

PARTITIONING A 2-EDGE-COLOURED GRAPH OF MINIMUM DEGREE $2n/3 + o(n)$ INTO THREE MONOCHROMATIC CYCLES

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ABSTRACT. Lehel conjectured in the 1970s that every red and blue edge-coloured complete graph can be partitioned into two monochromatic cycles. This was confirmed in 2010 by Bessy and Thomassé. However, the host graph G does not have to be complete. It suffices to require that G has minimum degree at least $3n/4$, where n is the order of G , as was shown recently by Letzter, confirming a conjecture of Balogh, Barát, Gerbner, Gyárfás and Sárközy. This degree condition is asymptotically tight.

Here we continue this line of research, by proving that for every red and blue edge-colouring of an n -vertex graph of minimum degree at least $2n/3 + o(n)$, there is a partition of the vertex set into three monochromatic cycles. This approximately verifies a conjecture of Pokrovskiy and is essentially tight.

1. INTRODUCTION

Influentially, Lehel conjectured that every colouring of the edges of the complete graph K_n with two colours admits two monochromatic cycles of distinct colours whose vertex sets partition $V(K_n)$. Here, an edge, a single vertex, and the empty set each count as a cycle. Łuczak, Rödl and Szemerédi [31] confirmed this conjecture for sufficiently large n after preliminary work of Gyárfás [13]. Their result was improved by the first author [1], and finally, Bessy and Thomassé [6] resolved the conjecture for all n .

Since the early 1990s the field of cycle partitioning has evolved in many directions. One natural question extensively studied is to what extent Lehel's conjecture generalizes to more than two colours. Here, a landmark result is due to Erdős, Gyárfás and Pyber [11], who showed that any r -edge-coloured complete graph K_n admits a partition into $25r^2 \log r$ monochromatic cycles and conjectured that r cycles suffice. The upper bound was later improved to $100r \log r$ by Gyárfás, Ruszinkó, Sárközy and Szemerédi [15] for large n . Pokrovskiy [33] disproved the conjecture of Erdős, Gyárfás and Pyber by providing an r -edge-colouring of K_n that needs $r+1$ cycles for every $r \geq 3$ and infinitely many n . So far, it is far from clear what is the best possible number of cycles, even for small r , showing that this problem is difficult.

Other variants of the problem that received much attention in the past decade include the appearance of subgraphs other than cycles, the use of colourings with an unbounded number of colours (but local restrictions on the colourings), and analogues for hypergraphs, to highlight just a few. We refer the reader to the recent survey of Gyárfás [14] and the references therein for more detail. Another important direction, which is the variant we are most interested in here, concerns the replacement of the complete graph K_n by a non-complete host graph. This has been investigated for almost complete graphs [29], complete bipartite graphs [16, 26, 27], graphs with fixed independence number [36], infinite graphs [10, 35] and random graphs [22, 24] among others. We are interested in the problem for graphs of large minimum degree.

The study of minimum degree conditions for spanning substructures has a long tradition in extremal graph theory (see for example the survey [23] for an overview), and so it is natural to seek a variant of Lehel's conjecture for host graphs of large minimum degree. Balogh, Barát, Gerbner, Gyárfás and Sárközy [4] conjectured the following: For any 2-edge-colouring of the edges of any n -vertex graph G of minimum degree $3n/4$, there are two distinctly coloured monochromatic cycles which together partition the vertices of G . Examples show that this would be tight. In support of their conjecture, they proved an approximate version in which G has minimum degree $3n/4 + o(n)$ and the cycles are allowed to miss $o(n)$ vertices. DeBiasio and Nelsen [9] showed that under this (stronger) degree condition a complete partition is possible. Finally, Letzter [28] resolved the full conjecture for all sufficiently large n .

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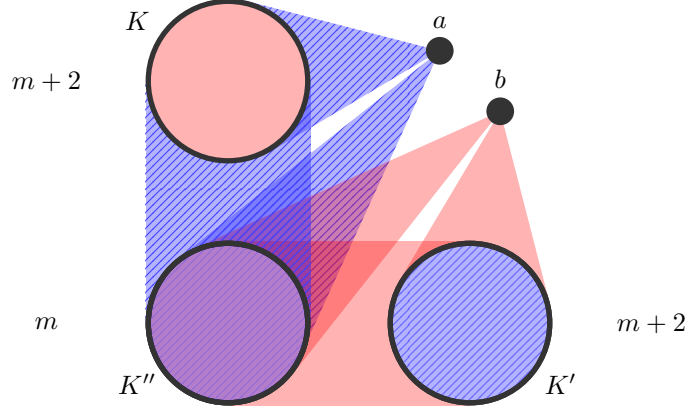


FIGURE 1. A graph on $3m + 6$ vertices, consisting of three vertex disjoint cliques K , K' and K'' and two additional vertices a and b . Here, K and K' have $m + 2$ vertices each, all edges of K are coloured red and all edges of K' are coloured blue. The third clique K'' is of size m and its edges can be coloured either red or blue. Further, all edges between K and K'' are present and coloured blue and all edges between K' and K'' are present and coloured red. The vertex a has blue edges to $K \cup K''$ and the vertex b has red edges to $K' \cup K''$.

Motivated by these advances, Pokrovskiy [34] conjectured that analogous results are true for graphs of lower minimum degree. In particular, he conjectured that any 2-edge-coloured graph of minimum degree $2n/3$ can be partitioned into 3 monochromatic cycles. The construction in Figure 1 shows that this bound is essentially tight. Our main result confirms Pokrovskiy's conjecture approximately.

Theorem 1.1. *For any $\beta > 0$, there is n_0 such that for any colouring with two colours of the edges of any graph G on $n \geq n_0$ vertices and minimum degree at least $(2/3 + \beta)n$, there is a partition of $V(G)$ into three monochromatic cycles.*

The proof of this theorem is partly based on tried and tested techniques such as graph regularity, the use of matchings, and absorption, but also draws from several new ideas which could prove useful for future research in this area. For an outline of our approach, see Section 2.

Organization. The remainder of this paper is organized as follows. In the next section we outline our proof method. In Section 3 we then introduce some notation and concepts related to the regularity method. In Section 4 we present the proof of Theorem 1.1 alongside the main lemmas needed for this proof. Sections 5, 6, 7 and 8 are dedicated to the Lemmas concerning finding the components, distributing the exceptional vertices and solving the cases of extremal colourings, respectively. Finally, we offer some open problems and conjectures in Section 9.

2. PROOF OUTLINE

In this section we shall briefly outline our proof, which was inspired by a method developed by Garbe, Lang, Lo, Mycroft and Sanhueza-Matamala [12] to solve cycle partitioning problems in hypergraphs.

Suppose that G is an n -vertex graph of minimum degree at least $2n/3 + o(n)$ and that each of its edges is coloured red or blue. We start by applying the regularity lemma (Lemma 3.8) to obtain a regular partition and the corresponding reduced graph $\mathcal{G}$. Roughly speaking, our strategy is to distinguish the case that $\mathcal{G}$ is close to an extremal configuration, which can be solved with arguments tailored to this particular structure (see Lemma 4.4), or we can find three monochromatic components C_1, C_2, C_3 in $\mathcal{G}$ whose union C has the following property (see Lemma 4.2). Every stable set S in C has linearly many more neighbours in $\mathcal{G}$ than $|S|$ (see Definition 3.3). From the LP duality of maximum fractional matchings and minimum vertex covers we obtain that C admits a *perfect 2-matching* in a robust way, where a perfect 2-matching is a spanning union of vertex disjoint edges and cycles (see Theorem 3.2 and the discussion preceding it).

As is canonical, we then connect the clusters of the components in the perfect 2-matching by short monochromatic paths (see Lemma 3.9), which we can further require to contain certain

exceptional vertices (see Lemma 4.6). Creating these paths will create some imbalances between the clusters of the matching edges, which we correct by solving a weighted matching problem in an auxiliary graph and using the robustness of the perfect 2-matching (see Lemma 4.7). To finish the proof, we apply the blow-up lemma (Lemma 3.10) to find monochromatic spanning paths in each of the components of the perfect 2-matching. Together with the short paths constructed earlier this yields the desired cycle partition.

We remark that our approach has the advantage that it avoids a number of technicalities from earlier approaches, and allows us to isolate the main steps (finding the components, distributing exceptional vertices and solving extremal configurations) cleanly in separate lemmas.

3. PRELIMINARIES

In this section we introduce some notation and tools, which we will need for the proof of Theorem 1.1.

3.1. Notation. We let $[n]$ denote the set $\{1, 2, \dots, n\}$. Let $G = (V, E)$ be a graph. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. We denote the *neighbourhood* of a vertex v by $N_G(v)$ and write $N_G(v, W) = N_G(v) \cap W$ for a set of vertices $W \subseteq V(G)$. We denote the *degree* of v by $\deg_G(v) = |N_G(v)|$ and set $\deg_G(v, W) = |N_G(v) \cap W|$. For a set of vertices $S \subseteq G$ we write $N_G(S) = \bigcup_{s \in S} N(s)$. If it is clear from the context, we often drop the index G . We write $G[W]$ for the subgraph of G induced by the vertices in W and for a set U disjoint from W we write $G[W, U]$ for the bipartite subgraph of G containing all edges in $E(G)$ with one end in U and one in W . For another graph H we denote by $G \cup H$ the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We write $H \subseteq G$ if H is a subgraph of G . A *stable set* in G is a set of vertices $S \subseteq V(G)$ such that there are no edges with both endpoints in S . For vertices v and w , a v, w -*path (walk)* is a path (walk) that starts in v and ends in w . Similarly, for vertex sets W and W' , a W, W' -*path (walk)* is a path (walk) that starts in W and ends in W' .

Suppose that G is a graph whose edges are coloured with red and blue. We denote by G_{red} (G_{blue}) the subgraph on $V(G)$ that contains the red (blue) edges. For a vertex v of G we also write $N_{\text{red}}(v)$, $N_{\text{blue}}(v)$, $\deg_{\text{red}}(v)$, $\deg_{\text{blue}}(v)$ for $N_{G_{\text{red}}}(v)$, $N_{G_{\text{blue}}}(v)$, $\deg_{G_{\text{red}}}(v)$, $\deg_{G_{\text{blue}}}(v)$ for the neighbourhood/degree of v in the respective colour. We will say that $w \in N_{\text{red}}(v)$ is a *red neighbour* of v and likewise for colour blue. A *red (blue) component* of G is a component of G_{red} (G_{blue}). In particular a vertex v with $\deg_{\text{red}}(v) = 0$ is in a red component of order 1. For convenience of notation, we also count the empty graph as a component.

In our coloured graphs we do occasionally allow parallel edges (more precisely, this is the case for our so-called reduced graphs) – however, there will always be at most one red edge and at most one blue edge between any pair of vertices. For such a multi-graph G we denote by $\delta(G)$ the minimum degree of the underlying simple graph, and by $\deg_G(u)$ the degree of u in the underlying simple graph.

3.2. Perfect 2-matchings. We make use of the concept of a 2-matching in a graph G .

Definition 3.1 (Perfect 2-matching). A *perfect 2-matching* of a graph G is a spanning subgraph of G which is the disjoint union of cycles and single edges.

The next theorem is a convenient analogue of Tutte’s classical characterization of perfect matchings for 2-matchings. It follows easily from the fact that maximum fractional matching and minimum vertex cover are LP dual problems both of which have optimal solutions which are half-integral, and the LP-duality theorem. For a proof of this result see, e.g., [37, Corollary 30.1a].

Theorem 3.2 (Tutte’s Theorem for 2-matchings). A graph G has a perfect 2-matching if and only if every stable set $S \subseteq V(G)$ satisfies $|N(S)| \geq |S|$.

In order to apply this in our proof, we will need a more robust version of the condition $|N(S)| \geq |S|$. This motivates the following definition.

Definition 3.3 (γ -robust Tutte). We say that a graph G on n vertices is γ -robust Tutte if for every non-empty stable subset S of $V(G)$ we have $|N(S)| \geq |S| + \gamma n$.

3.3. Hamiltonicity. A *Hamilton cycle (Hamilton path)* in a graph G is a cycle (path) that covers all the vertices of G . Chvátal’s theorem provides a degree sequence criterium for the existence of Hamilton cycles.

Theorem 3.4 (Chvátal [7]). *Let G be a graph with vertex degree sequence $d_1 \leq \dots \leq d_n$. If for every $1 \leq i < n/2$ we have $d_i \geq i+1$ or $d_{n-i} \geq n-i$, then G has a Hamilton cycle.*

As a corollary we get the following version for balanced bipartite graphs, which we shall use in our proof.

Corollary 3.5 (Bipartite Chvátal). *Let H be a bipartite graph with bipartition classes X and Y , each of size n . Let X have vertex degree sequence $x_1 \leq \dots \leq x_n$ and Y have vertex degree sequence $y_1 \leq \dots \leq y_n$. If for every $i \in [n-1]$ we have $x_i \geq i+1$ or $y_{n-i} \geq n-i+1$, then H is Hamiltonian.*

Proof. Obtain H' from H by adding edges between vertices of Y until $H'[Y]$ is complete. So every vertex in Y gains $n-1$ new neighbours. Let $d_1, \dots, d_{2n}$ be the degree sequence of H' . By assumption we have either $x_{n-1} \geq n$ or $y_1 \geq 2$. In either case, $y_i \geq 2$ for all $i \in [n]$. Hence, for each $i \in [n]$ we have $d_i = x_i$ and $d_{2n-i+1} = y_{n-i+1} + n-1$. It follows that $d_i \geq i+1$ or $d_{2n-i+1} \geq 2n-i$ for $i \in [n-1]$. By Theorem 3.4, H' has a Hamilton cycle C , which, as $|X| = |Y|$, has no edges within Y . Thus C is a Hamilton cycle of H . $\square$

This also implies the following result on Hamilton paths.

Corollary 3.6. *Let H be a bipartite graph with bipartition classes X_1, X_2 , each of size n . If for each $i \in \{1, 2\}$, and for each $u \in X_i$ we have $\deg_H(u, X_{3-i}) \geq \frac{1}{2}n + 1$, then for any $w \in X_1, w' \in X_2$ there is a Hamilton w, w' -path in H .*

Proof. Obtain the graph H' from H by adding a new vertex v to X_1 , a new vertex v' to X_2 and the edges $wv', v'v, vw'$. It is easy to verify that the conditions of Corollary 3.5 are satisfied, which gives a Hamilton cycle in H' . As this cycle clearly uses the edges $wv', v'v, vw'$ it contains a Hamilton w, w' -path in H . $\square$

We use Corollary 3.6 to obtain Hamilton paths in the following somewhat specialized setting, which appears in the analysis of our extremal cases.

Lemma 3.7. *Let $0 < \mu \leq 1$, $n \geq 100\mu^{-1}$, and let H be an n -vertex graph on $V(H) = A \dot{\cup} B$ with $A' \subseteq A$, $|B| \leq |A| \leq |B| + \mu n$ and $|A'| \leq \mu n$, such that*

- (i) *for each $v \in A$ we have $\deg_H(v, B) \geq (\frac{1}{4} + 3\mu)n$,*
- (ii) *for each $v \in B$ we have $\deg_H(v, A) \geq (\frac{1}{4} + 3\mu)n$, and*
- (iii) *for each $v \in A \setminus A'$ we have $\deg_H(v, A) \geq 3\mu n$.*

Let $u, u' \in V(H) \setminus A'$ be distinct vertices such that if both $u, u' \in A \setminus A'$ then $|A| \geq |B| + 1$. Then there exists a Hamilton u, u' -path in H .

Proof. We start paths P in u and P' in u' . If $u, u' \in B$, we extend P by adding an edge from u to an arbitrary neighbour in $A \setminus A'$. Similarly, if $u, u' \in A \setminus A'$, we extend P' by adding an edge from u' to an arbitrary neighbour in B . Finally, if $|A \cap \{u, u'\}| = |B \cap \{u, u'\}| = 1$, then we assume without loss of generality that $u \in A$ and $u' \in B$ and do not perform any extensions.

Observe that after this first step, we have $|A \setminus V(P \cup P')| \geq |B \setminus V(P \cup P')|$, because of the assumption in the last sentence of Lemma 3.7. We then greedily extend the path P inside the set $A \setminus A'$ until we have $|A \setminus V(P \cup P')| = |B \setminus V(P \cup P')|$, which is possible because $|A'| \leq \mu n$ and for $v \in A \setminus A'$ we have $\deg_H(v, A) \geq 3\mu n$. Since $|A| \leq |B| + \mu n$, we now have $V(P) \leq \mu n + 1$.

Remove from A and B all vertices from the paths P and P' except their endvertices w and w' , thus obtaining sets A^* and B^* . Note that $n^* := |A^*| = |B^*| \leq (\frac{1}{2} + \mu)n$. We apply Corollary 3.6 to the bipartite graph H^* between A^* and B^* (it is not difficult to see that the degree conditions are satisfied), in order to find a Hamiltonian path P'' in H^* connecting w and w' . We finally connect P', P'' and P to form the desired Hamilton u, u' -path in H . $\square$

3.4. Regularity. Given a graph G and disjoint vertex sets $U, W \subseteq V(G)$ we denote the number of edges between U and W by $e(U, W)$ and the *density* of (U, W) by $d(U, W) = e(U, W)/(|U||W|)$. The pair (U, W) is called ε -*regular* in G if, for all subsets $X \subseteq U, Y \subseteq W$ with $|X| \geq \varepsilon|U|$ and $|Y| \geq \varepsilon|W|$, we have $|d(U, W) - d(X, Y)| \leq \varepsilon$. We say that (U, W) is (ε, d) -*regular* if it is ε -regular and has density at least d . We say that a vertex $u \in U$ has *typical degree* in a regular pair (U, W) if $\deg(u, W) \geq (d(U, W) - \varepsilon)|W|$. It follows directly from the definition of ε -regularity that if (U, W) is ε -regular, then

$$(3.1) \quad \text{all but at most } \varepsilon|U| \text{ vertices in } U \text{ have typical degree in } (U, W).$$

Another straightforward consequence of the definition of ε -regularity concerns large subpairs of regular pairs: If (U, W) is an (ε, d) -regular pair and $U' \subseteq U$, $W' \subseteq W$ are of size $|U'| \geq \frac{1}{2}|U|$, $|W'| \geq \frac{1}{2}|W|$, then

$$(3.2) \quad (U', W') \text{ is a } (2\varepsilon, d/2)\text{-regular pair.}$$

Szemerédi's regularity lemma allows us to partition the vertex set of a graph into clusters of vertices such that most pairs of clusters are regular [38]. We will use the regularity lemma in its degree form and with 2 colours (see [20]). For a coloured graph G , a colour χ and two disjoint vertex sets U, W of G we write $(U, W)_\chi$ for the bipartite subgraph of $G[U, W]$ containing all edges of colour χ .

Lemma 3.8 (Regularity lemma). *For each $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ there is $M = M(\varepsilon, m_0)$ such that for any 2-edge colouring of any n -vertex graph G , and for each $d > 0$ there is a partition $V_0, V_1, \dots, V_m$ of $V(G)$ and a subgraph G' of G with vertex set $V(G') \setminus V_0$ such that:*

- (a) $m_0 \leq m \leq M$,
- (b) $|V_0| \leq \varepsilon n$ and $|V_1| = \dots = |V_m| \leq \lceil \varepsilon n \rceil$,
- (c) $\deg_{G'}(v) \geq \deg_G(v) - (2d + \varepsilon)n$ for each $v \in V(G) \setminus V_0$,
- (d) $G'[V_i]$ has no edges for $i \in [m]$,
- (e) for all pairs (V_i, V_j) and each colour χ the pair $(V_i, V_j)_\chi$ is ε -regular in G' and has density either 0 or at least d .

