SINGULAR STOCHASTIC CONTROL PROBLEMS MOTIVATED BY THE OPTIMAL SUSTAINABLE EXPLOITATION OF AN ECOSYSTEM[*]

GECHUN LIANG[†], ZHESHENG LIU[‡], AND MIHAIL ZERVOS[§]

Abstract. We derive the explicit solutions to singular stochastic control problems of the monotone follower type with (a) an expected discounted criterion, (b) an expected ergodic criterion and (c) a pathwise ergodic criterion. These problems have been motivated by the optimal sustainable exploitation of an ecosystem, such as a natural fishery. Under general assumptions on the diffusion coefficients, the discounting rate function, the running payoff function and the marginal profit of control action, we show that the optimal strategies are of a threshold type. We solve the three problems by first constructing suitable solutions to their associated HJB equations, which take the form of quasi-variational inequalities with gradient constraints. In the cases of the ergodic control problems, we also use a suitable new variational argument. Furthermore, we establish the convergence of the solution of the discounted control problem to the one of the ergodic control problems as the discounting rate function tends to 0 in an Abelian sense.

Key words. singular stochastic control, linear diffusions, optimal harvesting

MSC codes. 93E20, 60J60, 91B76

1. Introduction.

We consider a stochastic dynamical system with a positive state process that satisfies the stochastic differential equation

$$\mathrm{d}X_t^\zeta = b(X_t^\zeta)\,\mathrm{d}t - \mathrm{d}\zeta_t + \sigma(X_t^\zeta)\,\mathrm{d}W_t, \quad X_{0-}^\zeta = x > 0, \tag{1.1}$$

where W is a standard one-dimensional Brownian motion and ζ is a controlled càdlàg increasing process. With each controlled process ζ, we associate the expected discounted performance index

$$I_x(\zeta) = \mathsf{E}_x\left[\int_0^\infty e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \int_0^\infty e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t\right], \tag{1.2}$$

the expected long-term average performance index

$$J_x^{\mathrm{e}}(\zeta) = \limsup_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\left[\int_0^T h(X_t^\zeta)\,\mathrm{d}t + \int_0^T k(X_t^\zeta) \circ \mathrm{d}\zeta_t\right], \tag{1.3}$$

as well as the pathwise long-term average performance criterion

$$J_x^{\mathrm{p}}(\zeta) = \limsup_{T\uparrow\infty} \frac{1}{T}\left(\int_0^T h(X_t^\zeta)\,\mathrm{d}t + \int_0^T k(X_t^\zeta) \circ \mathrm{d}\zeta_t\right), \tag{1.4}$$

[*]Research supported by the LMS Grant Scheme 4: Research in Pairs (Ref: 41920).

[†]Department of Statistics, University of Warwick, Coventry, CV4 7AL, UK, g.liang@warwick.ac.uk.

[‡]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China, liuzhesheng@outlook.com.

[§]Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, UK, mihalis.zervos@gmail.com.

where

$$\Lambda_t^\zeta = \int_0^t r(X_u^\zeta)\,\mathrm{d}u \tag{1.5}$$

$$\text{and}\quad \int_0^T k(X_t^\zeta)\circ \mathrm{d}\zeta_t = \int_0^T k(X_t^\zeta)\,\mathrm{d}\zeta_t^{\mathrm{c}} + \sum_{0\le t\le T}\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u. \tag{1.6}$$

In the last of these definitions, ζ^{c} is the continuous part of the càdlàg increasing process ζ and $\Delta\zeta_t = \zeta_t - \zeta_{t-}$, with the convention that $\zeta_{0-} = 0$. The objective of the resulting singular stochastic control problems is to maximise each of the objective criteria (1.2), (1.3) and (1.4) over all admissible controlled processes ζ.

The control problems defined by (1.1)–(1.4) have been motivated by the sustainable exploitation of an ecosystem, such as a forest or a natural fishery. In such a context, X models the population level process of a harvested species, while ζ_t is the total amount of the species that has been harvested by time t. The function $k > 0$ in (1.2)–(1.4) models the profit made from each unit of the harvested species. On the other hand, the function h models the utility arising from having a population level X_t of the harvested species at time t, which could reflect the role that the species plays in the stability of the overall ecosystem. Alternatively, the function h can be used to model running costs.

Motivated by applications to the optimal harvesting of stochastically fluctuating populations, similar singular stochastic control problems with $h = 0$, constant k and with a discounted performance criterion with constant r have been studied by Alvarez [1, 2], Alvarez and Shepp [5], and Lungu and Øksendal [20]. Extensions of these earlier works have been studied by Framstad [9], who considers a state process X with jumps, Song, Stockbridge and Zhu [26], who consider a state process X with regime switching, Morimoto [22], who considers the finite time horizon case, Alvarez, Lungu and Øksendal [4] and Lungu and Øksendal [21], who consider multidimensional state processes X, Hening, Tran, Phan and Yin [12], who consider multidimensional state processes X as well as allow for the modelling of both seeding and harvesting, and Gaïgi, Ly Vath and Scotti [10], who consider constraints of no-take areas. On the other hand, control problems with an expected ergodic performance criterion, similar to the one that we study here with $h = 0$ and constant k, have been solved by Hening, Nguyen, Ungureanu and Wong [11], Alvarez and Hening [3], as well as Cohen, Hening and Sun [8], who consider a performance criterion with model ambiguity. Several other closely related contributions can be found in the literature of all these papers.

We solve the control problems that we consider by deriving explicit solutions to their corresponding HJB equations. In generalising the special cases arising when $h = 0$ and k is constant, our main contributions include (a) the determination of sufficiently general assumptions on the functions h and k that give rise to threshold optimal strategies without making extra assumptions on the data b and σ of the underlying diffusion, and (b) the derivation of explicit solutions to the problems' HJB equations that are way more complicated than the ones associated with the special case arising when $h = 0$ and k is constant (e.g., we are faced with integral equations, such as (5.2), instead of algebraic equations, such as the one in Remark 5.2 from Alvarez and Hening [3]).

We derive the solution to the discounted singular stochastic control problem in Section 4 using results from Liu and Zervos [19], who solve the corresponding discounted impulse control problem. On the other hand, we solve the ergodic singular stochastic control problems in Sections 5 and 6. The analysis of these problems, which are in the so-called monotone follower singular stochastic control setting, has been influenced by Karatzas [15], Menaldi, Robin and Taksar [23], Weerasinghe [28], and Jack and Zervos [14], who consider different formulations. In the solution to the ergodic control problems that we solve, a notable difficulty arises from the fact that the solution $(w, \lambda^\star)$ to their corresponding HJB equation may involve functions w that are unbounded from below (see Remark 5.3), which makes the establishment of a suitable verification theorem intractable. We overcome this complication by means of a variational argument involving suitable pairs (w_λ, λ) with bounded from below functions w_λ that converge to $(w, \lambda^\star)$ as $\lambda \downarrow \lambda^\star$. The introduction of this technique is a further contribution of this paper.

In Section 7, we establish the convergence of the solution to the discounted control problem to the one of the ergodic control problems as the discounting rate function r tends to 0 in an Abelian sense. In particular, we prove that, if r depends on a parameter $\iota > 0$ and tends to zero as $\iota \downarrow 0$ in the sense of Assumption 7.1, then

$$\lim_{\iota\downarrow 0} \beta^\star(\iota) = \beta^\star, \quad \lim_{\iota\downarrow 0} r(y;\iota) w(x;\iota) = \lambda^\star \quad \text{and} \quad \lim_{\iota\downarrow 0} w'(x;\iota) = w'(x) \tag{1.7}$$

for all $x, y > 0$, where $\beta^\star(\iota)$ (resp., $\beta^\star$) is the threshold point characterising the optimal strategy of the discounted problem (resp., the ergodic problems) and $w(\cdot;\iota)$ (resp., $(w, \lambda^\star)$) is the solution to the HJB equation of the discounted problem (resp., ergodic problems). In the context of singular stochastic control, Abelian limits, such as the first two ones in (1.7), have been obtained for constant r by Karatzas [15], Weerasinghe [29], Hynd [13], Alvarez and Hening [3], and Kunwai, Xi, Yin and Zhu [17], as well as by Cao, Dianetti and Ferrari [7] in a mean-field game setting, using different techniques. To the best of our knowledge, no results (a) with non-constant discounting rate functions $r(\cdot;\iota)$, or (b) such as the third limit in (1.7), exist in the singular stochastic control literature, with the exception of Karatzas [15, Proposition 6], who establishes a limit such as the third one in (1.7) for a model with a standard Brownian motion and constant r.

2. Problem formulation. Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a biological system, the uncontrolled stochastic dynamics of which are modelled by the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x > 0, \tag{2.1}$$

for some deterministic functions $b, \sigma :]0, \infty[\to \mathbb{R}$.

Assumption 2.1. The function b is C^1 and the limit $b(0) := \lim_{x\downarrow 0} b(x)$ exists in $\mathbb{R}$. On the other hand, the function σ is C^1, the limit $\sigma(0) := \lim_{x\downarrow 0} \sigma(x)$ exists in $\mathbb{R}$ and

$$0 < \sigma^2(x) \le C_1(1 + x^\eta) \quad \text{for all } x > 0, \tag{2.2}$$

for some constant $C_1, \eta > 0$.

This assumption implies that the scale function p and the speed measure m of the diffusion associated with the SDE (2.1), which are given by

$$p(1) = 0 \quad \text{and} \quad p'(x) = \exp\left(-2\int_1^x \frac{b(s)}{\sigma^2(s)}\,\mathrm{d}s\right), \quad \text{for } x > 0, \tag{2.3}$$

and

$$m(\mathrm{d}x) = \frac{2}{\sigma^2(x)p'(x)}\,\mathrm{d}x, \tag{2.4}$$

are well-defined. We also make the following assumption, which, together with Assumption 2.1, implies that the SDE (2.1) has a unique non-explosive strong solution (e.g., see Karatzas and Shreve [16, Proposition 5.5.22]).

Assumption 2.2. The scale function p and the speed measure m defined by (2.3) and (2.4) satisfy

$$\lim_{x\downarrow 0} p(x) = -\infty, \quad \lim_{x\uparrow\infty} p(x) = \infty \quad \text{and} \quad m\big(]0,1[\big) < \infty.$$

For the solution to the ergodic control problems, we need the following additional assumption, which implies that the diffusion associated with the SDE (2.1) is ergodic.

Assumption 2.3. The integrability condition

$$\int_0^\infty \big(s^\eta + 1\big)\, m(\mathrm{d}s) < \infty$$

holds true, where $\eta > 0$ is as in (2.2).

If the system is subject to harvesting, then its state process X satisfies the controlled one-dimensional SDE (1.1).

Definition 2.4. *An* admissible harvesting strategy *is any $(\mathcal{F}_t)$-adapted process ζ with càdlàg increasing sample paths such that $\zeta_{0-} = 0$ and the SDE (1.1) has a unique non-explosive strong solution. We denote by $\mathcal{A}$ the family of all admissible strategies.*

With each admissible harvesting strategy $\zeta \in \mathcal{A}$, we associate the expected discounted performance index $I_x(\zeta)$ given by (1.2), the expected ergodic performance index J_x^{e} given by (1.3) and the pathwise performance criterion J_x^{p} given by (1.4). The objective of the control problem that we consider is to maximise each of $I_x(\zeta)$, $J_x^{\mathrm{e}}(\zeta)$ and $J_x^{\mathrm{p}}(\zeta)$ over all $\zeta \in \mathcal{A}$.

