $(L, A^b - X; \mathbb{V}^{EN})$.*
2. *$\mathbb{P}(\mathcal{D}(E^{\mathcal{P};*}, L^{\mathcal{P}}, A^b - X; \mathbb{V}^{EN}) \cap \mathcal{S} = \emptyset) = 1$, i.e., contagion from i to $\mathcal{S}$ is impossible in $(L^{\mathcal{P}}, A^b - X; \mathbb{V}^{EN})$.*

Statement 1 of the proposition corresponds to the second part of Glasserman & Young (2015, Proposition 1). The key insight in our setting is that if condition (21) is satisfied which guarantees that there are no contagious defaults in the original network, this also implies that

[15] These assumptions correspond to the assumptions by Glasserman & Young (2015) but reflect our notation where the external node is part of the set $\mathcal{N}$.

[16] See the corresponding lemma in the Appendix (Lemma A.4).

there are no contagious defaults in the aggregate-conservative PTN-network. This result makes no distributional assumption on the shock vector X.

For specific shock distributions, Glasserman & Young (2015) have additional results that also carry over to the aggregate-conservative PTN-networks. In particular, according to Glasserman & Young (2015), contagion from a bank i to a set $\mathcal{S}$ is called *weak*, if the banks in $\mathcal{S}$ "are more likely to default through independent shocks than through contagion from i". Furthermore, the probability to default through independent shocks is $\mathbb{P}(X_i > w_i)\Pi_{i\in S}\mathbb{P}(X_j > w_j)$ Glasserman & Young (2015, equation (16)), which corresponds to the probability that banks in $\{i\} \cup S$ are in fundamental default through independent shocks. This probability is the same in the original and in the PTN-network.

For i.i.d. Beta distributed shocks and under the assumption that the net worth of all nodes prior to the shock is nonnegative, Glasserman & Young (2015, Theorem 1) states that contagion from a bank i to a set $\mathcal{S}$ that does not contain i (and we additional assume that it does not contain N due to our slightly different notation for the external node) is weak in $(L, A^b - X; \mathbb{V}^{\text{EN}})$ if

$$\tilde{\lambda}_S \bar{w}_S \geq w_i \theta_i (\lambda_i - 1), \tag{22}$$

where $\tilde{\lambda}_S = \left(\frac{1}{|S|}\sum_{j\in\mathcal{S}}\frac{1}{\lambda_j}\right)^{-1}$ is the harmonic mean of the outside leverage ratios and $\bar{w}_S = \frac{1}{|S|}\sum_{j\in\mathcal{S}} w_j$ is the average net worth. Hence, we see immediately if condition (22) is satisfied, then due to (20), contagion from i to $\mathcal{S}$ is then also weak in the aggregate-conservative PTN-network $(L^{\mathcal{P}}, A^b - X; \mathbb{V}^{\text{EN}})$. Glasserman & Young (2015, Theorem 2) provides a slightly stricter condition that implies that contagion is weak for any increasing failure rate distribution. Using the same arguments as before, it follows directly that if this stricter condition is satisfied in the original network then it is also satisfied in the aggregate-conservative PTN-network.

Hence, if the sufficient conditions derived in Glasserman & Young (2015) for contagion to be impossible or weak are satisfied in the original network, they are also satisfied in the aggregate-conservative PTN-network. These findings only use the aggregate information of the network and not the individual position sizes of the network.

5 Conclusion

We have provided a unifying mathematical characterisation of a post-trade netting (PTN-) exercise, that includes portfolio rebalancing and portfolio compression as special cases. We have established the mathematical link between portfolio rebalancing and portfolio compression (Theorem 2.10). We have then used our framework to analyse the consequences of a PTN-exercise for contagion in financial networks.

Our key result (Theorem 4.1) states that PTN-exercises with no defaults among the participants do not impose contagion risk to the system when considering the greatest fixed point. For the least fixed point, this statement only holds under additional assumptions.

As discussed before, the equilibrium associated with the least fixed point can be understood economically as the outcome of a decentralised clearing mechanism. In particular, we show that the situation in which banks respond strategically to liquidity stress and only make sequential payments corresponds to the least fixed point (Theorem 4.6). In this case, we show that conservative post-trade netting always reduces systemic risk, but this is not true for general PTN-exercises (Theorem 4.4).

We also show that if some sufficient conditions for contagion to be impossible or weak ex ante are satisfied in the original network, then they are also satisfied in the PTN-network.

The results in this paper have several implications for systemic risk in derivatives markets. Our analysis has focused on illiquidity propagation triggered by defaults on variation margin calls. While these margin calls are intended to reduce counterparty risk, the fact that they are inherently procyclical can increase the stress on market participants in adverse market conditions (ESRB, 2017). With this in mind, PTN-exercises might be used for macroprudential purposes to mitigate the procyclicality of margin requirements and address systemic risk. Notably, these services can reduce the variation margins that must be exchanged between counterparties when market conditions move. In this sense, they already mitigate the procyclical effect by reducing the overall magnitude of variation margins. Recently, in March 2020, the COVID-19 pandemic caused significant liquidity stress in financial markets, with considerable variation margins becoming due precisely when liquidity was already under strain (ISDA, 2022). Our analysis implies that PTN-exercises could reduce parts of these pressures without increasing contagion risk.

In addition, although regulatory reforms promote central clearing where possible, it can depend on the mandatory clearing obligation, standardisation of the product, and interests of market participants on a voluntary basis; therefore, institutions will inevitably have both centrally-cleared and non-centrally cleared portfolios. As pointed out by ESRB (2020), post-trade netting services designed for risk mitigation purposes could complement central clearing in the uncleared space. Our analysis supplements this view from the perspective of contagion. More specifically, the sufficient condition for systemic risk reduction in Theorem 4.1 suggests that post-trade netting services are not likely to increase contagion risk in normal times. Despite the potential benefits of reducing operational and counterparty risks, however, post-trade netting can lead to different loss propagation and hence is not guaranteed to reduce systemic risk.

A Proofs

A.1 Proofs for Section 2

Corollary A.1. *Let $(C, \mathcal{P}, C+R)$ be a rebalancing exercise, then $(\psi C, \mathcal{P}, \psi(C+R))$ and $(\psi C^{bi}, \mathcal{P}, \psi(C+R)^{bi})$ are PTN-exercises for all $\psi \geq 0$.*

Proof of Corollary A.1. Let $L = \psi C$ and $L^{\mathcal{P}} = \psi C^{\mathcal{P}} = \psi(C+R)$. Then,

$$\begin{aligned}\sum_{j\in\mathcal{P}}(L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) &= \psi\sum_{j\in\mathcal{P}}(C_{ji} + R_{ji} - C_{ij} - R_{ij}) = \psi\sum_{j\in\mathcal{P}}(C_{ji} - C_{ij}) + \psi\underbrace{\sum_{j\in\mathcal{P}}(R_{ji} - R_{ij})}_{=0 (\text{by} (7))} \\ &= \sum_{j\in\mathcal{P}}(L_{ji} - L_{ij}).\end{aligned}$$

Similarly, for the bilaterally netted case we set $L = \psi C^{bi}$ and $L^{\mathcal{P}} = \psi(C^{\mathcal{P}})^{bi} = \psi(C+R)^{bi}$ and then

$$
\begin{aligned}
\sum_{j\in\mathcal{P}}(L^{\mathcal{P}}_{ji}-L^{\mathcal{P}}_{ij}) &= \psi\sum_{j\in\mathcal{P}}\Big((C^{\mathcal{P}})^{bi}_{ji}-(C^{\mathcal{P}})^{bi}_{ij}\Big) = \psi\left(\sum_{j\in\mathcal{P}:C^{\mathcal{P}}_{ji}\geq C^{\mathcal{P}}_{ij}}(C^{\mathcal{P}}_{ji}-C^{\mathcal{P}}_{ij}) - \sum_{j\in\mathcal{P}:C^{\mathcal{P}}_{ji}<C^{\mathcal{P}}_{ij}}(C^{\mathcal{P}}_{ij}-C^{\mathcal{P}}_{ji})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C^{\mathcal{P}}_{ji}-C^{\mathcal{P}}_{ij}) = \psi\left(\sum_{j\in\mathcal{P}}(C_{ji}-C_{ij})+\sum_{j\in\mathcal{P}}(R_{ji}-R_{ij})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C_{ji}-C_{ij}) = \psi\left(\sum_{j\in\mathcal{P}:C_{ji}\geq C_{ij}}(C_{ji}-C_{ij}) - \sum_{j\in\mathcal{P}:C_{ji}<C_{ij}}(C_{ij}-C_{ji})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C^{bi}_{ji}-C^{bi}_{ij}) = \sum_{j\in\mathcal{P}}(L_{ji}-L_{ij}),
\end{aligned}
$$

where the third line again uses (7). □

Proof of Theorem 2.10. 1. First, we show that the K-compression-rebalancing-parity-matrix R is indeed a rebalancing matrix. By construction, we have $R_{ij}\geq 0\ \forall\, i,j\in\mathcal{N}$. Next, we show that R satisfies

$$\sum_{j\in\mathcal{P}}R_{ji}=\sum_{j\in\mathcal{P}}R_{ij}\quad\forall\, i\in\mathcal{P}.$$

By the definition of R it holds for all $i,j\in\mathcal{N}$ that

$$
\begin{aligned}
&R_{ij}=\max\{0,(K_{ij}-K_{ji})-(C_{ij}-C_{ji})\},\\
&R_{ji}=\max\{0,(K_{ji}-K_{ij})-(C_{ji}-C_{ij})]\}=\max\{0,-[(K_{ij}-K_{ji})-(C_{ij}-C_{ji})]\}.
\end{aligned}
$$

Therefore,

$$R_{ij}=r^{+}_{ij}=\max\{0,r_{ij}\}, \tag{23}$$

$$R_{ji}=r^{-}_{ij}=\max\{0,-r_{ij}\}, \tag{24}$$

$$r_{ij}=(K_{ij}-K_{ji})-(C_{ij}-C_{ji}). \tag{25}$$

Hence, for all $i\in\mathcal{P}$,

$$\sum_{j\in\mathcal{P}}(R_{ij}-R_{ji})=\sum_{j\in\mathcal{P}}(r^{+}_{ij}-r^{-}_{ij})=\sum_{j\in\mathcal{P}}r_{ij}=\sum_{j\in\mathcal{P}}(K_{ij}-K_{ji})-\sum_{j\in\mathcal{P}}(C_{ij}-C_{ji})=0,$$

where the last equality follows from (9). Also, note that for all $(i,j)\notin\mathcal{P}\times\mathcal{P}$ it holds that $R_{ij}=R_{ji}=0$ since $r_{ij}=0$.