Let G be a red and blue edge coloured graph with a partition $V_0, \dots, V_m$ obtained from Lemma 3.8 with parameters ε , m_0 and d . We then also call the family $\mathcal{V} = \{V_1, \dots, V_m\}$ an (ε, d) -regular partition of G with exceptional set V_0 . We define the (coloured) (ε, d) -reduced multi-graph $\mathcal{G}$ corresponding to this partition to be the multi-graph with vertex set $V(\mathcal{G}) = [m]$ in which two vertices i and j are connected by a red (or blue) edge, if (V_i, V_j) is an ε -regular pair of density at least d in red (blue, respectively). Note that it is possible that some i and j are connected by both a red and a blue edge.

It is easy to show that $\mathcal{G}$ inherits the minimum degree of G . More precisely, if G has minimum degree cn , it follows that

$$(3.3) \quad \delta(\mathcal{G}) \geq (c - 2d - \varepsilon)m.$$

Further, for a graph G , vertex disjoint sets $V_1, \dots, V_\ell$ of $V(G)$ and a graph $\mathcal{H}$ on vertex set $[\ell]$, we say that G is (ε, d) -regular on $\mathcal{H}$ if (V_i, V_j) is (ε, d) -regular whenever $ij \in E(\mathcal{H})$.

The next lemma allows us to connect clusters by short paths provided their counterparts in the reduced graph lie in the same component. Though the lemma as stated here does not explicitly appear in previous work, this technique is standard, and the lemma is straightforward to prove. Given a family of sets $\mathcal{V} = \{V_1, \dots, V_t\}$, we say that a set W (or a graph D with $V(D) = W$) is δ -sparse in $\mathcal{V}$ if $|W \cap V_i| \leq \delta|V_i|$ for each $i \in [t]$.

Lemma 3.9 (Connection lemma). *Let $\varepsilon, d > 0$ and let $m, \ell, n \in \mathbb{N}$ such that $10\varepsilon \leq d \leq \frac{1}{10}$ and $\frac{10m \cdot \ell}{d} \leq n$. Let G be a graph on n vertices, let $V_0 \subseteq V(G)$ with $|V_0| \leq \varepsilon n$, let $\mathcal{V} = \{V_1, \dots, V_m\}$ be an equipartition of $V(G) \setminus V_0$, and let $\mathcal{G}$ be a graph on $[m]$ so that G is (ε, d) -regular on $\mathcal{G}$. Let $W = (w_1, \dots, w_\ell)$ be a walk of order ℓ in $\mathcal{G}$. Let $X \subseteq V_{w_1}, Y \subseteq V_{w_\ell}$ be subsets of size $d|V_1|/2$. Let $S \subseteq V(G)$ be $(d/4)$ -sparse in $\mathcal{V}$. Then there is an X, Y -path $P \subseteq G - S$ of order ℓ .*

Finally, the blow-up lemma allows us to find almost spanning cycles or paths in certain regular partitions. A pair (U, W) is (ε, d) -super-regular if it is (ε, d) -regular and, in addition, for all vertices $u \in U, w \in W$, we have $\deg(u, W) \geq d|U|$ and $\deg(w, U) \geq d|W|$. The following version of the blow-up lemma can be found in [2, Lemma 7.1]. Note that we only require the special case $\Delta(\mathcal{G}) \leq 2$ here. We remark that one could replace our use of Lemma 3.10 with a use of the original blow-up lemma of Komlós, Sárközy and Szemerédi [19] and some additional technical work.

Lemma 3.10 (Blow-up lemma). *For all $\Delta \geq 2$, $\alpha, \zeta, d > 0$, $\kappa > 1$ there exist $\varepsilon, \rho > 0$ such that for all M the following holds for all sufficiently large n .*

Let $\mathcal{G}$ be a graph on $[m]$ with $m \leq M$ and let $\mathcal{G}'$ be a spanning subgraph of $\mathcal{G}$ with $\Delta(\mathcal{G}') \leq 2$. Let H, G be graphs whose vertex sets are partitioned into sets $X_1, \dots, X_m$ and $V_1, \dots, V_m$ respectively, such that $|X_i| = |V_i| \geq \frac{n}{\kappa m}$ and $|X_i| \leq \kappa|X_j|$ for each $i, j \in [m]$. Let $\tilde{X}_i \subseteq X_i$ satisfy $|\tilde{X}_i| \geq \alpha|X_i|$ for each $i \in [m]$, and let $I_x \subseteq V_i$ for each $i \in [m]$ and $x \in X_i$. Suppose that

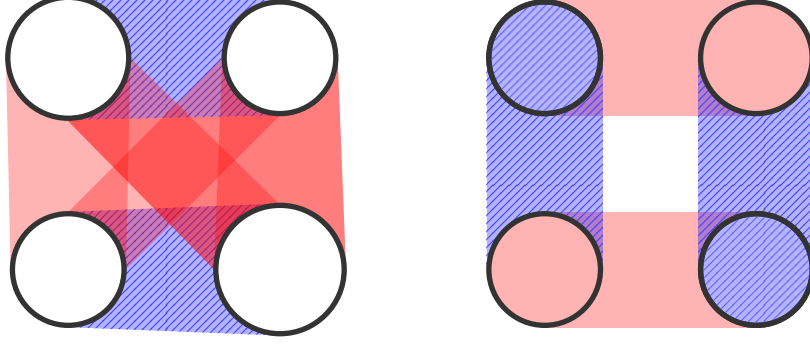


FIGURE 2. The extremal colourings: The configuration given in Definition 4.1(a) is depicted on the left and the one in Definition 4.1(b) on the right.

- (B1) $\Delta(H) \leq \Delta$, all edges of H lie on pairs $X_i X_j$ with $ij \in \mathcal{G}$, and for each $i \in [m]$, all edges of H incident with $\tilde{X}_i \cup N(\tilde{X}_i)$ lie on pairs $X_j X_{j'}$ with $jj' \in \mathcal{G}'$,
- (B2) G is (ε, d) -regular on $\mathcal{G}$, and for each $ij \in E(\mathcal{G}')$ the pair (V_i, V_j) is (ε, d) -super-regular, and
- (B3) for each $i \in [m]$ and $x \in X_i$ we have $|I_x| \geq \zeta|X_i|$ and there are at most $\rho|X_i|$ vertices $x \in X_i$ such that $I_x \neq V_i$.

Then there is an embedding $\psi: V(H) \rightarrow V(G)$ with $\psi(x) \in I_x$ for each $x \in H$.

4. MAIN LEMMAS AND PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1.1. We first sketch the proof and give the lemmas we need, and then give a formal proof.

For proving Theorem 1.1, we apply the regularity lemma to G to obtain a regular partition $\mathcal{V} = \{V_1, \dots, V_m\}$ with exceptional set V_0 and a corresponding coloured reduced multi-graph $\mathcal{G}$ on vertex set $[m]$. Note that $\mathcal{G}$ approximately inherits the (relative) minimum degree conditions as G . Our task is now to analyse the structure of monochromatic components in $\mathcal{G}$.

We begin by introducing the following extremal colourings (see also Figure 2), which require special treatment. Recall that in a coloured multi-graph there is at most one red edge and at most one blue edge between any pair of vertices.

Definition 4.1 (Extremal colourings). Let G be an m -vertex (multi-)graph whose edges are coloured with red and blue. We say that the colouring of G is γ -extremal if one of the following holds (modulo swapping colours).

- (a) G has a spanning bipartite red component with bipartition classes X_1, X_2 and such that $||X_1| - |X_2|| \leq \gamma m$. There are (exactly) two blue components B_1, B_2 , both bipartite, with $V(B_1) = X_1$ and $V(B_2) = X_2$.
- (b) The subgraph G has two red and two blue monochromatic components R_1, R_2, B_1, B_2 such that $V(G) = \bigcup_{i,j \in [2]} V(R_i) \cap V(B_j)$ and $|V(R_i) \cap V(B_j)| \leq (1/4 + \gamma)m$ for $1 \leq i, j \leq 2$. Moreover, all but at most γm^2 edges of $G[V(R_i) \cap V(B_j)]$ are blue if $i = j$ and red otherwise.

Now we present our main structural lemma, whose proof we defer to Section 5. It states that if $\mathcal{G}$ is not close to an extremal colouring, then we can choose three monochromatic components covering all vertices of $\mathcal{G}$ which have various good properties that allow us to complete the cycle partition.

Lemma 4.2 (Finding components). *Let $1/m \ll \gamma$. Let $\mathcal{G}$ be a multi-graph on m vertices with $\delta(\mathcal{G}) \geq (2/3 + 8\gamma)m$, whose edges are coloured in two colours. Suppose that the colouring of $\mathcal{G}$ is not (4γ) -extremal. Then there are distinct monochromatic components $C_1, C_2, C_3 \subseteq \mathcal{G}$ such that $C := \bigcup C_i$ spans $\mathcal{G}$ and all of the following hold.*

- (i) C is γ -robust Tutte.
- (ii) One of the following holds:
 - (a) $|\bigcup_{i \neq j} V(C_i) \cap V(C_j)| \geq (1/3 + \gamma)m$, or
 - (b) C_1 is spanning, C_1 or C_2 contains an odd cycle, and $C_3 = \emptyset$.

- (iii) If C_1, C_2, C_3 are each bipartite, then $C_3 = \emptyset$.
- (iv) C is connected.

Our next task is to make a case distinction, depending on the structure of $\mathcal{G}$ and of G . For this we need the following definition.

Definition 4.3 (Bridges). Suppose G is a graph whose edges are coloured in red and blue and which has an (ε, d) -regular partition $\mathcal{V} = \{V_1, \dots, V_m\}$ with corresponding coloured (ε, d) -reduced multi-graph $\mathcal{G}$. Let $H_1, H_2 \subseteq \mathcal{G}$ be monochromatic connected subgraphs of $\mathcal{G}$ of colour χ and let G_χ be the subgraph of G formed by all edges in $E(G)$ of colour χ . We say that (H_1, H_2) admits bridges if there are two distinct vertices u, u' in G and clusters $V_i, V_j, V_{i'}, V_{j'}$ in $\mathcal{V}$ with $i, i' \in V(H_1)$ and $j, j' \in V(H_2)$ such that

$$\begin{aligned} \deg_{G_\chi}(u, V_i) &\geq d|V_i|, & \deg_{G_\chi}(u, V_j) &\geq d|V_j|, \\ \deg_{G_\chi}(u', V_{i'}) &\geq d|V_{i'}| & \text{and} & \deg_{G_\chi}(u', V_{j'}) &\geq d|V_{j'}|. \end{aligned}$$

If the colour χ is blue, then we also talk of *blue bridges*, if χ is red we talk of *red bridges*.

Note that for any monochromatic connected $H \subseteq \mathcal{G}$ with at least one edge, the pair (H, H) admits bridges.

Continuing with our proof strategy, we separate three cases. First, $\mathcal{G}$ is not close to either extremal colouring. Second, $\mathcal{G}$ is close to an extremal colouring but there is a pair of red (blue) components of $\mathcal{G}$ which admit red (blue) bridges. Third, $\mathcal{G}$ is close to an extremal colouring and no such bridges exist. In the last case, the structure of G is so close to the respective extremal graph that we can find the desired three-cycle partition without the regularity lemma, as the following lemma states. We prove this lemma in Section 8.

Lemma 4.4 (Extremal cases). *Let $1/n \ll \varepsilon, 1/m \ll d \ll \gamma \ll \beta$. Suppose G is a graph on n vertices with minimum degree $\delta(G) \geq (2/3 + \beta)n$ whose edges are coloured with red and blue. Suppose G has a coloured (ε, d) -reduced multi-graph $\mathcal{G}$ on m vertices. Let the colouring of $\mathcal{G}$ be γ -extremal and such that one of the following is true:*

- $\mathcal{G}$ has a colouring as in Definition 4.1(a) with monochromatic components R_1, B_1, B_2 and (B_1, B_2) does not admit bridges.
- $\mathcal{G}$ has a colouring as in Definition 4.1(b) with monochromatic components R_1, R_2, B_1, B_2 and neither (R_1, R_2) nor (B_1, B_2) admits bridges.

Then G has a partition into three monochromatic cycles.

When $\mathcal{G}$ is not close to an extremal colouring, or when it is but some pair of components admits bridges, we work with the regular partition to find the desired three-cycle partition. We now describe the proof for the case that $\mathcal{G}$ is not close to either extremal colouring, and explain briefly at the end the modification required for the case that $\mathcal{G}$ is close to an extremal colouring but some pair of components admit bridges. We set $H_i := C_i$ for $i = 1, 2, 3$, where the C_i are the components provided by Lemma 4.2.

The first step is to fix a perfect 2-matching $\mathcal{M}$ of $\mathcal{G}$. We then move a few vertices from each of the clusters $V_1, \dots, V_m$ to the exceptional set in order to make the regular pairs of G corresponding to $\mathcal{M}$ super-regular. We denote the resulting exceptional set by V^{Exc} .

Our next step is to find three pairwise disjoint cycles in G which form the basis of our three-cycle partition. In each H_i we will find a cycle D_i , such that all the D_i together cover the vertices V^{Exc} . In addition, the cycles D_i will be ‘easy to extend to the pairs in H_i ’, which we make precise in next definition.

Definition 4.5 (Support path embeddings). Let G be a graph whose edges are coloured in red and blue and which has an (ε, d) -regular partition $\mathcal{V} = \{V_1, \dots, V_t\}$ with corresponding coloured (ε, d) -reduced multi-graph $\mathcal{G}$. Let $H \subseteq \mathcal{G}_\chi$ be a subgraph of colour χ . We say that $D \subseteq G_\chi$ supports path embeddings in H , if for each edge $ij \in E(H)$ there is an edge $f_{ij} = \{u, v\} \in D$ with $u \in V_i$ and $v \in V_j$ such that u and v have typical degree in the regular pair $(V_i, V_j)_\chi$. We also say that the edge f_{ij} witnesses that D supports path embeddings in H for ij .

Our next lemma constructs the desired cycles covering the exceptional vertices. We prove this lemma in Section 6.

Lemma 4.6 (Covering exceptional vertices). *Let $1/n \ll \varepsilon, \frac{1}{m} \ll \delta \ll d \ll \gamma \ll \beta$. Suppose G is a graph on n vertices of minimum degree $\deg(G) \geq (2/3 + \beta)n$ whose edges are coloured*

with two colours. Let G have an (ε, d) -regular partition $\mathcal{V} = \{V_1, \dots, V_m\}$ with exceptional set V_0 and corresponding coloured (ε, d) -reduced multi-graph $\mathcal{G}$. Let $W \subseteq V(G)$ be δ -sparse in $\mathcal{V}$. Let H_1, H_2, H_3 be subgraphs of $\mathcal{G}$ with the following properties.

(i) One of the following holds:

- (a) $|\bigcup_{i \neq j} V(H_i) \cap V(H_j)| \geq (1/3 + \gamma)m$ or
- (b) H_1 is spanning, H_1 or H_2 contains an odd cycle, and $H_3 = \emptyset$.

(ii) If H_1, H_2, H_3 are each bipartite, then $H_3 = \emptyset$.

(iii) H_1 and H_3 are monochromatic components and H_2 is the union of two monochromatic components which admit bridges.

(iv) If $V(H_i) \cap V(H_j) \neq \emptyset$, then H_i and H_j are of different colours.

Then there are disjoint monochromatic cycles D_1, D_2, D_3 such that D_i supports paths embeddings in H_i for each $i \in [3]$. Moreover, for $D = D_1 \cup D_2 \cup D_3$, the set $V(D)$ is $\sqrt{\delta}$ -sparse in $\mathcal{V}$, we have $W \subseteq V(D)$, and $|V(G) \setminus V(D)|$ is even.

To complete the proof, we aim to extend the cycles D_1, D_2, D_3 to monochromatic cycles partitioning $V(G)$. To do this, we will replace in each D_i some of the edges f_e (as in Definition 4.5) with longer paths P_e of the correct colour which are contained in the regular pair corresponding to $e \in E(\mathcal{G})$ such that, overall, we cover all vertices of G . Any such path uses the same number t'_e of vertices in each of the two clusters, which is the reason why we require in the conclusion of Lemma 4.6 that $|V(G) \setminus V(D)|$ is even.

But how do we choose the numbers t'_e ? For explaining this, we remark first that our strategy is to embed all the paths P_e vertex-disjointly using only one application of the blow-up lemma (Lemma 3.10) with $\mathcal{G}' = \mathcal{M}$. We need t'_e to be large for all edges $e \in \mathcal{M}$ (for satisfying condition (B1) of Lemma 3.10). More precisely, we let $z = \lfloor \frac{n}{4m} \rfloor$ and require for all edges e that $t'_e \geq \omega(e)z$ where $\omega(f) = 2$ if e is on an edge of $\mathcal{M}$, $\omega(f) = 1$ if e is on a cycle of $\mathcal{M}$, and $\omega(e) = 0$ otherwise. Changing variables by setting

$$(4.1) \quad t_e = t'_e - \omega(e)z = t'_e - \omega(e)\lfloor \frac{n}{4m} \rfloor,$$

this is equivalent to requiring $t_e \geq 0$. Moreover, as we need to cover all vertices in each cluster $V_i \in \mathcal{G}$, we require

$$(4.2) \quad |V_i \setminus V(D)| = \sum_{e \in E(\mathcal{G}): i \in e} t'_e = \sum_{e \in E(\mathcal{G}): i \in e} (t_e + \omega(e)z) = 2z + \sum_{e \in E(\mathcal{G}): i \in e} t_e.$$

The following lemma guarantees that such non-negative values t_e exist. When we apply this lemma, we shall set $t_i = |V_i \setminus V(D)| - 2z$, hence all these t_i will be very similar in size.

Lemma 4.7 (Balancing lemma). *Given m and $0 < \gamma \leq \frac{1}{2}$, if $t \geq 5m/\gamma$ the following holds. Let H be a connected graph on m vertices that is γ -robust Tutte. Let $(t_i)_{i \in V(H)}$ be integers such that $\sum_i t_i$ is even and such that $t_i = (1 \pm \frac{1}{5}\gamma)t$ for each $i \in V(H)$. Then there exist non-negative integers $(t_e)_{e \in E(H)}$ such that for each $i \in V(H)$ we have $\sum_{e \in E(H): i \in e} t_e = t_i$.*

We prove Lemma 4.7 in Section 7. This concludes our proof sketch in the case that $\mathcal{G}$ is not close to an extremal colouring.

The final case to be considered is that $\mathcal{G}$ is close to an extremal colouring but two components admit bridges. In this case we let H_2 be the union of these two components, and let H_1, H_3 be the other two components (it is possible that $H_3 = \emptyset$). It turns out that the above proof sketch also works in this setting.

We are now ready to give the details of the proof of Theorem 1.1.