Assumption 2.5. (i) The function h is C^1 as well as bounded from below and the limit $h(0) := \lim_{x\downarrow 0} h(x)$ exists in $\mathbb{R}$.
(ii) The function k is positive, bounded and C^2. Also, the limits $k(0) := \lim_{x\downarrow 0} k(x)$ and $k'(0) := \lim_{x\downarrow 0} k'(x)$ both exists in $\mathbb{R}_+$.

For the discounted control problem, we make the following additional assumption.

Assumption 2.6. (i) The discounting rate function r is bounded and C^1. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$ for all $x \geq 0$.
(ii) The limit $\lim_{x\downarrow 0} h(x)/r(x)$ exists in $\mathbb{R}$ and

$$\mathsf{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|h(X_t)\big|\,\mathrm{d}t\right] < \infty.$$

(iii) The conditions

$$\mathsf{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\Theta(X_t)\big|\,\mathrm{d}t\right] < \infty \quad \text{and} \quad \limsup_{x\uparrow\infty} \frac{1}{\psi(x)}\int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+$$

hold true, where ψ is as at the beginning of Section 4 and

$$\Theta(x) = \frac{1}{2}\sigma^2(x)k'(x) + b(x)k(x) - r(x)\int_0^x k(s)\,\mathrm{d}s, \quad \text{for } x > 0. \tag{2.5}$$

(iv) The limit $\lim_{x\downarrow 0} \Theta(x)/r(x)$ exists in $\mathbb{R}$ and there exists a constant $\xi \in]0,\infty[$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\Theta(x)+h(x)}{r(x)} \begin{cases} > 0, & \text{for } x \in]0,\xi[, \\ < 0, & \text{for } x \in]\xi,\infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{\Theta(x)+h(x)}{r(x)} < \frac{\Theta(0)+h(0)}{r(0)},$$

where Θ is defined by (2.5).

For the ergodic control problems, we make the following additional assumption.

Assumption 2.7. (i) The following integrability condition is satisfied:

$$\int_0^\infty |h(s)|\, m(\mathrm{d}s) < \infty.$$

(ii) If we define

$$K(x) = \frac{1}{2}\sigma^2(x)k'(x) + b(x)k(x), \quad \text{for } x > 0, \tag{2.6}$$

then there exists a constant $\xi \in]0,\infty[$ such that

$$K'(x) + h'(x) \begin{cases} > 0, & \text{for } x \in]0,\xi[, \\ < 0, & \text{for } x \in]\xi,\infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty}\bigl(K(x)+h(x)\bigr) < K(0)+h(0).$$

Remark 2.8. In the presence of Assumptions 2.2 and 2.5.(ii), the definitions of the scale function p and the speed measure m imply that

$$\int_0^x b(s)k(s)\, m(\mathrm{d}s) = \int_0^x k(s)\left(\frac{1}{p'}\right)'(s)\,\mathrm{d}s = \frac{k(x)}{p'(x)} - \frac{1}{2}\int_0^x \sigma^2(s)k'(s)\, m(\mathrm{d}s).$$

In turn, these identities and the definition (2.6) of K imply that

$$\int_0^x K(s)\, m(\mathrm{d}s) = \frac{k(x)}{p'(x)}. \tag{2.7}$$

Remark 2.9. In view of Assumption 2.7.(ii), we define

$$\underline{\lambda} = \lim_{x\uparrow\infty} K(x) + h(x) \quad \text{and} \quad \overline{\lambda} = K(\xi) + h(\xi), \tag{2.8}$$

and we note that the equation $K(x) + h(x) - \lambda = 0$ has

- no strictly positive solutions if $\lambda > \overline{\lambda}$,
- two strictly positive solutions if $\lambda \in \bigl]K(0)+h(0),\, \overline{\lambda}\bigr[$, and
- one strictly positive solution if $\lambda \in \bigl]\underline{\lambda},\, K(0)+h(0)\bigr]$ or $\lambda = \overline{\lambda}$

(see also Figure 1). In particular, there exists a unique function ϱ such that

$$\xi < \varrho(\lambda) \quad \text{and} \quad K\bigl(\varrho(\lambda)\bigr) + h\bigl(\varrho(\lambda)\bigr) - \lambda = 0 \quad \text{for all } \lambda \in]\underline{\lambda}, \overline{\lambda}[. \tag{2.9}$$

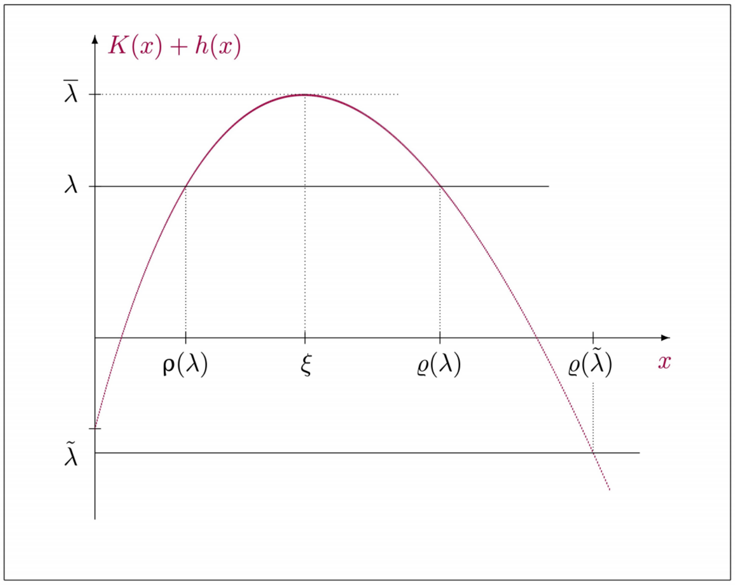

FIG. 1. *Notation associated with the graph of the function* $K + h$.

Furthermore, this function is such that

$$K(x) + h(x) - \lambda \begin{cases} > 0, & \text{for all } x \in \left[\xi, \varrho(\lambda)\right[, \\ < 0, & \text{for all } x > \varrho(\lambda), \end{cases} \tag{2.10}$$

$$\text{and} \quad K'\big(\varrho(\lambda)\big) + h'\big(\varrho(\lambda)\big) < 0 \quad \text{for all } \lambda \in \left]\underline{\lambda}, \overline{\lambda}\right[. \tag{2.11}$$

On the other hand, there is a unique function ρ such that

$$0 < \rho(\lambda) < \xi \quad \text{and} \quad K\big(\rho(\lambda)\big) + h\big(\rho(\lambda)\big) - \lambda = 0 \quad \text{for all } \lambda \in \left]K(0) + h(0),\, \overline{\lambda}\right[. \tag{2.12}$$

Given any $\lambda \in \left]K(0) + h(0),\, \overline{\lambda}\right[$, this function is such that

$$K(x) + h(x) - \lambda < 0 \quad \text{for all } x \in \left]0, \rho(\lambda)\right[. \tag{2.13}$$

We conclude this section with the following three examples.

Example 2.10. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t) X_t \, \mathrm{d}t + \sigma X_t^{\ell} \, \mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [1, \frac{3}{2}]$. Note that the celebrated stochastic Verhulst-Pearl logistic model of population growth arises in the special case

$\ell = 1$. The derivative of the scale function admits the expression

$$p'(x) = x^{-2\kappa\gamma/\sigma^2} \exp\left(\frac{2\kappa}{\sigma^2}(x-1)\right), \quad \text{if } \ell = 1,$$

$$p'(x) = \exp\left(\frac{\kappa\gamma}{(\ell-1)\sigma^2}\big(x^{-2(\ell-1)} - 1\big) + \frac{2\kappa}{(3-2\ell)\sigma^2}\big(x^{3-2\ell} - 1\big)\right), \quad \text{if } \ell \in \,]1, 1.5[,$$

$$\text{and} \quad p'(x) = x^{2\kappa/\sigma^2} \exp\left(\frac{2\kappa\gamma}{\sigma^2}(x^{-1} - 1)\right), \quad \text{if } \ell = 1.5.$$

Assumptions 2.1–2.3 hold true if $\ell \in \,]1, \frac{3}{2}]$ or if $\ell = 1$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if r, k are constant and either (a) $h = 0$ and $r < \kappa\gamma$,[1] or (b) h is a strictly concave function satisfying the Inada conditions

$$\lim_{x\downarrow 0} h'(x) = \infty \quad \text{and} \quad \lim_{x\uparrow\infty} h'(x) = 0, \tag{2.14}$$

as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 2.5–2.7 are satisfied.

Example 2.11. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \left(\kappa\gamma + \frac{1}{2}\sigma^2 - \kappa\ln(X_t)\right) X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constants $\kappa, \gamma, \sigma > 0$, namely, the logarithm of the uncontrolled state process is the Ornstein-Uhlenbeck process given by

$$\mathrm{d}\ln(X_t) = \kappa\big(\gamma - \ln(X_t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad \ln(X_0) = \ln(x) \in \mathbb{R}.$$

In this case, the derivative of scale function admits the expression

$$p'(x) = x^{\frac{\kappa}{\sigma^2}\ln(x) - \frac{2\kappa\gamma}{\sigma^2} - 1}$$

and all of Assumptions 2.1–2.3 hold true. Furthermore, if r, k are constant and either $h = 0$ or h is a strictly concave function satisfying the Inada conditions (2.14), as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 2.5–2.7 are satisfied.

Example 2.12. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)\,\mathrm{d}t + \sigma X_t^\ell \,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [\frac{1}{2}, 1]$. Note that, in the special case that arises for $\ell = \frac{1}{2}$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$, the process X identifies with the short rate process in the Cox-Ingersoll-Ross interest rate model. The derivative of the scale

[1]The inequality $r < \kappa\gamma$ is needed for Assumption 2.6.(iv) to hold true. In the absence of this condition, the harvesting strategy that drives the species to extinction at time 0 is optimal for the discounted version of the problem (see Alvarez and Shepp [5, Theorem 1.(i)]).

function admits the expression

$$p'(x) = x^{-2\kappa\gamma/\sigma^2} \exp\left(\frac{2\kappa}{\sigma^2}(x-1)\right), \quad \text{if } \ell = 0.5,$$

$$p'(x) = \exp\left(\frac{2\kappa\gamma}{(2\ell-1)\sigma^2}\left(x^{-(2\ell-1)}-1\right) + \frac{\kappa}{(1-\ell)\sigma^2}\left(x^{2(1-\ell)}-1\right)\right), \quad \text{if } \ell \in \left]0.5, 1\right[,$$

$$\text{and} \quad p'(x) = x^{2\kappa/\sigma^2} \exp\left(\frac{2\kappa\gamma}{\sigma^2}(x^{-1}-1)\right), \quad \text{if } \ell = 1.$$

Assumptions 2.1–2.3 hold true if $\ell \in \left]\frac{1}{2}, 1\right]$ or if $\ell = \frac{1}{2}$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if r, k are constant and h is a strictly concave function satisfying the Inada conditions (2.14), as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 2.5–2.7 are satisfied.