The statement that $(C+R)^{bi}_{ij}=K^{bi}_{ij}$ follows directly from the definition of R. In particular, for all $(i,j)\in\mathcal{N}\times\mathcal{N}$

$$
\begin{aligned}
(C+R)^{bi}_{ij} &= \max\{0,C_{ij}+R_{ij}-C_{ji}-R_{ji}\}=\max\{0,(C_{ij}-C_{ji})+(R_{ij}-R_{ji})\}\\
&= \max\{0,(C_{ij}-C_{ji})+r_{ij}\}=\max\{0,(C_{ij}-C_{ji})+(K_{ij}-K_{ji})-(C_{ij}-C_{ji})\}\\
&= \max\{0,K_{ij}-K_{ji}\}=K^{bi}_{ij}.
\end{aligned}
$$

Finally, let K be a super-conservative compression matrix, i.e., $0\leq K_{ij}\leq C^{bi}_{ij}\ \forall\, i,j\in\mathcal{N}$. From (12) we obtain that the corresponding K-compression-rebalancing-parity-matrix R

satisfies

$$(C^{\mathcal{P}})^{bi}_{ij} = (C+R)^{bi}_{ij} = K^{bi}_{ij} \leq C^{bi}_{ij} \quad \forall\, i,j \in \mathcal{N}$$

and hence, R is a net-conservative rebalancing matrix.

2. Let R be a rebalancing matrix. We show that $K = (C+R)^{bi}$ is a compression matrix. K is clearly nonnegative. We only need to check that for all $i \in \mathcal{P}$,

$$\sum_{j\in\mathcal{P}} (K_{ji} - K_{ij}) = \sum_{j\in\mathcal{P}} (C_{ji} - C_{ij}).$$

Let $i \in \mathcal{P}$. Then,

$$\begin{aligned}
\sum_{j\in\mathcal{P}} (K_{ji} - K_{ij}) &= \sum_{j\in\mathcal{P}} \left((C+R)^{bi}_{ji} - (C+R)^{bi}_{ij} \right) \\
&= \sum_{j\in\mathcal{P}} \left(\max\{0, C_{ji} + R_{ji} - C_{ij} - R_{ij}\} - \max\{0, C_{ij} + R_{ij} - C_{ji} - R_{ji}\} \right) \\
&= \sum_{j\in\mathcal{P}} (C_{ji} + R_{ji} - C_{ij} - R_{ij}) \\
&= \underbrace{\sum_{j\in\mathcal{P}} (R_{ji} - R_{ij})}_{=0 (\text{by } (7))} + \sum_{j\in\mathcal{P}} (C_{ji} - C_{ij}) \\
&= \sum_{j\in\mathcal{P}} (C_{ji} - C_{ij}).
\end{aligned}$$

If R is a net-conservative rebalancing matrix, then it follows from above that $K = (C+R)^{bi}$ is a compression matrix. Furthermore, since R is a net-conservative rebalancing matrix we obtain that

$$K_{ij} = (C+R)^{bi}_{ij} \leq C^{bi}_{ij} \quad \forall i,j \in \mathcal{P}.$$

Hence, K is a super-conservative compression matrix.

3. Let K^* be a solution to the compression optimisation problem. We show that the K^*-compression-rebalancing-parity-matrix denoted by R^* is a solution to the rebalancing optimisation problem.

We denote the set of feasible points for the compression optimisation problem by

$$F^{\text{Comp}} = \{K \in [0,\infty)^{N\times N} \mid K \text{ satisfies } (9)\}.$$

Hence, it holds that

$$\min_{K\in F^{\text{Comp}}} \sum_{i\in\mathcal{P}} \sum_{j\in\mathcal{P}} K_{ij} = \sum_{i\in\mathcal{P}} \sum_{j\in\mathcal{P}} K^*_{ij} = \sum_{i\in\mathcal{P}} \sum_{j\in\mathcal{P}} (K^*)^{bi}_{ij} = \sum_{i\in\mathcal{P}} \sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij},$$

where the last equality follows from part 1. of this theorem and the second equality follows from the fact that if K^* solves the compression optimisation problem, then $K^*_{ij} = (K^*)^{bi}_{ij}$ for all $i,j \in \mathcal{P}$. Suppose this does not hold, then $K^*_{ij} K^*_{ji} > 0$ for some fixed $i,j \in \mathcal{P}$. We define a new matrix $\hat{K} \in [0,\infty)^{N\times N}$ by setting $\hat{K}_{\nu\mu} = K^*_{\nu\mu}$ for all $(\nu,\mu) \in \mathcal{P} \times \mathcal{P} \setminus \{(i,j),(j,i)\}$ and we set $\hat{K}_{ij} = K^{bi}_{ij}$ and $\hat{K}_{ji} = K^{bi}_{ji}$. It follows that $\hat{K} \in F^{\text{Comp}}$ and $\sum_{i,j\in\mathcal{P}} \hat{K}_{ij} < \sum_{i,j\in\mathcal{P}} K^*_{ij}$, so K^* is not a solution to the compression optimisation problem. Therefore, we must have $K^*_{ij} = (K^*)^{bi}_{ij}$ for all $i, j \in \mathcal{P}$.

From part 1. of this theorem we know that R^* is a rebalancing matrix and hence a feasible point of the rebalancing optimisation problem. We need to show that

$$\min_{R \in F^{\text{Rebal}}} \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R)^{bi}_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R^*)^{bi}_{ij}, \tag{26}$$

where

$$F^{\text{Rebal}} = \{R \in [0,\infty)^{N \times N} \mid R \text{ satisfies (7) } \}$$

is the set of feasible points for the rebalancing optimisation problem. We prove (26) by contradiction. Suppose there exists an $\tilde{R} \in F^{\text{Rebal}}$ with

$$\sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+\tilde{R})^{bi}_{ij} < \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R^*)^{bi}_{ij}.$$

Then, $\tilde{K} = (C+\tilde{R})^{bi} \in F^{\text{Comp}}$ by part 2. of this theorem. Furthermore,

$$\sum_{i \in \mathcal{P}} \sum_{i \in \mathcal{P}} \tilde{K}_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+\tilde{R})^{bi}_{ij} < \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R^*)^{bi}_{ij} = \min_{K \in F} \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K_{ij}$$

which is a contradiction. Hence, (26) holds.

It remains to show the statement for the super-conservative case. Let K^* be a solution to the super-conservative compression optimisation problem, then by part 1. of this theorem, the K^*-compression-rebalancing-parity-matrix R^* is a net-conservative rebalancing matrix. We can use exactly the same argument as before together with the feasible sets for the super-conservative compression optimisation problem and the net-conservative rebalancing optimisation problem given by

$$\begin{aligned} F^{\text{SC Comp}} &= \{K \in [0,\infty)^{N \times N} \mid K \text{ satisfies (9) and (10)}\}, \\ F^{\text{NC Rebal}} &= \{R \in [0,\infty)^{N \times N} \mid R \text{ satisfies (7) and (8)}\}. \end{aligned} \tag{27}$$

Then, it follows that R^* is a solution to the net-conservative rebalancing optimisation problem.

4. Let R^* be a solution to the rebalancing optimisation problem. Then, from part 2. of this theorem $K^* = (C+R^*)^{bi}$ is a compression matrix. Furthermore,

$$\min_{R \in F^{\text{Rebal}}} \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R)^{bi}_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (C+R^*)^{bi}_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K^*_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} (K^*)^{bi}_{ij}.$$

We show that

$$\min_{K \in F^{\text{Comp}}} \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K_{ij} = \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K^*_{ij}. \tag{28}$$

We prove this by contradiction. Suppose there exists a $\tilde{K} \in F^{\text{Comp}}$ such that

$$\sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} \tilde{K}_{ij} < \sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}} K^*_{ij}.$$

Then, the corresponding $\tilde{K}$-compression-rebalancing-parity-matrix $\tilde{R} \in F^{\text{Rebal}}$ satisfies by part 1. of this theorem

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}(C+\tilde{R})^{bi}_{ij}=\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}\tilde{K}^{bi}_{ij}\leq\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}\tilde{K}_{ij}<\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}K^*_{ij}=\min_{R\in F^{\text{Rebal}}}\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}}(C+R)^{bi}_{ij}.$$

This is a contradiction and hence such a $\tilde{K}$ does not exist and (28) holds.

Now, let R^* be a solution to the net-conservative rebalancing optimisation problem. Then, from part 2. of this theorem $K^*=(C+R^*)^{bi}$ is a super-conservative compression matrix. We can then use the same argument as for the general case using the feasible sets for the net- and super-conservative case $F^{\text{SC Comp}}$ and $F^{\text{NC Rebal}}$ defined in (27).

□

A.2 Proofs for Section 4

A.2.1 Additional properties of a PTN-exercise

We use the following lemmas to prove the main results in Section 4.

Lemma A.2. *Let $(L,\mathcal{P},L^{\mathcal{P}})$ be a PTN-exercise. Then,*

1. *$\bar{L}_i=\bar{L}^{\mathcal{P}}_i$ for all $i\in\mathcal{N}\setminus\mathcal{P}$;*
2. *for all $i\in\mathcal{N}$,*

$$\sum_{j\in\mathcal{P}}L_{ji}-\bar{L}_i=\sum_{j\in\mathcal{P}}L^{\mathcal{P}}_{ji}-\bar{L}^{\mathcal{P}}_i. \tag{29}$$

Proof of Lemma A.2. 1. This is a direct consequence of Definition 2.1.