Proof of Theorem 1.1. We will start by fixing some constants. Given $\beta > 0$ we may assume that $\beta \leq 1/3$. We choose $\gamma, d, \delta > 0$, and m_0 such that

$$(4.3) \quad \frac{1}{m_0} \ll \delta \ll d \ll \gamma \ll \beta$$

are small enough so that we can apply Lemma 4.2 with input $\frac{1}{m_0} \ll \gamma$, Lemma 4.4 with input $\frac{1}{m_0} \ll d \ll 4\gamma \ll \beta$ and Lemma 4.6 with input $\frac{1}{m_0} \ll \delta \ll d \ll \beta$. The blow-up lemma (Lemma 3.10) with input

$$\Delta = 2, \quad \alpha = \frac{1}{5}, \quad \zeta = d/2, \quad d/2 \quad \text{and} \quad \kappa = 2$$

provides us with constants $\varepsilon' > 0$ and $\rho > 0$. We choose ε such that $0 < 2\varepsilon \leq \varepsilon'$, and

$$(4.4) \quad \varepsilon \ll \delta.$$

Note that our choices guarantee we can use Lemma 4.4 with input $\varepsilon, \frac{1}{m_0} \ll d \ll 4\gamma \ll \beta$ and Lemma 4.6 with input $\varepsilon, \frac{1}{m_0} \ll \delta \ll d \ll \beta$. The regularity lemma (Lemma 3.8) provides us with a constant M for input ε and m_0 . Finally, we choose n_0 such that

$$(4.5) \quad n_0^{-1} \ll M^{-1} \leq m_0^{-1}$$

is sufficiently small for Lemma 3.10, Lemma 4.4, and Lemma 4.6 (with inputs as set above, but with M instead of m_0). Assume that $n \geq n_0$. We further assume that our choice of constants in (4.3), (4.4), and (4.5) is such that these constants are chosen sufficiently small in terms of each other for various estimates we will use in the proof, for example for the application of Lemma 4.7 below.

Given an n -vertex graph G with minimum degree $\delta(G) \geq (\frac{2}{3} + \beta)n$, whose edges are coloured with red and blue, let $\mathcal{V} = \{V_1, \dots, V_m\}$ be a regular partition of G with exceptional set V_0 , as provided by Lemma 3.8 for input ε, m_0, d . We have $m_0 \leq m \leq M$. Let $\mathcal{G}$ be the corresponding coloured (ε, d) -reduced multi-graph of G . By (3.3) and by the choice of γ we have $\delta(\mathcal{G}) \geq (\frac{2}{3} + 8\gamma)m$. Hence we can apply Lemma 4.2 with input m and γ to the coloured multi-graph $\mathcal{G}$, which tells us that $\mathcal{G}$ is (4γ) -extremal or there are monochromatic components $C_1, C_2, C_3 \subseteq \mathcal{G}$ such that $C := \bigcup C_i$ spans $\mathcal{G}$ and (i)–(iv) of Lemma 4.2 hold. We distinguish three cases and define subgraphs $H_1, H_2, H_3 \subseteq \mathcal{G}$ as follows:

- (a) $\mathcal{G}$ is not (4γ) -extremal. In this case we set $H_i := C_i$ for each $i = 1, 2, 3$.
- (b) $\mathcal{G}$ is (4γ) -extremal and some pair of components admit bridges.
 - (1) If $\mathcal{G}$ is close to the colouring of Definition 4.1(a), with one red component R_1 and two blue components B_1, B_2 which admit bridges, we set $H_1 := R_1$, $H_2 := B_1 \cup B_2$ and $H_3 = \emptyset$.
 - (2) If $\mathcal{G}$ is close to the colouring of Definition 4.1(a), with one blue component B_1 and two red components R_1, R_2 which admit bridges, we set $H_1 := B_1$, $H_2 := R_1 \cup R_2$ and $H_3 = \emptyset$.
 - (3) If $\mathcal{G}$ is close to the colouring of Definition 4.1(b), with two red components R_1, R_2 and two blue components B_1, B_2 such that (B_1, B_2) admits bridges, we set $H_1 := R_1$, $H_2 := B_1 \cup B_2$ and $H_3 = R_2$.
 - (4) If $\mathcal{G}$ is close to the colouring of Definition 4.1(b), with two red components R_1, R_2 and two blue components B_1, B_2 such that (R_1, R_2) admits bridges, we set $H_1 := B_1$, $H_2 := R_1 \cup R_2$ and $H_3 = B_2$.
- (c) None of the above cases apply.

In case (c), by Lemma 4.4 there exist three monochromatic cycles which partition the vertices of G , as desired. We may therefore assume one of the other two cases holds. Let H be the uncoloured spanning subgraph of $\mathcal{G}$ whose edges are all pairs lying in at least one of H_1, H_2 and H_3 . We claim that (in either of the two cases) all of the following hold.

- (H: i) H is γ -robust Tutte.
- (H: ii) One of the following holds
 - (a) $|\bigcup_{i \neq j} V(H_i) \cap V(H_j)| \geq (1/3 + \gamma)m$ or
 - (b) H_1 is spanning, H_1 or H_2 contains an odd cycle, and $H_3 = \emptyset$.
- (H: iii) If H_1, H_2, H_3 are each bipartite, then $H_3 = \emptyset$.
- (H: iv) H_1 and H_3 are monochromatic components of $\mathcal{G}$, while H_2 is the union of two monochromatic components of $\mathcal{G}$ of the same colour which admit bridges.
- (H: v) If $V(H_i) \cap V(H_j) \neq \emptyset$, then H_i and H_j are of different colours.
- (H: vi) H is connected.

These statements hold in case (a) by Lemma 4.2, with each H_i being a distinct monochromatic component of $\mathcal{G}$ (and so $H_2 = C_2 \cup C_3$, and as observed after Definition 4.3, (C_2, C_3) admits bridges trivially). In case (b), H_1 and H_3 are distinct monochromatic components of $\mathcal{G}$ and H_2 is by construction the union of two monochromatic components of $\mathcal{G}$ of the same colour which admit bridges, hence we get (H: iv). Further, there are no other other monochromatic components in $\mathcal{G}$ by Definition 4.1, hence we get (H: vi) and $H = \mathcal{G}$ has minimum degree $\delta(H) \geq 2m/3$. It follows that no stable set has more than $m/3$ vertices and every stable set has a neighbourhood of size at

least $2m/3$, giving (H:i). It is easy to verify that also (H:ii)(a) and (H:iii) hold. From this point on, we do not need to distinguish cases (a) and (b); the above list of properties is all we need.

By (H:i) and Theorem 3.2, there is an (uncoloured) perfect 2-matching $\mathcal{M}$ in H . Let V^{Exc} consist of V_0 together with, for each $i \in V(\mathcal{G})$, all vertices $v \in V_i$ such that v does not have typical degree in any pair (V_i, V_j) with $ij \in E(\mathcal{M})$ in any colour such that $\mathcal{G}$ has an edge ij of this colour. By (3.1), there are at most $4\varepsilon|V_i|$ such vertices in each V_i , so that V^{Exc} is δ -sparse in $\mathcal{V} = \{V_1, \dots, V_m\}$.

We apply Lemma 4.6 to G with the partition $\mathcal{V}$ with reduced multi-graph $\mathcal{G}$, and with $W = V^{\text{Exc}}$. The conditions of the lemma are satisfied by (H:ii), (H:iii) and (H:iv). We obtain pairwise vertex disjoint monochromatic cycles D_1, D_2, D_3 in G , such that for each i the colour of D_i is the same as the colour of H_i , and D_i supports path embeddings in H_i . Furthermore, for $D := D_1 \cup D_2 \cup D_3$ the set $V(D)$ is $\sqrt{\delta}$ -sparse in $\mathcal{V}$, we have $V^{\text{Exc}} \subseteq V(D)$, and

$$(4.6) \quad |V(G) \setminus V(D)| \text{ is even.}$$

For each $e \in H$, choose some $p(e) \in \{1, 2, 3\}$ such that $e \in E(H_{p(e)})$, and let f_e be the edge of $D_{p(e)}$ witnessing that $D_{p(e)}$ supports path embeddings for e .

It remains to extend the cycles D_1, D_2, D_3 to cycles covering all of G , for which we shall use the blow-up lemma. This lemma requires a reduced graph without multi-edges, and hence we fix a (coloured) subgraph $\mathcal{G}^*$ of $\mathcal{G}$ by deleting for each multi-edge in $\mathcal{G}$ one of the two parallel edges, with the constraint that if $e \in H$ then the edge of the same colour as f_e is kept. Observe that this does not create problems with connectivity: Since D_i supports path-embeddings in H_i and we shall use this to extend D_i by merely replacing some of its edges by longer paths, the resulting graph will still be a cycle. Hence it only remains to check that we can do these extensions so that all vertices of G get covered. For this we shall use the blow-up lemma with reduced graph $\mathcal{G}^*$ and with $\mathcal{G}' = \mathcal{M}$.

Let us next check that the vertices not covered by D_1, D_2 , and D_3 indeed form a regular partition which is super-regular on the pairs corresponding to $\mathcal{M}$, so that we can apply the blow-up lemma. For each $i \in [m]$ let $V_i^* = V_i \setminus V(D)$ and let G^* be a graph on vertex set $V(G) \setminus V(D)$ that contains for each $ij \in \mathcal{G}^*$ of some colour χ the edges of G in the regular pair (V_i, V_j) of colour χ that have one endpoint in V_i^* and the other in V_j^* . Consider the partition $\mathcal{V}^* = \{V_1^*, \dots, V_m^*\}$. Observe that since $V^{\text{Exc}} \subseteq V(D)$, for each $ij \in \mathcal{M}$ and for any vertex in $v \in V_i^*$ we have

$$\deg_{G^*}(v, V_j^*) \geq (d - \varepsilon)|V_j| - \sqrt{\delta}|V_j| > d|V_j^*|/2.$$

Moreover, by (3.2) for each edge $ij \in E(\mathcal{G}^*)$ the pair (V_i^*, V_j^*) in G^* is $(2\varepsilon, d/2)$ -regular. In other words, G^* with the partition $\mathcal{V}^*$ is $(2\varepsilon, d/2)$ -regular on $\mathcal{G}^*$ and (V_i^*, V_j^*) is $(2\varepsilon, d/2)$ -super-regular for each $ij \in \mathcal{M}$. Hence G^* , with the partition $\mathcal{V}^*$ and the graph $\mathcal{G}^*$ with subgraph $\mathcal{G}' = \mathcal{M}$, satisfies Lemma 3.10(B2).

As explained before, for edges $e \in H$ we shall replace the edge f_e (which is part of some D_i) by a path in the same colour of length t'_e . Let us now choose the values t'_e with the help of Lemma 4.7. To this end, set $z := \lfloor \frac{n}{4m} \rfloor$ and for each $i \in [m]$, set $t_i := |V_i^*| - 2z$. Observe that $\sum_{i \in [m]} t_i = |V(G^*)| - 2mz$, which is even by (4.6). Moreover, since $(1 - \varepsilon)n/m \leq |V_i| \leq n/m$ holds by Lemma 3.8(b), and since $V(D)$ is $\sqrt{\delta}$ -sparse in $\mathcal{V}$, we have $(1 - \varepsilon)n/m - \sqrt{\delta}n/m \leq |V_i^*| \leq n/m$ and hence $t_i = (1 \pm 3\sqrt{\delta})n/(2m) = (1 \pm \frac{1}{5}\gamma)n/(2m)$ for each $i \in V(H)$. Since H is γ -robust Tutte by (H:i) and connected by (H:vi), we can apply Lemma 4.7 to obtain non-negative integers $(t_e)_{e \in E(H)}$ such that $\sum_{e \in E(H): i \in e} t_e = t_i$ holds for each i . Setting $t'_e := t_e + \omega(e)z$, where $\omega(f) = 2$ if e is on an edge of $\mathcal{M}$, $\omega(f) = 1$ if e is on a cycle of $\mathcal{M}$, and $\omega(e) = 0$ otherwise, we obtain

$$(4.7) \quad \sum_{e \in E(H): i \in e} t'_e = |V_i^*| \quad \text{and} \quad t'_e \geq z \geq \frac{n}{5m}$$

for each $i \in [m]$ and each $e \in \mathcal{M}$.

We now construct a graph F containing the paths that we shall embed with the help of the blow-up lemma, and define a partition $V(F) = X_1 \cup \dots \cup X_m$ as follows. For each $e = ij \in H$ with $t'_e > 0$, we add a path P_e with $2t'_e$ vertices to F , whose vertices we put alternately to X_i and X_j . Thus all edges of F lie on pairs (X_i, X_j) such that $ij \in H$. For each $i \in V(H)$, we let $\tilde{X}_i \subseteq X_i$ be the vertices lying on paths between X_i and some X_j such that $ij \in \mathcal{M}$. By (4.7) we have $|\tilde{X}_i| \geq \frac{n}{5m} \geq |V_i^*|/5$ and $|X_i| = |V_i^*|$ for each $i \in [m]$. Hence F and the partition $X_1 \cup \dots \cup X_m$ satisfy Lemma 3.10(B1).

Finally, we set up image restrictions I_x for $x \in V(F)$ as follows, to guarantee that when we embed the paths in F they connect appropriately to the corresponding D_i . If $x \in X_i$ has degree 2 in F , we set $I_x = V_i^*$. Otherwise, x is an end-vertex of a path whose vertices are in $X_i \cup X_j$ and by definition, the edge $e = ij$ lies in $H_{p(e)}$. In this case let z be the end-vertex of $f_e \in E(D_{p(e)})$ that is in X_j and let I_x be the G^* -neighbours z' of z in X_i^* . Observe that by definition of G^* each such edge zz' has the same colour as f_e . Since D_i supports path embeddings in H_i , the vertex z is typical in (V_i, V_j) and hence has at least $(d - \varepsilon)|V_i| - \sqrt{\delta}|V_i| \geq \frac{1}{2}d|V_i^*|$ neighbours in V_i^* . Further, since for each edge e of $\mathcal{G}^*$ we choose a unique $f_e \in E(D)$, for each $i \in [m]$ there are at most $m - 1 \leq \rho|X_i|$ vertices $x \in X_i$ with $I_x \neq V_i^*$. Hence the I_x satisfy Lemma 3.10(B3).

Thus, applying Lemma 3.10, we find an embedding ϕ of F into G^* such that for each $x \in V(F)$ we have $x \in I_x$. We combine the embedding of F with the cycles D_1, D_2, D_3 , by replacing for each $e \in H$ such that $t'_e > 0$ the edge $f_e = uv$ on the cycle $D_{p(e)}$ with a path as follows. Assume without loss of generality that $u \in V_i$ and $v \in V_j$ and that the first vertex w of $\phi(P_e)$ is in V_j and the last vertex w' of $\phi(P_e)$ is in V_i . Then we replace f_e by the path $u, \phi(P_e), v$ to obtain a longer monochromatic cycle. This is possible since $\phi(P_e)$ has the same colour χ as f_e by the definition of G^* , and since, by the definition of the sets I_x , also uw and $w'v$ have colour χ . We thus obtain three monochromatic cycles D'_1, D'_2, D'_3 , whose vertices partition $V(G)$. $\square$

5. FINDING COMPONENTS: THE PROOF OF LEMMA 4.2

This section is dedicated to the proof of Lemma 4.2. Given $\gamma > 0$, let $G := \mathcal{G}$ be a red and blue edge coloured graph on $n := m$ vertices and of minimum degree at least $(2/3 + 8\gamma)n$. Suppose that the colouring of G is not (4γ) -extremal. Our goal is to find monochromatic components $C_1, C_2, C_3 \subseteq G$ whose union spans G and satisfies (i)–(iv) of Lemma 4.2.

We start by showing that G is spanned by two monochromatic components.

Claim 5.1. *There are monochromatic components C_1, C_2 that together span G .*

Proof. Let $\mathcal{C}$ be the set of monochromatic components of G . Let $v_1, \dots, v_q$ be a maximal number of vertices such that for distinct $i, j \in [q]$, vertices v_i and v_j are both in distinct red and in distinct blue components. Then

$$\frac{2}{3}nq \leq \sum_{i \in [q]} \deg(v_i) = \sum_{i \in [q]} d_{\text{red}}(v_i) + \sum_{i \in [q]} d_{\text{blue}}(v_i) \leq 2(n - q),$$

and therefore, $q \leq 2$. Now consider the bipartite graph H whose partition classes are the red and blue components respectively and which has an edge CC' whenever $V(C) \cap V(C') \neq \emptyset$. Observe that the size of any matching in H is at most $q \leq 2$. So by König's theorem, the edges of H are covered by two vertices. These vertices are monochromatic components spanning G . $\square$

We fix monochromatic components C_1 and C_2 that together span the vertices of G and distinguish three cases:

Case 1: One of C_1 and C_2 is spanning,

Case 2: C_1 and C_2 have distinct colours,

Case 3: C_1 and C_2 have the same colour.

We introduce the notion of contracting sets, which will be convenient when arguing about part (i) of Lemma 4.2.

Definition 5.2. For a spanning subgraph $H \subseteq G$, we call a stable set $S \subseteq V(G)$ *contracting* in H , if $|N_H(S)| < |S| + \gamma n$.

Note that if S is contracting in H , then $2|N_H(S)| - \gamma n < |S| + |N_H(S)| \leq n$ and, therefore,

$$(5.1) \quad |N_H(S)| < (1/2 + \gamma/2)n.$$

5.1. Case 1: One of C_1 and C_2 is spanning. We assume that $R := C_1$ is spanning and red. We will show that either there exist one or two blue components which together with R satisfy the conditions of Lemma 4.2, or the colouring is (4γ) -extremal as in Definition 4.1(a) in contradiction to our assumption.

Let us begin with the following observation. For a collection of blue components $\{B_i, i \in I\}$, consider a subgraph $H = R \cup \bigcup_{i \in I} B_i \subseteq G$, and assume that some stable set S is contracting in H . If B is a blue component with $v \in S \cap V(B)$, then

$$(5.2) \quad |S| + \gamma n > |N_H(S)| \geq \deg_H(v) \geq (2/3 + 8\gamma)n - |V(B)|.$$

Furthermore, we must have that $B \neq B_i$, $i \in I$, since otherwise $|N_H(S)| \geq \deg_H(v) = \deg_G(v) \geq (2/3 + 8\gamma)n$, in contradiction to (5.1). Hence, for any B_i , $i \in I$, we have

$$(5.3) \quad S \cap V(B_i) = \emptyset.$$

The next claim summarizes a few observations on contracting sets.

Claim 5.3. *Let $H = R \cup \bigcup_{i \in I} C_i$ for blue components C_i , $i \in I$. Suppose S is a contracting set in H , and let $B_1, \dots, B_t$ be the blue components that have non-empty intersection with S . Then*

$$(a) \quad t \leq 2 \text{ and } |V(\bigcup_{j \in [t]} B_j)| \geq (1/3 + 3\gamma)n.$$

In addition, we have the following:

$$(b) \quad \text{If } S \text{ intersects with a bipartite blue component, then } t = 1.$$

$$(c) \quad \text{If } t = 2, \text{ then } V(G) \setminus N_H(S) \subseteq V(\bigcup_{j \in [t]} B_j) \text{ and } V(G) \setminus V(\bigcup_{j \in [t]} B_j) \subseteq N_H(S).$$

Before we prove Claim 5.3 let us note that, by (5.3), the sets $V(B_j)$, $j \in [t]$, are distinct from the sets $V(C_i)$, $i \in I$.