3. The HJB equations of the control problems. We will solve the discounted control problem by deriving a suitable C^2 solution w to the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x),\ k(x) - w'(x)\right\} = 0 \tag{3.1}$$

that identifies with the control problem's value function. In particular, we will construct a solution to this HJB equation such that

$$\sup_{\zeta \in \mathcal{A}} I_x(\zeta) = w(x) \quad \text{for all } x > 0.$$

On the other hand, we will solve both of the ergodic control problems by constructing a C^2 function w and finding a constant λ such that the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda,\ k(x) - w'(x)\right\} = 0 \tag{3.2}$$

holds true for all $x > 0$. Given such a solution, we will prove that

$$\sup_{\zeta \in \mathcal{A}} J_x^{\mathrm{e}}(\zeta) = \lambda \quad \text{and} \quad \sup_{\zeta \in \mathcal{A}} J_x^{\mathrm{p}}(\zeta) = \lambda \quad \text{for all } x > 0.$$

Given suitable solutions to the HJB equations, the optimal strategies can be characterised as follows. In the discounted control problem, the controller should wait and take no action for as long as the state process X takes values in the interior of the set in which the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) = 0 \tag{3.3}$$

is satisfied. On the other hand, the "no-action" region of the ergodic control problems is the interior of the set in which the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0 \tag{3.4}$$

is satisfied. In all problems, the controller should take the minimal action required so that the state process is kept outside the interior of the set defined by $w'(x) = k(x)$ at all times.

We are going to establish that, in all of the problems that we consider, the optimal strategy takes the following qualitative form. There exists a point β in the state space $]0,\infty[$ such that it is optimal to push in an impulsive way the state process down to level β if the initial state x is strictly greater than β and otherwise take minimal action so that the state process X is kept inside the set $]0,\beta]$ at all times, which amounts at reflecting X in β in the negative direction. In view of the discussion in the previous paragraph, the optimality of such a strategy is associated with a solution w to the HJB equation (3.1) such that

$$\begin{aligned}\frac{1}{2}\sigma^2(x)w''(x)+b(x)w'(x)-r(x)w(x)+h(x)=0, &\quad \text{for } x\in\,]0,\beta[,\\ w'(x)=k(x), &\quad \text{for } x\in[\beta,\infty[,\end{aligned}$$

and a solution (w,λ) to the HJB equation (3.2) such that

$$\begin{aligned}\frac{1}{2}\sigma^2(x)w''(x)+b(x)w'(x)+h(x)-\lambda=0, &\quad \text{for } x\in\,]0,\beta[,\\ w'(x)=k(x), &\quad \text{for } x\in[\beta,\infty[.\end{aligned}$$

In both cases, we will determine the free-boundary point β using the so-called "smooth pasting condition" of singular stochastic control, which requires that w should be C^2, in particular, at the free-boundary point β. This condition suggests the free-boundary equations

$$\lim_{x\uparrow\beta} w'(x)=k(\beta) \quad\text{and}\quad \lim_{x\uparrow\beta} w''(x)=k'(\beta). \tag{3.5}$$

4. The solution to the discounted harvesting problem. It is well-known that, in the presence of Assumptions 2.1, 2.2, 2.5.(i) and 2.6.(i)-(ii), every solution to the ODE (3.3) is given by

$$w(x)=A\varphi(x)+B\psi(x)+R_h(x), \tag{4.1}$$

for some constants $A,B\in\mathbb{R}$. The functions $\varphi,\psi:\,]0,\infty[\,\to\mathbb{R}$ are C^2, unique up to a multiplicative constant and satisfying

$$\begin{aligned}0<\varphi(x) \quad\text{and}\quad \varphi'(x)<0 \quad \text{for all } x>0,\\ 0<\psi(x) \quad\text{and}\quad \psi'(x)>0 \quad \text{for all } x>0\\ \text{and}\quad \lim_{x\downarrow 0}\varphi(x)=\lim_{x\uparrow\infty}\psi(x)=\infty,\end{aligned}$$

as well as

$$\varphi(x)\psi'(x)-\varphi'(x)\psi(x)=\big(\varphi(1)\psi'(1)-\varphi'(1)\psi(1)\big)p'(x)=:Cp'(x), \tag{4.2}$$

$$\frac{\varphi(y)}{\varphi(x)}=\mathsf{E}_y\big[e^{-\Lambda_{T_x}}\big] \quad\text{and}\quad \frac{\psi(x)}{\psi(y)}=\mathsf{E}_x\big[e^{-\Lambda_{T_y}}\big], \tag{4.3}$$

for all $0<x\le y$, where Λ is defined by (1.5) for X^ζ being the solution X to the SDE (2.1) and T_y is the first hitting time of $\{y\}$ by X (e.g., see Borodin and Salminen [6, II.10]). The function $R_h:\,]0,\infty[\,\to\mathbb{R}$ is C^2 and admits the expressions

$$\begin{aligned}R_h(x)&=\mathsf{E}_x\left[\int_0^\infty e^{-\Lambda_t}h(X_t)\,\mathrm{d}t\right]\\ &=\varphi(x)\int_0^x h(s)\Psi(s)\,\mathrm{d}s+\psi(x)\int_x^\infty h(s)\Phi(s)\,\mathrm{d}s,\end{aligned} \tag{4.4}$$

in which, X is the solution to the SDE (2.1),

$$\Phi(x) = \frac{2\varphi(x)}{C\sigma^2(x)p'(x)} = \frac{1}{C}\varphi(x)\frac{m(\mathrm{d}x)}{\mathrm{d}x} \tag{4.5}$$

$$\text{and} \quad \Psi(x) = \frac{2\psi(x)}{C\sigma^2(x)p'(x)} = \frac{1}{C}\psi(x)\frac{m(\mathrm{d}x)}{\mathrm{d}x}. \tag{4.6}$$

In view of the definition (2.5) of Θ, the conditions in Assumptions 2.6.(iii) imply that

$$R_h(x) = R_{\Theta+h}(x) + \int_0^x k(s)\,\mathrm{d}s - \Theta_\infty\psi(x), \tag{4.7}$$

where $R_{\Theta+h}$ is given by (4.4) with $\Theta + h$ in place of h and

$$\Theta_\infty = \lim_{x\uparrow\infty} \frac{1}{\psi(x)}\int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+. \tag{4.8}$$

All of these claims follow from the results in Lamberton and Zervos [18, Section 4] (see also Liu and Zervos [19, Section 3]).

In (4.1), we choose $A = 0$ because, otherwise, a solution to the HJB equation (3.1) of the form we have discussed in the previous section cannot satisfy the so-called "transversality condition", which is required for a solution w to the HJB equation to identify with the control problem's value function. In view of this observation and the identity (4.7), we consider the solution to the ODE (3.3) that is given by

$$w(x) = R_{\Theta+h}(x) + \int_0^x k(s)\,\mathrm{d}s + (B - \Theta_\infty)\psi(x),$$

for some constant $B \in \mathbb{R}$. Substituting B for its unique choice for which this function satisfies the first boundary condition in (3.5), we obtain

$$\begin{aligned} w(x) &= \int_0^x k(s)\,\mathrm{d}s + R_{\Theta+h}(x) - \frac{R'_{\Theta+h}(\beta)}{\psi'(\beta)}\psi(x) \\ &= R_h(x) + \left(\Theta_\infty - \frac{R'_{\Theta+h}(\beta)}{\psi'(\beta)}\right)\psi(x), \quad \text{for } x \in \,]0, \beta]. \end{aligned} \tag{4.9}$$

Furthermore, this function satisfies the second boundary condition in (3.5) if and only if $\beta > 0$ is a solution to the algebraic equation

$$R''_{\Theta+h}(\beta) - \frac{R'_{\Theta+h}(\beta)}{\psi'(\beta)}\psi''(\beta) = 0. \tag{4.10}$$

Lemma 4.1. *Consider the discounted control problem formulated in Section 2. There exists a unique point $\beta^\star \in \,]\xi, \infty[$ satisfying equation (4.10), which is equivalent to*

$$\int_0^\beta \big(\Theta(s) + h(s)\big)\Psi(s)\,\mathrm{d}s = \big(\Theta(\beta) + h(\beta)\big)\frac{1}{r(\beta)}\int_0^\beta r(s)\Psi(s)\,\mathrm{d}s. \tag{4.11}$$

Furthermore, the function w defined by

$$w(x) = \begin{cases} R_h(x) + \left(\Theta_\infty - \frac{R'_{\Theta+h}(\beta^\star)}{\psi'(\beta^\star)}\right)\psi(x), & \text{for } x \in \,]0, \beta^\star], \\ w(\beta^\star) + \int_{\beta^\star}^x k(s)\,\mathrm{d}s, & \text{for } x \in \,]\beta^\star, \infty[, \end{cases} \tag{4.12}$$

is a C^2 solution to the HJB equation (3.1) that is bounded from below.

Proof. In the presence of the assumptions that we have made, Lemma 4 in Liu and Zervos [19] proves that that there exists a unique point $\beta^\star \in \,]\xi, \infty[$ (denoted by $\underline{x}$ in that lemma) such that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_{\Theta+h}(x)}{\psi'(x)} \begin{cases} < 0 & \text{for all } x \in \,]0, \beta^\star[, \\ > 0 & \text{for all } x \in \,]\beta^\star, \infty[, \end{cases} \quad \text{and} \quad \lim_{x \uparrow \infty} \frac{R'_{\Theta+h}(x)}{\psi'(x)} = 0. \tag{4.13}$$

The first set of these inequalities and the continuity of the derivative $(R'_{\Theta+h}/\psi')'$ imply that $\beta^\star$ is the unique solution to equation (4.10).

To establish the equivalence of (4.10) and (4.11), we use the definitions (4.4)–(4.6) to see that (4.10) is equivalent to

$$\int_0^\beta \big(\Theta(s) + h(s)\big) \Psi(s) \, \mathrm{d}s = \big(\Theta(\beta) + h(\beta)\big) \frac{\psi'(\beta)}{C r(\beta) p'(\beta)}.$$

In the presence of Assumption 2.2, 0 is an inaccessible boundary point of the solution X to the SDE (2.1), therefore, $\lim_{x \downarrow 0} \psi'(x)/p'(x) = 0$ (e.g., see Borodin and Salminen [6, Section II.10]). Combining this limit with the calculation

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\psi'(x)}{p'(x)} = \frac{2}{\sigma^2(x) p'(x)} \left(\frac{1}{2} \sigma^2(x) \psi''(x) + b(x) \psi'(x) \right) = \frac{2 r(x) \psi(x)}{\sigma^2(x) p'(x)} = C r(x) \Psi(x),$$

we obtain

$$\int_0^x r(s) \Psi(s) \, \mathrm{d}s = \frac{\psi'(x)}{C p'(x)} \quad \text{for all } x > 0, \tag{4.14}$$

and the equivalence of (4.10) and (4.11) follows.