2. This also follows directly from Definition 2.1, since $\forall i\in\mathcal{N}$

$$\begin{aligned}\sum_{j\in\mathcal{P}}(L_{ji}-L_{ij})&=\sum_{j\in\mathcal{P}}(L^{\mathcal{P}}_{ji}-L^{\mathcal{P}}_{ij})\\ \Leftrightarrow\sum_{j\in\mathcal{P}}L_{ji}-\bar{L}_i+\sum_{j\in\mathcal{N}\setminus\mathcal{P}}L_{ji}&=\sum_{j\in\mathcal{P}}L^{\mathcal{P}}_{ji}-\bar{L}^{\mathcal{P}}_i+\sum_{j\in\mathcal{N}\setminus\mathcal{P}}\underbrace{L^{\mathcal{P}}_{ji}}_{=L_{ji}}\\ \Leftrightarrow\sum_{j\in\mathcal{P}}L_{ji}-\bar{L}_i&=\sum_{j\in\mathcal{P}}L^{\mathcal{P}}_{ji}-\bar{L}^{\mathcal{P}}_i.\end{aligned}$$

□

Lemma A.3. *Let $(L,\mathcal{P},L^{\mathcal{P}})$ be a PTN-exercise. Set*

$$\mathcal{M}=\{i\in\mathcal{N}\,|\,\bar{L}_i>0\},\qquad\mathcal{M}^{\mathcal{P}}=\{i\in\mathcal{N}\,|\,\bar{L}^{\mathcal{P}}_i>0\}.$$

Let $j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{P}}$. Then, $j\in\mathcal{P}$ and $L_{ji}=0\ \forall\, i\in\mathcal{N}\setminus\mathcal{P}$.

Proof of Lemma A.3. Let $j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{P}}$. It follows that $\bar{L}_j>0$ and $\bar{L}^{\mathcal{P}}_j=0$. Therefore, $\bar{L}_j\neq\bar{L}^{\mathcal{P}}_j$, which implies that $j\in\mathcal{P}$ by Lemma A.2.

Now suppose there exists an $i\in\mathcal{N}\setminus\mathcal{P}$ such that $L_{ji}>0$. Then $\bar{L}^{\mathcal{P}}_j=\sum_{k\in\mathcal{N}}L^{\mathcal{P}}_{jk}\geq L_{ji}>0$, which contradicts the assumption that $j\notin\mathcal{M}^{\mathcal{P}}$. □

A.2.2 Proofs of the main results in Section 4

We first introduce some additional notation useful for later proofs.

Lemma A.4. *For all $i \in \mathcal{N}$, let $E_i^{(0)}$ and $E_i^{\mathcal{P}(0)}$ be the initial equities defined in* (15). *For $n \in \mathbb{N}$, we define two sequences recursively by*

$$
\begin{aligned}
E^{(n)} &= \Phi\left(E^{(n-1)}\right), \\
E^{\mathcal{P}(n)} &= \Phi^{\mathcal{P}}\left(E^{\mathcal{P}(n-1)}\right),
\end{aligned}
\tag{30}
$$

where the functions Φ and $\Phi^{\mathcal{P}}$ are defined in (13) *and* (14)*, respectively. Then*

1. $E_i^{(0)} = E_i^{\mathcal{P}(0)} \quad \forall\, i \in \mathcal{N}$;
2. *the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{P}(n)}\right)$ are non-increasing, i.e., for all $i \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$, it holds that*
$$E_i^{(n)} \geq E_i^{(n+1)}, \qquad E_i^{\mathcal{P}(n)} \geq E_i^{\mathcal{P}(n+1)};$$
3. *for all $i \in \mathcal{N}$, the sequences $\left(E_i^{(n)}\right)$ and $\left(E_i^{\mathcal{P}(n)}\right)$ converge to the greatest fixed points of Φ and $\Phi^{\mathcal{P}}$, respectively, i.e.,*
$$\lim_{n\to\infty} E_i^{(n)} = E_i^*, \qquad \lim_{n\to\infty} E_i^{\mathcal{P}(n)} = E_i^{\mathcal{P};*}.$$

Proof of Lemma A.4.

1. We can rewrite the initial equities as
$$
\begin{aligned}
E_i^{(0)} &= A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{P}} (L_{ji} - L_{ij}), \\
E_i^{\mathcal{P}(0)} &= A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) + \sum_{j\in\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) = A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}).
\end{aligned}
$$
Finally, it follows directly from Definition 2.1 that $\sum_{j\in\mathcal{P}}(L_{ji} - L_{ij}) = \sum_{j\in\mathcal{P}}(L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}})$.
2. We know from Veraart (2020, Lemma A.1) that the functions Φ and $\Phi^{\mathcal{P}}$ are non-decreasing, so the statement follows from Veraart (2020, Theorem 2.6).
3. The convergence of the two sequences defined by (30) to the greatest re-evaluated equities in the corresponding network follows from Theorem 2.6 in Veraart (2020).

□

Proof of Theorem 4.1. Let

$$
\begin{aligned}
\mathcal{M} &= \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \\
\mathcal{M}^{\mathcal{P}} &= \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{P}} > 0\}.
\end{aligned}
\tag{31}
$$

1. Recall that for the financial networks $(L, A^b; \mathbb{V})$ and $(L^{\mathcal{P}}, A^b; \mathbb{V})$ we consider the functions
$$
\begin{aligned}
\Phi_i(E) &= A_i^b + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \quad \forall\, i \in \mathcal{N}, \\
\Phi_i^{\mathcal{P}}(E) &= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{E_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \quad \forall\, i \in \mathcal{N}
\end{aligned}
$$
on $\mathcal{E} = [-\bar{L}, A^b + \bar{A} - \bar{L}]$, and $\mathcal{E}^{\mathcal{P}} = [-\bar{L}^{\mathcal{P}}, A^b + \bar{A}^{\mathcal{P}} - \bar{L}^{\mathcal{P}}]$, respectively.

It holds that $\tilde{E} \in \mathcal{E}$. Before we show that $\tilde{E}$ is a fixed point of $\Phi^{\mathcal{P}}$ we show that $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$. First, note that since $L^{\mathcal{P}}$ is a PTN-exercise, the PTN-contraint (1) implies that $\mathcal{E}^{\mathcal{P}} =$

$[-\bar{L}^{\mathcal{P}}, A^b + \bar{A} - \bar{L}]$. Hence, $\mathcal{E}$ and $\mathcal{E}^{\mathcal{P}}$ have the same upper bound but different lower bounds. To see that $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$, note that for all $i \in \mathcal{N} \setminus \mathcal{P}$ it holds that $\bar{L}_i = \bar{L}_i^{\mathcal{P}}$ and hence the corresponding lower bound for $\tilde{E}_i$ is the same in $\mathcal{E}$ and $\mathcal{E}^{\mathcal{P}}$. Furthermore, for all $i \in \mathcal{P}$ by assumption (16) $\tilde{E}_i \geq 0$ and hence the lower bound does not matter. Hence, indeed $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$.

Since $\tilde{E}$ is a fixed point of Φ it holds that for all $i \in \mathcal{N}$

$$\tilde{E}_i = \Phi_i(\tilde{E}) = A_i^b + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i.$$

We show that $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$. Let $i \in \mathcal{N} \setminus \mathcal{P}$. Then,

$$\begin{aligned}
\tilde{E}_i &= A_i^b + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= A_i^b + \sum_{j \in \mathcal{M}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i^{\mathcal{P}} \qquad \text{(since } i \in \mathcal{N} \setminus \mathcal{P}\text{)} \\
&= A_i^b + \sum_{j \in \mathcal{M} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) + \sum_{j \in \mathcal{M} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right)}_{=1 \text{(no defaults in } \mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j \in \mathcal{M} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j \in \mathcal{M}^{\mathcal{P}} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{=1 \text{(no defaults in } \mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \\
&= \Phi_i^{\mathcal{P}}(\tilde{E}).
\end{aligned}$$

Let $i \in \mathcal{P}$. Then,

$$
\begin{aligned}
\tilde{E}_i &= A_i^b + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= A_i^b + \sum_{j\in\mathcal{M}\setminus\mathcal{P}} \underbrace{L_{ji}}_{=L_{ji}^{\mathcal{P}}}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j}{\bar{L}_j}\right) + \sum_{j\in\mathcal{M}\cap\mathcal{P}} L_{ji}\underbrace{\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j}{\bar{L}_j}\right)}_{=1(\text{no defaults in }\mathcal{P})} - \bar{L}_i \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \underbrace{\sum_{j\in\mathcal{P}} L_{ji} - \bar{L}_i}_{=\sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}(\text{by (29)})} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j\in\mathcal{M}^{\mathcal{P}}\cap\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j\in\mathcal{M}^{\mathcal{P}}\cap\mathcal{P}} L_{ji}^{\mathcal{P}}\underbrace{\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{=1(\text{no defaults in }\mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{\tilde{E}_j+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \\
&= \Phi_i^{\mathcal{P}}(\tilde{E}).
\end{aligned}
$$

Hence, $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$.

Furthermore, $\mathcal{D}(\tilde{E}, L, A^b; \mathbb{V}) = \{i \in \mathcal{N} \mid \tilde{E}_i < 0\} = \mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V})$ and hence $\mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, i.e., systemic risk is reduced but $\mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V}) \not\subsetneq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, i.e., there is no strong reduction of systemic risk.

2. Let $\tilde{E}$ be the greatest fixed point of Φ satisfying (16). We show that it is also the greatest fixed point of $\Phi^{\mathcal{P}}$. By part 1. of this theorem, $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$. Let $\tilde{E}^{\mathcal{P}}$ be the greatest fixed point of $\Phi^{\mathcal{P}}$. Since $\tilde{E}$ is a fixed point of $\Phi^{\mathcal{P}}$, we have $\tilde{E} \le \tilde{E}^{\mathcal{P}}$ and in particular $0 \le \tilde{E}_i \le \tilde{E}_i^{\mathcal{P}}$ for all $i \in \mathcal{P}$ by (16) and hence $\{i \in \mathcal{P} \mid \tilde{E}_i^{\mathcal{P}} < 0\} = \emptyset$. Therefore, it follows from part 1. of Proposition 4.3 that $\tilde{E}^{\mathcal{P}}$ is also a fixed point of Φ, implying $\tilde{E}^{\mathcal{P}} \le \tilde{E}$. This leads to $\tilde{E} = \tilde{E}^{\mathcal{P}}$.