Proof. Set $W = V(G) \setminus (N_H(S) \cup S)$ and note that

$$(5.4) \quad H \text{ has no edges in } S \text{ and no edges between } S \text{ and } W.$$

For each $j \in [t]$, fix a vertex $v_j \in V(B_j) \cap S$ and observe that

$$(5.5) \quad |(S \cup W) \cap V(B_j)| \geq \deg_{B_j}(v_j, S \cup W) \stackrel{(5.4)}{\geq} (2/3 + 8\gamma)n - |N_H(S)|.$$

To prove $t \leq 2$, let us assume that $t \geq 3$ and obtain a contradiction. Since B_1, B_2, B_3 are vertex disjoint, we have By (5.4) we have

$$|S \cup W| \geq \sum_{i \in [3]} |(S \cup W) \cap V(B_i)| \stackrel{(5.5)}{\geq} (2 + 24\gamma)n - 3|N_H(S)|.$$

This, together with the fact that $V(G) = S \cup N_H(S) \cup W$, contradicts (5.1):

$$\begin{aligned} 3|N_H(S)| + |S \cup W| &\geq (6/3 + 24\gamma)n && \Leftrightarrow \\ 2|N_H(S)| &\geq (3/3 + 24\gamma)n && \Leftrightarrow \\ |N_H(S)| &\geq (1/2 + 12\gamma)n. \end{aligned}$$

Hence $t \leq 2$.

In preparation for proving parts (b) and (c), we show the following:

$$(5.6) \quad \text{If } t = 2, \text{ then } |S| \geq (1/3 + 15\gamma)n.$$

Indeed, assuming that $t = 2$, we have

$$\begin{aligned} |W| &\geq |W \cap V(B_1)| + |W \cap V(B_2)| \\ &\geq |(W \cup S) \cap V(B_1)| - |S \cap V(B_1)| + |(W \cup S) \cap V(B_2)| - |S \cap V(B_2)| \\ &\stackrel{(5.5)}{\geq} 2(2/3 + 8\gamma)n - 2|N_H(S)| - |S \cap (V(B_1) \cup V(B_2))| \\ &\geq (4/3 + 16\gamma)n - 2|N_H(S)| - |S|. \end{aligned}$$

This, together with $V(G) = W \cup S \cup N_H(S)$, gives $|S| + \gamma n > |N_H(S)| \geq (1/3 + 16\gamma)n$, which implies (5.6).

To obtain part (c) suppose that $t = 2$. It follows from $\delta(G) \geq (2/3 + 8\gamma)n$ and (5.6) that any vertex in $V(G) \setminus (N_H(S) \cup S)$ has an edge to S . All such edges are not in H by (5.4), so they are blue and in B_1 or B_2 . Hence $V(G) \setminus N_H(S) \subseteq V(\bigcup_{j \in [t]} B_j)$ and $V(G) \setminus V(\bigcup_{j \in [t]} B_j) \subseteq N_H(S)$, as desired.

To show part (b), let us assume that $t = 2$ and, without loss of generality, B_1 is bipartite. Consider any vertex $v_1 \in S \cap V(B_1)$. By (5.6) and since S is stable in H , the vertex v_1 has a blue neighbour $v_2 \in S$. Clearly v_1, v_2 belong to distinct bipartition classes of B_1 and their neighbourhoods in B_1 are disjoint. Hence

$$|(S \cup W) \cap V(B_1)| \geq \sum_{j=1}^2 \deg_{B_1}(v_j, S \cup W) \stackrel{(5.5)}{\geq} 2(2/3 + 8\gamma)n - 2|N_H(S)|.$$

We combine the last inequality with (5.5) and obtain

$$|S \cup W| \geq \sum_{i \in [2]} |(S \cup W) \cap V(B_i)| \geq (2 + 24\gamma)n - 3|N_H(S)|.$$

Using the fact that $V(G) = S \cup N_H(S) \cup W$, we get again

$$3|N_H(S)| + |S \cup W| \geq (6/3 + 24\gamma)n \stackrel{(5.5)}{\Rightarrow} |N_H(S)| \geq (1/2 + 12\gamma)n,$$

which is a contradiction to (5.1). This proves part (b).

Finally, to complete the proof of part (a), we must show that $|V(\bigcup_{j \in [t]} B_j)| \geq (1/3 + 3\gamma)n$. If $|S| \geq (1/3 + 3\gamma)n$, this follows from $S \subseteq V(\bigcup_{j \in [t]} B_j)$. Otherwise, we have $|S| < (1/3 + 3\gamma)n$, and hence $t = 1$ by (5.6). Then

$$(1/3 + 4\gamma)n > |S| + \gamma n \stackrel{(5.2)}{>} (2/3 + 8\gamma)n - |V(B_1)|$$

from which we obtain $|V(B_1)| > (1/3 + 4\gamma)n$ as desired. $\square$

Now we show that we can select two blue components which together with R satisfy the conditions of Lemma 4.2.

Remark 5.4. *Note that since we always select R and R is spanning, part (iv) of Lemma 4.2 holds trivially.*

Claim 5.5. *If there are two blue components B_1, B_2 with $|V(B_1)|, |V(B_2)| \geq (1/3 + 3\gamma)n$, then Lemma 4.2 is true.*

Proof. Consider blue components B_1, B_2 with $|V(B_1)|, |V(B_2)| \geq (1/3 + 3\gamma)n$. First, suppose that one of R, B_1, B_2 contains an odd cycle. Let $H_2 = R \cup B_1 \cup B_2$. We will show that H_2 satisfies the conditions of Lemma 4.2. If H_2 has a contracting set, then by Claim 5.3 there are (up to) two more blue components which together contain at least $(1/3 + 3\gamma)n$ vertices. But this contradicts $|V(B_1)|, |V(B_2)| \geq (1/3 + 3\gamma)n$. Hence H_2 satisfies Lemma 4.2(i). Moreover, H_2 also trivially satisfies (ii)(a). Since one of R, B_1, B_2 contains an odd cycle, H_2 satisfies Lemma 4.2(iii), and we are done.

Hence we can assume that R, B_1, B_2 are each bipartite. Denote the bipartite partition classes of R by X and Y , where $|X| \geq |Y|$. We claim that either $V(B_1) = X$ and $V(B_2) = Y$ or vice versa. Observe that $\delta(G_{\text{blue}}[X]) \geq (2/3 + 8\gamma)n - |Y| \geq (1/6 + 8\gamma)n$. Since B_1 and B_2 have each order at least $(1/3 + 3\gamma)n$ and $|Y| \leq n/2$, there must be a vertex $x \in X$ which is also in B_1 or B_2 , say in B_1 .

Let us show that X is contained in $V(B_1)$. Since $\delta(G_{\text{blue}}[X]) \geq (1/6 + 8\gamma)n$, the component B_1 has an edge in X . It follows, using $\delta(G_{\text{blue}}[X]) \geq (2/3 + 8\gamma)n - |Y|$ and the bipartiteness of B_1 , that B_1 has at least $(4/3 + 16\gamma)n - 2|Y|$ vertices in X . Take any vertex $x' \in X$. Since

$$(4/3 + 16\gamma)n - 2|Y| + \delta(G_{\text{blue}}[X]) \geq (2 + 24\gamma)n - 3|Y| > |X|,$$

x' must have a blue neighbour in $V(B_1)$. This proves $X \subseteq V(B_1)$.

Since X is contained in $V(B_1)$, it follows that $V(B_2)$ is contained in Y . In fact, we have $V(B_2) = Y$. To see this, recall that $|V(B_2)| \geq (1/3 + 3\gamma)n$. As G has minimum degree $(2/3 + 8\gamma)n$ and Y contains no red edges, every vertex $y \in Y$ must have a blue neighbour in $V(B_2)$, and therefore also belongs to B_2 . Hence $Y \subseteq V(B_2)$. So in fact, we have $V(B_1) = X$ and $V(B_2) = Y$.

Set $H_1 = R \cup B_1$. If H_1 has no contracting sets, then it satisfies the conditions of Lemma 4.2 and we are done. So suppose that S is a contracting set in H_1 . Since vertices in X have degree at least $(2/3 + 8\gamma)n$ in H_1 , the set S must be contained in $Y = V(B_2)$. In the following, we distinguish two cases depending on the size of S .

Let us begin with the case, where $|S| > (1/3 - 8\gamma)n$. Then every vertex of G has a neighbour in S . In particular, every vertex in X has a red neighbour in S , hence $X \subseteq N_{H_1}(S)$. So $|Y| + \gamma n \geq |S| + \gamma n \geq |N_{H_1}(S)| \geq |X|$, and hence $|X| \geq |Y| \geq |X| - \gamma n$. It follows that the colouring is γ -extremal as in Definition 4.1(a), which contradicts our initial assumption.

Now suppose that $|S| \leq (1/3 - 8\gamma)n$. We show that this results a contradiction. Suppose first that $G[S]$ contains an edge vw . Such an edge must be blue and in B_2 . Using the bipartiteness of

B_2 , we obtain that

$$\begin{aligned} |V(B_2)| &\geq \deg_G(v, V(G) \setminus X) + \deg_G(w, V(G) \setminus X) \\ &\geq (4/3 + 16\gamma)n - 2|N_{H_1}(S)| \\ &\geq (4/3 + 14\gamma)n - 2|S| \\ &\geq (2/3 + 30\gamma)n, \end{aligned}$$

where we used that S is contracting in the penultimate line. Now suppose that $G[S]$ contains no edge. It follows, for any $v \in S$, that

$$\begin{aligned} |V(B_2)| &\geq |S| + \deg_G(v, V(G) \setminus N_{H_1}(S)) \\ &\geq |S| + (2/3 + 8\gamma)n - |N_{H_1}(S)| \\ &\geq (2/3 + 7\gamma)n. \end{aligned}$$

In both cases, we have $|V(B_2)| \geq 2n/3$, which contradicts $|V(B_1)| \geq (1/3 + 3\gamma)n$. $\square$

Denote the three blue components of largest order by B_1, B_2, B_3 , where $|B_1| \geq |B_2| \geq |B_3|$. Let $H_1 = R \cup R_1$ and, for $i \in \{2, 3\}$, let $H_i = R \cup B_1 \cup B_i = H_1 \cup B_i$.

Claim 5.6. *Let $i \in \{2, 3\}$. If H_i has no contracting set, then Lemma 4.2 holds.*

Proof. Suppose that H_i has no contracting set for some $i \in \{2, 3\}$.

First we deal with the case, when $|V(B_1) \cup V(B_2)| < (1/3 + \gamma)n$. Here, we shall show that H_1 satisfies the conditions of Lemma 4.2. To begin, observe that all blue components distinct from B_1 have order at most $(1/6 + \gamma/2)n$. It follows that H_1 has minimum degree at least $\delta(G) - (1/6 + \gamma/2)n > (1/2 + 7\gamma)n$ and, therefore, contains no contracting sets (see (5.1)). So H_1 satisfies Lemma 4.2(i). If R contains an odd cycle, then H_1 also satisfies Lemma 4.2(ii)(b) and (iii), and we are done. Thus we may assume that R is bipartite. Let $V(R) = X \cup Y$ be a bipartition of R , where $|X| \geq |Y|$. Note that $\delta_{\text{blue}}(G[X]) \geq (2/3 + 8\gamma)n - |Y| \geq (1/6 + 8\gamma)n$. Since B_1 is the only blue component of order (possibly) more than $(1/6 + \gamma)n$, it follows that $X \subseteq V(B_1)$. Thus $|V(B_1)| \geq |X| \geq n/2$ in contradiction to the assumption of $|V(B_1) \cup V(B_2)| < (1/3 + \gamma)n$.

Hence we may assume that $|V(B_1) \cup V(B_2)| \geq (1/3 + \gamma)n$. This means that H_2 satisfies Lemma 4.2(ii)(a). If $|V(B_1) \cup V(B_3)| \geq (1/3 + \gamma)n$, then also H_3 satisfies Lemma 4.2(ii)(a). Otherwise, $|V(B_1) \cup V(B_3)| < (1/3 + \gamma)n$ and all blue components distinct from B_1 and B_2 have order at most $(1/6 + \gamma/2)n$. It follows that H_2 has minimum degree at least $\delta(G) - (1/6 + \gamma/2)n > (1/2 + 7\gamma)n$ and, therefore, contains no contracting sets (see (5.1)). So, we may assume $i = 2$. In either case, H_i satisfies Lemma 4.2(ii)(a).

If one of R, B_1 and B_i is not bipartite, then H_i satisfies Lemma 4.2(iii), and Lemma 4.2 holds. Hence assume that R, B_1 and B_i are each bipartite. As before, let $V(R) = X \cup Y$ be a bipartition of R with $|X| \geq |Y|$.

We need to distinguish between two sub-cases: either $B_1 \cap X \neq \emptyset$ or $V(B_1) \subseteq Y$. Suppose first that B_1 has at least one vertex in X . We will show that $X \subseteq V(B_1)$. To this end, note that $\delta_{\text{blue}}(G[X]) \geq \delta(G) - |Y| \geq (1/6 + 8\gamma)n$. Since B_1 is bipartite and has at least one edge in X , it has at least $2\delta_{\text{blue}}(G[X]) \geq (1/3 + 16\gamma)n$ vertices in X . Using that $\delta(G) \geq (2/3 + 8\gamma)n$, we obtain that any vertex x in X has a neighbour x' in $V(B_1) \cap X$. Since R is bipartite, xx' must be blue, hence $x \in V(B_1)$ and $X \subseteq V(B_1)$.

The situation, where $|X| < (1/2 + 2\gamma)n$ is easily resolved. Indeed, if $|X| < (1/2 + \gamma)n$, then $|V(B_i)| \geq 2\delta(G_{\text{blue}}[Y]) \geq 2((2/3 + 8\gamma)n - |X|) \geq (1/3 + 12\gamma)n$, where we used the bipartiteness of B_i . Since $|V(B_1)| \geq |V(B_i)|$, we are done by Claim 5.5.

Thus we can assume that $|X| \geq (1/2 + 2\gamma)n$. Now it is not hard to see that $H_1 = R \cup B_1$ satisfies the conditions of Lemma 4.2. Note that (ii)(a) and (iii) are trivially satisfied as $X \subseteq V(B_1)$. If H_1 has no contracting sets, then (i) also holds, and we are done. So assume that S is a contracting set in H_1 . By Claim 5.3 there are blue components $Q_1, \dots, Q_t$ with $t \leq 2$ such that $|V(\bigcup_{j \in [t]} Q_j)| \geq (1/3 + 3\gamma)n$. Note that $t = 2$, since otherwise $|V(B_1)| \geq |V(Q_1)| \geq (1/3 + 3\gamma)n$ and we are done by Claim 5.5 again. So, by Claim 5.3(c), we have $X \subseteq V(B_1) \subseteq N_{H_1}(S)$. Together with $|X| \geq (1/2 + 2\gamma)n$, this contradicts (5.1). This concludes the analysis of the case when B_1 intersects with X .

Now suppose that $V(B_1) \subseteq Y$. We claim that no bipartite blue component has a vertex in X . Indeed, assume otherwise. Then such a component has order at least $(1/3 + 16\gamma)n$ due to being bipartite and $\delta(G_{\text{blue}}[X]) \geq (2/3 + 8\gamma)n - |Y| \geq (1/6 + 8\gamma)n$. Hence $|V(B_1)| \geq (1/3 + 16\gamma)n$ by

the maximality of B_1 , and we are done again by Claim 5.5. So blue components with vertices in X are not bipartite. In particular, $V(B_i) \subseteq Y \setminus V(B_1)$.

Next, we show that B_{5-i} has a vertex in X . Indeed, assume otherwise. Owing to $\delta(G_{\text{blue}}[X]) \geq (1/6 + 8\gamma)n$ and B_3 being the third largest component, it follows that $|V(B_1)| \geq |V(B_2)| \geq |V(B_3)| \geq (1/6 + 8\gamma)n$. Hence $|Y| \geq |V(B_1)| + |V(B_2)| + |V(B_3)| > n/2$, which contradicts $|X| \geq |Y|$. So B_{5-i} intersects with X , which implies in particular that B_{5-i} is not bipartite.

To finish, we show that $H_{5-i} = R \cup B_1 \cup B_{5-i}$ satisfies the conditions of Lemma 4.2. Note that $|V(B_{5-i})| \geq (1/6 + 8\gamma)n$ as $\delta(G_{\text{blue}}[X]) \geq (1/6 + 8\gamma)n$. Hence $|V(B_1) \cup V(B_{5-i})| \geq (1/3 + 16\gamma)n$, which gives (ii)(a). Moreover, (iii) holds as B_{5-i} is not bipartite. If H_{5-i} has no contracting sets, then we also have the condition (i). Hence assume that S is a contracting set in H_{5-i} .

By Claim 5.3 applied to H_{5-i} , there are blue components $Q_1, \dots, Q_t$ with $t \leq 2$ such that $|V(\bigcup_{j \in [t]} Q_j)| \geq (1/3 + 3\gamma)n$. As before, we observe that $t = 2$, since otherwise $|V(B_1)| \geq |V(Q_1)| \geq (1/3 + 3\gamma)n$, and we are again done by Claim 5.5. As B_i is bipartite, it follows that $B_i \neq Q_1, Q_2$ by Claim 5.3(b). Moreover, by Claim 5.3(c), we have $V(B_1) \cup V(B_2) \cup V(B_3) = V(B_1) \cup V(B_{5-i}) \cup V(B_i) \subseteq N_{H_{5-i}}(S)$. On the other hand, $|V(B_3)| \geq (1/6 + \gamma)n$ since one of Q_1 and Q_2 has size at least $(1/6 + \gamma)n$ and B_3 is the blue component of third largest order. Hence, $|N_{H_{5-i}}(S)| \geq |V(B_1 \cup B_2 \cup B_3)| \geq (1/2 + 3\gamma)n$ in contradiction to (5.1). $\square$

By Claim 5.6, we can assume that H_2 and H_3 have each a contracting set, which we denote by S_2 and S_3 , respectively.

Claim 5.7. *We have $|S_2|, |S_3| \geq (1/2 - \gamma)n$.*

Proof. Fix $i \in \{2, 3\}$. By Claim 5.3 applied to H_i , there are blue components $Q_1, \dots, Q_t$ for $t \leq 2$, which together cover at least $(1/3 + 3\gamma)n$ vertices. If $t = 1$, then Q_1 has order at least $(1/3 + 3\gamma)n$. In this case, B_1 (as the largest blue component) has also order at least $(1/3 + 3\gamma)n$, and we are done by Claim 5.5. Hence we have $t = 2$. Now Claim 5.3(c) yields $V(G) \setminus N_{H_i}(S_i) \subseteq V(\bigcup_{j \in [t]} Q_j)$ and, therefore, we must have $V(B_1 \cup B_i) \subseteq N_{H_i}(S_i)$. Since B_1, B_2, B_3 are three blue components of largest order and $|V(B_1)| \geq |V(B_2)| \geq |V(B_3)|$, we have $|V(Q_1)| + |V(Q_2)| \leq |V(B_1)| + |V(B_i)|$. Together, this gives

$$\begin{aligned} n - |S_i| - \gamma n &\leq |V(G) \setminus N_{H_i}(S_i)| \leq |V(Q_1)| + |V(Q_2)| \leq |V(B_1)| + |V(B_i)| \\ &\leq |N_{H_i}(S_i)| \leq |S_i| + \gamma n. \end{aligned}$$

Hence $|S_i| \geq (1/2 - \gamma)n$. $\square$

Claim 5.8. *We have $|V(B_2) \cap S_3|, |V(B_3) \cap S_2| \geq (1/4 - 4\gamma)n$.*

Proof. Since $|S_2| \geq (1/2 - \gamma)n$ by Claim 5.7, the vertices in S_2 have at least $(1/6 + 7\gamma)n$ blue neighbours in S_2 . It follows from Claim 5.3 that S_2 is contained in at most two blue components, one of which has order at least $(1/4 - \gamma)n$. Thus $|V(B_1)| \geq |V(B_2)| \geq |V(B_3)| \geq (1/4 - \gamma)n$ by the maximality of B_1, B_2, B_3 . Recall that $H_2 = R \cup B_1 \cup B_2$ and, therefore, $S_2 \cap B_1 = \emptyset$ and $S_2 \cap B_2 = \emptyset$ by (5.3). As $|S_2| \geq (1/2 - \gamma)n$, $|V(B_1) \cup V(B_2)| \geq (1/2 - 2\gamma)n$ and $V(B_1) \cup V(B_2) \subseteq V(G) \setminus S_2$, the component B_3 has at most $3\gamma n$ vertices outside of S_2 . Consequently, $|V(B_3) \cap S_2| \geq (1/4 - 4\gamma)n$.