The assumption that h is bounded from below implies that

$$\inf_{x>0} R_h(x) = \inf_{x>0} \mathsf{E}_x \left[\int_0^\infty e^{-\Lambda_t} h(X_t) \, \mathrm{d}t \right] \geq \inf_{x>0} h(x) \, \mathsf{E}_x \left[\int_0^\infty e^{-\Lambda_t} \, \mathrm{d}t \right] > -\infty. \tag{4.15}$$

Combining this observation with the fact that $\Theta_\infty \in \mathbb{R}_+$ (see the remark after (4.8)) and the positivity of k, we can see that w is bounded from below.

By construction, we will establish all of the lemma's other claims if we prove that

$$k(x) - w'(x) \leq 0 \quad \text{for all } x \in \,]0, \beta^\star[\tag{4.16}$$

$$\text{and} \quad \frac{1}{2} \sigma^2(x) w''(x) + b(x) w'(x) - r(x) w(x) + h(x) \leq 0 \quad \text{for all } x \in \,]\beta^\star, \infty[. \tag{4.17}$$

The inequality (4.16) holds true because the first expression of w in (4.9) and (4.13) imply that

$$k(x) - w'(x) = \psi'(x) \left(\frac{R'_{\Theta+h}(\beta^\star)}{\psi'(\beta^\star)} - \frac{R'_{\Theta+h}(x)}{\psi'(x)} \right) < 0 \quad \text{for all } x \in \,]0, \beta^\star[.$$

To show (4.17), we first use the expression of w given by (4.12) for $x > \beta^\star$, the definition the definition (2.5) of Θ and the first expression in (4.9) to calculate

$$\begin{aligned} &\frac{1}{2} \sigma^2(x) w''(x) + b(x) w'(x) - r(x) w(x) + h(x) \\ &\qquad = \Theta(x) + h(x) - r(x) \left(w(\beta^\star) - \int_0^{\beta^\star} k(s) \, \mathrm{d}s \right) \\ &\qquad = \Theta(x) + h(x) - r(x) G_{\Theta+h}(\beta^\star), \end{aligned} \tag{4.18}$$

where

$$G_{\Theta+h}(x) = R_{\Theta+h}(x) - \frac{R'_{\Theta+h}(x)}{\psi'(x)}\psi(x) = \frac{Cp'(x)}{\psi'(x)} \int_0^x \big(\Theta(s) + h(s)\big)\Psi(s)\,\mathrm{d}s. \tag{4.19}$$

In view of the calculations

$$\begin{aligned} G'_{\Theta+h}(x) &= -\psi(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{R'_{\Theta+h}(x)}{\psi'(x)} \\ &= -\frac{2Cr(x)p'(x)\psi(x)}{(\sigma(x)\psi'(x))^2}\left(\int_0^x \big(\Theta(s) + h(s)\big)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x) + h(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)}\right), \end{aligned}$$

and the inequalities (4.13), we can see that

$$G_{\Theta+h}(x) < G_{\Theta+h}(\beta^\star) \tag{4.20}$$

$$\text{and} \quad \int_0^x \big(\Theta(s) + h(s)\big)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x) + h(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)} \tag{4.21}$$

for all $x > \beta^\star$. The inequality (4.21) and the second expression of $G_{\Theta+h}$ in (4.19) imply that

$$G_{\Theta+h}(x) = \frac{Cp'(x)}{\psi'(x)} \int_0^x \big(\Theta(s) + h(s)\big)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x) + h(x)}{r(x)} \quad \text{for all } x > \beta^\star.$$

However, this result, (4.18) and the inequality (4.20) yield

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) < r(x)\left(\frac{\Theta(x) + h(x)}{r(x)} - G_{\Theta+h}(x)\right) < 0$$

for all $x > \beta^\star$, and (4.17) follows. □

THEOREM 4.2. *Consider the discounted control problem formulated in Section 2. If the point* $\beta^\star \in \,]\xi, \infty[$ *and the function* w *are as in Lemma 4.1, then*

$$w(x) = \sup_{\zeta \in \mathcal{A}} J_x(\zeta) \quad \textit{for all } x > 0, \tag{4.22}$$

while the harvesting strategy $\zeta^\star \in \mathcal{A}$ *that has a jump of size* $\Delta\zeta^\star_0 = (x - \beta^\star)^+$ *at time 0 and then reflects the state process* $X^\star$ *at the level* $\beta^\star$ *in the negative direction is optimal.*

Proof. Fix any initial value $x > 0$, consider any admissible controlled process $\zeta \in \mathcal{A}$ and denote by X^ζ the associated solution to the SDE (1.1). Using Itô's formula, we obtain

$$\begin{aligned} &e^{-\Lambda^\zeta_T} w(X^\zeta_T) \\ &\quad = w(x) + \int_0^T e^{-\Lambda^\zeta_t}\left(\frac{1}{2}\sigma^2(X^\zeta_t)w''(X^\zeta_t) + b(X^\zeta_t)w'(X^\zeta_t) - r(X^\zeta_t)w(X^\zeta_t)\right)\mathrm{d}t + M^\zeta_T \\ &\qquad - \int_{[0,T]} e^{-\Lambda^\zeta_t} w'(X^\zeta_{t-})\,\mathrm{d}\zeta_t + \sum_{0 \le t \le T} e^{-\Lambda^\zeta_t}\Big(w(X^\zeta_t) - w(X^\zeta_{t-}) - w'(X^\zeta_{t-})\,\Delta X^\zeta_t\Big), \end{aligned}$$

where

$$M^\zeta_T = \int_0^T e^{-\Lambda^\zeta_t}\sigma(X^\zeta_t)w'(X^\zeta_t)\,\mathrm{d}W_t.$$

Since $\Delta X_t^\zeta = X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \leq 0$, we can see that

$$w(X_t^\zeta) - w(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u = \int_{X_t^\zeta}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u.$$

In view of the facts that ζ^c is an increasing process, $X_t^\zeta < X_{t-}^\zeta$ and w satisfies the HJB equation (3.1), we can see that these observations imply that

$$\begin{aligned}
&\int_0^T e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \int_0^T e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t \\
&\qquad = w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + M_T^\zeta \\
&\qquad\quad + \int_0^T e^{-\Lambda_t^\zeta} \left(\frac{1}{2}\sigma^2(X_t^\zeta) w''(X_t^\zeta) + b(X_t^\zeta) w'(X_t^\zeta) - r(X_t^\zeta) w(X_t^\zeta) + h(X_t^\zeta)\right) \mathrm{d}t \\
&\qquad\quad + \int_0^T e^{-\Lambda_t^\zeta} \big(k(X_t^\zeta) - w'(X_t^\zeta)\big)\,\mathrm{d}\zeta_t^c + \sum_{0\leq t\leq T} e^{-\Lambda_t^\zeta} \int_{X_t^\zeta}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u \\
&\qquad \leq w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + M_T^\zeta.
\end{aligned} \tag{4.23}$$

We next consider any sequence (τ_n) of bounded localising stopping times for the local martingale M^ζ. Recalling that h and w are both bounded from below, k is positive and ζ is an increasing process, we use the dominated and the monotone convergence theorems to observe that (4.23) implies that

$$\begin{aligned}
I_x(\zeta) &= \lim_{n\uparrow\infty} \mathsf{E}_x\left[\int_0^{\tau_n} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \int_0^{\tau_n} e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t\right] \\
&\leq \lim_{n\uparrow\infty} \mathsf{E}_x\Big[w(x) + e^{-\Lambda_{\tau_n}^\zeta} w^-(X_{\tau_n}^\zeta)\Big] = w(x),
\end{aligned} \tag{4.24}$$

where $w^-(x) = -\min\big\{0, w(x)\big\}$.

Consider the harvesting strategy $\zeta^\star \in \mathcal{A}$ that is as in the statement of the theorem: such a strategy indeed exists (see Tanaka [27, Theorem 4.1]). This strategy is such that (4.23) holds true with equality, namely,

$$\int_0^T e^{-\Lambda_t^\star} h(X_t^\star)\,\mathrm{d}t + \int_0^T e^{-\Lambda_t^\star} k(X_t^\star) \circ \mathrm{d}\zeta_t^\star = w(x) - e^{-\Lambda_T^\star} w(X_T^\star) + M_T^\star.$$

Furthermore, the processes $h(X^\star)$, $k(X^\star)$ and $w(X^\star)$ are all bounded because $X_t^\star \in\,]0, \beta^\star]$ for all $t > 0$. In view of these observations, we can use the dominated convergence theorem to obtain

$$\begin{aligned}
I_x(\zeta^\star) &= \lim_{n\uparrow\infty} \mathsf{E}_x\left[\int_0^{\tau_n^\star} e^{-\Lambda_t^\star} h(X_t^\star)\,\mathrm{d}t + \int_0^{\tau_n^\star} e^{-\Lambda_t^\star} k(X_t^\star) \circ \mathrm{d}\zeta_t^\star\right] \\
&= \lim_{n\uparrow\infty} \mathsf{E}_x\Big[w(x) - e^{-\Lambda_{\tau_n^\star}^\star} w\big(X_{\tau_n^\star}^\star\big)\Big] = w(x),
\end{aligned}$$

where $(\tau_n^\star)$ is a sequence of bounded localising stopping times for the local martingale $M^\star$. However, these identities and (4.24) imply (4.22) as well as the optimality of $\zeta^\star$. ∎

5. The solution to the ergodic problem's HJB equation. In view of (2.7) in Remark 2.8, we can verify that a solution to the ODE (3.4) is given by

$$w'(x) = w'(x;\lambda) = p'(x) \int_0^x \big(\lambda - h(s)\big)\, m(\mathrm{d}s) = k(x) - p'(x)\Xi(x,\lambda), \tag{5.1}$$

where

$$\Xi(x,\lambda) = \int_0^x \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s). \tag{5.2}$$

This function satisfies the boundary conditions (3.5) if and only if

$$\Xi(\beta,\lambda) = 0 \quad \text{and} \quad K(\beta) + h(\beta) - \lambda = 0. \tag{5.3}$$

In the next result, we derive a unique solution to the system of equations in (5.3) as well as a solution to the HJB equation (3.2). It turns out that the solution to the HJB equation (3.2) may be unbounded from below (see Remark 5.3 at the end of the section), which gives rise to a non-trivial complication in the verification arguments we use for the proof of Theorem 6.1. The introduction of the auxiliary function w_λ in part (III) of the next result provides a way to overcome this problem. To overcome the same complication, Cohen, Hening and Sun [8] followed a similar but much more complex approach in the presence of stronger assumptions on the problem data. In particular, the auxiliary function that they construct for the proof of their Proposition 3.4 is C^2. In contrast, the function w_λ that we consider is C^1 but loses the C^2 regularity at two points.