3. We consider the network presented in Figure 4 for $\mathbb{V} = \mathbb{V}^{\mathrm{RV}}$ with $\beta = 0.1$, where

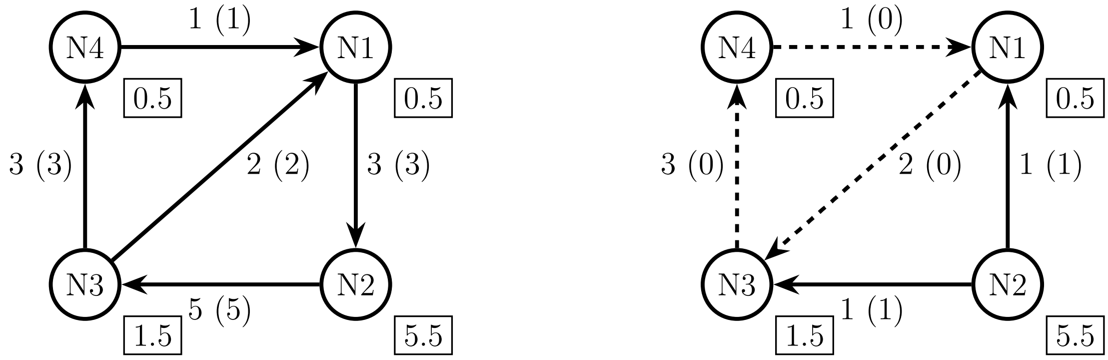
$$
\mathbb{V}^{\mathrm{RV}}(y) = \begin{cases} 1, & \text{if } y \ge 1, \\ \beta y^+, & \text{if } y < 1, \end{cases}
$$

which corresponds to a special case of the model by Rogers & Veraart (2013).

In particular, here $N = 4$, $\mathcal{P} = \{1, 2, 3\}$, and

$$
A^b = \begin{pmatrix} 0.5 \\ 5 \\ 1.5 \\ 0.5 \end{pmatrix}, L = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}, L^{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

One can check that $E^* = (0.5, 3.5, 1.5, 2.5)^\top = E_*$ is the greatest and least fixed point in the original network, hence there are no defaults in the original network. Furthermore,



(a) The original network.

(b) The rebalanced network.

Figure 4: Networks of variation margin payments corresponding to the least fixed point.

Note. The figure shows a harmful rebalancing exercise with respect to the least fixed point. The liabilities are next to the arrows, and the numbers in boxes and brackets represent the liquid assets and actual payments, respectively. The dashed lines indicate the liabilities that are not settled.

$E^{\mathcal{P};*} = E^*$ is the greatest fixed point in the PTN-network (no defaults). But the least fixed point in the PTN-network is given by $E_*^{\mathcal{P}} = (-47/111, 3.5, -38/111, -26/111)^\top \ngeq E_*$, i.e., banks $1, 3, 4$ default and hence under the least fixed point this PTN-exercise is harmful.

4. Let $\tilde{E}$ be the least fixed point of Φ. Let $E_*^{\mathcal{P}}$ be the least fixed point of $\Phi^{\mathcal{P}}$. Since the conditions of Theorem 4.4 are satisfied, we can conclude with Theorem 4.4 result (19) that $\tilde{E} \leq E_*^{\mathcal{P}}$. From part 1. of this theorem it follows that $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$ and since $E_*^{\mathcal{P}}$ is the least fixed point of $\Phi^{\mathcal{P}}$ it follows that $\tilde{E} = E_*^{\mathcal{P}}$.

□

Proof of Corollary 4.2. Since (16) holds, $\tilde{E}_i \geq 0 \ \forall i \in \mathcal{P}$, and hence,

$$\mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) = 1 = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right) \quad \forall i \in \mathcal{P}. \tag{32}$$

The payments in the original network are

$$p_{ij}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) L_{ij} \quad \forall i, j \in \mathcal{N},$$

and because of (32)

$$p_{ij}(\tilde{E}) = L_{ij} \quad \forall i \in \mathcal{P}, \forall j \in \mathcal{N}.$$

The payments in the PTN-network are

$$p_{ij}^{\mathcal{P}}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right) L_{ij}^{\mathcal{P}} \quad \forall i, j \in \mathcal{N},$$

which reduces to (using the fact that nonparticipants have the same payment obligations in both networks and (32))

$$\begin{aligned} p_{ij}^{\mathcal{P}}(\tilde{E}) &= \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) L_{ij} && \forall i \in \mathcal{N} \setminus \mathcal{P}, \forall j \in \mathcal{N}, \\ p_{ij}^{\mathcal{P}}(\tilde{E}) &= L_{ij}^{\mathcal{P}} && \forall i \in \mathcal{P}, \forall j \in \mathcal{N}. \end{aligned}$$

Hence,

$$\begin{aligned} p_{ij}(\tilde{E}) - p_{ij}^{\mathcal{P}}(\tilde{E}) &= 0 = L_{ij} - L_{ij}^{\mathcal{P}}, && \forall i \in \mathcal{N} \setminus \mathcal{P}, \forall j \in \mathcal{N}, \\ p_{ij}(\tilde{E}) - p_{ij}^{\mathcal{P}}(\tilde{E}) &= L_{ij} - L_{ij}^{\mathcal{P}} && \forall i \in \mathcal{P}, \forall j \in \mathcal{N}. \end{aligned}$$

□

We use Lemma A.5 (whose second and third statements are similar to Veraart (2022, Lemma B.4)) to prove Proposition 4.5.

Lemma A.5. *Let $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < 0\}$ and $\mathcal{F}^{\mathcal{P}} = \{i \in \mathcal{N} \mid E_i^{\mathcal{P}(0)} < 0\}$ be the fundamental default set in the original network and in the PTN-network, respectively. Let $\tilde{E}$ be a fixed point of Φ and let $\tilde{E}^{\mathcal{P}}$ be a fixed point of $\Phi^{\mathcal{P}}$.*
Then 1.) $\mathcal{F}^{\mathcal{P}} = \mathcal{F}$, 2.) $\mathcal{F} \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, and 3.) $\mathcal{F}^{\mathcal{P}} \subseteq \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$.

Proof of Lemma A.5.

1. First, note that the definition of $\mathcal{F}$ is indeed the same as our earlier Definition 3.3. We have $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < 0\}$ and $\mathcal{F}^{\mathcal{P}} = \{i \in \mathcal{N} \mid E_i^{\mathcal{P}(0)} < 0\}$. Since $E_i^{(0)} = E_i^{\mathcal{P}(0)}$ for all $i \in \mathcal{N}$ by Lemma A.4, we obtain $\mathcal{F}^{\mathcal{P}} = \mathcal{F}$.
2. We consider the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{P}(n)}\right)$ defined by (30). Fix $i \in \mathcal{F}$. Lemma A.4 implies that $\forall m \in \mathbb{N}$, $0 > E_i^{(0)} \geq E_i^{(m)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$. Therefore, $i \in \mathcal{D}(E^*, L, A^b; \mathbb{V})$ where E^* is the greatest fixed point of Φ. Since, $\tilde{E}_i \leq E_i^* < 0$, this implies that $i \in \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$.
3. Fix $i \in \mathcal{F}^{\mathcal{P}}$. Again by Lemma A.4, $\forall m \in \mathbb{N}$, $0 > E_i^{\mathcal{P}(0)} \geq E_i^{\mathcal{P}(m)} \geq \lim_{n\to\infty} E_i^{\mathcal{P}(n)} = E_i^{\mathcal{P};*}$, hence $i \in \mathcal{D}(E^{\mathcal{P};*}, L^{\mathcal{B}}, A^b; \mathbb{V})$, where $E^{\mathcal{P};*}$ is the greatest fixed point of $\Phi^{\mathcal{P}}$. Since, $\tilde{E}_i^{\mathcal{P}} \leq E_i^{\mathcal{P};*} < 0$, this implies that $i \in \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$.

□

Proof of Proposition 4.5. Let $\tilde{E}$ be a fixed point of Φ and let $\tilde{E}^{\mathcal{P}}$ be a fixed point of $\Phi^{\mathcal{P}}$. We know from Lemma A.5 that $\mathcal{F}^{\mathcal{P}} \subseteq \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$. We prove $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$ by showing that $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V}) \subseteq \mathcal{F}^{\mathcal{P}}$. Let $i \in \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$. Then

$$0 > \tilde{E}_i^{\mathcal{P}} = A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j^{\mathcal{P}} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}}.$$

Hence, $\bar{L}_i^{\mathcal{P}} > 0$. Since the graph corresponding to the optimal PTN-network (when $\mathcal{P} = \mathcal{N}$) is bipartite by D'Errico & Roukny (2021, Lemma 1), this implies that $L_{ji}^{\mathcal{P}} = 0$ for all $j \in \mathcal{M}^{\mathcal{P}}$. To be more precise, we can apply D'Errico & Roukny (2021, Lemma 1) because the optimal PTN-optimisation problem is identical to the non-conservative compression problem in D'Errico & Roukny (2021) when $\mathcal{P} = \mathcal{N}$. It follows that

$$\sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j^{\mathcal{P}} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) = 0 = \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}},$$

and hence

$$0 > \tilde{E}_i^{\mathcal{P}} = A_i^b - \bar{L}_i^{\mathcal{P}} = E_i^{\mathcal{P}(0)}.$$

This implies that $i \in \mathcal{F}^{\mathcal{P}}$, therefore $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$. Hence, $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$ by Lemma A.5. □

We will use Lemma A.6 to prove Theorem 4.4. The statement of the Lemma and its proofs is only a small modification of ideas used in the proof of Veraart (2022, Proposition 4.12).