Analogously, as $|S_3| \geq (1/2 - \gamma)n$ by Claim 5.7 and $V(B_1) \cup V(B_3) \subseteq V(G) \setminus S_3$, the component B_2 has at most $3\gamma n$ vertices outside of S_3 . As a result, we have $|V(B_2) \cap S_3| \geq (1/4 - 4\gamma)n$. $\square$

Note that $V(B_1)$ has vertices in neither S_2 nor S_3 by (5.3) and $|V(B_1)| \geq |V(B_2)| \geq |V(B_2) \cap S_3| \geq (1/4 - 4\gamma)n$. So, by Claim 5.7, there is a vertex $v \in S_2 \cap S_3$. By (5.3) and the definition of H_2 and H_3 , the vertex v is not in $V(B_2 \cup B_3)$. At the same time, v has no red neighbours in $S_2 \cup S_3$ since S_2, S_3 are stable in H_2, H_3 respectively. Therefore, v has no neighbours in $S_2 \cap V(B_3)$ and no neighbours $S_3 \cap V(B_2)$. Since B_2 and B_3 are distinct blue components, these sets are disjoint. Moreover, by Claim 5.8, B_2 and B_3 have each size at least $(1/4 - 4\gamma)n$. Hence v has degree at most $n - 2(1/4 - 4\gamma)n = (1/2 + 8\gamma)n$ in contradiction with $\delta(G) \geq (2/3 + 8\gamma)n$. This concludes the analysis of the case, where one of C_1 and C_2 is spanning.

5.2. Case 2: C_1 and C_2 have distinct colours. Suppose that $R := C_1$ is red and $B := C_2$ is blue. If R or B is spanning, we proceed as in Subsection 5.1. We will show that R, B and another red component together satisfy the conditions of Lemma 4.2.

Since neither R nor B is spanning, there are vertices $v_R \in V(R) \setminus V(B)$ and $v_B \in V(B) \setminus V(R)$. Note that

$$(5.7) \quad |N_G(v_R) \cap N_G(v_B)| \geq (1/3 + 16\gamma)n.$$

Moreover, any common neighbour of v_R and v_B must lie in $V(R) \cap V(B)$ (if for instance, v_R has a neighbour $v \in V(B) \setminus V(R)$ then the edge $v_R v$ cannot be red and it cannot be blue, a contradiction). It follows that

$$(5.8) \quad |V(R) \cap V(B)| \geq (1/3 + 16\gamma)n.$$

Similarly, observe that, for $i \in \{1, 2\}$,

$$(5.9) \quad |V(R) \setminus V(B)| \leq |V(G) \setminus N_G(v_B)| < (1/3 - 8\gamma)n$$

and hence $|V(B)| \geq (2/3 + 8\gamma)n$. By symmetry, the same is true for colours flipped.

Let R' be the red component of largest order in $B - V(R)$. Note that any vertex v'_B in $B - V(R \cup R')$ is incident to fewer than $(1/6 - 4\gamma)n$ red edges. Indeed, otherwise $|V(R')| \geq (1/6 - 4\gamma)n$ (by maximality of R'), and so v'_B sends a red edge to R' by (5.9) (with colours flipped).

We claim that $H := R \cup B \cup R'$ satisfies items (i)–(iv) of Lemma 4.2. As by Claim 5.1, $R \cup B$ spans G , part (iv) is trivial and part (ii) holds because of (5.8). By (5.7), and because of our assumption on the minimum degree of G , we know that $G[N(v_R) \cap N(v_B)]$ contains an edge. If this edge is red, v_R is in a red triangle, and if it is blue, v_B is in a blue triangle. Hence part (iii) follows.

It remains to show that part (i) is satisfied. To this end, we assume there is a contracting set S in H and show that this results in a contradiction. Observe that any vertex v in $V(R) \cap V(B)$ or in $V(R')$ has degree

$$\deg_H(v) \geq (2/3 + 8\gamma)n.$$

Also, recall that by definition of R' , any vertex $v'_B \in V(B) \setminus V(R \cup R')$ has $\deg_{\text{red}}(v'_B) < (1/6 - 4\gamma)n$ and so

$$\deg_H(v'_B) > (2/3 + 8\gamma)n - (1/6 - 4\gamma)n = (1/2 + 12\gamma)n.$$

On the other hand, since S is contracting, we have $|N_H(S)| \leq (1/2 + \gamma/2)n$ by (5.1). Since $|N_H(S)| \geq \deg_H(v)$ for any $v \in S$, this means that $S \subseteq V(R) \setminus V(B)$. By (5.9) this gives $|S| \leq (1/3 - 8\gamma)n$. However, any vertex $v'_R \in V(R) \setminus V(B)$ has

$$\deg_H(v'_R) \geq (2/3 + 8\gamma)n - |V(R) \setminus V(B)| \stackrel{(5.9)}{\geq} (1/3 + 16\gamma)n.$$

Thus $|N_H(S)| \geq |S| + 24\gamma n$. So, contrary to our assumption, S is not contracting.

5.3. Case 3: C_1 and C_2 have the same colour. Suppose that $R_1 := C_1$ and $R_2 := C_2$ are both red with

$$(5.10) \quad |V(R_1)| \geq n/2 \geq |V(R_2)|.$$

Let $B_1, \dots, B_t$ be the blue components that intersect with R_2 . If $t = 1$, or if one of the B_i contains R_1 , we proceed as in Subsection 5.2. Hence we can assume that $t \geq 2$, and $R_1 \not\subseteq B_i$, for $1 \leq i \leq t$.

We begin by showing that $t = 2$. To this end, let $v_i \in V(B_i) \cap V(R_2)$, for $i \in [t]$. If $t \geq 3$, then (using (5.10) for the third inequality),

$$\begin{aligned} |V(R_1)| &\geq \sum_{i \in [3]} \deg(v_i, V(R_1)) \geq 3((2/3 + 8\gamma)n - |V(R_2)|) \\ &> 2n - (n + |V(R_2)|) \\ &= n - |V(R_2)|, \end{aligned}$$

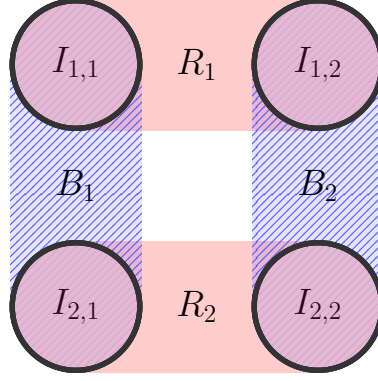
a contradiction. So indeed $t = 2$.

Now let us partition $V(G)$ into non-empty sets

$$I_{i,j} := V(R_i) \cap V(B_j)$$

for $i, j \in [2]$ as illustrated in Figure 3. Since vertices in $I_{i,j}$ have no neighbours in $I_{3-i,3-j}$ and as $\delta(G) \geq (2/3 + 8\gamma)n$, we obtain

$$(5.11) \quad |I_{i,j}| < (1/3 - 8\gamma)n \text{ for } i, j \in [2].$$


 FIGURE 3. The partition of G in Case 3.

In particular,

$$(5.12) \quad \text{each of } R_1, R_2, B_1, B_2 \text{ has at least } (1/3 + 16\gamma)n \text{ vertices,}$$

and moreover,

$$(5.13) \quad \begin{aligned} \delta(G[I_{i,j}]) &\geq \delta(G) - |I_{3-i,j} \cup I_{i,3-j}| \\ &\stackrel{(5.11)}{\geq} (2/3 + 8\gamma)n - 2(1/3 - 8\gamma)n \geq 24\gamma n. \end{aligned}$$

We will show that three components among R_1, R_2, B_1, B_2 can be selected so that they satisfy the conditions of Lemma 4.2 unless the colouring is (4γ) -extremal as in Definition 4.1(b) in contradiction to our assumption. Note that for any such choice, Lemma 4.2(iv) holds since $R_1 \cup R_2, B_1 \cup B_2$ span G and part (ii)(a) follows by (5.12). The next claim yields part (iii).

Claim 5.9. *If one colour has a bipartite component, then the other colour has no bipartite component.*

Proof. Suppose that there is a bipartite component, say R_1 . We claim that

$$(5.14) \quad \text{the bipartition classes of } R_1 \text{ are } I_{1,1} \text{ and } I_{1,2}.$$

Indeed, otherwise one of the two sets, say $I_{1,1}$, contains vertices x, y from different bipartition classes. Clearly, the neighbourhoods of x and y in R_1 are disjoint. Note that the neighbours of x and y in G are in $I_{1,2} \cup B_1$. Thus, each of x and y has at least $(2/3 + 8\gamma)n - |V(B_1)|$ neighbours in $I_{1,2}$. Thus by (5.11),

$$2((2/3 + 8\gamma)n - |V(B_1)|) \leq |I_{1,2}| < (1/3 - 8\gamma)n,$$

implying that $|V(B_1)| > n/2$. Applying the same argument to neighbours x', y' of x, y in $I_{1,2}$, we obtain that $|V(B_2)| > n/2$ as well. This is a contradiction, which proves (5.14).

If B_1 say, is bipartite as well, then the same reasoning shows that one of its bipartition classes is $I_{1,1}$. Consequently $I_{1,1}$ contains no edges, in contradiction to (5.13). This proves the claim. $\square$

It remains to show that either there is a choice of three components among R_1, R_2, B_1, B_2 satisfying Lemma 4.2(i) or the colouring is (4γ) -extremal as in Definition 4.1(b). To this end, assume that no choice of three components among R_1, R_2, B_1, B_2 fulfils Lemma 4.2(i).

For each $i \in [2]$, let S_{R_i} be a contracting set of minimum size in $G - E(R_i)$, and let S_{B_i} be a contracting set of minimum size in $G - E(B_i)$. If $v \in V(R_{3-i})$ then $\deg_G(v) = \deg_{G-E(R_i)}(v)$, and so, $v \notin S_{R_i}$ by (5.1). It follows by symmetry that for $i \in [2]$,

$$(5.15) \quad S_{R_i} \subseteq V(R_i) \text{ and } S_{B_i} \subseteq V(B_i).$$

In fact, we can show slightly more.

Claim 5.10. *For each $i \in [2]$ there is $j \in [2]$ such $S_{R_i} \subseteq I_{i,j}$. Similarly, for each $i \in [2]$ there is $j \in [2]$ such that $S_{B_i} \subseteq I_{j,i}$.*

Proof. Suppose that the claim does not hold for, say S_{R_2} . So, by (5.15), $S_{R_2} \cap V(B_i) \neq \emptyset$ for $i = 1, 2$. Then

$$\begin{aligned} |S_{R_2} \cap V(B_1)| + |S_{R_2} \cap V(B_2)| + 2\gamma n &= |S_{R_2}| + 2\gamma n \\ &> |N_{G-E(R_2)}(S_{R_2})| + \gamma n \\ &> |N_{G-E(R_2)}(S_{R_2})| \\ &\geq |N_{G-E(R_2)}(S_{R_2} \cap V(B_1))| + |N_{G-E(R_2)}(S_{R_2} \cap V(B_2))|, \end{aligned}$$

where the last line follows because the red edges incident with S_{R_2} belong to R_2 and are therefore not present in $G-E(R_2)$. And, moreover, there are no blue edges between $V(B_1)$ and $V(B_2)$. Thus by Definition 5.2, at least one of $S_{R_2} \cap V(B_1)$, $S_{R_2} \cap V(B_2)$ is a contracting set, in contradiction to the minimality of S_{R_2} . $\square$

Claim 5.11. *Let $i, j \in [2]$. If $S_{R_i} \cap V(B_j) \neq \emptyset$, then $I_{3-i,j} \subseteq N_{G-E(R_i)}(S_{R_i})$. Similarly, if $S_{B_i} \cap V(R_j) \neq \emptyset$, then $I_{j,3-i} \subseteq N_{G-E(B_i)}(S_{B_i})$.*

Proof. Suppose the claim is wrong for S_{R_2} and B_1 . Let $v \in S_{R_2} \cap V(B_1)$ and $w \in I_{1,1} \setminus N_{G-E(R_2)}(S_{R_2})$. Then by Claim 5.10, $S_{R_2} \subseteq I_{2,1}$. Since S_{R_2} is contracting in $G-E(R_2)$, we have

$$|S_{R_2}| + \gamma n > |N_{G-E(R_2)}(S_{R_2})| \geq \deg_{G-E(R_2)}(v) \geq (2/3 + 8\gamma)n - |V(R_2)|.$$

Moreover, as w has its neighbours in G in $I_{2,1} \cup R_1$,

$$|I_{2,1} \setminus S_{R_2}| \geq \deg_{G-E(R_2)}(w, I_{2,1}) \geq (2/3 + 8\gamma)n - |V(R_1)|.$$

Summing the two inequalities (recalling that $S_{R_2} \subseteq I_{2,1}$) gives

$$|I_{2,1}| > (4/3 + 15\gamma)n - |V(R_1 \cup R_2)| = (1/3 + 15\gamma)n,$$

in contradiction to (5.11). $\square$

Claim 5.12. *For each $i, j \in [2]$, the set $I_{i,j}$ contains at most one of the sets $S_{R_1}, S_{R_2}, S_{B_1}, S_{B_2}$.*

Proof. Suppose the claim is wrong for $i = j = 2$, namely, we suppose that $S_{R_2}, S_{B_2} \subseteq I_{2,2}$. Then by Claim 5.11,

$$(5.16) \quad I_{2,1} \subseteq N_{G-E(B_2)}(S_{B_2}) \text{ and } I_{1,2} \subseteq N_{G-E(R_2)}(S_{R_2}).$$

As S_{R_2} and S_{B_2} are each contracting, this gives

$$|I_{2,1}| < |S_{B_2}| + \gamma n \text{ and } |I_{1,2}| < |S_{R_2}| + \gamma n.$$

So, for any $v \in I_{2,2}$, we have

$$\begin{aligned} \deg_G(v, I_{2,2}) &\geq (2/3 + 8\gamma)n - |I_{1,2}| - |I_{2,1}| \\ &\geq |I_{2,2}| - |I_{1,2}| + |I_{2,2}| - |I_{2,1}| + 24\gamma n \\ &> |I_{2,2}| - |S_{R_2}| + |I_{2,2}| - |S_{B_2}| + 22\gamma n \end{aligned}$$

where the second inequality follows from (5.11). Hence v satisfies at least one of the following:

$$\deg_{\text{blue}}(v, I_{2,2}) > |I_{2,2}| - |S_{R_2}| + 11\gamma n \quad \text{or} \quad \deg_{\text{red}}(v, I_{2,2}) > |I_{2,2}| - |S_{B_2}| + 11\gamma n.$$

Let us call T_B the set of all vertices in $I_{2,2}$ satisfying the first inequality and let T_R be the set of all vertices in $I_{2,2}$ satisfying the second inequality. Note that $T_R \cup T_B = I_{2,2}$. Moreover, vertices in T_B must each have a blue edge to S_{R_2} , and similarly vertices in T_R have each a red edge to S_{B_2} . Putting this together with (5.16), we obtain

$$T_R \cup I_{2,1} \subseteq N_{G-E(B_2)}(S_{B_2}) \text{ and } T_B \cup I_{1,2} \subseteq N_{G-E(R_2)}(S_{R_2}).$$

By definition of T_R and since the set S_{R_2} is stable in $G-E(R_2)$, we have $S_{R_2} \subseteq T_R$. So, because of (5.13),

$$|S_{R_2}| \leq |T_R| \leq |N_{G-E(B_2)}(S_{B_2})| - |I_{2,1}| \leq |S_{B_2}| + \gamma n - 24\gamma n < |S_{B_2}|,$$

and similarly, $|S_{B_2}| < |S_{R_2}|$, a contradiction. $\square$

By Claim 5.10 and Claim 5.12 we can assume without loss of generality that $S_{R_2} \subseteq I_{2,1}$, $S_{B_1} \subseteq I_{1,1}$, $S_{R_1} \subseteq I_{1,2}$ and $S_{B_2} \subseteq I_{2,2}$. Thus Claim 5.11 yields the following inequalities

$$\begin{aligned} |I_{1,1}| &\leq |N_{G-E(R_2)}(S_{R_2})| < |S_{R_2}| + \gamma n \leq |I_{2,1}| + \gamma n; \\ |I_{2,1}| &\leq |N_{G-E(B_2)}(S_{B_2})| < |S_{B_2}| + \gamma n \leq |I_{2,2}| + \gamma n; \\ |I_{2,2}| &\leq |N_{G-E(R_1)}(S_{R_1})| < |S_{R_1}| + \gamma n \leq |I_{1,2}| + \gamma n; \text{ and} \\ |I_{1,2}| &\leq |N_{G-E(B_1)}(S_{B_1})| < |S_{B_1}| + \gamma n \leq |I_{1,1}| + \gamma n. \end{aligned}$$

Hence the colouring is (4γ) -extremal as in Definition 4.1(b), which contradicts our initial assumption. This finishes the proof of Lemma 4.2.

6. COVERING EXCEPTIONAL VERTICES: THE PROOF OF LEMMA 4.6

The proof of Lemma 4.6 goes roughly as follows. We first pick two sparse and pairwise disjoint families of paths P^{Edge} and P^{Exc} such that P^{Edge} supports paths embeddings in the H_i 's and P^{Exc} contains the vertices W . We choose the families such that each of their paths connects to one of the H_i 's. This allows Lemma 3.9 to connect these families up to three monochromatic cycles $C = C_1 \cup C_2 \cup C_3$. Finally, we adjust the parity of $V(G) \setminus V(C)$ using either an odd cycle in one of the H_i 's or the fact that otherwise $H_3 = \emptyset$, in which case we can set C_3 to be a single vertex.

Before we start, let us formalize what we mean by a path connecting to one of the H_i 's.

Definition 6.1 (δ -connect). Suppose G is a 2-edge-coloured graph on n vertices and let $\mathcal{V} = \{V_1, \dots, V_m\}$ be an (ε, d) -regular partition of G . Let $\mathcal{G}$ be the corresponding 2-edge-coloured (ε, d) -reduced multi-graph. For $\delta > 0$, we say that a path $P \subseteq G_\chi$ δ -connects to $H \subseteq \mathcal{G}_\chi$, if both ends of P have at least $\delta|V_x|$ edges of colour χ leading to the same cluster V_x for some $x \in V(H)$.

Proof of Lemma 4.6. Let $1/n \ll \varepsilon, 1/m \ll d \ll \beta \ll 1$ such that Lemma 3.9 holds with input $\varepsilon, d, m, \ell = m + 2, n$. Let t be the size of the clusters $V_1, \dots, V_m$.