PROPOSITION 5.1. *In the presence of Assumptions 2.1, 2.2, 2.5 and 2.7, the following statements hold true:*
(I) *There exists a unique pair $(\beta^\star, \lambda^\star)$ with $\beta^\star > 0$ satisfying the system of equations (5.3). This pair is such that*

$$K(0) + h(0) < \lambda^\star = \frac{1}{m\big(]0,\beta^\star[\big)} \int_0^{\beta^\star} \big(K(s) + h(s)\big)\, m(\mathrm{d}s) < \overline{\lambda} \text{ and } \beta^\star = \varrho(\lambda^\star), \tag{5.4}$$

where $\overline{\lambda}$ and ϱ are given by (2.8) and (2.9).
(II) *The unique, modulo an additive constant, function w that is defined by*

$$w'(x) = \begin{cases} k(x) - p'(x)\Xi(x,\lambda^\star), & \text{for } x \in \,]0,\beta^\star[, \\ k(x), & \text{for } x \geq \beta^\star, \end{cases} \tag{5.5}$$

is a C^2 solution to the HJB equation (3.2).
(III) *Given any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$, there exists a point $\alpha(\lambda) \in \big]0, \mathsf{\rho}(\lambda)\big[$, where $\mathsf{\rho}$ is as in (2.12), such that the unique, modulo an additive constant, function w_λ that is defined by*

$$w'_\lambda(x) = \int_{\alpha(\lambda)}^x \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s), \quad \text{for } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \tag{5.6}$$

$$\text{and} \quad w'_\lambda(x) = k(x), \quad \text{for } x \in \big]0, \alpha(\lambda)\big] \cup \big[\varrho(\lambda), \infty\big[, \tag{5.7}$$

is C^1 in $\mathbb{R}_+$ and C^2 in $\mathbb{R}_+ \setminus \big\{\alpha(\lambda), \varrho(\lambda)\big\}$,

$$\lim_{\lambda \downarrow \lambda^\star} \alpha(\lambda) = 0, \quad \lim_{\lambda \downarrow \lambda^\star} \varrho(\lambda) = \beta^\star \quad \text{and} \quad \lim_{\lambda \downarrow \lambda^\star} w'_\lambda(x) = w'(x) \text{ for all } x > 0. \tag{5.8}$$

Furthermore, this function is such that

$$w'_\lambda(x) \geq k(x), \quad \text{if } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \tag{5.9}$$

$$\frac{1}{2}\sigma^2(x)w''_\lambda(x) + b(x)w'_\lambda(x) + h(x) - \lambda \begin{cases} = 0, & \text{if } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \\ < 0, & \text{if } x \in \big]0, \alpha(\lambda)\big[\cup \big]\varrho(\lambda), \infty\big[, \end{cases} \tag{5.10}$$

$$\text{and} \quad \big|w'_\lambda(x)\big| \leq C_2 \quad \text{for all } x > 0, \tag{5.11}$$

for some constant $C_2 = C_2(\lambda) > 0$.

Proof. We develop the proof in four steps.

Preliminary results. Given any $\beta > 0$, the definition (5.2) of Ξ implies that

$$\Lambda(\beta) = \frac{1}{m\big(]0, \beta[\big)} \int_0^\beta \big(K(s) + h(s)\big)\, m(\mathrm{d}s)$$

is the unique solution to the equation $\Xi(\beta, \lambda) = 0$. In light of Assumption 2.7.(ii) (see also Figure 1), a straightforward inspection of the definition of Ξ reveals that this solution is such that one of the following two cases holds true:

$$\begin{aligned} &\text{(i) } K(0) + h(0) < \Lambda(\beta) < \overline{\lambda} \\ \text{or} \quad &\text{(ii) } \underline{\lambda} < \Lambda(\beta) \leq K(0) + h(0) \text{ and } \varrho\big(\Lambda(\beta)\big) < \beta, \end{aligned} \tag{5.12}$$

where $\underline{\lambda} < \overline{\lambda}$ are defined by (2.8) and ϱ is introduced by (2.9). In particular, we note that $\Lambda(\beta) \in]\underline{\lambda}, \overline{\lambda}[$, which is the domain of the function ϱ.

Differentiating the identity

$$\Xi\big(\beta, \Lambda(\beta)\big) = 0, \quad \text{for } \beta > 0, \tag{5.13}$$

which defines Λ, with respect to β, we calculate

$$\Lambda'(\beta) = \frac{2}{\sigma^2(\beta)p'(\beta)m\big(]0, \beta[\big)} \big(K(\beta) + h(\beta) - \Lambda(\beta)\big).$$

On the other hand, differentiating the identity

$$K\Big(\varrho\big(\Lambda(\beta)\big)\Big) + h\Big(\varrho\big(\Lambda(\beta)\big)\Big) - \Lambda(\beta) = 0,$$

which follows from (2.9), with respect to β, we derive the expression

$$\Lambda'(\beta) = \Big(K'\Big(\varrho\big(\Lambda(\beta)\big)\Big) + h'\Big(\varrho\big(\Lambda(\beta)\big)\Big)\Big) \frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\Lambda(\beta)\big).$$

Combining these calculations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\Lambda(\beta)\big) = \frac{2\big(K(\beta) + h(\beta) - \Lambda(\beta)\big)}{\sigma^2(\beta)p'(\beta)m\big(]0, \beta[\big)\Big(K'\Big(\varrho\big(\Lambda(\beta)\big)\Big) + h'\Big(\varrho\big(\Lambda(\beta)\big)\Big)\Big)}.$$

In view of this result and the inequality

$$K'\Big(\varrho\big(\Lambda(\beta)\big)\Big) + h'\Big(\varrho\big(\Lambda(\beta)\big)\Big) < 0 \quad \text{for all } \beta > 0,$$

which follows from (2.11), we can see that

$$\operatorname{sgn}\Big(\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\Lambda(\beta)\big)\Big) = -\operatorname{sgn}\Big(K(\beta)+h(\beta)-\Lambda(\beta)\Big) \quad \text{for all } \beta>0, \tag{5.14}$$

where sgn is the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|}, & \text{for } x\neq 0, \\ 0, & \text{for } x=0. \end{cases}$$

Proof of (I). In view of (5.13), we can see that there exists a pair $(\beta^\star,\lambda^\star)$ with $\beta^\star>0$ satisfying the system of equations (5.3) if and only if

$$K(\beta^\star)+h(\beta^\star)-\lambda^\star=0 \quad \text{and} \quad \lambda^\star=\Lambda(\beta^\star). \tag{5.15}$$

The structure of the function $K+h$, which we have discussed in Remark 2.9 (see also Figure 1), implies that there exists no $\beta^\star$ satisfying (5.15) if $\Lambda(\beta^\star)$ is as in case (ii) of (5.12). We therefore need to show that there exists $\beta^\star>0$ such that, if we define $\lambda^\star=\Lambda(\beta^\star)$, then $\lambda^\star$ satisfies the inequalities (5.4), and

$$\text{either} \quad \beta^\star=\mathsf{p}\big(\Lambda(\beta^\star)\big)=\mathsf{p}(\lambda^\star) \quad \text{or} \quad \beta^\star=\varrho\big(\Lambda(\beta^\star)\big)=\varrho(\lambda^\star), \tag{5.16}$$

where ϱ, p are as in (2.9), (2.12). Furthermore, the resulting solution $(\beta^\star,\lambda^\star)$ to the system of equations (5.3) is unique if and only if only one of the two equations in (5.16) has a unique solution and the other one has no solution.

If the equation $\beta=\mathsf{p}\big(\Lambda(\beta)\big)$ had a solution $\beta^\star>0$, then (2.13) would imply that

$$K(s)+h(s)-\Lambda(\beta^\star)<0 \quad \text{for all } s<\mathsf{p}\big(\Lambda(\beta^\star)\big)=\beta^\star,$$

which would contradict the identity

$$\Xi\big(\beta^\star,\Lambda(\beta^\star)\big)=\int_0^{\beta^\star}\big(K(s)+h(s)-\Lambda(\beta^\star)\big)\,m(\mathrm{d}s)=0.$$

To establish part (I) of the theorem, we therefore have to prove that there exists a unique point $\beta^\star>0$ such that

$$\Lambda(\beta^\star)\in\,\big]K(0)+h(0),\,\overline{\lambda}\big[\quad \text{and} \quad \beta^\star=\varrho\big(\Lambda(\beta^\star)\big). \tag{5.17}$$

To prove that there exists a unique $\beta^\star>0$ satisfying (5.17), we first observe that the inequality in (2.9) implies that

$$\beta<\varrho\big(\Lambda(\beta)\big) \quad \text{for all } \beta\leq\xi. \tag{5.18}$$

We next argue by contradiction and we assume that there is no $\beta^\star>0$ satisfying the equation in (5.17). In view of (5.18) and the continuity of the functions ϱ and Λ, we can see that such an assumption implies that

$$\beta<\varrho\big(\Lambda(\beta)\big) \quad \text{for all } \beta>\xi. \tag{5.19}$$

In turn, this inequality and (2.10) imply that

$$K(\beta)+h(\beta)-\Lambda(\beta)>0 \quad \text{for all } \beta>\xi.$$

Combining this observation with (5.14), we obtain $\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\Lambda(\beta)\big) < 0$ for all $\beta > \xi$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big(\beta - \varrho\big(\Lambda(\beta)\big)\Big) > 1 \quad \text{for all } \beta > \xi,$$

which contradicts (5.19). It follows that there exists $\beta^\star > 0$ satisfying the equation in (5.17).

To see that the solution $\beta^\star > \xi$ to the equation in (5.17) is indeed unique, we note that (2.9) implies that $K(\beta) + h(\beta) - \Lambda(\beta) = 0$ for all $\beta > \xi$ such that $\beta = \varrho\big(\Lambda(\beta)\big)$. This observation and (5.14) imply

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\big(\beta - \varrho(\beta)\big) = 1 \quad \text{for all } \beta > \xi \text{ such that } \beta = \varrho\big(\Lambda(\beta)\big).$$

Based on this result, we can develop a simple contradiction argument to show that the equation in (5.17) has at most one solution $\beta^\star > \xi$.

We conclude this part of the proof by noting that the first statement in (5.17) can be seen by a straightforward inspection of the equation (5.2) that $\big(\beta^\star, \Lambda(\beta^\star)\big)$ satisfies in light of the identity in (5.17) and Figure 1.

Proof of (II). By construction, we will show that the function w given by (5.5) is a C^2 solution to the HJB equation (3.2) if we prove that

$$w'(x) \geq k(x) \quad \text{for all } x \in \,]0, \beta^\star[$$
$$\text{and} \quad \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda^\star \leq 0 \quad \text{for all } x \in \,]\beta^\star, \infty[.$$

In view of the identities $\lambda^\star = \Lambda(\beta^\star)$ and $\beta^\star = \varrho\big(\Lambda(\beta^\star)\big)$, the second of these inequalities is equivalent to

$$K(x) + h(x) - \Lambda(\beta^\star) \leq 0 \quad \text{for all } x > \beta^\star = \varrho\big(\Lambda(\beta^\star)\big),$$

which is true thanks to (2.10). On the other hand, the first of these inequalities follows immediately from the expression of w' in (5.5) and the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s) \begin{cases} < 0 & \text{for all } x \in \big]0, \rho(\lambda^\star)\big[, \\ > 0 & \text{for all } x \in \big]\rho(\lambda^\star), \beta^\star\big[, \end{cases}$$

which hold true thanks to the identities $\lambda^\star = \Lambda(\beta^\star)$ and $\beta^\star = \varrho\big(\Lambda(\beta^\star)\big)$, the inequalities in (5.4) and Assumption 2.7.(ii) (see also Figure 1).