Lemma A.6. *Consider a PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$ that satisfies (3). Let the valuation function be $\mathbb{V} = \mathbb{V}^{zero}$. Let $E^{(n)} \in \mathcal{E}, E^{\mathcal{P}(n)} \in \mathcal{E}^{\mathcal{P}}$ be such that $E^{\mathcal{P}(n)} \geq E^{(n)}$. Then,*

$$\Phi_i(E^{\mathcal{P}(n)}) = \Phi_i(E^{\mathcal{P}(n)}; \mathbb{V}^{zero}) \geq \Phi_i^{\mathcal{P}}(E^{(n)}; \mathbb{V}^{zero}) = \Phi_i^{\mathcal{P}}(E^{(n)}) \quad \forall i \in \mathcal{N}.$$

Proof of Lemma A.6. Let $E^{\mathcal{P}(n)} \geq E^{(n)}$ and $\mathbb{V} = \mathbb{V}^{\text{zero}}$. Then, for any $\bar{L}, \bar{L}^{\mathcal{P}} \in [0, \infty)^N$

$$\mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) = \mathbb{I}_{\left\{E_i^{\mathcal{P}(n)} \geq 0\right\}} \geq \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} = \mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right). \tag{33}$$

First, let $i \in \mathcal{N} \setminus \mathcal{P}$. Then,

$$\begin{aligned}
\Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}) = \Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) &= A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_j^{\mathcal{P}} > 0} L_{ji}^{\mathcal{P}} \mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_j^{\mathcal{P}} > 0} L_{ji} \mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i \quad (\text{since } i \notin \mathcal{P}) \\
&= A_i^b + \sum_{j \in \mathcal{N}} L_{ji} \mathbb{I}_{\left\{E_i^{\mathcal{P}(n)} \geq 0\right\}} - \bar{L}_i \\
&\geq A_i^b + \sum_{j \in \mathcal{N}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i \quad (\text{by } (33)) \\
&= A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_j > 0} L_{ji} \mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= \Phi_i(E^{(n)}; \mathbb{V}^{\text{zero}}) = \Phi_i(E^{(n)}).
\end{aligned}$$

Second, let $i \in \mathcal{P}$. Then,

$$\begin{aligned}
\Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}) &= \Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_j^{\mathcal{P}} > 0} L_{ji}^{\mathcal{P}} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}} \right) - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{N}} L_{ji}^{\mathcal{P}} \mathbb{I}_{\left\{E_i^{\mathcal{P}(n)} \geq 0\right\}} - \bar{L}_i^{\mathcal{P}} \\
&\geq A_i^b + \sum_{j \in \mathcal{N}} L_{ji}^{\mathcal{P}} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i^{\mathcal{P}} \quad \text{(by(33))} \\
&= A_i^b + \sum_{j \in \mathcal{N} \setminus \mathcal{P}} \underbrace{L_{ji}^{\mathcal{P}}}_{= L_{ji}} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} + \sum_{j \in \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{N} \setminus \mathcal{P}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} + \sum_{j \in \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i + \sum_{j \in \mathcal{P}} L_{ji} - \sum_{j \in \mathcal{P}} L_{ji}^{\mathcal{P}} \quad \text{(by(29))} \\
&= A_i^b + \sum_{j \in \mathcal{N}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \sum_{j \in \mathcal{P}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} + \sum_{j \in \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i \\
&\quad + \sum_{j \in \mathcal{P}} L_{ji} - \sum_{j \in \mathcal{P}} L_{ji}^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{N}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i + \sum_{j \in \mathcal{P}} \underbrace{(L_{ji} - L_{ji}^{\mathcal{P}})}_{\geq 0 \text{(by (3))}} \underbrace{(1 - \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}})}_{\geq 0} \\
&\geq A_i^b + \sum_{j \in \mathcal{N}} L_{ji} \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} - \bar{L}_i = \sum_{j \in \mathcal{N}: \bar{L}_j > 0} L_{ji} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i \\
&= \Phi_i(E^{(n)}; \mathbb{V}^{\text{zero}}) = \Phi_i(E^{(n)}).
\end{aligned}$$

□

Proof of Theorem 4.4. 1. The result for the greatest fixed points, i.e., (18) has already been shown in Veraart (2022, Proposition 4.12, Proposition 4.13). As shown there, one can again consider a fixed point iteration. For the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{P}(n)}\right)$ defined by (30) one can show by induction that if $\mathbb{V} = \mathbb{V}^{\text{zero}}$, then $E_i^{\mathcal{P}(n)} \geq E_i^{(n)}$ $\forall i \in \mathcal{N}$ holds for all $n \in \mathbb{N}_0$. In particular, the statement for $n = 0$, $E_i^{\mathcal{P}(0)} = E_i^{(0)}$ for all $i \in \mathcal{N}$ holds by Lemma A.4. If we assume that $E_i^{\mathcal{P}(n)} \geq E_i^{(n)}$ holds for a fixed $n \in \mathbb{N}_0$, then, by Lemma A.6 $E_i^{\mathcal{P}(n+1)} = \Phi_i^{\mathcal{P}}\left(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}\right) \geq \Phi_i^{\mathcal{P}}\left(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}\right) = E_i^{\mathcal{P}(n+1)}$ which completes the proof by induction. Therefore, $E_i^{\mathcal{P};*} = \lim_{n \to \infty} E_i^{\mathcal{P}(n)} \geq \lim_{n \to \infty} E_i^{(n)} = E_i^* \quad \forall i \in \mathcal{N}$ by Lemma A.4 which proves (18).

Next, we prove the result for the least fixed point, i.e., that (19) holds. We set

$$\begin{aligned}
(E_{(0)})_i &= A_i^b - \bar{L}_i, \quad \forall i \in \mathcal{N}, \\
(E_{\mathcal{P}(0)})_i &= A_i^b - \bar{L}_i^{\mathcal{P}} \quad \forall i \in \mathcal{N},
\end{aligned}$$

and we define the sequences

$$\begin{aligned}
(E_{(n+1)})_i &= \Phi_i(E_{(n)}; \mathbb{V}^{\text{zero}}) \quad \forall i \in \mathcal{N}, \\
(E_{\mathcal{P}(n+1)})_i &= \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) \quad \forall i \in \mathcal{N},
\end{aligned}$$

for $n \geq 0$. We show by induction that $(E_{(n)})_{n \in \mathbb{N}_0}$ and $(E_{\mathcal{P}(n)})_{\mathbb{N}_0}$ are non-decreasing sequences, i.e.,

$$(E_{(n+1)})_i \geq (E_{(n)})_i, \quad \forall i \in \mathcal{P}$$
$$(E_{\mathcal{P}(n+1)})_i \geq (E_{\mathcal{P}(n)})_i \quad \forall i \in \mathcal{P}.$$

Let $n = 0$. It follows directly from the definition of Φ and $\Phi^{\mathcal{P}}$ that $\forall i \in \mathcal{N}$

$$\begin{aligned}
(E_{(1)})_i = \Phi_i(E_{(0)}; \mathbb{V}^{\text{zero}}) &= A_i^b + \underbrace{\sum_{j \in \mathcal{N}: \bar{L}_i > 0} L_{ij} \mathbb{V}^{\text{zero}} \left(\frac{(E_{(0)})_j + \bar{L}_j}{\bar{L}_j} \right)}_{\geq 0} - \bar{L}_i \\
&\geq A_i^b - \bar{L}_i = (E_{(0)})_i \\
(E_{\mathcal{P}(1)})_i = \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(0)}; \mathbb{V}^{\text{zero}}) &= A_i^b + \underbrace{\sum_{j \in \mathcal{N}: \bar{L}_i^{\mathcal{P}} > 0} L_{ij}^{\mathcal{P}} \mathbb{V}^{\text{zero}} \left(\frac{(E_{\mathcal{P}(0)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}} \right)}_{\geq 0} - \bar{L}_i^{\mathcal{P}} \\
&\geq A_i^b - \bar{L}_i^{\mathcal{P}} = (E_{\mathcal{P}(0)})_i.
\end{aligned}$$

Now fix $n \in \mathbb{N}_0$ and assume that $E_{(n)} \geq E_{(n-1)}$ and $E_{\mathcal{P}(n)} \geq E_{\mathcal{P}(n-1)}$ holds. Then,

$$\begin{aligned}
(E_{(n+1)})_i = \Phi_i(E_{(n)}; \mathbb{V}^{\text{zero}}) &= A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_i > 0} L_{ij} \underbrace{\mathbb{V}^{\text{zero}} \left(\frac{(E_{(n)})_j + \bar{L}_j}{\bar{L}_j} \right)}_{\geq \mathbb{V}^{\text{zero}} \left(\frac{(E_{(n-1)})_j + \bar{L}_j}{\bar{L}_j} \right)} - \bar{L}_i \\
&\geq \Phi_i(E_{(n-1)}; \mathbb{V}^{\text{zero}}) = (E_{(n)})_i \\
(E_{\mathcal{P}(n+1)})_i = \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) &= A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_i^{\mathcal{P}} > 0} L_{ij}^{\mathcal{P}} \underbrace{\mathbb{V}^{\text{zero}} \left(\frac{(E_{\mathcal{P}(n)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}} \right)}_{\geq \mathbb{V}^{\text{zero}} \left(\frac{(E_{\mathcal{P}(n-1)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}} \right)} - \bar{L}_i^{\mathcal{P}} \\
&\geq \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n-1)}; \mathbb{V}^{\text{zero}}) = (E_{\mathcal{P}(n)})_i
\end{aligned}$$

which completes the proof by induction.

Hence, $(E_{(n)})_{n \in \mathbb{N}_0}$ and $(E_{(n)})_{\mathbb{N}_0}$ are non-decreasing sequences. They are also bounded from above, since for all $i \in \mathcal{N}$ $(E_{(n)})_i \leq A_i^b + \sum_{j \in \mathcal{N}} L_{ij} - \bar{L}_i$ and $(E_{\mathcal{P}(n)})_i \leq A_i^b + \sum_{j \in \mathcal{N}} L_{ij}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}$. Hence, both sequences converge to a limit.