Claim 6.2. *Let $S \subseteq V(G)$ be δ -sparse in $\mathcal{V}$. Then there is a family P^{Edge} of at most $2m^2$ monochromatic paths with the following properties:*

- the paths of P^{Edge} are pairwise disjoint and disjoint from S ;
- For each i , the paths of P^{Edge} with the same colour as H_i support path embeddings in H_i ;
- each path of P^{Edge} $(d/2)$ -connects to one of the H_i 's.

Proof. We choose the paths P^{Edge} greedily as follows. For each $i \in [3]$, we order arbitrarily the edges of H_i . When we come to the edge pq of H_i , we choose a 2-edge path with both ends in V_p and the middle vertex in V_q , which is disjoint from previously chosen paths and from S , whose end-vertices both have at least $dt/2$ neighbours in the colour of H_i in V_q . Since at most $2\varepsilon t$ vertices of V_p do not have degree at least $dt/2$ to V_q in the colour of H_i , and at most $6m^2 + \delta|V_p|$ vertices of V_p are contained in previously chosen paths or in S , we are left with at least $\frac{1}{2}|V_p|$ vertices of V_p to choose the endpoints from, and similarly at least $\frac{1}{2}|V_q|$ vertices of V_q to choose the middle vertex from. Since $V_p V_q$ is an edge of H_i , by definition of ε -regularity there is such a 2-edge path in the colour of H_i as desired. $\square$

The next two claims take care of the exceptional vertices.

Claim 6.3. *Suppose that $|\bigcup_{i \neq j} V(H_i) \cap V(H_j)| \geq m/3$. Let S be δ -sparse in $\mathcal{V}$. Then there is a family P^{Exc} of at most β^{-2} monochromatic paths with the following properties:*

- the paths of P^{Exc} are pairwise disjoint and disjoint from S ;
- the set $V(P^{\text{Exc}})$ contains W and is $\sqrt{\delta}$ -sparse in $\mathcal{V}$;
- each path of P^{Exc} $(d/2)$ -connects to one of the H_i 's.

Proof. Let us define

$$\mathcal{Z} = \bigcup_{1 \leq i \neq j \leq 3} V(H_i) \cap V(H_j) \subseteq V(\mathcal{G}), \quad \mathcal{Z} = \bigcup_{x \in \mathcal{Z}} V_x \subseteq V(G).$$

We set $r = \lceil 4/\beta \rceil$. By assumption we have $|\mathcal{Z}| \geq \frac{m}{3}$ and hence

$$|\mathcal{Z}| \geq (n - |V_0|)|\mathcal{Z}|/m \geq n/3 - \varepsilon n.$$

So as $\delta(G) \geq (2/3 + \beta)n$, it follows that $\deg_G(w, \mathcal{Z}) \geq \beta n/2 \geq 2n/r$ for any $w \in V(G)$. We partition W by setting $W_1 := \{v \in W : \deg_{\text{red}}(v, \mathcal{Z}) \geq n/r\}$ and $W_2 := W \setminus W_1$. For $i \in [2]$

define an auxiliary graph A_i on vertex set W_i , by connecting two vertices $v, w \in W_i$ if $|N_i(v, Z) \cap N_i(w, Z)| \geq n/r^3$. We claim that

$$(6.1) \quad \text{the independence number of } A_i \text{ is bounded by } r.$$

Indeed, suppose otherwise and let $w_1, \dots, w_{r+1}$ be pairwise disjoint non-adjacent vertices in A_i . We then obtain the following contradiction.

$$\begin{aligned} |Z| &\geq \left| \bigcup_{p \in [r+1]} N_i(w_p, Z) \right| \\ &\geq (r+1) \frac{n}{r} - \sum_{1 \leq q < p \leq r+1} |N_i(w_q, Z) \cap N_i(w_p, Z)| \\ &\geq |Z| \left(\frac{r+1}{r} - \frac{\binom{r+1}{2}}{r^3} \right) > |Z|. \end{aligned}$$

This proves (6.1). Now a classic result of Pósa (see [30]) guarantees that A_i can be partitioned into $r_i \leq r$ disjoint cycles $C_{i,1}, \dots, C_{i,r_i}$, where we consider edges and vertices to be cycles of lengths respectively 2 and 1.

To finish, we build from each $C_{i,j}$ a path $P_{i,j}$ of colour i that $(d/2)$ -connects to one of H_1, H_2, H_3 , such that the union of these paths is $\sqrt{\delta}$ -sparse in $\mathcal{V}$. In addition to this, we require the paths $P_{i,j}$ to be pairwise disjoint and disjoint from S . This can be done greedily as follows. We break $C_{i,j}$ into a path by removing one edge, and then create $P_{i,j}$ by connecting one after another two adjacent vertices of this path with a vertex in Z .

To see that we never run out of available vertices in this process, note that by definition of r each two adjacent vertices in one of the $C_{i,j}$'s share $(\beta/4)^3 n$ common neighbours of colour i in Z . So each two adjacent vertices in one of the $C_{i,j}$'s have at least $(\beta^3/8)t \geq (d/2)t$ common neighbours in some cluster of $\mathcal{Z}$. Moreover, the restriction of $\sqrt{\delta}$ -sparseness and disjointness renders at most

$$\frac{|W|}{\sqrt{\delta}t} \cdot t + 2|W| + |S| \leq 3\sqrt{\delta}n$$

vertices unavailable at any point of embedding.

Finally, note that since there is an edge in $C_{i,j}$ between the first and last vertices w and z of $P_{i,j}$, so w and z have at least n/r^3 common neighbours in Z in colour i . By averaging, some cluster V_p of $\mathcal{Z}$ contains at least t/r^3 of these common neighbours. Picking $H_{i,j}$ to be one of H_1, H_2, H_3 which contains V_p and is of colour i (which must exist since V_p is in two of the H_1, H_2, H_3 and these two have different colours by condition (iv) of Lemma 4.6) we see that $P_{i,j}$ is $(d/2)$ -connected to $H_{i,j}$. $\square$

Claim 6.4. *Suppose that (i)(b) of the assumptions of Lemma 4.6 holds. Let S be δ -sparse in $\mathcal{V}$. Then there is a family P^{Exc} of at most β^{-2} monochromatic paths and a monochromatic cycle C_3 with the following properties:*

- *the paths of P^{Exc} are pairwise disjoint and disjoint from S ;*
- *the set $V(P^{Exc}) \cup V(C_2)$ contains W and is $\sqrt{\delta}$ -sparse in $\mathcal{V}$;*
- *each path of P^{Exc} $(d/2)$ -connects to H_1 .*

Proof. The proof is very similar to the one of Claim 6.3. Without loss of generality we can assume that H_1 is red. We define

$$\mathcal{Z} = V(H_1) = V(\mathcal{G}), \quad Z = \bigcup_{x \in \mathcal{Z}} V_x \subseteq V(G).$$

By the assumptions we have

$$|Z| \geq (n - |V_0|)|\mathcal{Z}|/t \geq (1 - \varepsilon)n.$$

It follows that $\deg_G(w, Z) \geq (2/3 + \beta/2)$ for any $w \in V(G)$. As before, we use set $r = \lceil 4/\beta \rceil$. We can partition W by setting $W_1 := \{v \in W : \deg_{\text{red}}(v, Z) \geq n/r\}$ and $W_2 := W \setminus W_1$. For the set W_1 we proceed as in Claim 6.3. For W_2 , however, note we now have $\deg_2(v, Z) \geq (2/3 + \beta/4)n$ for every $v \in W_2$. Hence we can greedily construct a single blue cycle C_3 for the vertices W_2 , which has the desired properties. $\square$

Now we can finish the proof of Lemma 4.6. If H_i has no edges for some $i \in [3]$ we replace it with $H_i = \emptyset$. Since H_1 and H_3 are monochromatic components, and H_2 is the union of at most two monochromatic components, if H_i has no edges then it has at most two vertices. So after this step the assumptions of the lemma still hold with (i)(a) relaxed to $|\bigcup_{i \neq j} V(H_i) \cap V(H_j)| \geq (1/3 + \gamma/2)m$. Next, if $H_i = \emptyset$ for some $i \in [2]$, we take C_i to be the empty set. If $H_3 = \emptyset$ and we are not in case (i)(b), we also take C_3 to be the empty set. (Note that in case (i)(b), C_3 plays a more active role as detailed in Claim 6.4.) In all other cases we construct the C_i 's as follows.

Let us first handle the case, where (i)(a) of the assumptions of Lemma 4.6 holds. By assumption (iii), if $H_2 \neq \emptyset$, then H_2 is the union of monochromatic components F_1 and F_2 , which admit bridges (Definition 4.3). Hence there are distinct vertices u, u' in G , which each d -connect to both F_1 and F_2 , and we let $P^{\text{Bridge}} = \{u, u'\}$ be the set of these two one-vertex paths. If $H_2 = \emptyset$ then we set $P^{\text{Bridge}} = \emptyset$. We fix a family of paths P^{Edge} obtained from an application of Claim 6.2 with $S = V(P^{\text{Bridge}})$. Next we fix a family of paths P^{Exc} of size at most β^{-2} obtained from an application of Claim 6.3 with $S = V(P^{\text{Edge}} \cup P^{\text{Bridge}})$. Note that $P^{\text{Bridge}} \cup P^{\text{Edge}} \cup P^{\text{Exc}}$ is $2\sqrt{\delta}$ -sparse in $\mathcal{V}$ and contains at most $2 + 2t^4 + \beta^{-2}$ paths.

We partition the paths $P^{\text{Edge}} \cup P^{\text{Exc}} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ such that each path in $\mathcal{P}_i$ is $(d/2)$ -connected to H_i . For H_2 we partition $\mathcal{P}_2 = \mathcal{P}_{2,1} \cup \mathcal{P}_{2,2}$ further such that each path in $\mathcal{P}_{2,j}$ is $(d/2)$ -connected to F_j . To finish we apply Lemma 3.9 to connect the $\mathcal{P}_i$'s up to three monochromatic cycles. At this point it is important that H_1, H_3, F_1, F_3 are monochromatic connected subgraphs.

For H_1 we orient the paths of $\mathcal{P}_1$ arbitrarily and then choose an arbitrary cyclic order of $\mathcal{P}_1$. We then use Lemma 3.9 to connect one of the dm neighbours of the end of each path of $\mathcal{P}_1$ with one of the dm neighbours of the start of the next path of $\mathcal{P}_1$ (in the chosen order). Note that since H_1 is a connected graph on at most m vertices, any such connection requires a path with at most $m + 1$ vertices. This way we obtain the cycle C_1 and similarly C_3 . For H_2 we want to use the paths P^{Bridge} . So for each $j \in [2]$ we orient the paths of $\mathcal{P}_{2,j}$ arbitrarily, then choose an order on the paths of $\mathcal{P}_{2,j}$ which starts in a cluster with many colour- H_2 -neighbours of u . We then proceed as before to obtain the cycle C_2 , except that after connecting the paths of $\mathcal{P}_{2,j}$ we add one further connecting path to a colour- H_2 -neighbour of u' .

Recall that $P^{\text{Bridge}} \cup P^{\text{Edge}} \cup P^{\text{Exc}}$ is $2\sqrt{\delta}$ -sparse in $\mathcal{V}$ and contains at most $2 + 2t^4 + \beta^{-2}$ paths. Since each of the application of Lemma 3.9 yields a path of order at most $2t^2$ and $(2 + 2t^4 + \beta^{-2})2t^2 \ll \sqrt{\delta}m$, it follows that $C = C_1 \cup C_2 \cup C_3$ is $(3\sqrt{\delta})$ -sparse in $\mathcal{V}$. Hence by setting S to contain the vertices of all paths chosen so far in each of the above applications of Lemma 3.9, we can guarantee that the C_i 's are vertex disjoint.

It remains to ensure that $V(G) \setminus C$ has even size. So suppose this is not the case. If each of the H_i 's is bipartite, then $H_3 = \emptyset$ by assumption and so $C_3 = \emptyset$. In this case we set $C_3 = v$ for some unused vertex $v \in V(G) \setminus V(C)$. On the other hand, suppose, F_1 say, has an odd cycle. Note that this implies that F_1 contains walks of any parity between any two vertices. Let P be one of the paths obtained by Lemma 3.9 and which connects two paths of $\mathcal{P}_{2,1}$. We use Lemma 3.9 to swap P for a path P' whose order has the opposite parity to that of P . Then $C' = C_1 \cup C_2 \cup C_3 \setminus P \cup P'$ has the desired parity. The cases where H_1, H_3 , or F_2 have an odd cycle are handled the same way. This finishes the proof for the case (a).

Now assume that (i)(b) of the assumptions of Lemma 4.6 holds. We proceed very similarly. The main difference is that the paths returned by Lemma 6.4 correspond only to H_1 and H_2 , i.e. $\mathcal{P}_3 = \emptyset$. So for H_1, H_2 we can proceed as in the case of (a) and obtain cycles C_1, C_2 . Finally, we take C_3 to be the cycle returned by Lemma 6.4, which is possible as by (b) we have $H_3 = \emptyset$. It remains to ensure that $V(G) \setminus C$ has even size. As in case (b) we are guaranteed that H_1 or H_2 contains an odd cycle, we may swap a path P for a path P' whose order is of the opposite parity as we did in case (a) to obtain a set of cycles of the desired parity. $\square$

7. BALANCING: THE PROOF OF LEMMA 4.7

In this section we prove Lemma 4.7. The idea is simple. First we set some t_e equal to one and others to zero such that the shortfall at each vertex is even. Then we create an auxiliary graph H' obtained by blowing up each vertex of H to exactly half its shortfall number of vertices, and find a 2-matching in H' ; this tells us how much to increment each t_e by.

Proof of Lemma 4.7. Suppose $t \geq m/\gamma$ and let H be a connected m -vertex graph that is γ -robust Tutte. For a vertex i in H denote by E_i the set of edges in H such that $i \in e$.

We start with adjusting parity for i by defining π_e for $e \in E(H)$ such that $t_i - \sum_{e \in E_i} \pi_e$ is even. For this purpose let T be an arbitrary spanning tree in H with root r . The idea is that for the leaves i of this tree, the value π_e for the single edge e containing i will adjust the parity of t_i . Then we will work upwards from the leaves to the root, iteratively adjusting the parity of t_i for a vertex i by appropriately setting π_e for the unique edge e containing i in the path from i to r . How we set this π_e depends on the value of the $\pi_{e'}$ for the other edges containing i that we set previously.

Formally, for each $ij = e \in E(T)$ with j closer to the root than i , let $X(e)$ denote the set of vertices of the component of $T - e$ that contains i . If $e \notin E(T)$ then we set $\pi_e = 0$. For $e \in E(T)$ let $\pi_e = (\sum_{j \in X(e)} t_j) \bmod 2$. We have for each $i \in V(H)$ that

$$(7.1) \quad t_i - \sum_{e \in E_i} \pi_e = t_i - \sum_{e \in E(T) \cap E_i} \pi_e = t_i - \sum_{e \in E(T) \cap E_i} \left(\left(\sum_{j \in X(e)} t_j \right) \bmod 2 \right).$$

Now, let $e_1, \dots, e_s$ be the edges in E_i , where e_1 is the unique edge among these such that j is closer to the root than i . Observe that we have $X(e_\ell) \cap X(e_{\ell'}) = \emptyset$ for each $2 \leq \ell < \ell' \leq s$ and $X(e_1) = \{i\} \cup \bigcup_{2 \leq \ell \leq s} X(e_\ell)$. This implies that in the last sum in (7.1) the terms appearing are exactly the following: The term t_i appears once, and for each $j \in X(e_1) \setminus \{i\}$ the term t_j appears twice. We conclude that the quantity $2n_i := t_i - \sum_{e \in E_i} \pi_e$ is an even integer, so that n_i is an integer. Since $t \geq 5m/\gamma$ we have $|E_i| \leq m \leq \gamma t/5$ for each i , and so because $t_i = (1 \pm \frac{1}{5}\gamma)t$ we get $n_i = (1 \pm \frac{2}{5}\gamma)t/2$.

In the second part of this proof, we will define non-negative integers t''_e such that

$$(7.2) \quad \sum_{e \in E_i} t''_e = 2n_i.$$

This suffices to prove the lemma since setting $t_e := t''_e + \pi_e$ gives non-negative values such that $\sum_{e \in E_i} t_e = 2n_i + \sum_{e \in E_i} \pi_e = t_i$ as desired.

For obtaining these t''_e , consider the graph H' obtained from H by blowing up each i into a stable set W_i of n_i vertices, and replacing edges with complete bipartite graphs. Let S' be a stable set in H' . We aim to show $|N_{H'}(S')| \geq |S'|$. Without loss of generality, we can assume S' is a union of the blow-ups of some vertices $S \in V(H)$ (otherwise we can add any vertices missing from such a blow-up to S' without changing the left hand side), where S must be a stable set in H . Then, since H is γ -robust Tutte, we have

$$\begin{aligned} |N_{H'}(S')| &= \sum_{i \in N_H(S)} n_i \geq (1 - \tfrac{2}{5}\gamma) \tfrac{t}{2} |N_H(S)| \\ &\geq (1 - \tfrac{2}{5}\gamma)(1 + \gamma) \tfrac{t}{2} |S| \geq \frac{1 - \frac{2}{5}\gamma}{1 + \frac{2}{5}\gamma} (1 + \gamma) |S'| \geq |S'|, \end{aligned}$$

where the last inequality uses $\gamma \leq \frac{1}{2}$. Thus by Theorem 3.2 there is a perfect 2-matching $\mathcal{M}$ in H' . For $e = ij \in E(H)$, recall that $W_i \times W_j$ is the blow-up of this edge in H' . Now set $t''_e := \sum_{f \in W_i \times W_j} \omega(f)$, where $\omega(f) = 2$ if e is on an edge of $\mathcal{M}$, $\omega(f) = 1$ if e is on a cycle of $\mathcal{M}$, and $\omega(e) = 0$ otherwise. We have

$$\sum_{e \in E_i} t''_e = \sum_{ij \in E_i} \sum_{f \in W_i \times W_j} \omega(f) = \sum_{w \in W_i} \sum_{f \in E(H') : w \in f} \omega(f) = n_i \cdot 2,$$

where the last equality uses that $|W_i| = n_i$, the definition of ω and the fact that $\mathcal{M}$ is a perfect 2-matching. Thus we obtain (7.2), as required. $\square$

8. EXTREMAL CASES: THE PROOF OF LEMMA 4.4

For the proof of Lemma 4.4 it is natural to distinguish two cases corresponding to the two cases in the definition of extremal colourings. For both cases we choose

$$1/n \ll \varepsilon, 1/m \ll d \ll \gamma \ll \beta,$$

suitable for various estimates and we suppose G is a graph on n vertices with minimum degree $\delta(G) \geq (2/3 + \beta)n$ whose edges are coloured with red and blue, and with a coloured (ε, d) -reduced multi-graph $\mathcal{G}$ on m vertices corresponding to an (ε, d) -regular partition $\mathcal{V} = \{V_1, \dots, V_m\}$ of G with exceptional set V_0 . By (3.3) we have

$$\delta(\mathcal{G}) \geq (2/3 + \beta/2)m.$$

Further, we let G' be the “cleaned” coloured subgraph of G on vertex set $V(G) \setminus V_0$ corresponding to the (ε, d) -reduced graph $\mathcal{G}$ with properties as guaranteed by Lemma 3.8: In particular, each red (blue) edge of G' is in some pair (V_i, V_j) such that ij is a red (blue) edge of $\mathcal{G}$. Further, let G'' be the spanning subgraph of G which coincides with G' on $V(G) \setminus V_0$ and has all G -edges incident to vertices in V_0 . Clearly,

$$\delta(G'), \delta(G'') \geq (2/3 + \beta - 2d - \varepsilon)n \geq (2/3 + \beta/2)n.$$

8.1. Case 1: A colouring as in Definition 4.1(a). We start with the case that the coloured reduced multi-graph has one spanning bipartite connected red component, and exactly two blue components and these are bipartite as well and do not admit bridges.