Proof of (III). Fix any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$. In view of Assumption 2.7.(ii) and the properties of the functions ϱ, ρ in (2.9), (2.12), we can see that

$$\int_0^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) < \int_0^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s)$$
$$< \int_0^{\beta^\star} \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s) = 0$$
$$\text{and} \quad \int_{\rho(\lambda)}^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) > 0,$$

which imply that there exists a unique point $\alpha(\lambda) \in \left]0, \mathsf{\rho}(\lambda)\right[$ such that

$$\int_{\alpha(\lambda)}^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) = 0.$$

For this choice of $\alpha(\lambda)$, we can see that the function w'_λ defined by (5.6) and (5.7) is indeed C^1. In particular, the limits in (5.8) all hold true. Furthermore,

$$\int_{\alpha(\lambda)}^{x} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) < 0 \quad \text{for all } x \in \left]\alpha(\lambda), \varrho(\lambda)\right[. \tag{5.20}$$

The inequality (5.9) follows immediately from (5.20). On the other hand, it is straightforward to check that the function w'_λ defined by (5.6) and (5.7) satisfies the equality in (5.10), while the inequality in (5.10) is equivalent to

$$K(x) + h(x) - \lambda < 0, \quad \text{for } x \in \left]0, \alpha(\lambda)\right[\cup \left]\varrho(\lambda), \infty\right[,$$

which is true thanks to the inequalities (2.10), (2.13) and the fact that $\alpha(\lambda) \in \left]0, \mathsf{\rho}(\lambda)\right[$.

Finally, (5.11) follows immediately from the continuity of w'_λ and the boundedness of k. □

Remark 5.2. The model studied by Alvarez and Hening [3] is the special case that arises when $h = 0$ and $k = 1$. In this case, the identity (2.7) in Remark 2.8 impies that

$$\int_0^x K(s)\, m(\mathrm{d}s) = \int_0^x b(s)\, m(\mathrm{d}s) = \frac{1}{p'(x)}. \tag{5.21}$$

In view of this identity, we can see that the system of equations in (5.3), which determines $(\beta^\star, \lambda^\star)$, and (5.5) reduce to

$$\lambda = b(\beta), \quad \lambda = \frac{1}{p'(\beta) m\big(]0, \beta[\big)} \quad \text{and} \quad w'(x) = \begin{cases} \lambda^\star p'(x) m\big(]0, x[\big), & \text{for } x \in]0, \beta^\star[, \\ 1, & \text{for } x \geq \beta^\star, \end{cases}$$

which are precisely the expressions (8) and (9) in Alvarez and Hening [3].

Remark 5.3. Consider the function w' defined by (5.5) and suppose that X is as in Example 2.10. Using L'Hôpital's formula, we calculate

$$\begin{aligned} \lim_{x \downarrow 0} \big(x w'(x)\big) &= -\lim_{x \downarrow 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\Big(x \int_0^x \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s)\Big)}{\frac{\mathrm{d}}{\mathrm{d}x}\big(1/p'(x)\big)} \\ &\geq -\lim_{x \downarrow 0} \frac{K(x) + h(x) - \lambda^\star}{\mathsf{\kappa}(\mathsf{\gamma} - x)} = \frac{\lambda^\star - K(0) - h(0)}{\mathsf{\kappa}\mathsf{\gamma}} > 0. \end{aligned}$$

It follows that, in the context of Example 2.10, $\lim_{x \downarrow 0} w(x) = -\infty$.

6. The solution to the ergodic harvesting problem.

THEOREM 6.1. *Consider the ergodic control problems formulated in Section 2, and let $(\beta^\star, \lambda^\star)$ be as in Proposition 5.1. Given any $x > 0$, the following statements hold true:*

(I) $J_x^{\mathrm{e}}(\zeta) \leq \lambda^\star$ *and* $J_x^{\mathrm{p}}(\zeta) \leq \lambda^\star$ *for all admissible harvesting strategies* $\zeta \in \mathcal{A}$.

(II) *If $\zeta^\star \in \mathcal{A}$ is the harvesting strategy that has a jump of size $\Delta\zeta_0^\star = (x-\beta^\star)^+$ at time 0 and then reflects the state process $X^\star$ at the level $\beta^\star$ in the negative direction, then*

$$J_x^{\mathrm{e}}(\zeta^\star) = \lim_{T\uparrow\infty} \frac{1}{T} \mathsf{E}\biggl[\int_0^T h(X_t^\star)\,\mathrm{d}t + \int_0^T k(X_t^\star)\circ \mathrm{d}\zeta_t^\star\biggr] = \lambda^\star$$

$$\textit{and} \quad J_x^{\mathrm{p}}(\zeta^\star) = \lim_{T\uparrow\infty} \frac{1}{T} \biggl(\int_0^T h(X_t^\star)\,\mathrm{d}t + \int_0^T k(X_t^\star)\circ \mathrm{d}\zeta_t^\star\biggr) = \lambda^\star.$$

Proof. Fix any initial state $x>0$, let $\zeta\in\mathcal{A}$ be any admissible harvesting strategy and let X be the associated solution to the SDE (1.1). Also, consider the function w_λ defined by (5.6) and (5.7) for $\lambda \in \,]\lambda^\star, \overline{\lambda}[$. Using Itô's formula, we calculate

$$\begin{aligned} w_\lambda(X_T^\zeta) = w_\lambda(x) &+ \int_0^T \biggl(\frac{1}{2}\sigma^2(X_t^\zeta)w_\lambda''(X_t^\zeta) + b(X_t^\zeta)w_\lambda'(X_t^\zeta)\biggr)\,\mathrm{d}t - \int_{[0,T]} w_\lambda'(X_{t-}^\zeta)\,\mathrm{d}\zeta_t \\ &+ \sum_{0\le t\le T} \bigl(w_\lambda(X_t^\zeta) - w_\lambda(X_{t-}^\zeta) - w_\lambda'(X_{t-}^\zeta)\,\Delta X_t^\zeta\bigr) + M_T^\zeta, \end{aligned}$$

where

$$M_T^{\lambda,\zeta} = \int_0^T \sigma(X_t^\zeta)w_\lambda'(X_t^\zeta)\,\mathrm{d}W_t.$$

Since $\Delta X_t^\zeta = X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \le 0$ and

$$w_\lambda(X_t^\zeta) - w_\lambda(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u = \int_{X_t^\zeta}^{X_{t-}^\zeta} \bigl(k(u) - w_\lambda'(u)\bigr)\,\mathrm{d}u,$$

it follows that

$$\begin{aligned} &\int_0^T h(X_t^\zeta)\,\mathrm{d}t + \int_0^T k(X_t^\zeta)\circ\mathrm{d}\zeta_t \\ &\qquad = \lambda T + w_\lambda(x) - w_\lambda(X_T^\zeta) \\ &\qquad\quad + \int_0^T \biggl(\frac{1}{2}\sigma^2(X_t^\zeta)w_\lambda''(X_t^\zeta) + b(X_t^\zeta)w_\lambda'(X_t^\zeta) + h(X_t^\zeta) - \lambda\biggr)\mathrm{d}t \\ &\qquad\quad + \int_0^T \bigl(k(X_t^\zeta) - w_\lambda'(X_t^\zeta)\bigr)\,\mathrm{d}\zeta_t^c + \sum_{0\le t\le T} \int_{X_t^\zeta}^{X_{t-}^\zeta} \bigl(k(u) - w_\lambda'(u)\bigr)\,\mathrm{d}u + M_T^{\lambda,\zeta}. \end{aligned}$$

Since ζ^c is an increasing process, $X_t^\zeta < X_{t-}^\zeta$ and the pair $(w_{\lambda^\star}, \lambda^\star)$ (resp., (w_λ, λ)) satisfies the HJB equation (3.2) (resp., the inequalities (5.9) and (5.10)), we can see that

$$\text{(6.1)} \qquad \int_0^T h(X_t^\zeta)\,\mathrm{d}t + \int_0^T k(X_t^\zeta)\circ\mathrm{d}\zeta_t \le \lambda T + w_\lambda(x) - w_\lambda(X_T^\zeta) + M_T^{\lambda,\zeta}.$$

Proof of the inequality $J_x^{\mathrm{e}}(\zeta) \le \lambda^\star$. Fix any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$ and let (τ_n) be a sequence of localising times for the corresponding local martingale $M^{\lambda,\zeta}$. Recalling the assumptions that h is bounded from below and k is positive, as well as the facts that ζ is an increasing process and w_λ is bounded from below (see (5.11) in Proposition 5.1.(III)), we take expectations in (6.1) and we use the monotone and the dominated convergence theorems to calculate

$$\begin{aligned}\frac{1}{T}\,\mathsf{E}\biggl[\int_0^T h(X_t^\zeta)\,\mathrm{d}t+\int_0^T k(X_t^\zeta)\circ\mathrm{d}\zeta_t\biggr]&\\ &=\frac{1}{T}\lim_{n\uparrow\infty}\mathsf{E}\biggl[\int_0^{\tau_n\wedge T}h(X_t^\zeta)\,\mathrm{d}t+\int_0^{\tau_n\wedge T}k(X_t^\zeta)\circ\mathrm{d}\zeta_t\biggr]\\ &\le\frac{1}{T}\lim_{n\uparrow\infty}\mathsf{E}\Bigl[\lambda(\tau_n\wedge T)+w_\lambda(x)+w_\lambda^-(X_{\tau_n\wedge T}^\zeta)\Bigr]\\ &=\lambda+\frac{w_\lambda(x)}{T}+\frac{1}{T}\,\mathsf{E}\Bigl[w_\lambda^-(X_T^\zeta)\Bigr],\end{aligned}$$

where $w_\lambda^-(x)=-\min\bigl\{w_\lambda(x),0\bigr\}$. Using the fact that w_λ^- is bounded once again, we can pass to the limit as $T\uparrow\infty$ to obtain the inequality $J_x^{\mathrm{e}}(\zeta)\le\lambda$, which implies the required inequality $J_x^{\mathrm{e}}(\zeta)\le\lambda^\star$ by passing to the limit as $\lambda\downarrow\lambda^\star$.