Next, we show that $\forall i \in \mathcal{N}$ and $\forall n \in \mathcal{N}_0$

$$(E_{\mathcal{P}(n)})_i \geq (E_{(n)})_i. \tag{34}$$

This follows directly by induction. For $n = 0$, since $(L, \mathcal{P}, L^{\mathcal{P}})$ satisfies (3) it holds that

$$(E_{\mathcal{P}(0)})_i = A_i^b - \bar{L}_i^{\mathcal{P}} \geq A_i^b - \bar{L}_i = (E_{(0)})_i \quad \forall i \in \mathcal{N}$$

and the induction step follows directly with Lemma A.6.

Hence, we obtain that

$$\lim_{n \to \infty} (E_{\mathcal{P}(n)})_i \geq \lim_{n \to \infty} (E_{(n)})_i \quad \forall i \in \mathcal{N}.$$

If $\lim_{n\to\infty}(E_{\mathcal{P}(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$ and $\lim_{n\to\infty}(E_{(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$, then there is nothing left to show.

But since $\mathbb{V}^{\text{zero}}$ is not left-continuous, there is no guarantee that $\lim_{n\to\infty}(E_{(n)})$ is a fixed point of Φ or that $\lim_{n\to\infty}(E_{\mathcal{P}(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$. Then, as discussed in Rogers & Veraart (2013, Section 3.1), if at least one of these limits is not a fixed point, then one will need to restart the iteration from these limits, i.e., set for $n \in \mathbb{N}_0$

$$\hat{E}_{(0)} = \lim_{m\to\infty}(E_{(m)}), \quad \hat{E}_{(n+1)} = \Phi(\hat{E}_{(n)}),$$
$$\hat{E}_{\mathcal{P}(0)} = \lim_{m\to\infty}(E_{\mathcal{P}(m)}), \quad \hat{E}_{\mathcal{P}(n+1)} = \Phi(\hat{E}_{\mathcal{P}(n)}),$$

and repeat the previous arguments. If the initial vector of such a sequence is a fixed point, then the sequence is just constant.

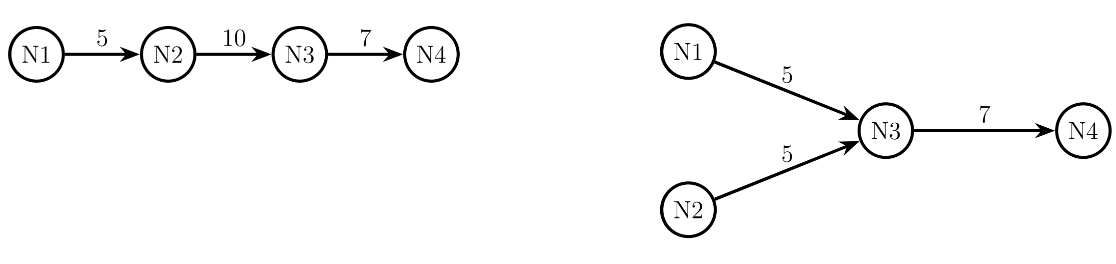
The situation that the limit is not a fixed point can only occur at a point where a bank just becomes solvent in the limit. This can happen at most N times since there are N banks, meaning at most $N-1$ restarts of this fixed point iteration could become necessary as discussed in Rogers & Veraart (2013). Then, after at most $N-1$ restarts, the limits of the iterations are indeed the least fixed points. If we need to restart the iteration, the same argument can be used to show the equivalence of (34) for the next two sequences. This sequence of arguments can be repeated until the fixed points are obtained.

2. In the following we provide an example for the situation mentioned in the statement. We consider a PTN-exercise with

$$L = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L^{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $A^b = (1, 10, 1, 0)^\top$. An illustration is provided in Figure 5.

One can check that $E^* = (-4, 0, 4, 7)^\top = E_*$, $E^{\mathcal{P};*} = (-4, 5, -1, 0)^\top = E^{\mathcal{P}}_*$. Hence, neither (18) nor (19) are satisfied and this PTN-exercise is harmful under the greatest and the least re-evaluated equity. In particular, in the original network, bank N1 is the only bank in fundamental default, which does not trigger further defaults. In the PTN-network, bank N1 remains the only one in fundamental default, but it now triggers the contagious default of bank N3. This difference is due to the fact that bank N3 now faces bank N1 directly and receives no payment from N1.



(a) The original network. (b) The PTN-network.

Figure 5: Harmful PTN-exercise for zero recovery rates.

Note. The figure shows a harmful PTN-exercise (that does not satisfy (3)) for $\mathbb{V} = \mathbb{V}^{\text{zero}}$. The liquidity buffers are $A^b = (1, 10, 1, 0)^\top$. The liabilities L_{ij} and $L^{\mathcal{P}}_{ij}$ are next to the arrows.

□

A.2.3 Clearing algorithms

Before proceeding to the proof of Theorem 4.6, we first introduce the clearing equilibrium for the Rogers & Veraart (2013) model. Given a financial network (L, A^b), the relative liabilities matrix Π is defined by $\Pi_{ij} = L_{ij}/\bar{L}_i$ if $\bar{L}_i > 0$, and $\Pi_{ij} = 0$ otherwise for all $i, j \in \mathcal{N}$. A *clearing vector* in the Rogers & Veraart (2013) model is a vector $L \in [0, \bar{L}]$ satisfying

$$L = \Psi^{RV}(L),$$

where the function $\Psi^{RV} : [0, \bar{L}] \to [0, \bar{L}]$ is given by

$$\Psi_i^{RV}(L) = \begin{cases} \bar{L}_i, & \text{if } A_i^b + \sum_{j \in \mathcal{N}} \Pi_{ji} L_j \geq \bar{L}_i, \\ \alpha A_i^b + \beta \sum_{j \in \mathcal{N}} \Pi_{ji} L_j, & \text{otherwise,} \end{cases}$$

with default cost parameters $\alpha, \beta \in [0, 1]$.

In Figure 6, we present the two clearing algorithms that are related to the modelling assumption in Section 4.1.1.

Algorithm 1 corresponds to the Full Payment Algorithm (FPA) by Bardoscia et al. (2019). It computes a vector $\tilde{l}_*$ that corresponds to the payments made by all banks in the network. The mechanism in the algorithm can be understood as follows. At the time t, $e(t)$ consists of the available liquid assets including received payments, and $\mathcal{A}(t)$ comprises banks that can pay in full. The assumption that banks either make full payment or pay nothing is incorporated into step 7. Note that an important difference between the FPA and the hard default in Paddrik et al. (2020) is that in the former setting, it is assumed that—unlike the Eisenberg & Noe (2001) model in finding an equilibrium payment vector—there is no coordination among banks in the FPA to determine the payments. Therefore, the modelling assumption results in a sequence of payments, and banks can only pay in full if they have received sufficient liquidity.

Algorithm 2 considers the least clearing vector in the Rogers & Veraart (2013) model with $\alpha = \beta = 0$. It corresponds to the case in which the defaulting banks make zero payments. We refer to Algorithm 2 as the Least Clearing Vector Algorithm (LA). The algorithm starts by assuming that initially there is no solvent bank that would be able to make any payment. $\mathcal{S}^{(0)}$ is the set of banks that would be able to pay liabilities in full even if all other banks did not meet their obligations. Similar to the construction in Rogers & Veraart (2013, Theorem 3.7), as the algorithm terminates, the output is the least clearing vector.

We use Lemma A.7 to prove Theorem 4.6.

Lemma A.7. *Consider the FPA (Algorithm 1) and the LA (Algorithm 2) described in Figure 6. Fix an iteration $t \in \mathbb{N}_0$. Then*

$$\bigcup_{s=0}^{t+1} \mathcal{A}(s) = \mathcal{S}^{(t)}. \tag{39}$$

In particular, the banks that make payments in the FPA up to time $t+1$ are identical to those that make payments in the LA up to time $t+1$.

Proof of Lemma A.7. We prove the result by induction. Let $t = 0$. By plugging the initial values of Algorithm 1 into (35) and (36), we obtain that $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{A}(0) \cup (\mathcal{A}(1) \setminus \mathcal{A}(0)) = \mathcal{A}(1) = \{i \in \mathcal{N} \,|\, e_i(1) \geq \bar{L}_i\} = \{i \in \mathcal{N} \,|\, A_i^b \geq \bar{L}_i\}$ and $\mathcal{S}^{(0)} = \{i \in \mathcal{N} \,|\, A_i^b - \bar{L}_i \geq 0\}$. Therefore, $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{S}^{(0)}$.

Now suppose (39) holds for a fixed t. We show that it also holds for $t+1$, i.e., $\bigcup_{s=0}^{t+2} \mathcal{A}(s) =$

Figure 6: Clearing algorithms.

Algorithm 1 Full Payment Algorithm (FPA) in Bardoscia et al. (2019)

1: Set $e(0) := A^b$, $l(0) := \mathbf{0}$, and $\mathcal{A}(0) := \emptyset$. Set $t = 1$.

2: For all $i \in \mathcal{N}$, set

$$e_i(t) = e_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1). \tag{35}$$

3: Determine

$$\mathcal{A}(t) = \{i \in \mathcal{N} \mid e_i(t) \geq \bar{L}_i\} \setminus \bigcup_{s=0}^{t-1} \mathcal{A}(s). \tag{36}$$

4: **if** $\mathcal{A}(t) \equiv \emptyset$ **then**

5:     **return** $\tilde{l}_* = \sum_{s=0}^{t-1} l(s)$.

6: **else**

7:     set $l_i(t) = \bar{L}_i$ for all $i \in \mathcal{A}(t)$, and $l_i(t) = 0$ otherwise.

8: **end if**

9: Set $t = t + 1$ and go back to step 2.

Algorithm 2 Least Clearing Vector Algorithm (LA) for Rogers & Veraart (2013) model with $\alpha = \beta = 0$

1: Set $t = 0$, $l^{(0)} := \mathbf{0}$, and $\mathcal{D}^{(-1)} := \mathcal{N}$.

2: For all $i \in \mathcal{N}$, determine

$$v_i^{(t)} := A_i^b + \sum_{j \in \mathcal{N}} l_j^{(t)} \Pi_{ji} - \bar{L}_i. \tag{37}$$

3: Define

$$\mathcal{D}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} < 0\} \text{ and } \mathcal{S}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} \geq 0\}. \tag{38}$$

4: **if** $\mathcal{D}^{(t)} \equiv \mathcal{D}^{(t-1)}$ **then**

5:     **return** $l_* = l^{(t-1)}$.