Proof of Lemma 4.4 when $\mathcal{G}$ has a colouring as in Definition 4.1(a). Let R be the spanning, red say, bipartite component of $\mathcal{G}$ with bipartition classes $\mathcal{X}_R$ and $\mathcal{Y}_R$. As the colouring is (4γ) -extremal we have $||\mathcal{X}_R| - |\mathcal{Y}_R|| \leq 4\gamma m$ and hence

$$(8.1) \quad m/2 - 4\gamma m \leq |\mathcal{X}_R|, |\mathcal{Y}_R| \leq m/2 + 4\gamma m.$$

Set $X_R := \bigcup_{i \in \mathcal{X}_R} V_i$ and $Y_R := \bigcup_{i \in \mathcal{Y}_R} V_i$. In particular, any vertex x in X_R (in Y_R , respectively) sends red edges in G' only to vertices in Y_R (in X_R , respectively) and blue edges in G' only to vertices in X_R (in Y_R , respectively). Let B_j with $j = 1, 2$ be the bipartite blue components of $\mathcal{G}$ with bipartition classes $\mathcal{X}_{B_j}, \mathcal{Y}_{B_j}$. Suppose that $V(B_1) = \mathcal{X}_R$ and $V(B_2) = \mathcal{Y}_R$. Let $X_{B_j} := \bigcup_{i \in \mathcal{X}_{B_j}} V_i$ and $Y_{B_j} := \bigcup_{i \in \mathcal{Y}_{B_j}} V_i$ for $j = 1, 2$. Thus $\{X_{B_1}, Y_{B_1}\}$ is a partition of X_R and $\{X_{B_2}, Y_{B_2}\}$ is a partition of Y_R .

Let us next estimate the sizes of the $\mathcal{X}_{B_j}, \mathcal{Y}_{B_j}$. By (8.1), we have

$$\deg_{G'_{\text{blue}}}(x, \mathcal{X}_R), \deg_{G'_{\text{blue}}}(y, \mathcal{X}_R) \geq (1/6 + 4\gamma)m$$

for any $x, y \in \mathcal{X}_R$. If xy is additionally a (blue) edge, then bipartiteness implies $N_{G'_{\text{blue}}}(x, \mathcal{X}_R) \cap N_{G'_{\text{blue}}}(y, \mathcal{X}_R) = \emptyset$ and hence

$$(8.2) \quad |\mathcal{X}_{B_1}|, |\mathcal{Y}_{B_1}| \leq |\mathcal{X}_R| - (1/6 + 4\gamma)m \stackrel{(8.1)}{\leq} m/3.$$

By symmetry we also have $|\mathcal{X}_{B_2}|, |\mathcal{Y}_{B_2}| \leq m/3$. We conclude for $j = 1, 2$ that

$$(8.3) \quad |X_{B_j}|, |Y_{B_j}| \leq n/3.$$

We next distribute the exceptional vertices in V_0 as follows. We let

$$\begin{aligned} X'_R &= \{v \in V(G) : \deg_{G''_{\text{blue}}}(v, Y_R) \leq \frac{1}{10}\beta n\}, \\ Y'_R &= \{v \in V(G) : \deg_{G''_{\text{blue}}}(v, X_R) \leq \frac{1}{10}\beta n\}. \end{aligned}$$

Clearly, $X_R \subseteq X'_R$ since vertices in X_R send no blue edges to Y_R in G' (and hence in G''); analogously, $Y_R \subseteq Y'_R$. By (8.1) this implies

$$(8.4) \quad n/2 - 5\gamma n \leq |X'_R|, |Y'_R| \leq n/2 + 5\gamma n.$$

Moreover, since by assumption (B_1, B_2) does not admit bridges, $V(G) \setminus (X'_R \cup Y'_R)$ contains at most one vertex; call this vertex z^* if it exists, and otherwise we say z^* does not exist.

Without loss of generality suppose that $|X'_R| \geq |Y'_R|$. Our goal is to use two blue cycles to cover $|X'_R| - |Y'_R|$ vertices in X'_R and the vertex z^* , and to use a red cycle to cover all remaining vertices. We now prepare for this by establishing some further degree bounds. By definition, any $x \in X'_R$ and $y \in Y'_R$ satisfy

$$\begin{aligned} \deg_{G''_{\text{red}}}(x, Y'_R) &\geq \left(\frac{2}{3} + \frac{1}{2}\beta\right)n - |X'_R| - \frac{1}{10}\beta n - 1 \\ (8.5) \quad &\geq \left(\frac{2}{3} + \frac{1}{2}\beta\right)n - \left(\frac{1}{2} + 5\gamma\right)n - \frac{1}{10}\beta n - 1 \geq \left(\frac{1}{6} + \frac{1}{4}\beta\right)n, \\ \deg_{G''_{\text{red}}}(y, X'_R) &\geq \left(\frac{1}{6} + \frac{1}{4}\beta\right)n. \end{aligned}$$

For $x \in X_R$ and $y \in Y_R$ we know even more: The vertex x is either in X_{B_1} or in Y_{B_1} . In the former (latter) case it sends all its G'' -edges in X_R to Y_{B_1} (to X_{B_1}). This together with (8.3) (and an analogous argument for y) gives

$$\begin{aligned} \deg_{G''_{\text{red}}}(x, Y'_R) &\geq \left(\frac{2}{3} + \frac{1}{2}\beta\right)n - |Y_{B_1}| - |V_0| - \frac{1}{10}\beta n - 1 \\ (8.6) \quad &\geq \left(\frac{1}{3} + \frac{1}{4}\beta\right)n. \\ \deg_{G''_{\text{red}}}(y, X'_R) &\geq \left(\frac{1}{3} + \frac{1}{4}\beta\right)n. \end{aligned}$$

Concerning blue edges incident to $x \in X_R$ and $y \in Y_R$ we know

$$(8.7) \quad \deg_{G'_{\text{blue}}}(x, X_R) \geq (\frac{2}{3} + \frac{1}{2}\beta)n - |Y_R| - |V_0| \geq (\frac{1}{6} + \frac{1}{4}\beta)n.$$

Let us now construct our three cycles. We first construct a blue cycle C_2 with $V(C_2) \subseteq X_R$, of length

$$2\ell = 2\lceil(|X'_R| - |Y'_R|)/2\rceil \leq 10\gamma n + 1 \leq 11\gamma n.$$

For this, observe that the blue bipartite graph $F = G'_{\text{blue}}[X_{B_1}, Y_{B_1}]$ has partition classes of size at most $n/3$ by (8.3) and minimum degree at least $(\frac{1}{6} + \frac{1}{4}\beta)n$ by (8.7). Hence, picking an arbitrary vertex $x \in X_{B_1}$ we can greedily construct two paths P_1, P_2 with $\ell - 1$ edges starting in x which are otherwise vertex-disjoint. Let y_1 be the endpoint of P_1 and y_2 the endpoint of P_2 . Clearly, y_1 and y_2 are in the same partition class of F and thus have a common neighbour x' in $V(F) \setminus (V(P_1) \cup V(P_2))$ since $\gamma \ll \beta$. Thus, P_1, P_2 and x' give the desired blue cycle C_2 on 2ℓ vertices.

Our second cycle is also blue, and takes care of parity and the vertex z^* if it exists. If $|X'_R \setminus V(C_2)| = |Y'_R \setminus V(C_2)|$, we set $C_3 := z^*$ (a one-vertex cycle) if z^* exists, and if z^* does not exist we let C_3 be the zero-vertex cycle. If $|X'_R \setminus V(C_2)| = |Y'_R \setminus V(C_2)| + 1$ and z^* exists, then we choose $y \in X'_R \setminus V(C_2)$ such that yz^* is a blue edge (which exists by definition of z^*) and let C_3 be the 2-vertex blue cycle (y, z^*) . Finally if $|X'_R \setminus V(C_2)| = |Y'_R \setminus V(C_2)| + 1$ and z^* does not exist, we choose any $y \in X'_R \setminus V(C_2)$ and let C_3 be the one-vertex blue cycle z^* . Hence we have $|X'_R \setminus V(C_2 \cup C_3)| = |Y'_R \setminus V(C_2 \cup C_3)|$.

To finish we claim that $H = G''_{\text{red}}[X'_R \setminus V(C_2 \cup C_3), Y'_R \setminus V(C_2 \cup C_3)]$ has a Hamilton cycle C_1 . This follows from Corollary 3.5. Note that the degree conditions are fulfilled, because $\delta(H) \geq (1/6 + \beta/10)n$ by (8.5), and all but at most εn vertices of H have degree at least $(1/3 + \beta/10)n$ by (8.6). $\square$

8.2. Case 2: A colouring as in Definition 4.1(b). In the second case, the reduced graph has two red and two blue components, and there are neither bridges in red nor in blue.

Proof of Lemma 4.4 when $\mathcal{G}$ has a colouring as in Definition 4.1(b). We denote the red and blue components of the reduced multi-graph $\mathcal{G}$ by $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{B}_1, \mathcal{B}_2$ and suppose that there are no red or blue bridges. We set $\mathcal{I}_{i,j} = V(\mathcal{R}_i) \cap V(\mathcal{B}_j)$ for $i, j \in [2]$. We may assume that all but at most $\sqrt{\gamma}m$ vertices $\mathcal{Q}_{i,j} \subseteq V(\mathcal{R}_i) \cap V(\mathcal{B}_i)$ have more than $\sqrt{\gamma}m$ red neighbours in $V(\mathcal{R}_i) \cap V(\mathcal{B}_i)$ if $i = j$ and more than $\sqrt{\gamma}m$ blue neighbours otherwise. Let $I_{i,j} = \bigcup_{x \in \mathcal{I}_{i,j} \setminus \mathcal{Q}_{i,j}} V_x$ for $i, j \in [2]$ and $V'_0 = V_0 \cup \bigcup_{x \in \mathcal{Q}_{i,j}} V_x$. We have

$$(8.8) \quad (1/4 - 5\sqrt{\gamma})n \leq (1 - \varepsilon)|\mathcal{I}_{i,j}|n/m \leq |I_{i,j}| \leq |\mathcal{I}_{i,j}|n/m \leq (1/4 + 5\sqrt{\gamma})n,$$

$$(8.9) \quad |V'_0| \leq |V_0| + 4|\mathcal{Q}_{i,j}|n/m \leq \varepsilon n + 4\sqrt{\gamma}n \leq 5\sqrt{\gamma}n.$$

For $i, j \in [2]$ and any vertex $v \in I_{i,j}$, all G' -neighbours of v in $I_{3-i,j}$ are blue, all G' -neighbours of v in $I_{i,3-j}$ are red, v has no G' -neighbours in $I_{3-j,3-j}$, and all G' -neighbours of v in $I_{i,j}$ are blue if $i = j$ and red otherwise. This implies the following neighbourhood profiles of vertices in $\bigcup_{i,j \in [2]} I_{i,j}$.

Claim 8.1. *For each $i, j \in [2]$, every vertex $v \in I_{i,j}$ we have*

$$\deg_{G'_{\text{blue}}}(v, I_{3-i,j}) \geq (1/6 + \beta/2)n,$$

$$\deg_{G'_{\text{red}}}(v, I_{i,3-j}) \geq (1/6 + \beta/2)n,$$

$$\deg_{G'_c}(v, I_{i,j}) \geq (1/6 + \beta/2)n,$$

where c is blue if $i = j$, and c is red otherwise.

Proof. We will show the third part for a vertex $v \in I_{1,1}$. The other cases follow by analogous arguments. By (8.9), all but $|V'_0| \leq 5\sqrt{\gamma}n$ vertices of G are covered by $\bigcup_{i,j \in [2]} I_{i,j}$. Suppose that $v \in V_x$ for some $x \in \mathcal{I}_{i,j} \setminus \mathcal{Q}_{1,1}$. By definition of $\mathcal{Q}_{1,1}$, it follows that $|N_{\mathcal{G}_{\text{red}}}(x) \cap \mathcal{I}_{1,1}| \leq \sqrt{\gamma}n$. Moreover, we have $|V_y| \leq n/m$ for each $y \in \mathcal{I}_{1,1}$. This and (8.8) implies that

$$\deg_{G'_{\text{blue}}}(v, I_{1,1}) = (2/3 + \beta)n - |I_{1,2}| - |I_{2,1}| - 5\sqrt{\gamma}n - \sqrt{\gamma}m \cdot n/m$$

$\square$

$$\geq (1/6 + \beta/2)n.$$

Note that since $(\mathcal{R}_1, \mathcal{R}_2)$ does not admit red or blue bridges, there is at most one vertex $x_{\text{red}}^* \in V'_0$ with dn red G'' -neighbours in $I_{1,1} \cup I_{1,2}$ and dn red G'' -neighbours in $I_{2,1} \cup I_{2,2}$. If there is no such vertex, we say x_{red}^* does not exist. Similarly, there is at most one vertex $x_{\text{blue}}^* \in V'_0$

with dn blue G'' -neighbours in $I_{1,1} \cup I_{2,1}$ and dn blue G'' -neighbours in $I_{1,2} \cup I_{2,2}$, and if there is no such vertex we say x_{blue}^* does not exist. Let X^* be the set containing the existing vertices among x_{red}^* and x_{blue}^* (so $0 \leq |X^*| \leq 2$).

We shall next distribute the exceptional vertices in V_0' .

Claim 8.2. *We can partition the vertices of $V_0' \setminus X^*$ into sets $\{I'_{i,j}\}_{i,j \in [2]}$ (of size at most $5\sqrt{\gamma}n$) such that every $v \in I'_{i,j}$ satisfies*

$$\deg_{G''_{\text{blue}}}(v, I_{3-i,j}) \geq (1/6 + \beta/2)n, \quad \deg_{G''_{\text{red}}}(v, I_{i,3-j}) \geq (1/6 + \beta/2)n.$$

Proof. Let $v \in V_0' \setminus X^*$. Suppose that v has at least $2dn$ neighbours in each $I_{i,j}$ with $i, j \in [2]$, so that v sends at least dn edges in one or the other (or possibly both) colours to each $I_{i,j}$. If there are three or more $I_{i,j}$ such that v has dn or more red neighbours in each, then by definition we would have $v \in X^*$; the same holds for three or more $I_{i,j}$ such that v has dn or more blue neighbours in each. It follows that there are some two $I_{i,j}$ to which v sends less than dn red edges, and because v is not in X^* these two sets come from a blue component, namely they are $I_{1,j}$ and $I_{2,j}$ for some $j \in [2]$. Now v must send more than dn red edges to each of the sets $I_{1,3-j}$ and $I_{2,3-j}$. But these two sets are in different red components, so $v \in X^*$. This contradiction shows that there are $i, j \in [2]$, such that $\deg(v, I_{i,j}) < 2dn$. Moreover, all but $|V_0'| \leq 5\sqrt{\gamma}n$ vertices are covered by $\bigcup_{i,j \in [2]} I_{i,j}$. By (8.8), this implies that v has at least

$$(2/3 + \beta)n - 2(1/4 + 5\sqrt{\gamma})n - 2dn - 5\sqrt{\gamma}n \geq (1/6 + \beta/2)n$$

neighbours in each of $I_{3-i,j}$ and $I_{i,3-j}$. As $v \notin X^*$, we cannot have that v sends dn red edges to both $I_{i,3-j}$ and $I_{3-i,j}$, nor at least dn blue edges to both. Thus all but at most dn edges from v to $I_{i,3-j}$ are of one colour, and all but at most dn edges from v to $I_{3-i,j}$ are of the other colour, and we can add v either to $I'_{i,j}$ or to $I'_{3-i,3-j}$. $\square$

Now set $J_{i,j} = I_{i,j} \cup I'_{i,j}$ for $i, j \in [2]$. Note that by definition the sets $J_{i,j}$, together with X^* , partition the vertices of G . Observe that by (8.8) and Claim 8.2 we have

$$(8.10) \quad (1/4 - 5\sqrt{\gamma})n \leq |J_{i,j}| = |I_{i,j}| + |I'_{i,j}| \leq (1/4 + 5\sqrt{\gamma})n + 5\sqrt{\gamma}n \leq (1/4 + 10\sqrt{\gamma})n.$$

Suppose $|J_{1,1}| \geq |J_{2,1}|$. Then either we find a partition of $V(G)$ into three cycles, or G has one of two rather special structures. This claim, together with a short case distinction, is enough for Case 2: the claim (with sets $J_{i,j}$ and colours changed appropriately) applies equally if $|J_{2,1}| \geq |J_{2,2}|$, and so on; and we will argue that in at least one of these four situations we do find the cycle partition. Let us now state precisely what structural information we obtain.

Claim 8.3. *Suppose $|J_{1,1}| \geq |J_{2,1}|$. Then one of the following three statements holds.*

- (i) *There are three cycles in G , two blue and one red, whose vertex sets partition $V(G)$.*
- (ii) *x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$, and $|J_{1,1}| = |J_{2,1}|$.*
- (iii) *x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$, and $|J_{1,1}| = |J_{2,1}| + 1$, and either x_{blue}^* does not exist or it exists and has less than $\frac{1}{2}dn$ blue neighbours in $J_{2,1}$.*

Before we prove this claim, we explain why it is enough to complete the proof of Case 2 of Lemma 4.4. To begin with, we may suppose without loss of generality that $J_{1,1}$ is of maximum size among the $J_{i,j}$. Then Claim 8.3 (as written above) applies. Either we are done, or G has one of the structures described in (ii) and (iii) above. We distinguish these two separate cases. Note that in both situations x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$. We will only need to use the extra structure given by (iii) at this point; within the following two cases, when we apply Claim 8.3 we will simply conclude that either we obtain a cycle partition, or the two sets differ in size by at most one and x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$.

Case 2.1: Suppose x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$, and $|J_{1,1}| = |J_{2,1}|$. Because $J_{1,1}$ is maximum size, we have $|J_{2,1}| \geq |J_{2,2}|$. This means we can again apply Claim 8.3. Either we obtain the desired cycle partition, or $|J_{2,1}| = |J_{2,2}|$, or $|J_{2,1}| = |J_{2,2}| + 1$. Either way, we have that x_{blue}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{2,2}$. If $|J_{2,2}| \geq |J_{1,2}|$ we can yet again apply Claim 8.3, and it necessarily returns a partition into three cycles, since otherwise x_{red}^* must have less than $\frac{1}{2}dn$ neighbours in both $J_{1,2}$ and $J_{2,1}$, which violates the minimum degree condition of G . We have $|J_{2,2}| \geq |J_{1,2}|$ except when $|J_{2,1}| - 1 = |J_{2,2}| = |J_{1,2}| - 1$, so it remains to consider this final possibility. In this case we have $|J_{1,2}| \geq |J_{1,1}|$ (in fact, with equality). That means we can yet again apply Claim 8.3. Again it necessarily returns the desired partition into

three cycles since otherwise we are told that x_{blue}^* has less than $\frac{1}{2}dn$ neighbours in $J_{2,2}$. Since we already know it has less than $\frac{1}{2}dn$ neighbours in $J_{1,1}$ this violates the minimum degree condition of G .