Proof of the inequality $J_x^{\mathrm{p}}(\zeta)\le\lambda^\star$. Making a slight modification of the proof of the comparison Theorem V.43 in Rogers and Williams [25], we can show that $X_t^\zeta\le X_t$ for all $t\ge 0$, P-a.s., where X is the solution to the SDE (2.1). In view of this observation, we can see that, given any $\lambda\in\,]\lambda^\star,\overline{\lambda}[$,

$$\begin{aligned}\bigl\langle M^{\lambda,\zeta}\bigr\rangle_T&=\int_0^T\Bigl(\sigma(X_t^\zeta)w_\lambda'(X_t^\zeta)\Bigr)^2\,\mathrm{d}t\\ &\le C_1C_2^2\int_0^T\Bigl(1+\bigl(X_t^\zeta\bigr)^\eta\Bigr)\,\mathrm{d}t\le C_1C_2^2\biggl(T+\int_0^T X_t^\eta\,\mathrm{d}t\biggr),\end{aligned}$$

where C_1, η and $C_2=C_2(\lambda)$ are the constants in (2.2) and (5.11). Furthermore, the ergodic Theorem V.53 in Rogers and Williams [25] implies that

$$\begin{aligned}\limsup_{T\uparrow\infty}\frac{\bigl\langle M^{\lambda,\zeta}\bigr\rangle_T}{T}&\le C_1C_2^2\biggl(1+\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T X_t^\eta\,\mathrm{d}t\biggr)\\ &=C_1C_2^2\biggl(1+\frac{1}{m\bigl(]0,\infty[\bigr)}\int_0^\infty s^\eta\,m(\mathrm{d}s)\biggr)=:C_3<\infty,\end{aligned}\tag{6.2}$$

with the second inequality following thanks to Assumption 2.3.

The Dambis, Dubins and Schwarz theorem (e.g., see Revuz and Yor [24, Theorem V.1.7]) asserts that there exists a standard Brownian motion B, which may be defined on a possible enlargement of the probability space $(\Omega,\mathcal{F},\mathsf{P})$, such that $M^{\lambda,\zeta}=B_{\langle M^{\lambda,\zeta}\rangle}$. Using this representation, (6.2) and the fact that $\lim_{T\uparrow\infty}B_T/T=0$, we can see that

$$\lim_{T\uparrow\infty}\frac{\bigl|M_T^{\lambda,\zeta}\bigr|}{T}\mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty=\infty\}}\le C_3\lim_{T\uparrow\infty}\frac{\bigl|B_{\langle M^{\lambda,\zeta}\rangle_T}\bigr|}{\bigl\langle M^{\lambda,\zeta}\bigr\rangle_T}\mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty=\infty\}}=0.$$

On the other hand,

$$\lim_{T\uparrow\infty}\frac{\bigl|M_T^{\lambda,\zeta}\bigr|}{T}\mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty<\infty\}}=0$$

because $M^{\lambda,\zeta}$ converges in $\mathbb{R}$ on the event $\{\bigl\langle M^{\lambda,\zeta}\bigr\rangle_\infty<\infty\}$. In view of these results, we can pass to the limit as $T \uparrow \infty$ in (6.1) to obtain

$$J_x^{\mathrm{p}}(\zeta) \le \lim_{T\uparrow\infty} \left(\lambda + \frac{w_\lambda(x)}{T} + \frac{w_\lambda^-(X_T^\zeta)}{T} + \frac{M_T^{\lambda,\zeta}}{T} \right) = \lambda.$$

The inequality $J_x^{\mathrm{p}}(\zeta) \le \lambda^\star$ now follows by passing to the limit as $\lambda \downarrow \lambda^\star$.

Proof of (II). Let the harvesting strategy $\zeta^\star \in \mathcal{A}$ be as in the statement of the theorem: such a strategy indeed exists (see Tanaka [27, Theorem 4.1]). If we define

$$N_T = \int_0^T \sigma(X_t^\star)\, \mathrm{d}W_t,$$

then $\langle N \rangle_T / T \le \max_{s\in[0,\beta^\star]} \sigma(s) < \infty$. Therefore, N is a square integrable martingale and $\mathsf{E}\big[N_T\big] = 0$ for all $T > 0$. Furthermore, reasoning as above, we can see that $\lim_{T\uparrow\infty} N_T/T = 0$. In view of these observations, the expression

$$\frac{\zeta_T^\star}{T} = \frac{x}{T} - \frac{X_t^\star}{T} + \frac{1}{T}\int_0^T b(X_t^\star)\,\mathrm{d}t + \frac{N_T}{T}, \tag{6.3}$$

and the fact that, beyond its possible initial jump, $\zeta^\star$ increases on the set $\{X_t^\star = \beta^\star\}$, we can see that

$$\begin{aligned} J_x^{\mathrm{e}}(\zeta^\star) &= \lim_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\left[\int_0^T h(X_t^\star)\,\mathrm{d}t + k(\beta^\star)\zeta_T^\star \right] \\ &= \lim_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\left[\int_0^T \Big(h(X_t^\star) + k(\beta^\star) b(X_t^\star) \Big)\,\mathrm{d}t \right] \end{aligned}$$

and

$$J_x^{\mathrm{p}}(\zeta^\star) = \lim_{T\uparrow\infty} \frac{1}{T} \left(\int_0^T h(X_t^\star)\,\mathrm{d}t + k(\beta^\star)\zeta_T^\star \right) = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T \Big(h(X_t^\star) + k(\beta^\star) b(X_t^\star) \Big)\,\mathrm{d}t.$$

These expressions and standard ergodic theorems (e.g., see Borodin and Salminen [6, Section II.6] and Rogers and Williams [25, Theorem V.53]) imply that

$$J_x^{\mathrm{e}}(\zeta^\star) = J_x^{\mathrm{p}}(\zeta^\star) = \frac{1}{m\big(]0,\beta^\star[\big)} \int_0^{\beta^\star} \big(h(s) + k(\beta^\star) b(s) \big)\, m(\mathrm{d}s).$$

Combining these observations with the identities

$$k(\beta^\star) \int_0^{\beta^\star} b(s)\, m(\mathrm{d}s) \overset{(5.21)}{=} \frac{k(\beta^\star)}{p'(\beta^\star)} \overset{(2.7)}{=} \int_0^{\beta^\star} K(s)\, m(\mathrm{d}s),$$

we obtain

$$J_x^{\mathrm{e}}(\zeta^\star) = J_x^{\mathrm{p}}(\zeta^\star) = \frac{1}{m\big(]0,\beta^\star[\big)} \int_0^{\beta^\star} \big(K(s) + h(s) \big)\, m(\mathrm{d}s) \overset{(5.4)}{=} \lambda^\star.$$

□

7. Abelian limits. In this section, we allow for the discounting rate function r to depend on a parameter $\iota > 0$ and we establish the convergence of the solution to the discounted control problem to the one of the ergodic control problems in an Abelian sense. In particular, we make the following assumption, which is the same as Assumption 2.6.(i) for each individual $\iota > 0$.

Assumption 7.1. The discounting rate function $(x,\iota) \mapsto r(x;\iota)$ is continuous. Also, given any $\iota > 0$, the function $r(\cdot;\iota)$ is C^1 and such that

$$\underline{r}(\iota) \le r(x;\iota) \le \overline{r}(\iota) \quad \text{for all } x \ge 0, \tag{7.1}$$

for some $\underline{r}(\iota)$ and $\overline{r}(\iota)$ such that

$$0 < \underline{r}(\iota) < \overline{r}(\iota) < \infty \text{ for all } \iota > 0, \quad \lim_{\iota \downarrow 0} \frac{\overline{r}(\iota)}{\underline{r}(\iota)} = 1 \quad \text{and} \quad \lim_{\iota \downarrow 0} \overline{r}(\iota) = 0. \tag{7.2}$$

The dependence of r on the parameter ι implies that the functions Θ, φ, ψ and R_h that we have considered in our analysis also depend on ι. Throughout this section, we will make such dependences explicit for clarity of the arguments.

The functions φ and ψ introduced at the beginning of Section 4 are unique up to a multiplicative constant. In this section, we assume that they have been scaled so that

$$\varphi(1;\iota) = 1 \quad \text{and} \quad \psi(1;\iota) = 1 \quad \text{for all } \iota > 0, \tag{7.3}$$

without loss of generality.

LEMMA 7.2. *In the presence of Assumptions 2.1 and 2.2, the scaled as in (7.3) functions $(x,\iota) \mapsto \varphi(x;\iota)$ and $(x,\iota) \mapsto \psi(x;\iota)$ are continuous,*

$$\lim_{\iota \downarrow 0} \frac{\varphi(x;\iota)}{\varphi(y;\iota)} = \lim_{\iota \downarrow 0} \frac{\psi(x;\iota)}{\psi(y;\iota)} = 1 \quad \textit{for all } x, y > 0 \tag{7.4}$$

$$\textit{and} \quad \lim_{\iota \downarrow 0} \frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)} = p'(x) m\big(]0,x[\big) \quad \textit{for all } x, y > 0. \tag{7.5}$$

In the presence of the assumptions we have made in Section 2 and Assumption 7.1,

$$\lim_{\iota \downarrow 0} r(y;\iota) \left(R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)} \psi(x;\iota) \right) = \frac{1}{m\big(]0,\beta[\big)} \int_0^\beta h(s)\, m(\mathrm{d}s) \tag{7.6}$$

for all $\beta > 0$, $x \in]0,\beta]$ and $y > 0$.

Proof. The continuity of the functions φ and ψ, as well as (7.4), follow immediately from (4.3) and the dominated convergence theorem. In turn, (7.5) follows from the definition (4.6), the identity (4.14), the limit (7.4) and Assumption 7.1, which imply that

$$\lim_{\iota \downarrow 0} \frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)} = p'(x) \lim_{\iota \downarrow 0} \int_0^x \frac{r(s;\iota)}{r(y;\iota)} \frac{\varphi(s;\iota)}{\varphi(x;\iota)}\, m(\mathrm{d}s) = p'(x) m\big(]0,x[\big).$$

Using the definitions (4.4)–(4.6), we can see that

$$\begin{aligned} R_h(x;\iota) &- \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)} \psi(x;\iota) \\ &= \frac{1}{C} \widetilde{\varphi}_\beta(x;\iota) \int_0^x h(s)\psi(s;\iota)\, m(\mathrm{d}s) + \frac{1}{C} \psi(x;\iota) \int_x^\beta h(s) \widetilde{\varphi}_\beta(s;\iota)\, m(\mathrm{d}s), \end{aligned}$$

where

$$\widetilde{\varphi}_\beta(x;\iota) = \varphi(x;\iota) - \frac{\varphi'(\beta;\iota)}{\psi'(\beta;\iota)} \psi(x;\iota).$$

In view of the observation that

$$\frac{\widetilde{\varphi}_\beta(x;\iota)}{\widetilde{\varphi}_\beta(\beta;\iota)} = 1 + \left(\frac{\varphi(x;\iota)}{\varphi(\beta;\iota)} - 1\right) \frac{\varphi(\beta;\iota)\psi'(\beta;\iota)}{\varphi(\beta;\iota)\psi'(\beta;\iota) - \varphi'(\beta;\iota)\psi(\beta;\iota)} + \left(\frac{\psi(x;\iota)}{\psi(\beta;\iota)} - 1\right) \frac{-\varphi'(\beta;\iota)\psi(\beta;\iota)}{\varphi(\beta;\iota)\psi'(\beta;\iota) - \varphi'(\beta;\iota)\psi(\beta;\iota)}$$

and the fact that the two long fractions on the right-hand side of this expression take values in $]0,1[$, we can see that $\lim_{\iota\downarrow 0} \widetilde{\varphi}_\beta(x;\iota)/\widetilde{\varphi}_\beta(\beta;\iota) = 1$, thanks to (7.4). On the other hand, we can use the probabilistic expression in (4.4) to obtain

$$R_{r(\cdot;\iota)}(x) = \mathsf{E}_x\left[\int_0^\infty \exp\biggl(-\int_0^t r(X_u;\iota)\,\mathrm{d}u\biggr) r(X_t;\iota)\,\mathrm{d}t\right] = 1 \quad \text{for all } x, \iota > 0.$$