6: **else**

7:     set $l_i^{(t+1)} = \bar{L}_i$ for all $i \in \mathcal{S}^{(t)}$, and $l_i^{(t+1)} = 0$ for all $i \in \mathcal{D}^{(t)}$.

8: **end if**

9: Set $t = t + 1$ and go back to step 2.

$\mathcal{S}^{(t+1)}$. We rewrite

$$\bigcup_{s=0}^{t+2} \mathcal{A}(s) = \bigcup_{s=0}^{t+1} \mathcal{A}(s) \cup \mathcal{A}(t+2) \text{ and } \mathcal{S}^{(t+1)} = \mathcal{S}^{(t)} \cup \left(\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}\right),$$

so it is sufficient to prove that $\mathcal{A}(t+2) = \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$.

First, note that according to the definition of Algorithm 1, for all $i \in \mathcal{N}$, we can write $e_i(t+1)$ as

$$\begin{aligned}
e_i(t+1) &= e_i(t) + \sum_{j\in\mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= \underbrace{e_i(t-1) + \sum_{j\in\mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1)}_{=e_i(t)} + \sum_{j\in\mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= \cdots \\
&= \underbrace{A_i^b}_{=e_i(0)} + \sum_{j\in\mathcal{N}} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star)}{=} A_i^b + \sum_{j\in\bigcup_{s=0}^{t}\mathcal{A}(s)} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star\star)}{=} A_i^b + \sum_{j\in\bigcup_{s=0}^{t}\mathcal{A}(s)} \bar{L}_j\Pi_{ji} - \sum_{s=0}^{t} l_i(s),
\end{aligned} \tag{40}$$

where $(\star)$ follows from the fact that $\sum_{s=0}^{t} l_j(s) > 0$ implies $j \in \bigcup_{s=0}^{t}\mathcal{A}(s)$ (see step 3-7 in Algorithm 1), and $(\star\star)$ holds since $\sum_{s=0}^{t} l_j(s) = \bar{L}_j$ for all $j \in \bigcup_{s=0}^{t}\mathcal{A}(s)$.

In addition, we can rewrite (37) at $t+1$ as

$$v_i^{(t+1)} = A_i^b + \sum_{j\in\mathcal{N}} l_j^{(t+1)}\Pi_{ji} - \bar{L}_i = A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \bar{L}_i. \tag{41}$$

First, we show that $\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)} \subseteq \mathcal{A}(t+2)$. Let $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$, then $i \notin \bigcup_{s=0}^{t+1}\mathcal{A}(s)$ by the induction hypothesis $\mathcal{S}^{(t)} = \bigcup_{s=0}^{t+1}\mathcal{A}(s)$. From the definition of $l_i(s)$, $s \in \{0,\ldots,t+1\}$, this implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining this with equation (41) gives

$$\begin{aligned}
0 \le v_i^{(t+1)} = A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \bar{L}_i + \underbrace{0}_{=\sum_{s=0}^{t+1} l_i(s)} = A_i^b + \sum_{j\in\bigcup_{s=0}^{t+1}\mathcal{A}(s)} \bar{L}_j\Pi_{ji} + \sum_{s=0}^{t+1} l_i(s) - \bar{L}_i \\
= e_i(t+2) - \bar{L}_i.
\end{aligned}$$

Hence, $i \in \mathcal{A}(t+2)$.

Second, we show that $\mathcal{A}(t+2) \subseteq \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$. Let $i \in \mathcal{A}(t+2)$, then

$$\begin{aligned}
\bar{L}_i \le e_i(t+2) = A_i^b + \sum_{j\in\bigcup_{s=0}^{t+1}\mathcal{A}(s)} \bar{L}_j\Pi_{ji} - \sum_{s=0}^{t+1} l_i(s) \overset{\text{ind.hyp.}}{=} A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \underbrace{\sum_{s=0}^{t+1} l_i(s)}_{\overset{(\diamond)}{=}0} \\
= A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji},
\end{aligned} \tag{42}$$

where ($\diamond$) holds because the definition of $\mathcal{A}(t+2)$ implies that $i \notin \bigcup_{s=0}^{t+1} \mathcal{A}(s) \overset{\text{ind.hyp}}{=} \mathcal{S}^{(t)}$, which implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining this with (42) implies that $0 \leq A_i^b + \sum_{j \in \mathcal{S}^{(t)}} \bar{L}_j \Pi_{ji} - \bar{L}_i = v_i^{(t+1)}$ and hence $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$.

□

Now we can prove the result that the outcomes of Algorithms 1 and 2 in Figure 6 coincide.

Proof of Theorem 4.6. Suppose for a fixed $t > 0$ at one iteration it holds that $\mathcal{A}(t) = \emptyset$. Then $\tilde{l}_\star = \sum_{s=0}^{t-1} l(s)$, where $\tilde{l}_{\star,i} = \bar{L}_i$ if $i \in \bigcup_{s=0}^{t-1} \mathcal{A}(s)$ and 0 otherwise. In addition, since $\bigcup_{s=0}^{t-1} \mathcal{A}(s) = \mathcal{S}^{(t-2)}$ by Lemma A.7, $\mathcal{A}(t) = \emptyset$ is equivalent to $\mathcal{D}^{(t)} = \mathcal{D}^{(t-1)}$ in the LA. Furthermore, the LA returns $l_\star = l^{(t-1)}$, where $l_{\star,i} = \bar{L}_i$ if $i \in \mathcal{S}(t-2)$ and 0 otherwise.

Therefore, both algorithms terminate when the same banks are selected, and all their payments are identical. According to Rogers & Veraart (2013), the LA generates a sequence of vectors increasing to the least clearing vector, so the statement follows immediately. □

B More background on PTRR services

This section provides more background on post-trade netting services drawn from various sources. It is worth highlighting that the services develop fast and update frequently; we refer interested readers to the website of service providers such as OSTTRA[17] for the latest information. At the time of writing, CME's TriOptima—a leading PTRR service provider—is part of OSTTRA, a joint venture formed on 1st September 2021 between IHS Markit and CME Group. (IHS Markit was acquired by S&P Global on 1st March 2022.)

We split the development into three stages (which may have some overlapping). First, before the Global Financial Crisis, the volumes of outstanding derivatives contracts grew rapidly. In particular, concerns about counterparty risk drove the increase in the CDS market near the crisis; the notional amount then fell significantly, which according to Vause (2010), can be partly attributed to portfolio compression. The second stage follows the mandatory clearing of standardised derivatives contracts. For example, TriOptima introduced its triReduce service in 2003 for the interest rate swap (IRS) market, which now compresses bilateral swaps as well as products in centrally cleared markets. TriOptima collaborates with LCH.Clearnet on SwapClear.[18] ISDA (2012) reports that the progress on eliminating outstanding IRS notional positions since 2011 is significant.

While regulatory reforms are intended to make the financial markets more resilient, the implementation is costly. As a result, the most recent stage contributing to the development of post-trade netting services is attributable to those reforms that introduce higher costs, such as the Uncleared Margin Rules (UMRs), capital requirements (for example, the SA-CCR and G-SIBs' capital surcharges), Leverage Ratio (LR) requirements, and so on. [19] With the leverage ratio that uses gross exposures being an exception, margin and capital costs are often aligned with risk metrics based on net exposures. Therefore, portfolio rebalancing that optimises counterparty exposures could potentially reduce the all-in cost of trading derivatives.[20] Figure 7 shows the example of portfolio rebalancing provided in ESMA (2020b, Annex 1).

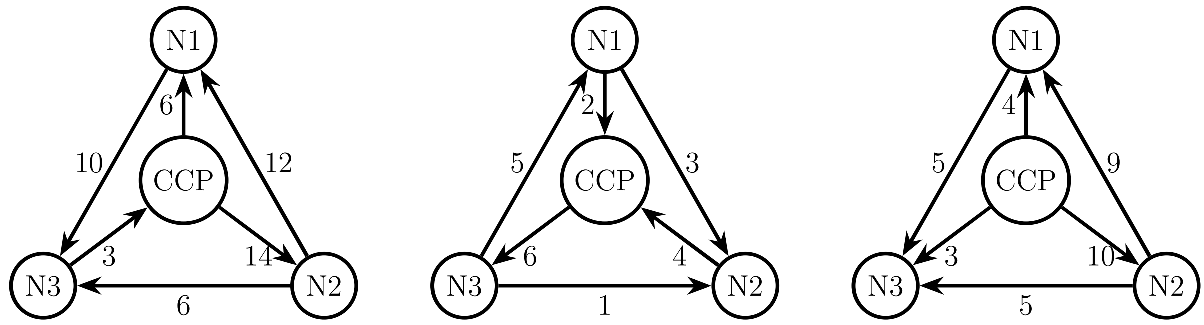
In summary, the regulatory regime spurs the development of post-trade netting services. Which feature leads to the wide use of post-trade netting services today, in addition to incentives from regulations? One answer is technology. Equipped with advanced optimisation techniques and data processing skills, the Fintech vendors who have access to all information submitted by

[17]See `https://osttra.com/`.

[18]See `https://www.lch.com/services/swapclear/enhancements/compression`.

[19]See `https://osttra.com/press_releases/osttras-compression-service-unlocks-additional-compression-potential-for-g-sibs/`.

[20]For OSTTRA's triBalance, see `https://osttra.com/articles/managing-ccr-to-reduce-the-all-in-cost-of-otc-derivatives-portfolios/`.



(a) The original notional matrix C.

(b) The rebalancing notional matrix R.

(c) The bilaterally netted rebalanced matrix $(C+R)^{bi}$.

Figure 7: An illustration of portfolio rebalancing given in ESMA (2020b, Annex 1) with three counterparties and a CCP.

Note. Note that the initial network in the example only involves a portfolio for risk reduction purposes (i.e., a subset of all transactions) made up of four counterparties. Therefore, the CCP does not need to have a matched book.

a large network of market participants could, in principle, achieve desirable outcomes more efficiently[21] compared to the fairs in preindustrial Europe which rely on the decentralised searching and matching procedure, see the descriptions in Börner & Hatfield (2017).