Case 2.2: Suppose that x_{red}^* exists and has less than $\frac{1}{2}dn$ neighbours in $J_{1,2}$, and $|J_{1,1}| = |J_{2,1}| + 1$, and either x_{blue}^* does not exist or it does exist and has fewer than $\frac{1}{2}dn$ blue neighbours in $J_{2,1}$. If x_{blue}^* exists, then it links B_1 and B_2 in blue, so it must have at least $\frac{1}{2}dn$ blue neighbours in $J_{1,1}$. The point is that either x_{blue}^* does not exist, or it does but has at least $\frac{1}{2}dn$ neighbours in $J_{1,1}$. This means that if $|J_{2,1}| \geq |J_{2,2}|$, then Claim 8.3 necessarily returns the desired three-cycle partition and we are done. Thus we have $|J_{2,1}| < |J_{2,2}|$, and this is only possible if $|J_{2,2}| = |J_{1,1}|$. So $J_{2,2}$ is of maximum size, and we have $|J_{2,2}| \geq |J_{1,2}|$, which is what we need to apply Claim 8.3. As in Case 1, we must obtain a three-cycle partition, since otherwise x_{red}^* violates the minimum degree condition on G .

What remains to do is to show that Claim 8.3 holds. There are a few ways we might obtain the required cycle decomposition; in any case, the first thing we need to do is construct a suitable red cycle, and the following claim provides several possibilities for this.

Claim 8.4. *The following three types of red cycle C_{red} are in G .*

- C_{red} is contained in $J_{1,2}$.
- if x_{red}^* has at least $\frac{1}{4}dn$ red neighbours in $J_{1,2}$, then C_{red} contains x_{red}^* and its remaining vertices are in $J_{1,2}$.
- if x_{red}^* has at least $\frac{1}{4}dn$ red neighbours in $J_{1,1}$, then C_{red} contains x_{red}^* , two vertices of $J_{1,1}$, and its remaining vertices are in $J_{1,2}$.

Furthermore, in each case we have $|V(C_{\text{red}})| \leq 20\sqrt{\gamma}n$, $|J_{1,2} \setminus V(C_{\text{red}})| \leq |J_{2,2}| - 2$. Additionally, for each $c \in \{\text{red}, \text{blue}\}$, if x_c^* exists and has more than 100 blue-neighbours in $J_{1,2}$, then $V(C_{\text{red}})$ covers at most half of the blue-neighbours of x_c^* in $J_{1,2}$.

Proof. For the first type of cycle, we pick an edge uu' in $I_{1,2}$ and let P be the path (u, u') . For the second, we choose two distinct red neighbours u, u' in $I_{1,2}$ of x_{red}^* and let $P = (u, x_{\text{red}}^*, u')$. For the third, we pick two distinct red neighbours v, v' of x_{red}^* in $I_{1,1}$, let u be a red-neighbour of v in $I_{1,2}$, and $u' \neq u$ be a red-neighbour of v' in $I_{1,2}$, and set $P = (u, v, x_{\text{red}}^*, v', u')$. In each case, we have a path P on at most five vertices whose ends are in $I_{1,2}$.

We now extend P to the desired cycle C_{red} as follows. We first let S^* be a vertex set containing $V(P)$, together with some other vertices as follows. For each $c \in \{\text{red}, \text{blue}\}$, if x_c^* has between 100 and $\frac{n}{1000}$ blue-neighbours in $J_{1,2}$, we add an arbitrary half of them, not contained in P , to S^* ; this is possible since P contains at most five vertices. In total we have $|S^*| \leq 2\frac{n}{2000} + 9 < \frac{n}{100}$. We let $\ell = \max(2, 2 + |J_{1,2}| - |J_{2,2}|)$. By Claim 8.1 we have $\deg_{G_{\text{red}}}''(u, I_{1,2}), \deg_{G_{\text{red}}}''(u', I_{1,2}) \geq (1/6 + \beta/2)n$. Since $|I_{1,2}| = (1/4 \pm 5\sqrt{\gamma})n$ by (8.8), this implies that u and u' have a common neighbour u_1 in $I_{1,2} \setminus (S^* \cup \{u, u'\})$, hence we get a path u, u_1, u' of length $\ell = 2$. We can then repeat this argument to get a common neighbour u_2 of u_1 and u' in $I_{1,2} \setminus (S^* \cup \{u\})$ and hence a path u, u_1, u_2, u' of length $\ell = 3$, and so on. In general we thus obtain a path $u, u_1, \dots, u_{\ell-1}, u'$ of length ℓ by extending the path of length $\ell - 1$ by finding a common neighbour $u_{\ell-1}$ of $u_{\ell-2}$ and u' in $I_{1,2} \setminus (S^* \cup \{u, u_1, \dots, u_{\ell-3}\})$, which we certainly can do since by (8.10) the difference in size between $J_{1,2}$ and $J_{2,2}$ is at most $15\sqrt{\gamma}n$, so $\ell \leq 17\sqrt{\gamma}n$.

We claim this C_{red} has all the required properties. What remains to check is that $|V(C_{\text{red}})| \leq 20\sqrt{\gamma}n$, that $|J_{1,2} \setminus V(C_{\text{red}})| \leq |J_{2,2}| - 2$, and that it does not cover too many blue-neighbours of the vertices of X^* . The first two of these follow by definition of ℓ . So we need to check C_{red} does not cover too many blue-neighbours of the vertices of X^* . If $x_c^* \in X^*$ has less than 100 blue-neighbours in $J_{1,2}$, there is nothing to check. If x_c^* has between 100 and $\frac{n}{1000}$ blue-neighbours in $J_{1,2}$, then at least half are in $S^* \setminus V(P)$ and hence not covered by C_{red} as desired. If x_c^* has more than $\frac{n}{1000}$ blue-neighbours in $J_{1,2}$, then at most $|V(C_{\text{red}})| \leq 20\sqrt{\gamma}n$ are covered by C_{red} , which is less than half as desired. $\square$

Next, we need to construct the two blue cycles covering the vertices left over after C_{red} is constructed. We need one cycle contained in $J_{1,1} \cup J_{2,1} \cup X^*$ and another in $J_{1,2} \cup J_{2,2} \cup X^*$. These two sets satisfy the same conditions, and the following lemma applies to both (though we state it for $J_{1,1} \cup J_{2,1} \cup X^*$). Since at this point C_{red} is not fixed (Claim 8.4 above gives several possible constructions) we simply avoid a vertex set S of size at most $20\sqrt{\gamma}n$.

Claim 8.5. *Let S be a set of at most $20\sqrt{\gamma}n$ vertices. Choose any (possibly empty) disjoint subsets $A, B \subseteq X^* \setminus S$ with the following property: x_c^* may only be in A if it has at least $\frac{1}{8}dn$ blue-neighbours in $J_{2,1} \setminus S$, and x_c^* may only be in B if it has at least $\frac{1}{8}dn$ blue-neighbours in $J_{1,1} \setminus S$. Suppose that*

$$|A \cup J_{1,1} \setminus S| \geq |B \cup J_{2,1} \setminus S|.$$

Then there is a blue cycle in G whose vertex set is $(A \cup B \cup J_{1,1} \cup J_{2,1}) \setminus S$.

Proof. We begin by constructing disjoint blue paths outside S greedily as follows. Given a vertex x of A , choose two blue-neighbours in $I_{2,1} \setminus S$, giving a three-vertex blue path containing x with ends in $I_{2,1} \setminus S$; extend this to a five-vertex blue path with ends in $I_{1,1} \setminus S$ by choosing blue neighbours in $I_{1,1} \setminus S$. This is possible by the condition on A , since $|J_{i,j}| \leq |I_{i,j}| + 5\sqrt{\gamma}n$, and by Claim 8.1. Given a vertex x of B , we create a three-vertex blue path containing x with ends in $I_{1,1} \setminus S$ by simply choosing two blue-neighbours of x in $I_{1,1} \setminus S$. Observe that at each step we have more than 10 choices for the next vertex to pick; since $|A \cup B| \leq 2$ the paths we construct have in total at most 10 vertices, so we can perform the construction vertex-disjointly.

At this point we have $|A \cup B|$ disjoint blue paths with ends in $I_{1,1} \setminus S$. If this is zero paths, choose a blue edge uu' from $I_{1,1} \setminus S$ to $I_{2,1} \setminus S$ and let $P = (u, u')$. If it is one path, let u be one end and let u' be a blue-neighbour in $I_{2,1} \setminus S$ not used on the path: this gives a blue path P from u to u' . If it is two paths, suppose they go from u to v and v' to v'' . Choose a common blue-neighbour w of v and v' in $I_{2,1} \setminus S$, and a different blue neighbour u' of v'' in $I_{2,1} \setminus S$, which are not used on the paths: this gives a blue path P from u to u' . Each of these constructions is possible by Claim 8.1. In any case, P uses the same number of vertices in $J_{1,1} \cup A$ as in $J_{2,1} \cup B$.

We now apply Lemma 3.7 to find a blue path from u to u' covering what remains of $J_{1,1} \setminus S$ and $J_{2,1} \setminus S$. Specifically, the two sets we need to find a Hamilton path in are $(J_{1,1} \setminus (S \cup V(P))) \cup \{u\}$ and $(J_{2,1} \setminus (S \cup V(P))) \cup \{u'\}$, with $I'_{1,1} \setminus (S \cup V(P))$ the vertices which do not necessarily have many blue neighbours in $J_{1,1}$. The required degree conditions hold by Claims 8.1 and 8.2, and since S is small. By assumption $V(P)$ covers an equal number of vertices in $A \cup J_{1,1} \setminus S$ and $B \cup J_{2,1} \setminus S$, so that

$$|(J_{1,1} \setminus S \cup V(P)) \cup \{u\}| \geq |(J_{2,1} \setminus S \cup V(P)) \cup \{u'\}|$$

as required. $\square$

We are now in a position to prove Claim 8.3, which we do by considering a few cases depending on which exist of, and the coloured neighbourhoods of, x_{red}^* and x_{blue}^* .

Proof of Claim 8.3. To begin with, suppose that x_{red}^* does not exist. We apply Claim 8.4 to find a red cycle C_{red} contained in $J_{1,2}$ satisfying the properties listed there. We apply Claim 8.5, with $S = A = B = \emptyset$, to find a blue cycle on vertices $J_{1,1} \cup J_{2,1}$. We now want to apply Claim 8.5 a second time, with $S = V(C_{\text{red}})$, to find a cycle on vertices $X^* \cup J_{2,2} \cup J_{1,2} \setminus S$. To do this, we assign x_{blue}^* , if it exists, to one of A and B such that the condition of Claim 8.5 is satisfied. This is possible for the following reason. If x_{blue}^* exists, it links the blue components and in particular has at least dn blue neighbours in $J_{1,2} \cup J_{2,2}$. So it has at least $\frac{1}{2}dn$ blue-neighbours in one of these sets, and so (by Claim 8.4) in must have at least $\frac{1}{4}dn$ blue-neighbours in one of $J_{1,2} \setminus S$ and $J_{2,2}$. These three cycles form the desired partition.

A very similar argument handles the case that x_{red}^* exists and has $\frac{1}{4}dn$ red neighbours in $J_{1,2}$; under this condition Claim 8.4 allows us to choose C_{red} contained in $J_{1,2} \cup \{x_{\text{red}}^*\}$ covering x_{red}^* , and we then obtain the remaining two blue cycles as above verbatim.

Again, if x_{red}^* exists and has $\frac{1}{4}dn$ blue neighbours in $J_{1,2}$, we can perform a similar argument to the first. We obtain C_{red} contained in $J_{1,2}$ and a blue cycle on vertices $J_{1,1} \cup J_{2,1}$ exactly as there; we obtain a blue cycle on vertices $X^* \cup J_{2,2} \cup J_{1,2} \setminus V(C_{\text{red}})$ by following the same argument as above, except that we add x_{red}^* to A .

If x_{red}^* exists and has $\frac{1}{2}dn$ neighbours in $J_{1,2}$, at least half are red or half are blue and so one of these two cases occur.

We are left to consider the case that x_{red}^* exists but has fewer than $\frac{2}{d}n$ neighbours in $J_{1,2}$. Since x_{red}^* by definition has at least dn red neighbours in $J_{1,1} \cup J_{1,2}$, in this case x_{red}^* must have at least $\frac{1}{2}dn$ red neighbours in $J_{1,1}$.

Suppose first that $|J_{1,1}| \geq |J_{2,1}| + 2$. We apply Claim 8.4 to find a red cycle C_{red} whose vertices are in $J_{1,2}$, apart from two in $J_{1,1}$ and x_{red}^* , satisfying the properties listed there. We apply Claim 8.5, with $S = V(C_{\text{red}})$ and $A = B = \emptyset$, to find a blue cycle on vertices $J_{1,1} \cup J_{2,1} \setminus V(C_{\text{red}})$.

We now want to apply Claim 8.5 a second time, with $S = V(C_{\text{red}})$, to find a cycle on vertices $(X^* \setminus \{x_{\text{red}}^*\}) \cup J_{2,2} \cup J_{1,2} \setminus S$. To do this, we assign x_{blue}^* , if it exists, to one of A and B such that the condition of Claim 8.5 is satisfied; this is possible for the same reason as in the first case above. These three cycles form the desired partition.

If $|J_{1,1}| = |J_{2,1}|$, then we have the structure described in (ii).

Finally, suppose $|J_{1,1}| = |J_{2,1}| + 1$. If x_{blue}^* does not exist, or it exists and has less than $\frac{1}{2}dn$ blue neighbours in $J_{2,1}$, we have the structure of (iii). So we only need to show that if x_{blue}^* exists and has at least $\frac{1}{2}dn$ blue neighbours then we obtain the desired three-cycle partition. We do this as follows. We first use Claim 8.4 to find a red cycle C_{red} which contains x_{red}^* , two vertices of $J_{1,1}$, and vertices of $J_{1,2}$ such that $|J_{1,2} \setminus V(C_{\text{red}})| \leq |J_{2,2}| - 2$. We now apply Claim 8.5, with $A = B = \emptyset$, to find a blue cycle covering $(J_{1,2} \cup J_{2,2}) \setminus V(C_{\text{red}})$. We apply Claim 8.5 again, with $A = \{x_{\text{blue}}^*\}$ and $B = \emptyset$, to find a blue cycle covering $(A \cup J_{1,1} \cup J_{2,1}) \setminus V(C_{\text{red}})$; this second application is possible since $|A \cup J_{1,1} \setminus V(C_{\text{red}})| = |J_{1,1}| - 2 + 1 = |J_{2,1}|$. This gives the desired partition into three cycles. $\square$

As explained above, Claim 8.3 completes the proof. $\square$

9. CONCLUDING REMARKS

There are several natural questions arising from our result. First of all, we believe that the error term β in Theorem 1.1 can be removed, as conjectured by Pokrovskiy [34]. However, this will likely require a careful and technically difficult analysis of the underlying extremal cases.

One might also ask how many monochromatic cycles are necessary to partition a graph with minimum degree below $2n/3$. Korándi, Lang, Letzter and Pokrovskiy [21] showed that r -edge-coloured graphs of minimum degree $n/2 + \Omega(\log n)$ can be covered with at most $O(r^2)$ disjoint monochromatic cycles. This is essentially tight in terms of the degree condition, as with a minimum degree bound below $n/2 + O(\log n / \log \log n)$ there are colourings where the number of cycles can no longer be bounded in terms of r . On the other hand, it is not known up to which minimum degree one can always partition into four monochromatic cycles.

Problem 9.1. Determine the smallest $\delta > 0$ for which every 2-edge-coloured graph G on n vertices and $\delta(G) \geq \delta n$ can be partitioned into four monochromatic cycles.

An old theorem of Ore [32] states that every graph on n vertices contains a Hamilton cycle, provided the graph satisfies $\deg(x) + \deg(y) \geq n$ for every two non-adjacent vertices x and y . Barát and Sárközy [5] asked whether something similar is true for monochromatic cycle partitions. They conjectured that any red and blue edge-coloured graph G on n vertices can be partitioned into a red and a blue monochromatic cycle, provided that $\deg(x) + \deg(y) > 3n/2$ for every two non-adjacent vertices x and y . In support of their conjecture, they proved that under the stronger assumption $\deg(x) + \deg(y) > (3/2 + o(1))n$ almost all vertices can be covered. We believe that our techniques are suitable to provide a covering of the remaining vertices.

Problem 9.2. For $\beta > 0$ and sufficiently large n , prove that any red and blue edge-coloured graph G on n vertices can be partitioned into a red and a blue monochromatic cycle, provided that $\deg(x) + \deg(y) > (3/2 + \beta)n$ for every two non-adjacent vertices x and y .

Very recently, Arras [3] considered an approximate analogue of this problem for three cycles: He proved that if $\deg(x) + \deg(y) > 4/3n + o(n)$ for each pair of non-adjacent vertices x and y , then all but $o(n)$ vertices can be partitioned into three monochromatic cycles. Note that this is an (approximate) generalization of our main result.

It would also be interesting to determine what happens if we have more than 2 colours. That is, we would like to know which minimum degree conditions guarantee a partition of an r -edge-coloured graph into $f(r)$ monochromatic cycles, where $f(r)$ is some function. In this direction, Korándi, Lang, Letzter and Pokrovskiy [21] showed for any fixed $\delta > 1/2$ and growing r , there are families of n -vertex graphs of minimum degree δn that cannot be partitioned into fewer than $\Omega(r^2)$ monochromatic cycles. (In contrast to this, the complete graph can be partitioned into $O(r \log r)$ cycles [15].) However, it is not clear what happens for smaller r . For instance, for $r = 3$ it might be possible to show an analogue of our main theorem with one more colour and one more cycle.

Conjecture 9.3. Any 3-edge-coloured graph on n vertices and of minimum degree at least $3n/4$ can be partitioned into 4 cycles.

This would be tight, as the following easy example shows. For $m \in \mathbb{N}$, consider the graph on sets A, B, C, D , with $|A| = |C| = m + 2$, $|B| = m$, $|D| = m + 1$, having all edges except the ones inside B , inside D , and between A and C . Colour all edges inside A , inside C , and between B and D red, colour all edges between A and B , and between C and D green, and colour all edges between A and D , and between B and C blue. This graph has $n = 4m + 5$ vertices, minimum degree $3m + 2 = \lfloor 3n/4 \rfloor - 1$, and cannot be partitioned into less than four monochromatic cycles.

Pósa proved that a graph G with degree sequence $d_1 \leq \dots \leq d_n$ and $d_i > i$ for every $i < n/2$ contains Hamilton cycle. We believe that the following analogue of this result for monochromatic cycle partitions is true. Very recently, Arras [3] obtained an approximate solution for this in the case $r = 2$.

Conjecture 9.4. There is a function f with the following property. For $\beta > 0$, $r \geq 2$ and n sufficiently large, let G be a graph with degree sequence $d_1 \leq \dots \leq d_n$ such that $d_i \geq i + \beta n$ for every $i \leq n/2$. Then every r -edge-colouring of G admits a partition into at most $f(r)$ monochromatic cycles.

Finally, recent contributions in extremal combinatorics have been concerned with finding sufficient conditions for spanning substructures that are in some sense ‘closer’ to being necessary than assumptions on the minimum degree [8, 17, 18, 25]. It would be very interesting to determine whether similarly abstract properties can be identified for monochromatic cycle partitioning, although we do not have a concrete suggestions of how such properties may look like.

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