Combining these observations with (7.4) and the fact that

$$\lim_{\iota\downarrow 0} \frac{r(x;\iota)}{r(y;\iota)} = 1 \quad \text{for all } x, y > 0, \tag{7.7}$$

which follows from Assumption 7.1, and using the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\iota\downarrow 0} r(y;\iota)\biggl(R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)\biggr) &= \lim_{\iota\downarrow 0} r(y;\iota) \frac{R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)}{R_{r(\cdot;\iota)}(x;p) - \frac{R_{r(\cdot;\iota)}'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)} \\ &= \lim_{\iota\downarrow 0} \frac{\int_0^x h(s)\frac{\psi(s;\iota)}{\psi(x;\iota)}\,m(\mathrm{d}s) + \int_x^\beta h(s)\frac{\widetilde{\varphi}_\beta(s;\iota)}{\widetilde{\varphi}_\beta(x;\iota)}\,m(\mathrm{d}s)}{\int_0^x \frac{r(s;\iota)}{r(y;\iota)}\frac{\psi(s;\iota)}{\psi(x;\iota)}\,m(\mathrm{d}s) + \int_x^\beta \frac{r(s;\iota)}{r(y;\iota)}\frac{\widetilde{\varphi}_\beta(s;\iota)}{\widetilde{\varphi}_\beta(x;\iota)}\,m(\mathrm{d}s)} = \frac{\int_0^\beta h(s)\,m(\mathrm{d}s)}{m(]0,\beta[)}, \end{aligned}$$

namely, (7.6). ∎

THEOREM 7.3. *Consider the control problems formulated in Section 2 and suppose that Assumption 7.1 also holds. If $\beta^\star(\iota)$, $w(\cdot;\iota)$ are as in Lemma 4.1 and $(\beta^\star, \lambda^\star)$, w are as in Proposition 5.1, then*

$$\lim_{\iota\downarrow 0} \beta^\star(\iota) = \beta^\star, \quad \lim_{\iota\downarrow 0} r(y;\iota)w(x;\iota) = \lambda^\star \quad \textit{and} \quad \lim_{\iota\downarrow 0} w'(x;\iota) = w'(x) \tag{7.8}$$

for all $x, y > 0$.

Proof. In view of the definition (4.6), the equation (4.11) that $\beta^\star(\iota) > 0$ satisfies takes the form

$$\int_0^{\beta^\star(\iota)} \bigl(\Theta(s;\iota) + h(s)\bigr)\frac{\psi(s;\iota)}{\psi\bigl(\beta^\star(\iota);\iota\bigr)}\,m(\mathrm{d}s) = \Bigl(\Theta\bigl(\beta^\star(\iota);\iota\bigr) + h\bigl(\beta^\star(\iota)\bigr)\Bigr)\int_0^{\beta^\star(\iota)} \frac{r(s;\iota)}{r\bigl(\beta^\star(\iota);\iota\bigr)}\frac{\psi(s;\iota)}{\psi\bigl(\beta^\star(\iota);\iota\bigr)}\,m(\mathrm{d}s).$$

The functions r, Θ, h and ψ are all continuous, while $\lim_{\iota\downarrow 0}\Theta(x;\iota) = K(x)$ (see the definitions (2.5) and (2.6) of Θ and K). Therefore, we can use (7.4), (7.7) and the dominated convergence theorem to come to the conclusion that the limit $\beta^\star(0) = \lim_{\iota\downarrow 0}\beta^\star(\iota)$ exists and satisfies the equation

$$\int_0^{\beta^\star(0)} \big(K(s)+h(s)\big)\, m(\mathrm{d}s) = \Big(K\big(\beta^\star(0)\big) + h\big(\beta^\star(0)\big)\Big) m\big(\,]0,\beta^\star(0)[\,\big).$$

It follows that the first limit in (7.8) holds true because this is the equation that $\beta^\star$ uniquely satisfies (see (5.4)).

The second limit in (7.8) follows immediately from the first expression for $w(\cdot;\iota)$ in (4.9), Assumption 7.1 and (7.6) with $\Theta(\cdot;\iota)+h$ in place of h. Finally, we use (4.2) and (7.4) to note that

$$\begin{aligned}\lim_{\iota\downarrow 0}\left(R'_{\Theta+h} - \frac{\psi'}{\psi}R_{\Theta+h}\right)(x;\iota) &= -p'(x)\lim_{\iota\downarrow 0}\int_0^x \big(\Theta(s;\iota)+h(s)\big)\frac{\psi(s;\iota)}{\psi(x;\iota)}\, m(\mathrm{d}s)\\ &= -p'(x)\int_0^x \big(K(s)+h(s)\big)\, m(\mathrm{d}s).\end{aligned}$$

In light of this limit, the fact that $\lim_{\iota\downarrow 0}\beta^\star(\iota) = \beta^\star$, (7.5) and (7.6) with $\Theta(\cdot;\iota)+h$ in place of h, we can see that

$$\begin{aligned}&\lim_{\iota\downarrow 0}\left(R'_{\Theta+h}(x;\iota) - \frac{R'_{\Theta+h}\big(\beta^\star(\iota);\iota\big)}{\psi'\big(\beta^\star(\iota);\iota\big)}\psi'(x;\iota)\right)\\ &\quad= \lim_{\iota\downarrow 0}\left(R'_{\Theta+h} - \frac{\psi'}{\psi}R_{\Theta+h}\right)(x;\iota)\\ &\qquad+ \lim_{\iota\downarrow 0} r(y;\iota)\left(R_{\Theta+h}(x;\iota) - \frac{R'_{\Theta+h}\big(\beta^\star(\iota);\iota\big)}{\psi'\big(\beta^\star(\iota);\iota\big)}\psi(x;\iota)\right)\lim_{\iota\downarrow 0}\frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)}\\ &\quad= p'(x)\left(\frac{m\big(\,]0,x[\,\big)}{m\big(\,]0,\beta^\star[\,\big)}\int_0^{\beta^\star}\big(K(s)+h(s)\big)\, m(\mathrm{d}s) - \int_0^x \big(K(s)+h(s)\big)\, m(\mathrm{d}s)\right).\end{aligned}$$

The third limit in (7.8) follows from this result, the first expression for $w(\cdot;\iota)$ in (4.9) and the second expression for w' in (5.1). ∎

Acknowledgment. We are grateful to three anonymous referees whose constructive comments led to a substantial enhancement of the paper.

REFERENCES

[1] L. H. R. Alvarez, *On the option interpretation of rational harvesting planning*, J. Math. Biol., 40 (2000), pp. 383–405.
[2] L. H. R. Alvarez, *Singular stochastic control, linear diffusions, and optimal stopping: A class of solvable problems*, SIAM J. Control Optim., 39 (2001), pp. 1697–1710.
[3] L. H. R. Alvarez and A. Hening, *Optimal sustainable harvesting of populations in random environments*, Stochastic Process. Appl., 150 (2022), pp. 678–698.
[4] L. H. R. Alvarez, E. Lungu and B. Øksendal, *Optimal multi-dimensional stochastic harvesting with density-dependent prices*, Afr. Mat., 27 (2016), pp. 427–442.
[5] L. H. R. Alvarez and L. A. Shepp, *Optimal harvesting of stochastically fluctuating populations*, J. Math. Biol., 37 (1998), pp. 155–177.
[6] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion - Facts and Formulae*, Birkhäuser, 2002.
[7] H. Cao, J. Dianetti and G. Ferrari, *Stationary discounted and ergodic mean field games of singular control*, Math. Oper. Res., 48 (2023), pp. 1871–1898.
[8] A. Cohen, A. Hening and C. Sun, *Optimal ergodic harvesting under ambiguity*, SIAM J. Control Optim., 60 (2022), pp. 1039–1063.

[9] N. C. Framstad, *Optimal harvesting of a jump diffusion population and the effect of jump uncertainty*, SIAM J. Control Optim., 42 (2003), pp. 1451–1465.

[10] M. Gaïgi, V. Ly Vath and S. Scotti, *Optimal harvesting under marine reserves and uncertain environment*, European J. Oper. Res., 301 (2022), pp. 1181–1194.

[11] A. Hening, D. H. Nguyen, S. C. Ungureanu and T. K. Wong, *Asymptotic harvesting of populations in random environments*, J. Math. Biol., 78 (2019), pp. 293–329.

[12] A. Hening, K. Q. Tran, T. T. Phan and G. Yin, *Harvesting of interacting stochastic populations*, J. Math. Biol., 79 (2019), pp. 533–570.

[13] R. Hynd, *The eigenvalue problem of singular ergodic control*, Comm. Pure Appl. Math., 65 (2012), pp. 649–682.

[14] A. Jack and M. Zervos, *A singular control problem with an expected and a pathwise ergodic performance criterion*, J. Appl. Math. Stochastic Anal., (2006), article ID 82538, 19 pp.

[15] I. Karatzas, *A class of singular stochastic control problems*, Adv. in Appl. Probab., 15 (1983), pp. 225–254.

[16] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, 1988.

[17] K. Kunwai, F. Xi, G. Yin and C. Zhu, *On an ergodic two-sided singular control problem*, Appl. Math. Optim., 86 (2022), paper no. 26, 34 pp.

[18] D. Lamberton and M. Zervos, *On the optimal stopping of a one-dimensional diffusion*, Electron. J. Probab., 18 (2013), paper no. 34, 49 pp.

[19] Z. Liu and M. Zervos, *The solution to an impulse control problem motivated by optimal harvesting*, J. Math. Anal. Appl., 542 (2025), paper no. 128809, 33 pp.

[20] E. Lungu and B. Øksendal, *Optimal harvesting from a population in a stochastic crowded environment*, Math. Biosci., 145 (1997), pp. 47–75.

[21] E. Lungu and B. Øksendal, *Optimal harvesting from interacting populations in a stochastic environment*, Bernoulli, 7 (2001), pp. 527–539.

[22] H. Morimoto, *Optimal harvesting in the stochastic logistic growth model with finite horizon*, SIAM J. Control Optim., 51 (2013), pp. 2335–2355.

[23] J. L. Menaldi, M. Robin and M. I. Taksar, *Singular ergodic control for multidimensional Gaussian processes*, Math. Control Signals Systems, 5 (1992), pp. 93–114.

[24] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd ed., Springer, 1999.

[25] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, vol. 2, Cambridge University Press, 2000.

[26] Q. Song, R. H. Stockbridge and C. Zhu, *On optimal harvesting problems in random environments*, SIAM J. Control Optim., 49 (2011), pp. 859–889.

[27] H. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J., 9 (1979), pp. 163–177.

[28] A. Weerasinghe, *Stationary stochastic control for Itô processes*, Adv. in Appl. Probab., 34 (2002), pp. 128–140.

[29] A. Weerasinghe, *An Abelian limit approach to a singular ergodic control problem*, SIAM J. Control Optim., 46 (2007), pp. 714–737.