C Additional examples

Example C.1 ((Optimal) portfolio rebalancing**.** Figure 8 illustrates an example of portfolio rebalancing in a network of three banks. The notional matrix C, the rebalancing notional positions R, the rebalanced notional matrix $C+R$, and the bilaterally netted notional matrices C^{bi} and $(C+R)^{bi}$ are given, respectively, by

$$C = \begin{pmatrix} 0 & 8 & 7 \\ 5 & 0 & 10 \\ 9 & 5 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}, \qquad C+R = \begin{pmatrix} 0 & 8 & 11 \\ 9 & 0 & 10 \\ 9 & 9 & 0 \end{pmatrix},$$

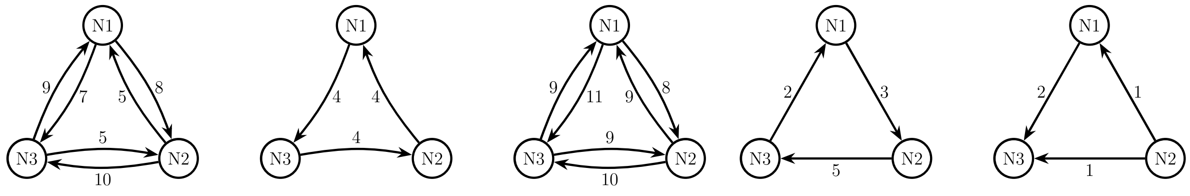
$$C^{bi} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 0 & 0 \end{pmatrix}, \qquad (C+R)^{bi} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In line with the definition of rebalancing, the net exposures of the three banks in the original network are given by $C^\top \mathbf{1} - C\mathbf{1} = (-1,-2,3)^\top$, and they coincide with the net exposures $(C+R)^\top\mathbf{1} - (C+R)\mathbf{1}$ after portfolio rebalancing.

Furthermore, portfolio rebalancing increases the gross exposures of the notional positions from $\mathbf{1}^\top(C^\top\mathbf{1}+C\mathbf{1}) = 88$ to $\mathbf{1}^\top((C+R)^\top\mathbf{1}+(C+R)\mathbf{1}) = 112$, but it decreases the gross exposures of the bilaterally netted positions from $\mathbf{1}^\top((C^{bi})^\top\mathbf{1} + C^{bi}\mathbf{1}) = 20$ to $\mathbf{1}^\top(((C+R)^{bi})^\top\mathbf{1} + ((C+R)^{bi})\mathbf{1}) = 8$.

This rebalancing exercise is not an optimal rebalancing exercise. It is easy to check that an optimal rebalancing exercise can be attained by choosing R such that $R_{13} = R_{21} = R_{32} = 3$, and $R_{ij} = 0$ otherwise. In this case, the sum of the gross exposures in the resulting network

[21] See `https://osttra.com/press_releases/osttra-streamlines-trade-reconciliation-with-connectivity-between-markitwire-and-triresolve/`.



(a) C (b) R (c) $C+R$ (d) C^{bi} (e) $(C+R)^{bi}$

Figure 8: Example of portfolio rebalancing.

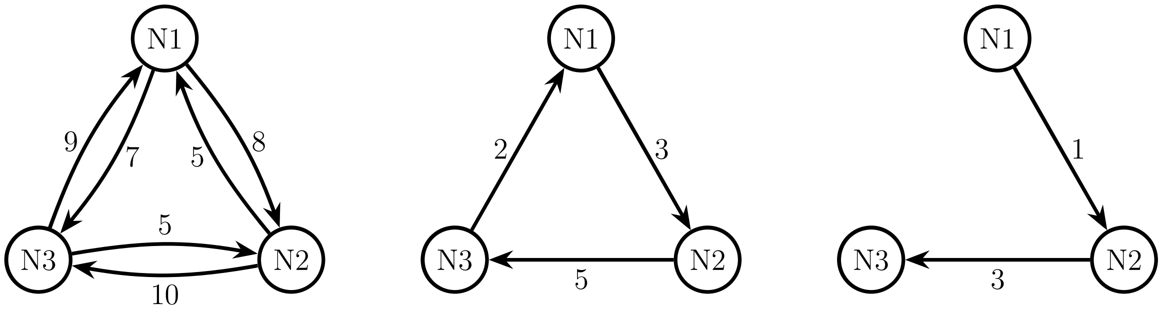
Note. Starting from the notional matrix C (Figure 8(a)), injecting the rebalancing notional matrix R (Figure 8(b)) results in the rebalanced notional matrix $C + R$ (Figure 8(c)). The bilaterally netted positions prior to rebalancing are C^{bi} (Figure 8(d)) and the bilaterally netted positions after rebalancing are $(C + R)^{bi}$ (Figure 8(e)).

is equal to 6, which is strictly less than the previous outcome. We also see that this matrix R is not a net-conservative rebalancing matrix since for example $1 = (C + R)^{bi}_{13} > C^{bi}_{13} = 0$.

It is easy to check that an optimal net-conservative rebalancing exercise can be attained by choosing R such that $R_{13} = R_{21} = R_{32} = 2,$ and $R_{ij} = 0$ otherwise. In this case, the sum of the bilaterally netted gross exposures in the resulting network is equal to 8, which is the same as in Figure 8(e) where R was such that $R_{13} = R_{21} = R_{32} = 4$ and $R_{ij} = 0$ otherwise. The difference now is that counterparty relationships are controlled, which was not the case in Figure 8 where for example bank $N1$ was a net borrower from bank $N2$ before rebalancing but became a net lender to bank $N2$ after the exercise.

Example C.2 ((Optimal) portfolio compression)**.** To provide intuition about the idea of compression, we present an example in Figure 9. This example follows the idea of O'Kane (2017), who proposes a loop compression algorithm by finding and eliminating all closed loops on the bilaterally netted notional matrix. Here, Figure 9(a) shows the original network which is the same as in Figure 8(a), Figure 9(b) shows the bilaterally compressed network and Figure 9(c) is then the outcome after removing all closed loops in the bilaterally compressed network which is in this example a solution to the super-conservative compression optimisation problem.

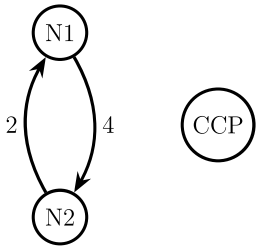
Example C.3 (Different use cases of portfolio compression and rebalancing)**.** We illustrate in Figure 10 different use cases of portfolio compression and portfolio rebalancing. In particular, it shows how a portfolio that can be compressed in the non-centrally cleared space (Figure 10(a)), can no longer be compressed in the partially centrally cleared space (Figure 10(d)). Still, portfolio rebalancing (Figure 10(f)) can achieve a comparable outcome to portfolio compression (Figure 10(c)) if new trades established as part of the rebalancing exercise are not subject to central clearing. Otherwise, Figure 10(i) shows that novation prevents the netting benefit in the non-cleared part of the portfolio and hence the overall netting benefit.



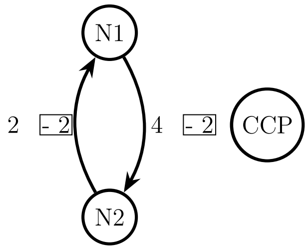
(a) C (b) $K = C^{bi}$ (c) K optimised

Figure 9: Examples of portfolio compression.

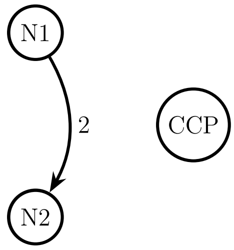
Note. The numbers in the figures represent notional positions. Figure 9(a) shows an illustration of the notional matrix C. Figure 9(b) shows an illustration of $K = C^{bi}$, i.e., the network obtained by removing cycles between pairs of counterparties. Figure 9(c) shows another example of multilateral compression where the cycle in C^{bi} between all three counterparties has been removed. This is also a solution to the super-conservative compression optimisation problem.



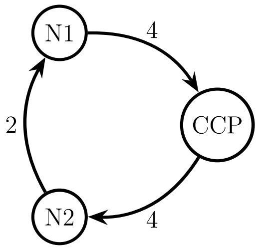
(a) Before compression.



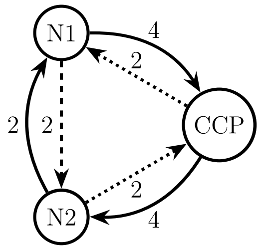
(b) Portfolio compression.



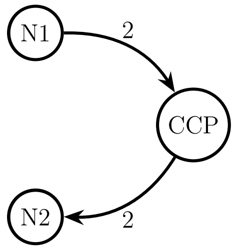
(c) After compression.



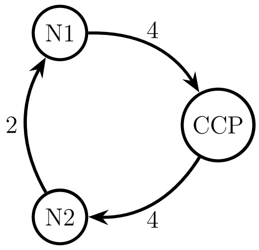
(d) Before rebalancing.



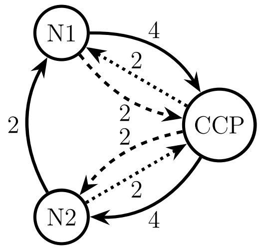
(e) Portfolio rebalancing without novation of new trades.



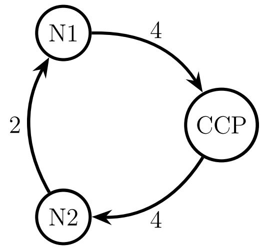
(f) After rebalancing.



(g) Before rebalancing.



(h) Portfolio rebalancing with novation of new trades.



(i) After rebalancing.

Figure 10: Different use cases of portfolio compression and portfolio rebalancing.

Note. The first row shows a portfolio that is not centrally cleared and subject to portfolio compression. The second and third row show the same positions under partial-central clearing and subject to portfolio rebalancing. The second row assumes that new trades are not novated to the CCP and the third row assumes that they are novated to the CCP.

Code and data disclosure

No empirical or simulated data were used in this paper. All data for the stylised examples are provided in the paper. The proofs of all theoretical results can be found in the Appendix.

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